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Abstract We consider a suitable weak solution to the three-dimensional Navier-Stokes equations in the space-time cylinder $\Omega\times]0,T[$. Let Σ be the set of singular points for this solution and $\Sigma(t)\equiv\{(x,t)\in\Sigma\}$. For a given open subset $\omega\subseteq\Omega$ and for a given moment of time $t\in]0,T[$, we obtain an upper bound for the number of points of the set $\Sigma(t)\cap\omega$.

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1 Introduction

The present paper deals with weak solutions to the three-dimensional Navier-Stokes equations for viscous incompressible fluids

$$\left.\begin{aligned} \partial_t v + \operatorname{div}(v\otimes v) - \Delta v = f - \nabla p, \\ \operatorname{div} v = 0 \end{aligned}\right\} \tag{1.1}$$

in the space-time cylinder $Q_T \equiv \Omega\times]0,T[$, where Ω is a domain in $\mathbb{R}^3$, T is a given positive parameter, v is the velocity field, p is the pressure and f is a given external force. We are interested in differentiability properties of functions v and p, assuming that:

$$\left.\begin{aligned} v\in L_\infty(0,T;L_2(\Omega;\mathbb{R}^3))\cap L_2(0,T;W^1_2(\Omega;\mathbb{R}^3)), \\ p\in L_{\frac32}(Q_T),\quad f\in L_2(Q_T;\mathbb{R}^3), \end{aligned}\right\} \tag{1.2}$$

and the local energy inequility holds. Weak solutions of such class are called suitable weak solutions. They were studied in [7]-[9], [1], [5] and [4]. As far as the author knows, the first precise and explicit definition of suitable weak solutions appeared in [1]. However, changing the space for the pressure in an approriate way, one can obtain other definitions of suitable weak solutions. We prefer the definition given in [5] (for discussions see [4]).

To show why the notion of suitable weak solutions is so important, let us recall two facts. At first, among of Hopf's solutions to the initial-boundary value problem for (1.1) with homogeneous Dirichlet boundary conditions there is at least one suitable weak solution (see [1]). For the definition of Hopf's solutions and historical remarks we refer the reader to monographs [2] and [3]. At second, every suitable weak solution possesses so-called partial regularity (see [1], and also [5] and [4]). Namely, let Σ be the set of singular points of a suitable weak solution, then the one-dimensional parabolic Hausdorff measure of Σ is equal to zero. As in [4], we say that a point of space-time cylinder Q_T is regular if the velocity field v is Hölder continuous in some neighborhood of this point. A point of Q_T is called singular if it is not regular.

The aim of our paper is to estimate the number of points in the set

$$\Sigma(t_0)\cap\omega$$

for any open subset $\omega \subseteq \Omega$ and for any moment of time $t_0 \in]0, T[$. Here

$$\Sigma(t_0) \equiv \{(x, t_0) \in \Sigma\}.$$

2 Notation and the Main Result

We denote by $\mathbb{M}^3$ the space of all real 3×3 matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we shall use the following notation

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3,\ v = (v_i) \in \mathbb{R}^3;$$

$$A : B = \mathrm{tr} A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$

$$A^* = (A_{ji}), \quad \mathrm{tr} A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3,\ B = (B_{ij}) \in \mathbb{M}^3;$$

$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3,\ A \in \mathbb{M}^3.$$

Let ω be a domain in some finite-dimensional space. We denote by $L_m(\omega; \mathbb{R}^n)$ and $W^l_m(\omega; \mathbb{R}^n)$ the known Lebesgue and Sobolev spaces of functions from ω into $\mathbb{R}^n$.

For summable in $Q_T = \Omega \times]0, T[$ scalar-valued, vector-valued and tensor-valued functions, we shall use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_{,i}), \quad \nabla\, u = (u_{i,j}),$$

$$\mathrm{div}\, v = v_{i,i}, \quad \mathrm{div}\, \tau = (\tau_{ij,j}), \quad \Delta\, u = \mathrm{div}\, \nabla\, u,$$

which are understood in the sense of distributions. Here x_i, $i = 1, 2, 3$, are Cartesian coordinates of a point $x \in \mathbb{R}^3$, and $t \in]0, T[$ is a moment of time. Space-time points are denoted by $z = (x, t)$, $z_0 = (x_0, t_0)$ and etc.

