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Abstract

In this paper we study the nonlinear equation of elastodynamics where the free energy functional is allowed to be nonconvex. We define the notion of Young measure solutions for this problem and prove an existence theorem in this class. This can be used as a model for the evolution of microstructures in crystals.
We furthermore introduce an optional coupling with a parabolic equation and prove existence of a Young measure solution for this system.

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1 Introduction

A crucial assumption to obtain existence of weak solutions for nonlinear elasticity equations in the static case is the polyconvexity of the underlying free energy potential (see [Bal77]). However in many cases the polyconvexity of the potential is not appropriate to reflect the physical situation. Therefore a weaker concept for solutions has been introduced, the so-called Young measure solutions (YM-solutions). This concept can be applied to crystals where nonconvex elasticity equations can be used to describe the development of microstructures (which are important especially for shape memory alloys) as has been pointed out in the fundamental paper [BJ87]. (For further information and references consider e.g. [Ped97], [Mül99].)
The equation of elastodynamics

$$u_{tt}(x,t) - \text{div}\, S(\nabla u(x,t)) = 0$$

is even more difficult to handle. Global existence results for weak solutions have been found only in one space dimension in [DiP83], [She94]. Under certain convexity assumptions Dafermos [Daf85] proved local existence of smooth solutions; for global solutions for small initial data consider [Rac92].
The concept of YM-solutions has been applied to dynamic problems in [Sle91], [KP92] and [Dem96] (in the context of the forward-backward heat equation) and was applied to the wave equation by [MNRR96] and [Dem97].
A first approach to the dynamic elasticity equation (with some additional assumptions on the free energy, valid in particular for antiplane sheer) was presented in [Rie00] using the method of discretization in time. A similiar result was obtained in [DST], where (in a different context) existence was proved in arbitrary space dimensions for the *polyconvex* case under some growth conditions.
In the first part of this article we prove the existence (globally in time, for large initial data) of Young measure solutions for nonconvex elasticity equations in arbitrary space dimensions under some growth conditions on the free energy. In contrast to [DST] we have to assume that the Andrews-Ball condition (see below) is satisfied, but we do *not* need polyconvexity.
In the second part we study a model problem where we couple a nonconvex elasticity equation with a parabolic equation (possibly of forward-backward type). The physical motivation is to study crystals consisting of different types of atoms, where solid state diffusion occurs and influences the elastic properties of the material. The mathematical structure is also similiar to thermoelastic problems. We extend the concept of YM-solutions to this hyperbolic-parabolic system and prove existence.

2 YM-solutions for an elasticity equation

In this section we prove the existence of Young measure solutions for nonconvex elasticity equations. Let $p \geq 2$ be a fixed constant. (Later p will denote the growth rate of the free energy at infinity.) By p' we denote its conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.
Throughout this article we denote by M a positive generic constant depending only on the initial data. By $||\cdot||$ we denote the $L^2(\Omega)$-norm.
For an open bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $T > 0$, $g \in W^{1,p}(\Omega, \mathbb{R}^m)$ and a function $u : \Omega \to \mathbb{R}^m$ we study the following initial boundary value problem:

$$
\begin{aligned}
u_{tt}(x,t) - \operatorname{div} S(\nabla u(x,t)) &= 0, \quad (x,t) \in \Omega \times [0,T) \\
u(\cdot,0) &= u_0, \\
u_t(\cdot,0) &= z_0, \\
u &= g \quad \text{on}\, \partial\Omega,
\end{aligned} \tag{1}
$$

with $S = \nabla\phi$ and $\phi \in \mathcal{C}^2(\mathbb{R}^{m\times n}, \mathbb{R}_+)$ satisfying the following growth conditions (for positive constants M_1, M_2):

$$
\begin{aligned}
|S(A)| &\le M_2(|A|^{p-1}+1), \\
M_1(|A|^p - 1) \le \quad \phi(A) &\le M_2(|A|^p + 1),
\end{aligned} \tag{2}
$$

and S satisfying the Andrews-Ball condition for some $R > 0$:

$$
(S(F_1) - S(F_2))\,(F_1 - F_2) \ge 0 \qquad \text{for all} \quad F_1 \in \mathbb{R}^{m\times n}, F_2 \in \mathbb{R}^{m\times n}, \quad |F_1|, |F_2| \ge R. \tag{3}
$$

