On the number of singular points of weak solutions to the Navier-Stokes equations

by

G. A. Seregin

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G. A. Seregin

V. A. Steklov Institute of Mathematics
at Saint Petersburg,
Fontanka 27, 191011 St.-Petersburg,
Russia
email: seregin@pdmi.ras.ru

Abstract We consider a suitable weak solution to the three-dimensional Navier-Stokes equations in the space-time cylinder $\Omega \times [0,T]$. Let $\Sigma$ be the set of singular points for this solution and $\Sigma(t) \equiv \{(x,t) \in \Sigma\}$. For a given open subset $\omega \subseteq \Omega$ and for a given moment of time $t \in ]0,T[$, we obtain an upper bound for the number of points of the set $\Sigma(t) \cap \omega$.

Key Words: Navier-Stokes equations, partial regularity, Hölder continuity, singular point.
1 Introduction

The present paper deals with weak solutions to the three-dimensional Navier-Stokes equations for viscous incompressible fluids

\[
\begin{align*}
\partial_t v + \text{div}(v \otimes v) - \Delta v &= f - \nabla p, \\
\text{div} v &= 0
\end{align*}
\]

(1.1)

in the space-time cylinder \( Q_T \equiv \Omega \times [0, T] \), where \( \Omega \) is a domain in \( \mathbb{R}^3 \), \( T \) is a given positive parameter, \( v \) is the velocity field, \( p \) is the pressure and \( f \) is a given external force. We are interested in differentiability properties of functions \( v \) and \( p \), assuming that:

\[
\begin{align*}
v &\in L_\infty(0, T; L_2(\Omega; \mathbb{R}^3)) \cap L_2(0, T; W^{1,2}_2(\Omega; \mathbb{R}^3)), \\
p &\in L^{\frac{3}{2}}(Q_T), \quad f \in L_2(Q_T; \mathbb{R}^3),
\end{align*}
\]

(1.2)

and the local energy inequality holds. Weak solutions of such class are called suitable weak solutions. They were studied in [7]-[9], [1], [5] and [4]. As far as the author knows, the first precise and explicit definition of suitable weak solutions appeared in [1]. However, changing the space for the pressure in an appropriate way, one can obtain other definitions of suitable weak solutions. We prefer the definition given in [5] (for discussions see [4]).

To show why the notion of suitable weak solutions is so important, let us recall two facts. At first, among of Hopf’s solutions to the initial-boundary value problem for (1.1) with homogeneous Dirichlet boundary conditions there is at least one suitable weak solution (see [1]). For the definition of Hopf’s solutions and historical remarks we refer the reader to monographs [2] and [3]. At second, every suitable weak solution possesses so-called partial regularity (see [1], and also [5] and [4]). Namely, let \( \Sigma \) be the set of singular points of a suitable weak solution, then the one-dimensional parabolic Hausdorff measure of \( \Sigma \) is equal to zero. As in [4], we say that a point of space-time cylinder \( Q_T \) is regular if the velocity field \( v \) is Hölder continuous in some neighborhood of this point. A point of \( Q_T \) is called singular if it is not regular.

The aim of our paper is to estimate the number of points in the set

\[
\Sigma(t_0) \cap \omega
\]
for any open subset $\omega \subseteq \Omega$ and for any moment of time $t_0 \in ]0, T[$. Here

$$\Sigma(t_0) \equiv \{(x, t_0) \in \Sigma\}.$$ 

## 2 Notation and the Main Result

We denote by $\mathbb{M}^3$ the space of all real $3 \times 3$ matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we shall use the following notation

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3, \quad v = (v_i) \in \mathbb{R}^3;$$

$$A : B = \text{tr} A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$

$$A^* = (A_{ji}), \quad \text{tr} A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3, \quad B = (B_{ij}) \in \mathbb{M}^3;$$

$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3, \quad A \in \mathbb{M}^3.$$

Let $\omega$ be a domain in some finite-dimensional space. We denote by $L^m(\omega; \mathbb{R}^n)$ and $W^m_p(\omega; \mathbb{R}^n)$ the known Lebesgue and Sobolev spaces of functions from $\omega$ into $\mathbb{R}^n$.

