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EXOTIC SPHERES AND THE SMOOTH RIGIDITY OF THE MAXIMAL DIAMETER SPHERE THEOREM FOR MANIFOLDS OF POSITIVE RICCI CURVATURE

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Abstract. The smooth diameter sphere theorem presented in this note shows that it is possible to isolate the standard sphere among all other complete Riemannian manifolds with positive Ricci curvature by using merely curvature and diameter assumptions, and that in fact any violation of smooth rigidity in Cheng's maximal diameter theorem must be accompanied by a blow-up of sectional curvatures:
For any given m and C there exists a positive constant $\epsilon = \epsilon(m, C) > 0$ such that any m-dimensional complete Riemannian manifold with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and diameter $\geq \pi - \epsilon$ is diffeomorphic to the standard m-sphere.

Introduction

Let M be a closed smooth m-dimensional manifold. Toponogov's maximal diameter sphere theorem, dating back to 1959, states that if M admits a Riemannian metric with sectional curvature $K \geq 1$ and diameter equal to π, then M is isometric to the unit m-sphere (see [To]). In 1977 Grove and Shiohama showed that Toponogov's theorem is topologically rigid by proving that if M carries a Riemannian metric with sectional curvature $K \geq 1$ and diameter $> \pi/2$, then M must be homeomorphic to the m-sphere (see [GS]). Since until now not a single example of an exotic sphere with positive sectional curvature everywhere has been discovered, one may moreover hope that Toponogov's theorem is even smoothly rigid, i.e., that the conclusion of Grove's and Shiohama's result can actually once be strengthened to a diffeomorphism statement.

There exists also a Ricci curvature analogue of Toponogov's theorem. It is Cheng's 1975 maximal diameter sphere theorem for manifolds of positive Ricci curvature, stating that a complete Riemannian m-manifold with Ricci curvature $Ricc \geq m-1$ and diameter equal to π is isometric to the unit m-sphere (see [Chg]).

However, the situation that one faces in positive Ricci curvature differs very strongly from the sectional curvature setting described above:

First of all, it is known that Cheng's maximal diameter sphere theorem is not topologically rigid: In every dimension $m \geq 4$ Anderson as well as Otsu (see [An1], [O1]) constructed on closed smooth m-manifolds whose homotopy type is distinct from that of the sphere Riemannian metrics with $Ricc \geq m-1$ and diameter arbitrarily close to π.

Secondly, as follows for example from work of Nash, Poor, Cheeger, Hérnandez, and Wraith (see [Nash], [Poor], [H], [Ch], [Wr]), in contrast to the sectional curvature case many exotic spheres are known to carry metrics of positive Ricci curvature. In dimension 4 the situation is especially intricate: It is till now unknown whether there exist exotic four-spheres, but if there existed only one such manifold, then by taking repeated connected sums one could produce an infinite number of pairwise non-diffeomorphic exotic four-spheres, and a priori all these manifolds could in addition also admit metrics with positive Ricci curvature.

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Thus additional assumptions are needed to obtain sphere theorems for positively Ricci curved manifolds, and because of the existence of positively Ricci curved exotic spheres it is of particular importance to find here conditions which go beyond topological rigidity and which yield differentiably rigid results.

The study of this field of questions was initiated by a 1983 paper of Shiohama (see [Shi1]), and has since then been pursued by many other authors. Contributions have here been made in particular by Anderson, Bessa, Cai, Cheeger-Colding, Colding, Eschenburg, Grove-Petersen, Itokawa, Katsuda, Nakamura, Otsu, Paeng, Perelman, Petersen, Petersen-Zhu, Wilhelm, Xia, and Yamaguchi (see [An2], [Bes], [Cai], [CC], [Co], [E], [GrP], [It], [Kat], [Na], [Ot], [Pa1], [Pa2], [Per], [Pet], [PZ], [Wi], [Xia], [Yam1]; compare also the survey [Shi2] and the references therein).

Most of these results, in particular all results on differentiable rigidity, use besides various curvature assumptions in addition conditions on further geometric invariants like, for example, volume, radius, injectivity radius, conjugate radius, or excess.

The only exception to this use of conditons other than curvature and diameter is Perelman's 1995 topological diameter sphere theorem (see [Per], and compare also [Wi]). It states that for any given m and C there exists a positive constant $\epsilon = \epsilon(m, C) > 0$ such that any m-dimensional complete Riemannian manifold with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \geq C$ and diameter $\geq \pi - \epsilon$ is a twisted sphere (and thus in particular homeomorphic to the standard m-sphere).

Perelman's result raises the question whether one can also isolate the standard sphere among all other topological spheres or, more generally, among all other complete Riemannian manifolds with positive Ricci curvature by using merely assumptions on curvature and diameter. The following differentiable diameter sphere theorem shows that this is indeed possible:

1.1 Theorem *For any given m and C there exists a positive constant $\epsilon = \epsilon(m, C) > 0$ such that any m-dimensional complete Riemannian manifold with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and diameter $\geq \pi - \epsilon$ is diffeomorphic to the standard m-sphere.*

In the above-mentioned examples of Anderson and Otsu, as the diameter of the metrics approach π, the upper (as well as the lower) sectional curvature bounds diverge.