An interpretation of this condition is that for "large" values the potential ϕ is assumed to be convex. We can relax this condition slightly: It is sufficient to assume that there exists a constant $M > 0$ such that for all F_1, $F_2 \in \mathbb{R}^{m\times n}$:

$$
(S(F_1) - S(F_2))\,(F_1 - F_2) \quad \ge \quad -M|F_1 - F_2|^2. \tag{4}
$$

We now want to define what we will call a Young measure solution. Therefore we introduce a measure ν expressing the probability distribution of the deformation gradient at a certain point $(x,t) \in \Omega \times (0,T)$. For "classical" solutions this measure will be a Dirac measure concentrated in ∇u.

Definition 2.1 (YM-solutions for elasticity) *A pair (u,ν) is a* Young measure solution *of the system (1) if for fixed $T > 0$:*

$$
\begin{aligned}
u \in W^{1,\infty}((0,T), L^2(\Omega)), \quad u - g \in L^\infty((0,T), W_0^{1,p}(\Omega)), \\
\nu = (\nu_{x,t})_{x,t} \text{ is a probability measure,} \\
\int_0^T \int_\Omega \langle \nu, S(\cdot)\rangle \nabla\zeta - u_t\zeta_t \,dx\,dt = 0 \quad \forall \zeta \in \mathcal{C}_0^\infty((0,T)\times\Omega), \\
\nabla u(x,t) = \langle \nu_{x,t}, Id\rangle \quad a.e.
\end{aligned}
$$

Here $\langle \nu, S(\cdot)\rangle$ is defined as dual pairing of S with the measure ν, i.e.: $\langle \nu, S(\cdot)\rangle := \int S(A)\,d\nu(A)$.

In this section we prove the following existence theorem:

Theorem 2.2 (Existence of YM–solutions) *Assume $\phi \in \mathcal{C}^2$, that the growth conditions (2) are satisfied, and that one of the conditions (3) or (4) are valid. Furthermore let $u_0 - g \in W_0^{1,p}(\Omega)$, $z_0 \in H_0^1(\Omega)$. Then there exists a Young measure solution (u,ν) of problem (1).*

To prove this we use a viscosity regularization, based on an idea of [ŠN]. Under the assumptions stated above the following viscoelastic equation (together with the standard initial and boundary conditions) has a weak solution (see [FD97] or consider [Dem] for more general viscosity terms):

$$
u^\varepsilon_{tt}(x,t) - \operatorname{div} S(\nabla u^\varepsilon(x,t)) - \varepsilon\Delta u^\varepsilon_t(x,t) = 0.
$$

More precisely there exists

$$u^\varepsilon \in W^{2,2}((0,T),W^{-1,p'}(\Omega)) \cap W^{1,2}((0,T),W^{1,2}(\Omega)) \cap W^{1,\infty}((0,T),L^2(\Omega)),$$
$$u^\varepsilon - g \in L^\infty((0,T),W_0^{1,p}(\Omega)),$$

such that for all $T>0$ and for all $\zeta \in \mathcal{C}_0^\infty((0,T)\times\Omega)$):

$$\int_0^T \int_\Omega (S(\nabla u^\varepsilon) + \varepsilon \nabla u_t^\varepsilon)\nabla\zeta - u_t^\varepsilon \zeta_t \, dx\, dt \;=\; 0. \tag{5}$$