For summable in $Q_T = \Omega \times ]0, T[$ scalar-valued, vector-valued and tensor-valued functions, we shall use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_i), \quad \nabla u = (u_{i,j}),$$

$$\text{div} v = v_{,i}, \quad \text{div} \tau = (\tau_{ij}), \quad \Delta u = \text{div} \nabla u,$$

which are understood in the sense of distributions. Here $x_i, i = 1, 2, 3,$ are Cartesian coordinates of a point $x \in \mathbb{R}^3,$ and $t \in ]0, T[$ is a moment of time. Space-time points are denoted by $z = (x, t), \ z_0 = (x_0, t_0)$ and etc.

For balls and parabolic cylinders, we shall use the notation

$$B(x_0, R) \equiv \{x \in \mathbb{R}^3 \mid |x - x_0| < R\},$$

$$Q(z_0, R) \equiv B(x_0, R) \times ]t_0 - R^2, t_0[.$$ 

We are going to use a "parabolic" variant of Morrey’s spaces. Given domain $\omega$ in $\mathbb{R}^3 \times \mathbb{R}$ and positive number $\gamma$, we define the space

$$M_{2,\gamma}(\omega; \mathbb{R}^3) \equiv \{f \in L^2_{\text{loc}}(\omega; \mathbb{R}^3) \mid d_\gamma(f; \omega) < +\infty\}.$$
Here
\[ d_\gamma(f; \omega) \equiv \sup \left\{ \frac{1}{R^{\gamma+\delta}} \left( \int_{Q(z, R)} |f|^2 \, dz' \right)^{\frac{1}{2}} \mid Q(z, R) \subseteq \omega, \, R > 0 \right\} \]

**Definition 2.1** Let \( \Omega \) be a domain in \( \mathbb{R}^3 \) and \( T \) be a positive parameter. Suppose that a function \( f \) satisfies the condition
\[ f \in M_{2, \gamma}(Q_T; \mathbb{R}^3) \]  
for some positive \( \gamma \). We say that a pair of functions \( v \) and \( p \) is a suitable weak solution to the Navier-Stokes equations in \( Q_T \) if \( v \) and \( p \) satisfy conditions (1.2) and meet equations (1.1) in the sense of distributions, and the inequality
\[
\begin{align*}
\int_{\Omega} |v(x, t)|^2 \phi(x, t) \, dx + 2 \int_{\Omega \times [0, t]} |\nabla v|^2 \phi \, dx \, dt' &\leq \\
\int_{\Omega \times [0, t]} \left\{ |v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p) v \cdot \nabla \phi + 2 f \cdot v \phi \right\} \, dx \, dt' 
\end{align*}
\]  
holds for a. a. \( t \in [0, T] \) and for all non-negative functions \( \phi \in C_0^\infty(Q_T) \).

Our aim is to prove the following fact.

**Theorem 2.2** Let \( \gamma \) be an arbitrary positive constant. Let \( \{\Omega, T, f, v, p\} \) be an arbitrary collection, satisfying Definition 2.1 with this constant \( \gamma \). There is a positive number \( \varepsilon_0 \), depending only on \( \gamma \), with the following property. For any open subset \( \omega \subseteq \Omega \) and for any moment of time \( t_0 \in [0, T] \), the inequality
\[ N(t_0, \omega) \leq \varepsilon_0(\gamma) \limsup_{R \to 0} \frac{1}{R^2} \int_{t_0 - R^2}^{t_0} dt \int_{\omega} |v(x, t)|^3 \, dx \]  
holds. Here \( N(t_0, \omega) = \text{card}\{\Sigma(t_0) \cap \omega\} \), i.e. the number of points in the set \( \Sigma(t_0) \cap \omega \).

We would like to mention interesting paper [6], containing some estimate in the spirit of (2.3). The author of [6] considered any Hopf’s solution \( v \) to the initial-boundary value problem for the Navier-Stokes equations with
homogeneous Dirichlet boundary conditions under the additional assumption \( v \in L_\infty(0, T; L_3(\Omega; \mathbb{R}^3)) \). His upper bound for \( \text{card}\{\Sigma(t_0)\} \) is proportional to

\[
\|v\|_{L_\infty(0,T;L_3(\Omega;\mathbb{R}^3))}^3.
\]

But, as it was shown in [11], from the assumption \( v \in L_\infty(0, T; L_3(\Omega; \mathbb{R}^3)) \) it follows that, for any Hopf’s solution \( v \) to the initial-boundary value problem mentioned above, one can define the associated pressure \( p \) so that the pair of functions \( v \) and \( p \) is a suitable weak solution to (1.1). Therefore, (2.3) implies the bound obtained in [6]. Moreover, even in this particular case our estimate is slightly better since

\[
\lim sup_{R \to 0} \frac{1}{R^2} \int_{t_0}^{t_0} dt \int_\Omega |v(x, t)|^3 dx \leq \text{ess sup}_{0 \leq t \leq t_0} \int_\Omega |v(x, t)|^3 dx.
\]

In what follows we shall denote by \( c_1, \ c_2, \) and etc all positive absolute constants, and by \( \varepsilon_1, \varepsilon_2, \) and etc all positive constants depending on \( \gamma \) only.