For balls and parabolic cylinders, we shall use the notation

$$B(x_0, R) \equiv \{x \in \mathbb{R}^3 \ \|\ |x - x_0| < R\},$$

$$Q(z_0, R) \equiv B(x_0, R) \times]t_0 - R^2, t_0[.$$

We are going to use a "parabolic" variant of Morrey's spaces. Given domain ω in $\mathbb{R}^3 \times \mathbb{R}$ and positive number γ, we define the space

$$M_{2,\gamma}(\omega; \mathbb{R}^3) \equiv \{f \in L_{2,\mathrm{loc}}(\omega; \mathbb{R}^3) \ \|\ d_\gamma(f; \omega) < +\infty\}.$$

Here

$$d_\gamma(f;\omega) \equiv \sup\{\frac{1}{R^{\gamma+\frac{1}{2}}}\Big(\int\limits_{Q(z,R)} |f|^2\, dz'\Big)^{\frac{1}{2}} \parallel Q(z,R) \Subset \omega,\ R>0\}.$$

Definition 2.1 *Let Ω be a domain in $\mathbb{R}^3$ and T be a positive parameter. Suppose that a function f satisfies the condition*

$$f \in M_{2,\gamma}(Q_T;\mathbb{R}^3) \tag{2.1}$$

for some positive γ. We say that a pair of functions v and p is a suitable weak solution to the Navier-Stokes equations in Q_T if v and p satisfy conditions (1.2) and meet equations (1.1) in the sense of distributions, and the inequality

$$\left.\begin{array}{c} \int\limits_{\Omega} |v(x,t)|^2\phi(x,t)\,dx + 2\int\limits_{\Omega\times]0,t[} |\nabla v|^2\phi\,dx\,dt' \leq \\ \leq \int\limits_{\Omega\times]0,t[} \{|v|^2(\partial_t\phi+\Delta\,\phi) + (|v|^2+2p)v\cdot\nabla\,\phi + 2f\cdot v\phi\}\,dx\,dt' \end{array}\right\} \tag{2.2}$$

holds for a. a. $t\in[0,T]$ and for all non-negative functions $\phi \in C_0^\infty(Q_T)$.

Our aim is to prove the following fact.

Theorem 2.2 *Let γ be an arbitrary positive constant. Let $\{\Omega, T, f, v, p\}$ be an arbitrary collection, satisfying Definition 2.1 with this constant γ. There is a positive number ε_0, depending only on γ, with the following property. For any open subset $\omega \subseteq \Omega$ and for any moment of time $t_0 \in]0,T[$, the inequality*

$$N(t_0,\omega) \leq \varepsilon_0(\gamma)\ \limsup_{R\to 0} \frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0} dt \int\limits_{\omega} |v(x,t)|^3 dx \tag{2.3}$$

holds. Here $N(t_0,\omega) = \mathrm{card}\{\Sigma(t_0)\cap\omega\}$, i.e. the number of points in the set $\Sigma(t_0)\cap\omega$.

We would like to mention interesting paper [6], containing some estimate in the spirit of (2.3). The author of [6] considered any Hopf's solution v to the initial-boundary value problem for the Navier-Stokes equations with homogeneous Dirichlet boundary conditions under the additional assumption $v \in L_\infty(0,T;L_3(\Omega;\mathbb{R}^3))$. His upper bound for card$\{\Sigma(t_0)\}$ is proportional to

$$\|v\|^3_{L_\infty(0,T;L_3(\Omega;\mathbb{R}^3))}.$$

But, as it was shown in [11], from the assumption $v \in L_\infty(0,T;L_3(\Omega;\mathbb{R}^3))$ it follows that, for any Hopf's solution v to the initial-boundary value problem mentioned above, one can define the associated pressure p so that the pair of functions v and p is a suitable weak solution to (1.1). Therefore, (2.3) implies the bound obtained in [6]. Moreover, even in this particular case our estimate is slightly better since

$$\limsup_{R\to 0} \frac{1}{R^2} \int\limits_{t_0-R^2}^{t_0} dt \int\limits_\Omega |v(x,t)|^3 dx \leq \operatorname{ess}\sup_{0\leq t\leq t_0} \int\limits_\Omega |v(x,t)|^3 dx.$$

In what follows we shall denote by c_1, c_2, and etc all positve absolute constants, and by ε_1, ε_2, and etc all positve constants depending on γ only.