Furthermore we have the inequality:

$$\frac{1}{2}||u_t^\varepsilon||^2 + ||\nabla u^\varepsilon||^p_{L^p(\Omega)} + \int_0^T ||\sqrt{\varepsilon}\nabla u_t^\varepsilon||^2\, dt \le M,$$

where $M>0$ is independent of ε and t. To get this estimate we can follow [FD97], where we simply add an ε to the viscosity term. Additionally we use the growth condition on ϕ.
These bounds on u^ε imply that there exists a subsequence, again denoted by u^ε with:

$$u^\varepsilon \overset{\star}{\rightharpoonup} u, \quad \text{in } L^\infty((0,T),W^{1,p}(\Omega)) \cap W^{1,\infty}((0,T),L^2(\Omega)),$$

and $(\nabla u^\varepsilon(\cdot,t))_\varepsilon$ generates for every fixed $t\in(0,T)$ a Young measure $\nu_{\cdot,t}$.
Now we claim, that (u,ν) is a Young measure solution of the elasticity equation. To prove this we consider the convergence of the terms in the viscoelastic equation (taking subsequences, if neccessary). First we observe that by the convergence proved above and the Hölder inequality:

$$u_t^\varepsilon \overset{\star}{\rightharpoonup} u_t \quad in \quad L^2((0,T),L^2(\Omega))).$$

Thus $\int_0^T\int_\Omega u_t^\varepsilon\zeta_t\,dx\,dt$ (the third term in the weak equation (5)) converges to $\int_0^T\int_\Omega u_t\zeta_t\,dx\,dt$. On the other hand $\int_0^T\int_\Omega \varepsilon\nabla u_t^\varepsilon\nabla\zeta\,dx\,dt$ converges for $\varepsilon\to 0$ to zero as the following calculation (using the Cauchy-Schwarz inequality) proves:

$$\int_0^T\int_\Omega \varepsilon\nabla u_t^\varepsilon\nabla\zeta\,dx\,dt \;\le\; \Bigg(\varepsilon \underbrace{\int_0^T ||\sqrt{\varepsilon}\nabla u_t^\varepsilon||^2\,dt}_{\le M}\Bigg)^{1/2} \Bigg(\underbrace{\int_0^T ||\nabla\zeta||^2\,dt}_{=const.}\Bigg)^{1/2} \to 0.$$

It remains to consider the term $\int_0^T\int_\Omega S(\nabla u^\varepsilon)\nabla\zeta\,dx\,dt$. If we define $\nu_{\cdot,t}$ for all $t\in(0,T)$ as the gradient Young measure generated by the sequence $\nabla u^\varepsilon(\cdot,t)$ (for a definition and an existence proof consider e.g. [KP94], [Mül99] or [Ped97]), we can see that $S(\nabla u^\varepsilon(\cdot,t))$ converges for all $t\in(0,T)$ weakly in $L^{p-1}(\Omega)$ to $\langle S,\nu_{\cdot,t}\rangle$.
On the other hand a subsequence of $S(\nabla u^\varepsilon)$ converges weakly-$\star$ in $L^\infty((0,T),L^{p'}(\Omega))$, since the bounds from the energy estimate together with the growth condition imply:

$$\begin{aligned}
\sup_t ||S(\nabla u^\varepsilon)||^{p'}_{L^{p'}(\Omega)} &\le M\sup_t \int_\Omega (1+|\nabla u^\varepsilon|^{p-1})^{p'}\,dx \\
&\le M\sup_t \left(1+\int_\Omega |\nabla u^\varepsilon|^{(p-1)p'}\right)\,dx \\
&= M\sup_t \left(1+||\nabla u^\varepsilon||^p_{L^p(\Omega)}\right)\,dx \\
&\le M.
\end{aligned}$$

Hence the term $S(\nabla u^\varepsilon)$ converges weakly-$\star$ in $L^\infty((0,T),L^{p'}(\Omega))$ to $\langle S,\nu\rangle$, and since $\nabla\zeta\in\mathcal{C}_0^\infty((0,T)\times\Omega)\subset L^1((0,T),L^p(\Omega))$ we have derived that (u,ν) is a Young measure solution of the elasticity equation, proving Theorem 2.2. □