### 3 The Main Lemma

**Lemma 3.1** Assume that

\( f \in M_{2,\gamma}(Q_T) \) \hspace{1cm} (3.1)

for some \( \gamma > -1 \). Let functions \( v \in L_3(Q_T; \mathbb{R}^3) \) and \( p \in L_\frac{3}{\gamma}(Q_T) \) satisfy equations (1.1) in \( Q_T \) in the sense of distributions. Suppose that

\( Q(z_0, \rho) \subset Q_T. \)

Then the following estimate

\[
D(z_0, r; p) \leq c_1 \left[ \frac{r}{\rho} D(z_0, \rho; p) + \left( \frac{\rho}{r} \right)^2 \left( C(z_0, \rho; v) + d_1 \rho^{3(\gamma+1)} \right) \right] \quad (3.2)
\]

holds for all \( r \in [0, \rho] \). Here \( d_1 \equiv d_1(f; Q_T) \) and

\[
C(z_0, R; v) \equiv \frac{1}{R^2} \int_{Q(z_0, R)} |v|^3 dz, \quad D(z_0, R; p) \equiv \frac{1}{R^2} \int_{Q(z_0, R)} |p|^{\frac{3}{\gamma}} dz,
\]
Proof. We use arguments of [10] and [4] (see Lemma 5.3).

By assumptions of the lemma,
\[
\int_{Q(z_0, \rho)} \left( -v \cdot \partial_t w - (v \otimes v) : \nabla w - v \cdot \Delta w \right) dz = \int_{Q(z_0, \rho)} \left( f \cdot w + p \text{div} w \right) dz
\]

for all \( w \in C_0^\infty(Q(z_0, \rho); \mathbb{R}^3) \). For any \( \chi \in C_0^\infty(t_0 - \rho^2, t_0) \) and \( q \in C_0^\infty(B(x_0, \rho)) \), we substitute \( \chi \nabla q \) for \( w \) in (3.3). As a result, we obtain
\[
- \int_{Q(z_0, \rho)} \chi p \Delta q dz = \int_{Q(z_0, \rho)} \chi \left( f \cdot \nabla q + (v \otimes v) : \nabla^2 q \right) dz.
\]

By the arbitrariness of \( \chi \), for a.a. \( t \in [t_0 - \rho^2, t_0] \), we have the identity
\[
- \int_{B(x_0, \rho)} p(x, t) \Delta q(x) dx = \int_{B(x_0, \rho)} \left( f(x, t) \cdot \nabla q(x) \right) dx + (v(x, t) \otimes v(x, t)) : \nabla^2 q(x) dx.
\]

for all \( q \in C_0^\infty(B(x_0, \rho)) \).

Let us define the function
\[
p_1 \in L_2^2(Q(z_0, \rho))
\]

in the following way. For a.a. \( t \in [t_0 - \rho^2, t_0] \), it satisfies the identity
\[
- \int_{B(x_0, \rho)} p_1(x, t) \Delta q(x) dx = \int_{B(x_0, \rho)} \left( f(x, t) \cdot \nabla q(x) \right) dx + (v(x, t) \otimes v(x, t)) : \nabla^2 q(x) dx
\]

for all \( q \in W_2^2(B(x_0, \rho)) \) such that \( q = 0 \) on \( \partial B(x_0, \rho) \). The existence of \( p_1 \), satisfying (3.5) and (3.6), can be proved with the help of a priori estimate for
\[
\| p_1(\cdot, t) \|_{L_2^2(B(x_0, \rho))}
\]
and suitable approximations for \( v(\cdot, t) \) and \( f(\cdot, t) \). To obtain a priori estimate, we solve, for a.a. \( t \in [t_0 - \rho^2, t_0] \), the following boundary value problem: to find the function

\[
q_0(\cdot, t) \in W^2_{\infty}(B(x_0, \rho))
\]
such that

\[
\Delta q_0(\cdot, t) = -|p_1(\cdot, t)|^{\frac{2}{3}} \text{sign} \{p_1(\cdot, t)\} \quad \text{in} \quad B(x_0, \rho),
\]

\[
q_0(\cdot, t) = 0 \quad \text{on} \quad \partial B(x_0, \rho).
\]