3 The Main Lemma

Lemma 3.1 *Assume that*

$$f \in M_{2,\gamma}(Q_T) \tag{3.1}$$

for some $\gamma > -1$. *Let functions* $v \in L_3(Q_T;\mathbb{R}^3)$ *and* $p \in L_{\frac{3}{2}}(Q_T)$ *satisfy equations (1.1) in* Q_T *in the sense of distributions. Suppose that*

$$Q(z_0,\rho) \subset Q_T.$$

Then the following estimate

$$D(z_0,r;p) \leq c_1 \Big[\frac{r}{\rho} D(z_0,\rho;p) + \Big(\frac{\rho}{r}\Big)^2 \Big(C(z_0,\rho;v) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)}\Big)\Big] \tag{3.2}$$

holds for all $r \in]0,\rho]$. *Here* $d_\gamma \equiv d_\gamma(f;Q_T)$ *and*

$$C(z_0,R;v) \equiv \frac{1}{R^2} \int\limits_{Q(z_0,R)} |v|^3 dz, \quad D(z_0,R;p) \equiv \frac{1}{R^2} \int\limits_{Q(z_0,R)} |p|^{\frac{3}{2}} dz,$$

Proof. We use arguments of [10] and [4] (see Lemma 5.3).

By assumptions of the lemma,

$$\left.\begin{aligned} &\int\limits_{Q(z_0,\rho)} \Big(-v\cdot\partial_t w-(v\otimes v):\nabla w-v\cdot\Delta w\Big)dz \\ &\qquad=\int\limits_{Q(z_0,\rho)}\Big(f\cdot w+p\,\mathrm{div}\,w\Big)dz \end{aligned}\right\}\tag{3.3}$$

for all $w\in C_0^\infty(Q(z_0,\rho);\mathbb{R}^3)$. For any $\chi\in C_0^\infty(t_0-\rho^2,t_0)$ and $q\in C_0^\infty(B(x_0,\rho))$, we substitute $\chi\,\nabla q$ for w in (3.3). As a result, we obtain

$$-\int\limits_{Q(z_0,\rho)}\chi\,p\,\Delta q\,dz=\int\limits_{Q(z_0,\rho)}\chi\,\Big(f\cdot\nabla q+(v\otimes v):\nabla^2 q\Big)\,dz.$$

By the arbitrariness of χ, for a.a. $t\in[t_0-\rho^2,t_0]$, we have the identity

$$\left.\begin{aligned} -\int\limits_{B(x_0,\rho)} p(x,t)\Delta q(x)\,dx &=\int\limits_{B(x_0,\rho)}\Big(f(x,t)\cdot\nabla q(x) \\ &+(v(x,t)\otimes v(x,t)):\nabla^2 q(x)\Big)\,dx \end{aligned}\right\}\tag{3.4}$$

for all $q\in C_0^\infty(B(x_0,\rho))$.

Let us define the function

$$p_1\in L_{\frac32}(Q(z_0,\rho))\tag{3.5}$$

in the following way. For a.a. $t\in[t_0-\rho^2,t_0]$, it satisfies the identity

$$\left.\begin{aligned} -\int\limits_{B(x_0,\rho)} p_1(x,t)\Delta q(x)\,dx &=\int\limits_{B(x_0,\rho)}\Big(f(x,t)\cdot\nabla q(x) \\ &+(v(x,t)\otimes v(x,t)):\nabla^2 q(x)\Big)\,dx \end{aligned}\right\}\tag{3.6}$$

for all $q\in W^2_3(B(x_0,\rho))$ such that $q=0$ on $\partial B(x_0,\rho)$. The existence of p_1, satisfying (3.5) and (3.6), can be proved with the help of a priori estimate for

$$\|\,p_1(\cdot,t)\|_{L_{\frac32}(B(x_0,\rho))}$$

and suitable approximations for $v(\cdot,t)$ and $f(\cdot,t)$. To obtain a priori estimate, we solve, for a.a. $t\in[t_0-\rho^2,t_0]$, the following boundary value problem: to find the function

$$q_0(\cdot,t)\in W^2_3(B(x_0,\rho))$$

such that

$$\Delta\, q_0(\cdot,t)=-|p_1(\cdot,t)|^{\frac12}\mathrm{sign}\,\{p_1(\cdot,t)\}\quad\text{in}\quad B(x_0,\rho),$$