The theorem of this note shows that this is not simply a coincidence, because it implies that that any violation of smooth rigidity, in particular any which is modelled on an exotic sphere, must necessarily be accompanied by a blow-up of sectional curvatures:

1.2 Corollary *Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of complete Riemannian m-manifolds with Ricci curvature $Ricc \geq m-1$ whose diameters converge to π as n goes to infinity. Then either all but finitely many M_n are diffeomorphic to the standard m-sphere, or $\limsup_{n\to\infty} K_{M_n} = +\infty$.*

Note that by results of Croke (see [Cr]) Theorem 1.1 is equivalent to the following eigenvalue sphere theorem for the first eigenvalue λ_1 of the Laplacian:

1.3 Theorem *For any given m and C there exists a positive constant $\epsilon' = \epsilon'(m, C) > 0$ such that any m-dimensional complete Riemannian manifold M with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and $\lambda_1(M) \leq m + \epsilon'$ is diffeomorphic to the standard m-sphere.*

The proof of Theorem 1.1 starts from Perelman's result and relies on the following injectivity radius estimate for positively Ricci pinched manifolds by Petrunin and the author (see [PT]):

1.4 Theorem ([PT]) *For each m and any $0 < \delta \leq 1$ there exists a positive constant $i_0(m, \delta) > 0$ such that the injectivity radius i_g of any Riemannian metric g with Ricci curvature $Ricc \geq (m-1)\delta$ and sectional curvature $K \leq 1$ on a simply connected closed m-dimensional manifold with finite second homotopy group is bounded from below by $i_g \geq i_0(m, \delta)$.*

2. Proof of Theorem 1.1

Suppose that the theorem is false. Then there exists for certain m and C an infinite sequence of complete m-dimensional Riemannian manifolds (M_n, g_n) with Ricci curvature $Ricc_{g_n} \geq m-1$, sectional curvature $K_{g_n} \leq C$ and diameter $diam(g_n) \geq \pi - \varepsilon_n$, where the numbers ε_n converge to zero as n tends to infinity.

Since an upper bound on sectional curvature and a lower bound on Ricci curvature imply a lower bound for sectional curvature, Perelman's topological diameter sphere theorem is applicable to this sequence and one can assume without loss of generality that all manifolds M_n are twisted spheres. Since these are in particular two-connected, the injectivity radius estimate of [PT] (see Theorem 1.4 above) can now be invoked. Thus the sequence is precompact in the Lipschitz distance (compare [G]) and contains (which is in dimension 4 not obvious, see above) in particular only finitely many diffeomorphism types.

Thus after passing to a subsequence if necessary one may assume that all manifolds M_n have fixed diffeomorphism type, different from that of the standard m-sphere, and that they converge with respect to the Lipschitz distance (see [G]) for $n \to \infty$ to a smooth manifold M, diffeomorphic to M_n, equipped with a Riemann metric g of Hölder class $C^{1,\alpha}$ and with diameter equal to π.

The Bishop-Gromov relative volume comparison theorem (see [G]) directly extends to bounded curvature Lipschitz limits of manifolds satisfying a fixed lower Ricci curvature bound (in fact much more general extensions are available, compare [CC] and [Yam2]).

Thus, in our situation, the limit (M, g) satisfies volume comparison with respect to the unit m-sphere $S^m(1)$ of constant sectional curvature 1, and by the equality case in the volume comparison theorem M will be isometric to $S^m(1)$ as soon as one can show that the volume of M equals the volume of the unit m-sphere. This can be done by using Shiohama's reasoning (compare [Shi1]):

Choose two points x and y in M which realize the diameter, i.e., which are at distance π apart, and consider the closed balls of radius $\pi/2$ and radius π, $B_{\pi/2}(x)$, $B_\pi(x)$, and $B_{\pi/2}(y)$, $B_\pi(y)$ with centers x and y.

The interiors of $B_{\pi/2}(x)$ and $B_{\pi/2}(y)$ are disjoint, since otherwise one would arrive at a contradiction to the triangle inequality.

Let $B(\pi/2)$ and B denote the balls of radius $\pi/2$ and radius π in the unit m-sphere of constant sectional curvature 1. Now by volume comparison

$$volM \geq volB_{\pi/2}(x) + volB_{\pi/2}(y) \geq volB_\pi(x) \cdot B(\pi/2)/volB + volB_\pi(y) \cdot B(\pi/2)/volB = volM.$$

Thus the volume of M is equal to that of the unit m-sphere, so that it follows that M is isometric to $S^m(1)$. Thus in particular M and all M_n are diffeomorphic to the standard m-sphere. This yiels a contradiction, and the proof of Theorem 1.1 is complete. □

2.1 Remark Once that one knows that the volume of M is close to that of the unit sphere, one can also invoke for instance Yamaguchi's volume pinching theorem (see [Yam1]) to obtain diffeomorphism to the standard sphere.

2.2 Remark The question remains whether it is possible to improve Perelman's original result to a diffeomorphism statement without making use of an upper sectional curvature bound.

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