This problem is uniquely solvable. Moreover, for its solution the estimate

\[
\left( \int_{B(x_0, \rho)} |\nabla^2 q_0(\cdot, t)|^3 \, dx \right)^{\frac{1}{3}} + \frac{1}{\rho} \left( \int_{B(x_0, \rho)} |\nabla q_0(\cdot, t)|^3 \, dx \right)^{\frac{1}{3}}
\]

\[
\leq c_2 \left( \int_{B(x_0, \rho)} |p_1(\cdot, t)|^\frac{3}{2} \, dx \right)^{\frac{1}{3}}, \quad t \in [t_0 - \rho^2, t_0],
\]

is valid. From identity (3.6) for \( q(\cdot) = q_0(\cdot, t) \) it follows that

\[
\left( \int_{B(x_0, \rho)} |p_1(\cdot, t)|^\frac{3}{2} \, dx \right)^{\frac{1}{3}}
\]

\[
\leq c_3 \left[ \left( \int_{B(x_0, \rho)} |v(\cdot, t)|^3 \, dx \right)^{\frac{2}{3}} + \rho \left( \int_{B(x_0, \rho)} |f(\cdot, t)|^3 \, dx \right)^{\frac{2}{3}} \right]
\]

\[
\leq c_3' \left[ \left( \int_{B(x_0, \rho)} |v(\cdot, t)|^3 \, dx \right)^{\frac{2}{3}} + \rho^2 \left( \int_{B(x_0, \rho)} |f(\cdot, t)|^2 \, dx \right)^{\frac{2}{3}} \right].
\]

After integration in \( t \) over the interval \( [t_0 - \rho^2, t_0] \), we arrive at the bound

\[
\int_{Q(z_0, \rho)} |p_1|^{\frac{3}{2}} \, d\tau \leq c_4 \left[ \int_{Q(z_0, \rho)} |v|^3 \, d\tau \right]
\]

\[
+ \rho^2 \int_{t_0 - \rho^2}^{t_0} dt \left( \int_{B(x_0, t)} |f(x, t)|^2 \, dx \right)^{\frac{2}{3}} \right] \right) \quad (3.7)
\]

\[
\leq c_4' \rho^2 \left\{ C(z_0, \rho; v) + d^2 \rho^3 (\gamma + 1) \right\},
\]
According to (3.4) and (3.6), for a.a. $t \in [t_0 - \rho^2, t_0]$, the function 

$$p_2 = p - p_1$$

is harmonic in $B(x_0, \rho)$, i.e.

$$\Delta p_2(\cdot, t) = 0 \quad \text{in} \quad B(x_0, \rho).$$

We therefore have

$$\frac{1}{r^3} \int_{B(x_0, r)} |p_2(\cdot, t)|^3 dx \leq c_5 \frac{1}{\rho^3} \int_{B(x_0, \rho)} |p_2(\cdot, t)|^3 dx$$

and after integration in $t$ we obtain

$$\frac{1}{r^3} \int_{Q(x_0, r)} |p_2|^3 dz \leq c_5 \frac{1}{\rho^3} \int_{Q(x_0, \rho)} |p_2|^3 dz. \quad (3.8)$$

On the other hand, by (3.7),

$$\int_{Q(x_0, \rho)} |p_2|^3 dz \leq c_6 \rho^2 \left[ D(z_0, \rho; p) + C(z_0, \rho; v) + d_7 \rho^{\frac{3}{2}(\gamma+1)} \right]. \quad (3.9)$$

Now, we have (see (3.7)-(3.9))

$$D(z_0, r; p) \leq c_7 \left[ \frac{1}{r^3} \int_{Q(x_0, r)} |p_1|^3 dz + \frac{1}{r^2} \int_{Q(x_0, r)} |p_2|^3 dz \right]$$

$$\leq c_7 \left[ \frac{1}{r^3} \int_{Q(x_0, r)} |p_1|^3 dz + \frac{r}{\rho} \frac{1}{r^2} \int_{Q(x_0, r)} |p_2|^3 dz \right]$$

$$\leq c_7 \left[ \frac{r}{\rho} D(z_0, \rho; p) + \left( \frac{2}{r} \right)^2 \left( C(z_0, \rho; v) + d_7 \rho^{\frac{3}{2}(\gamma+1)} \right) \right].$$

Lemma 3.1 is proved.