$$q_0(\cdot,t)=0\quad\text{on}\quad \partial B(x_0,\rho).$$

This problem is uniquely solvable. Moreover, for its solution the estimate

$$\Big(\int\limits_{B(x_0,\rho)}|\nabla^2\, q_0(\cdot,t)|^3\,dx\Big)^{\frac13}+\frac1\rho\Big(\int\limits_{B(x_0,\rho)}|\nabla\, q_0(\cdot,t)|^3\,dx\Big)^{\frac13}$$

$$\le c_2\Big(\int\limits_{B(x_0,\rho)}|p_1(\cdot,t)|^{\frac32}\,dx\Big)^{\frac13},\qquad t\in[t_0-\rho^2,t_0],$$

is valid. From identity (3.6) for $q(\cdot)=q_0(\cdot,t)$ it follows that

$$\Big(\int\limits_{B(x_0,\rho)}|p_1(\cdot,t)|^{\frac32}\,dx\Big)^{\frac23}$$

$$\le c_3\Big[\Big(\int\limits_{B(x_0,\rho)}|v(\cdot,t)|^3\,dx\Big)^{\frac23}+\rho\Big(\int\limits_{B(x_0,\rho)}|f(\cdot,t)|^{\frac32}\,dx\Big)^{\frac23}\Big]$$

$$\le c_3'\Big[\Big(\int\limits_{B(x_0,\rho)}|v(\cdot,t)|^3\,dx\Big)^{\frac23}+\rho^{\frac32}\Big(\int\limits_{B(x_0,\rho)}|f(\cdot,t)|^2\,dx\Big)^{\frac12}\Big].$$

After integration in t over the interval $]t_0-\rho^2,t_0[$, we arrive at the bound

$$\left.\begin{aligned}&\int\limits_{Q(z_0,\rho)}|p_1|^{\frac32}\,dz\le c_4\Big[\int\limits_{Q(z_0,\rho)}|v|^3\,dz\\&+\rho^{\frac94}\int\limits_{t_0-\rho^2}^{t_0}dt\Big(\int\limits_{B(x_0,\rho)}|f(x,t)|^2\,dx\Big)^{\frac34}\Big]\\&\le c_4'\rho^2\Big\{C(z_0,\rho;v)+d_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big\}.\end{aligned}\right\}\qquad(3.7)$$

According to (3.4) and (3.6), for a.a. $t \in [t_0 - \rho^2, t_0]$, the function

$$p_2 = p - p_1$$

is harmonic in $B(x_0, \rho)$, i.e.

$$\Delta p_2(\cdot, t) = 0 \qquad \text{in} \qquad B(x_0, \rho).$$

We therefore have

$$\frac{1}{r^3} \int\limits_{B(x_0,r)} |p_2(\cdot,t)|^{\frac{3}{2}}\, dx \le c_5 \frac{1}{\rho^3} \int\limits_{B(x_0,\rho)} |p_2(\cdot,t)|^{\frac{3}{2}}\, dx$$

and after integration in t we obtain

$$\frac{1}{r^3} \int\limits_{Q(z_0,r)} |p_2|^{\frac{3}{2}}\, dz \le c_5 \frac{1}{\rho^3} \int\limits_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}}\, dz. \tag{3.8}$$

On the other hand, by (3.7),

$$\int\limits_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}}\, dz \le c_6 \rho^2 \Big[D(z_0, \rho; p) + C(z_0, \rho; v) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \Big]. \tag{3.9}$$

Now, we have (see (3.7)-(3.9))

$$D(z_0, r; p) \le c_7 \Big[\frac{1}{r^2} \int\limits_{Q(z_0,r)} |p_1|^{\frac{3}{2}}\, dz + \frac{1}{r^2} \int\limits_{Q(z_0,r)} |p_2|^{\frac{3}{2}}\, dz \Big]$$

$$\le c_7 \Big[\frac{1}{r^2} \int\limits_{Q(z_0,r)} |p_1|^{\frac{3}{2}}\, dz + \frac{r}{\rho} c_5 \frac{1}{\rho^2} \int\limits_{Q(z_0,r)} |p_2|^{\frac{3}{2}}\, dz \Big]$$

$$\le c_7' \Big[\frac{r}{\rho} D(z_0, \rho; p) + \Big(\frac{\rho}{r}\Big)^2 \Big(C(z_0, \rho; v) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \Big) \Big].$$

Lemma 3.1 is proved.