3 Hyperbolic-parabolic systems

If we want to consider a coupling between elasticity and diffusion or if we want to study thermoelastic problems, we have to couple a parabolic equation (possibly of forward-backward type) to the elasticity equation. For this purpose we study the following model problem, where $\Omega \subset \mathbb{R}^n$ is a domain with Lipschitz boundary, $T > 0$, $(x,t) \in \Omega \times [0,T)$, $g \in H^1(\Omega, \mathbb{R}^m)$, $u : \Omega \times [0,T) \to \mathbb{R}^m$ and $c : \Omega \times [0,T) \to \mathbb{R}^d$:

$$\begin{aligned} u_{tt}(x,t) - \operatorname{div} S(\nabla u(x,t), c(x,t)) &= 0, \\ c_t(x,t) - \operatorname{div} K(\nabla c(x,t), u(x,t)) &= 0, \\ u(\cdot,0) &= u_0, \\ u_t(\cdot,0) &= z_0, \\ c(\cdot,0) &= c_0, \\ u = g \quad &\text{on } \partial\Omega, \\ \vec{n}K(\nabla c, u) = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{6}$$

with $S = \nabla_1 \phi$ and $K = \nabla_1 \psi$ (∇_1 denoting the derivative with respect to the first variable). By $\vec{n}$ we denote the outward normal on $\partial\Omega$.
To make things easier we only consider the case $p = 2$, i.e. we assume that S and K are of linear growth in the first variable and ϕ, $\psi \in \mathcal{C}^2$ are positive and of quadratic growth in the first variable. More precisely there are constants M_1, $M_2 > 0$, such that for all $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times d}$, $b \in \mathbb{R}^d$ and $a \in \mathbb{R}^m$ the following estimates hold:

$$\begin{aligned} M_1(|A|^2 - 1) \leq \quad & \phi(A,b) & \leq M_2(|A|^2 + |b|^2 + 1), \\ M_1(|B|^2 - 1) \leq \quad & \psi(B,a) & \leq M_2(|B|^2 + |a|^2 + 1), \\ & S(A,b) & \leq M_2(|A| + |b| + 1), \\ & K(B,a) & \leq M_2(|B| + |a| + 1). \end{aligned}$$

Furthermore we assume that S and K are globally Lipschitz continuous.
We want to remark, that (6) is only a model problem for studying some typical mathematical difficulties. A realistic model for diffusion phenomena should include at least a ∇u-dependence of the diffusion tensor K rather than a u-dependence.
We extend the notion of YM-solutions to the coupled system, where the measure ν describes the probability distribution of the gradient of u (in the same way as in the last section) and the measure μ describes the probability distribution of the gradient of c:

Definition 3.1 (YM-solutions for an hyperbolic-parabolic system) *We call the quadruple (u,ν,c,μ) a* Young measure solution *of the system (6) if for $T > 0$:*

$$\begin{aligned} u \in W^{1,\infty}((0,T), L^2(\Omega)), \quad u - g \in L^\infty((0,T), H_0^1(\Omega)), \\ c \in W^{1,2}((0,T), L^2(\Omega)) \cap L^\infty((0,T), H^1(\Omega)), \\ \nu = (\nu_{x,t})_{x,t},\ \mu = (\mu_{x,t})_{x,t},\ \textit{probability measures}, \\ \int_0^T \int_\Omega \langle \nu, S(\cdot,c)\rangle \nabla\zeta - u_t\zeta_t \, dx\, dt = 0 \quad \forall \zeta \in H_0^1((0,T)\times\Omega), \\ \int_0^T \int_\Omega \langle \mu, K(\cdot,u)\rangle \nabla\zeta + c_t\zeta \, dx\, dt = 0 \quad \forall \zeta \in H_0^1((0,T)\times\Omega), \\ \nabla u(x,t) = \langle \nu_{x,t}, Id\rangle \quad \textit{a.e.}, \\ \nabla c(x,t) = \langle \mu_{x,t}, Id\rangle \quad \textit{a.e.} \end{aligned}$$

In the rest of this section we prove the following existence theorem:

Theorem 3.2 (Existence of YM–solutions) *For $u_0 - g \in H_0^1$, $z_0 \in H_0^1$, $c_0 \in H^1$, $\vec{n}K(\nabla c_0, 0) = 0$ there exists a Young measure solution (u, ν, c, μ) of our problem under the assumptions stated above.*