**Corollary 3.2** Assume that all conditions of Lemma 3.1 hold. Let

$$Q(z_0, R) \subset Q_T$$

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and a number $\theta \in [0,1[$ is chosen so that
\[ c_1 \theta \leq \frac{1}{2}. \]  
(3.10)

Then, for any $k = 1,2,\ldots$, we have
\[ D(z_0, \theta^k R; p) \leq \frac{1}{2k} D(z_0, R; p) + \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \Phi(z_0, \theta^i R; v), \]  
(3.11)

where
\[ \Phi(z_0, \rho; v) \equiv C(z_0, \rho; v) + d^3 \rho^{\frac{3}{2} + 1}. \]

Indeed, we can use Lemma 2.1 for $r = \theta^{s+1} R$ and $\rho = \theta^s R$ and obtain
\[ D(z_0, \theta^{s+1} R; p) \leq \frac{1}{2} D(z_0, \theta^s R; p) + \frac{c_1}{\theta^2} \Phi(z_0, \theta^s R; v) \]
for all $s = 0,1,\ldots$. Iterating the latter inequality with respect to $s$, we establish (3.11).

4 Proof of Theorem 2.2

So, we assume that all conditions of the theorem hold.

We take an arbitrary point $z_0 \in Q_T$. It was proved in [4] (see Proposition 2.8) that there is a positive number $\varepsilon_0(\gamma)$ with the following property. If
\[ \liminf_{R \to 0} \left[ \left( \frac{3}{4\pi} C(z_0, R; v) \right)^{\frac{2}{3}} + \left( \frac{3}{4\pi} D(z_0, R; p) \right)^{\frac{2}{3}} \right] < \varepsilon_0(\gamma), \]  
(4.1)

then $z_0$ is a regular point, i.e. the function $z \mapsto v(z)$ is Hölder continuous in some neighborhood of $z_0$. But this immediately implies the following important statement.

Proposition 4.1 Let $\{\Omega, T, f, v, p\}$ be an arbitrary collection, satisfying Definition 2.1 with given positive constant $\gamma$. There is a positive number $\varepsilon_1(\gamma)$ with the following property. If $z_0 \in Q_T$ is a singular point of $v$, then there is a positive number $R_0$ such that
\[ C(z_0, R; v) + D(z_0, R; p) \geq \varepsilon_1(\gamma) \]  
(4.2)
for all $R \in [0, R_0]$. 

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Proof. Sufficient condition (4.1) allows us to conclude that if $z_0$ is a singular point, then there is a positive number $R_0$ such that

$$
\left( \frac{3}{4\pi} C(z_0, R; v) \right)^{\frac{1}{2}} + \left( \frac{3}{4\pi} D(z_0, R; p) \right)^{\frac{1}{2}} \geq \frac{1}{2} \varepsilon_0(\gamma)
$$

for all $R \in ]0, R_0[$. Therefore,

$$
\left( \frac{3}{4\pi} C(z_0, R; v) \right)^{\frac{1}{2}} + \frac{3}{4\pi} D(z_0, R; p) \geq \frac{1}{4} [\varepsilon_0(\gamma)]^{\frac{3}{2}}
$$

and, by Young’s inequality,

$$
\frac{3}{4\pi} \left( \varepsilon_0(\gamma) \right)^{\frac{3}{2}} C(z_0, R; v) + \frac{3}{4\pi} D(z_0, R; p) \geq \frac{1}{8} \left( \varepsilon_0(\gamma) \right)^{\frac{3}{2}}
$$

for all $R \in ]0, R_0[$. It remains to take

$$
\varepsilon_1(\gamma) \equiv \frac{\pi}{6} [\varepsilon_0(\gamma)]^{\frac{3}{2}} \min\{\frac{1}{2} [\varepsilon_0(\gamma)]^{\frac{3}{2}}, 1\}.
$$

Proposition 4.1 is proved.