Corollary 3.2 *Assume that all conditions of Lemma 3.1 hold. Let*

$$Q(z_0, R) \subset Q_T$$

and a number $\theta \in]0,1[$ *is chosen so that*

$$c_1\theta \leq \frac{1}{2}. \tag{3.10}$$

Then, for any $k = 1, 2, ...,$ *we have*

$$D(z_0, \theta^k R; p) \leq \frac{1}{2^k} D(z_0, R; p) + \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \Phi(z_0, \theta^i R; v), \tag{3.11}$$

where

$$\Phi(z_0, \rho; v) \equiv C(z_0, \rho; v) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)}$$

Indeed, we can use Lemma 2.1 for $r = \theta^{s+1}R$ and $\rho = \theta^s R$ and obtain

$$D(z_0, \theta^{s+1}R; p) \leq \frac{1}{2} D(z_0, \theta^s R; p) + \frac{c_1}{\theta^2} \Phi(z_0, \theta^s R; v)$$

for all $s = 0, 1, ...$ Iterating the latter inequality with respect to s, we establish (3.11).

4 Proof of Theorem 2.2

So, we assume that all conditions of the theorem hold.

We take an arbitraty point $z_0 \in Q_T$. It was proved in [4] (see Proposition 2.8) that there is a positive number $\bar{\varepsilon}_0(\gamma)$ with the following property. If

$$\liminf_{R \to 0} \Big[\Big(\frac{3}{4\pi} C(z_0, R; v) \Big)^{\frac{1}{3}} + \Big(\frac{3}{4\pi} D(z_0, R; p) \Big)^{\frac{2}{3}} \Big] < \bar{\varepsilon}_0(\gamma), \tag{4.1}$$

then z_0 is a regular point, i.e. the function $z \mapsto v(z)$ is Hölder continous in some neighborhood of z_0. But this immediately implies the following important statement.

Proposition 4.1 *Let* $\{\Omega, T, f, v, p\}$ *be an arbitrary collection, satisfying Definition 2.1 with given positive constant* γ. *There is a positive number* $\varepsilon_1(\gamma)$ *with the following property. If* $z_0 \in Q_T$ *is a singular point of* v, *then there is a positive number* R_0 *such that*

$$C(z_0, R; v) + D(z_0, R; p) \geq \varepsilon_1(\gamma) \tag{4.2}$$

for all $R \in]0, R_0[$.

Proof. Sufficient condition (4.1) allows us to conclude that if z_0 is a singular point, then there is a positive number R_0 such that

$$\Big(\frac{3}{4\pi}C(z_0,R;v)\Big)^{\frac{1}{3}}+\Big(\frac{3}{4\pi}D(z_0,R;p)\Big)^{\frac{2}{3}}\geq\frac{1}{2}\bar{\varepsilon}_0(\gamma)$$

for all $R\in]0,R_0[$. Therefore,

$$\Big(\frac{3}{4\pi}C(z_0,R;v)\Big)^{\frac{1}{2}}+\frac{3}{4\pi}D(z_0,R;p)\geq\frac{1}{4}[\bar{\varepsilon}_0(\gamma)]^{\frac{3}{2}}$$

and, by Young's inequality,

$$\frac{3}{4\pi}\frac{2}{[\bar{\varepsilon}_0(\gamma)]^{\frac{3}{2}}}C(z_0,R;v)+\frac{3}{4\pi}D(z_0,R;p)\geq\frac{1}{8}[\bar{\varepsilon}_0(\gamma)]^{\frac{3}{2}}$$

for all $R\in]0,R_0[$. It remains to take

$$\varepsilon_1(\gamma)\equiv\frac{\pi}{6}[\bar{\varepsilon}_0(\gamma)]^{\frac{3}{2}}\min\{\frac{1}{2}[\bar{\varepsilon}_0(\gamma)]^{\frac{3}{2}},1\}.$$

Proposition 4.1 is proved.

Without loss of generality it can be assumed that

$$A\equiv\limsup_{R\to 0}\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}dt\int\limits_{\omega}|v(x,t)|^3dx<+\infty.$$

Let us take any finite subset σ of $\Sigma(t_0)\cap\omega$. We let $M=\mathrm{card}\{\sigma\}<+\infty$. Theorem 2.2 is proved if we show that

$$M\leq\varepsilon_0(\gamma)\,A. \tag{4.3}$$

So, we have

$$\sigma\equiv\{z_l\}_{l=1}^M\equiv\{(x_l,t_0)\}_{l=1}^M\subseteq\Sigma(t_0)\cap\omega.$$

By Proposition 4.1, for each $l=1,2,...,M$, there is a number $R_{0l}>0$ such that

$$C(z_l,R;v)+D(z_l,R;p)\geq\varepsilon_1(\gamma) \tag{4.4}$$

for all $R\in]0,R_{0l}[$.