To prove this theorem we apply the same methods as in the previous section: We first prove the existence of a weak solution for our system equipped with additional dissipation terms, i.e. we study (for $\varepsilon > 0$):

$$\begin{aligned}
u_{tt}^\varepsilon(x,t) - \operatorname{div} S(\nabla u^\varepsilon(x,t), c^\varepsilon(x,t)) - \varepsilon \Delta u_t^\varepsilon(x,t) &= 0,\\
c_t^\varepsilon(x,t) - \operatorname{div} K(\nabla c^\varepsilon(x,t), u^\varepsilon(x,t)) - \varepsilon \Delta c_t^\varepsilon(x,t) &= 0,\\
u^\varepsilon(\cdot,0) = u_0,&\\
u_t^\varepsilon(\cdot,0) = z_0,&\\
c^\varepsilon(\cdot,0) = c_0,&\\
u^\varepsilon = g \quad \text{on}\,\partial\Omega,&\\
\vec{n}(K(\nabla c^\varepsilon, u^\varepsilon) + \varepsilon \nabla c_t^\varepsilon) = 0 \quad \text{on}\,\partial\Omega,&
\end{aligned} \tag{7}$$

For this system we can prove the following theorem:

Theorem 3.3 *For every $T > 0$ and $u_0 - g \in H_0^1$, $z_0 \in H_0^1$, $c_0 \in H^1$, $\vec{n}K(\nabla c_0, 0) = 0$ there exists a weak solution $(u^\varepsilon, c^\varepsilon)$ of the system (7), i.e.:*

$$\begin{aligned}
u^\varepsilon &\in L^\infty((0,T), H_0^1(\Omega)) \cap W^{1,\infty}((0,T), L^2(\Omega)) \cap W^{1,2}((0,T), H^1(\Omega))\\
&\quad \cap W^{2,2}((0,T), H^{-1}(\Omega)),\\
c^\varepsilon &\in W^{1,2}(I\!R^+, L^2(\Omega)) \cap L^\infty(I\!R^+, H^1(\Omega))
\end{aligned}$$

and:

$$\int_0^T \int_\Omega S(\nabla u^\varepsilon, c^\varepsilon)\nabla\zeta + \varepsilon \nabla u_t^\varepsilon \nabla\zeta - u_t^\varepsilon \zeta_t \, dx\, dt = 0 \quad \forall \zeta \in H_0^1((0,T)\times\Omega), \tag{8}$$

$$\int_0^T \int_\Omega K(\nabla c^\varepsilon, u^\varepsilon)\nabla\zeta + \varepsilon \nabla c_t^\varepsilon \nabla\zeta + c_t^\varepsilon \zeta \, dx\, dt = 0 \quad \forall \zeta \in H^1((0,T)\times\Omega). \tag{9}$$

Furthermore we have the following inequality:

$$\frac{1}{2}||u_t^\varepsilon||^2 + ||u^\varepsilon||_{H^1}^2 + ||c^\varepsilon||_{H^1}^2 + \int_0^T ||\sqrt{\varepsilon}\nabla u_t^\varepsilon||^2\, dt + \int_0^T ||\sqrt{\varepsilon}\nabla c_t^\varepsilon||^2\, dt \le M, \tag{10}$$

For the proof of this theorem we apply the methods introduced by [KP92] for the heat equation and [Dem97] for the wave equation. These methods were used for viscoelasticity by [Dem] and [FD97].

First we discretize with respect to time. To make life easier we drop the ε in the notation of u^ε and c^ε and use u and c instead. We denote the discretized variables by $(u^{h,j})_{h,j}, (c^{h,j})_{h,j}$. (Often we will drop the h.) For $j = 0, 1, \dots$ we will construct (weak) solutions of these discretized equations (together with the standard boundary conditions):

$$\frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2} - \operatorname{div} S(\nabla u^{h,j}, c^{h,j-1}) - \varepsilon \frac{\Delta u^{h,j} - \Delta u^{h,j-1}}{h} = 0, \tag{11}$$

$$\frac{c^{h,j} - c^{h,j-1}}{h} - \operatorname{div} K(\nabla c^{h,j}, u^{h,j-1}) - \varepsilon \frac{\Delta c^{h,j} - \Delta c^{h,j-1}}{h} = 0, \tag{12}$$