Without loss of generality it can be assumed that

$$
A \equiv \limsup_{R \to 0} \frac{1}{R^2} \int_{t_0 - R^2}^{t_0} dt \int_{\omega} |v(x, t)|^3 dx < +\infty.
$$

Let us take any finite subset $\sigma$ of $\Sigma(t_0) \cap \omega$. We let $M = \text{card}\{\sigma\} < +\infty$. Theorem 2.2 is proved if we show that

$$
M \leq \varepsilon_0(\gamma) A. \quad (4.3)
$$

So, we have

$$
\sigma \equiv \{z_l\}_{l=1}^{M} \equiv \{(x_l, t_0)\}_{l=1}^{M} \subseteq \Sigma(t_0) \cap \omega.
$$

By Proposition 4.1, for each $l = 1, 2, ..., M$, there is a number $R_{q_l} > 0$ such that

$$
C(z_l, R; v) + D(z_l, R; p) \geq \varepsilon_1(\gamma) \quad (4.4)
$$

for all $R \in ]0, R_{q_l}[$.
Since \( \omega \) is an open set, one can choose a positive number \( R_+ > 0 \) so that
\[
\bigcup_{l=1}^M B(x_l, R_+) \subseteq \omega, \tag{4.5}
\]
and
\[
B(x_l, R_+) \cap B(x_m, R_+) = \phi \tag{4.6}
\]
for all \( l, m = 1, 2, ..., M \) such that \( l \neq m \). If we let
\[
R_* \equiv \frac{1}{2} \min \{ R_+, R_{01}, ..., R_{0M} \},
\]
then from (4.4) we obtain
\[
C(z_l, R; v) + D(z_l, R; p) \geq \varepsilon_1(\gamma) \tag{4.7}
\]
for all \( R \in [0, R_*] \) and for all \( l = 1, 2, ..., M \).

Now, we are going to use Corollary 3.2 and inequality (4.7) for \( R = \theta^k R_* \). We therefore have (see (3.11))
\[
\begin{align*}
C(z_l, \theta^k R_*; v) &+ \frac{1}{2 \kappa} D(z_l, R_*; p) \\
&+ \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \Phi(z_l, \theta^i R_*; v)
\end{align*}
\]
\[
\geq C(z_l, \theta^k R_*; v) + D(z_l, \theta^k R_*; p) \geq \varepsilon_1(\gamma)
\]
for all \( l = 1, 2, ..., M \) and and for all \( k = 1, 2, ..., M \).

Summing the latter inequalities with respect to \( l \) and taking into account (4.5) and (4.6), we arrive at the estimate
\[
M \varepsilon_1(\gamma) \leq \sum_{l=1}^M C(z_l, \theta^k R_*; v) + \frac{1}{2 \kappa} \sum_{l=1}^M D(z_l, R_*; p)
\]
\[
+ \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \sum_{l=1}^M \Phi(z_l, \theta^i R_*; v)
\]
\[
\leq \Psi(t_0, \theta^k R_*; v) + \frac{1}{2 \kappa R_*^2} \int_{t_0 - R_*^2}^{t_0} dt \int_{\omega} |p(x, t)|^2 dx
\]
+ \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-i}} \left[ \Psi(t_0, \theta^i R_* ; v) + M d_{7,1}^2 (\theta^i R_*)^{\frac{3}{2} (1+\gamma)} \right],

where

\Psi(t_0, \rho; v) \equiv \frac{1}{\rho^2} \int_{t_0}^{t_0} \int_{|x| \leq \rho} |v(x, t)|^2 \, dx.

Passing to the limit as \( k \to +\infty \), we obtain

\begin{align*}
M \varepsilon_1 (\gamma) &\leq A \\
&\quad + \frac{c_1}{\theta^2} \lim sup_{k \to +\infty} \frac{1}{2^k} \sum_{i=0}^{k} 2^i \left[ \Psi(t_0, \theta^i R_* ; v) + M d_{7,1}^2 (\theta^i R_*)^{\frac{3}{2} (1+\gamma)} \right].
\end{align*}

It is easy to show that

\begin{align*}
\lim sup_{k \to +\infty} \frac{1}{2^k} \sum_{i=0}^{k} 2^i \left[ \Psi(t_0, \theta^i R_* ; v) + M d_{7,1}^2 (\theta^i R_*)^{\frac{3}{2} (1+\gamma)} \right] &\leq \lim sup_{k \to +\infty} \left[ \Psi(t_0, \theta^k R_* ; v) + M d_{7,1}^2 (\theta^k R_*)^{\frac{3}{2} (1+\gamma)} \right] \\
&\leq A.
\end{align*}

Now, from the latter inequality we deduce that

\begin{align*}
M \varepsilon_1 (\gamma) &\leq A + \frac{c_1}{\theta^2} A.
\end{align*}

So, it remains to let

\begin{align*}
\varepsilon_0 (\gamma) = \frac{1}{\varepsilon_1 (\gamma)} (1 + \frac{c_1}{\theta^2}).
\end{align*}

Theorem 2.2 is proved.

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References