Since ω is an open set, one can choose a positive number $R_+ > 0$ so that

$$\cup_{l=1}^{M} B(x_l, R_+) \subseteq \omega, \tag{4.5}$$

and

$$B(x_l, R_+) \cap B(x_m, R_+) = \phi \tag{4.6}$$

for all $l, m = 1, 2, ..., M$ such that $l \neq m$. If we let

$$R_\star \equiv \frac{1}{2} \min\{R_+, R_{01}, ..., R_{0M}\},$$

then from (4.4) we obtain

$$C(z_l, R; v) + D(z_l, R; p) \geq \varepsilon_1(\gamma) \tag{4.7}$$

for all $R \in]0, R_\star[$ and for all $l = 1, 2, ..., M$.

Now, we are going to use Corollary 3.2 and inequality (4.7) for $R = \theta^k R_\star$. We therefore have (see (3.11))

$$C(z_l, \theta^k R_\star; v) + \frac{1}{2^k} D(z_l, R_\star; p)$$

$$+ \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \Phi(z_l, \theta^i R_\star; v)$$

$$\geq C(z_l, \theta^k R_\star; v) + D(z_l, \theta^k R_\star; p) \geq \varepsilon_1(\gamma)$$

for all $l = 1, 2, ..., M$ and and for all $k = 1, 2, ...$

Summing the latter inequalities with respect to l and taking into account (4.5) and (4.6), we arrive at the estimate

$$M\,\varepsilon_1(\gamma) \leq \sum_{l=1}^{M} C(z_l, \theta^k R_\star; v) + \frac{1}{2^k} \sum_{l=1}^{M} D(z_l, R_\star; p)$$

$$+ \frac{c_1}{\theta^2} \sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \sum_{l=1}^{M} \Phi(z_l, \theta^i R_\star; v)$$

$$\leq \Psi(t_0, \theta^k R_\star; v) + \frac{1}{2^k\, R_\star^2} \int\limits_{t_0 - R_\star^2}^{t_0} dt \int\limits_{\omega} |p(x,t)|^{\frac{3}{2}} dx$$

$$+\frac{c_1}{\theta^2}\sum_{i=0}^{k-1}\frac{1}{2^{k-1-i}}\Big[\Psi(t_0,\theta^i R_\star;v)+M\,d_\gamma^{\frac{3}{2}}(\theta^i R_\star)^{\frac{3}{2}(1+\gamma)}\Big],$$

where

$$\Psi(t_0,\rho;v)\equiv\frac{1}{\rho^2}\int\limits_{t_0-\rho^2}^{t_0}dt\int\limits_{\omega}|v(x,t)|^3dx.$$

Passing to the limit as $k\to+\infty$, we obtain

$$M\,\varepsilon_1(\gamma)\le A$$

$$+\frac{c_1}{\theta^2}\lim\sup_{k\to+\infty}\frac{1}{2^k}\sum_{i=0}^{k}2^i\Big[\Psi(t_0,\theta^i R_\star;v)+M\,d_\gamma^{\frac{3}{2}}(\theta^i R_\star)^{\frac{3}{2}(1+\gamma)}\Big].$$

It is easy to show that

$$\lim\sup_{k\to+\infty}\frac{1}{2^k}\sum_{i=0}^{k}2^i\Big[\Psi(t_0,\theta^i R_\star;v)+M\,d_\gamma^{\frac{3}{2}}(\theta^i R_\star)^{\frac{3}{2}(1+\gamma)}\Big]$$

$$\le\lim\sup_{k\to+\infty}\Big[\Psi(t_0,\theta^k R_\star;v)+M\,d_\gamma^{\frac{3}{2}}(\theta^k R_\star)^{\frac{3}{2}(1+\gamma)}\Big]$$

$$\le A.$$

Now, from the latter inequality we deduce that

$$M\,\varepsilon_1(\gamma)\le A+\frac{c_1}{\theta^2}A.$$

So, it remains to let

$$\varepsilon_0(\gamma)=\frac{1}{\varepsilon_1(\gamma)}(1+\frac{c_1}{\theta^2}).$$

Theorem 2.2 is proved.

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