$$u^{h,0} = u_0, \quad u^{h,-1} = u_0 - hz_0, \quad c^{h,0} = c_0.$$

It is convenient to define the 'discretized velocity':

$$v^{h,j} := \frac{u^{h,j} - u^{h,j-1}}{h}.$$

We now want to derive an a priori estimate for the discrete energy:

$$E_j := E^{h,j} := \int_\Omega \phi(\nabla u^{h,j}, c^{h,j-1}) + \eta\psi(\nabla c^{h,j}, u^{h,j-1}) + \frac{1}{2}|v^{h,j}|^2\, dx,$$

where $\eta > 0$ will be choosen later.
We formulate the following lemma:

Lemma 3.4 (Discrete energy) *Let $T > 0$, $jh < T$ and $\delta \in (0,1)$. Then for every positive $h < h_0(\delta)$ the following inequality holds:*

$$E_j + \sum_{j=1}^{\infty} \frac{h}{2} \int_\Omega (\varepsilon - \delta)|\nabla v^{h,j}|^2 dx + \sum_{j=1}^{\infty} \frac{h}{2} \int_\Omega (\varepsilon - \delta) \left| \frac{\nabla c^{h,j} - \nabla c^{h,j-1}}{h} \right|^2 dx \le M$$

To prove this we exploit that the nonconvex energy densities ϕ and ψ are "convexified" by the viscosity term. We start by considering the energy difference in one time step:

$$\begin{aligned}
\Delta E_j &:= E_{j+1} - E_j \\
&= \int_\Omega \Big(\phi(\nabla u^{j+1}, c^j) + \frac{1}{2}|v^{j+1}|^2 + \eta\psi(\nabla c^{j+1}, u^j) \Big) \\
&\quad - \Big(\phi(\nabla u^j, c^{j-1}) + \frac{1}{2}|v^j|^2 + \eta\psi(\nabla c^j, u^{j-1}) \Big)\, dx \\
&= \int_\Omega \Big(\phi(\nabla u^{j+1}, c^j) - \phi(\nabla u^j, c^{j-1}) \\
&\quad + \frac{\delta}{h}|\nabla u^{j+1} - \nabla u^j|^2 - \frac{\delta}{h}|\nabla u^{j+1} - \nabla u^j|^2 - \frac{\delta}{h}\underbrace{|\nabla u^j - \nabla u^j|^2}_{=0} \\
&\quad + \eta\psi(\nabla c^{j+1}, u^j) - \eta\psi(\nabla c^j, u^{j-1}) \\
&\quad + \eta\frac{\delta}{h}|\nabla c^{j+1} - \nabla c^j|^2 - \eta\frac{\delta}{h}|\nabla c^{j+1} - \nabla c^j|^2 - \eta\frac{\delta}{h}\underbrace{|\nabla c^j - \nabla c^j|^2}_{=0} \\
&\quad + \frac{1}{2}|v^{j+1}|^2 - \frac{1}{2}|v^j|^2 \Big)\, dx
\end{aligned}$$

Before we proceed by estimating this expression, we first state the following auxiliary lemma:

Lemma 3.5 *Let $r, s \ge 1$ and $\omega \in \mathcal{C}^2(I\!R^{r\times s}, I\!R_+)$. Assume that either ω satisfies for every F_1, $F_2 \in I\!R^{r\times s}$ the inequality:*

$$(\nabla\omega(F_1) - \nabla\omega(F_2))(F_1 - F_2) \;\; \ge \;\; -M|F_1 - F_2|^2, \tag{13}$$

where $M > 0$ is a constant,
or that $\nabla\omega$ satisfies the Andrews-Ball condition, see (3).
Then for every $A \in I\!R^{r\times s}$ the function

$$g : F \longmapsto \omega(F) + \frac{\delta}{h}|F - A|^2$$

is convex for every $h \le h_0(\delta)$.
Furthermore for every F_1, $F_2 \in I\!R^{r\times s}$ and $h \le h_0(\delta)$ the following estimate holds:

$$\begin{aligned}
&\Big(\omega(F_1) + \frac{\delta}{h}|F_1 - A|^2\Big) - \Big(\omega(F_2) + \frac{\delta}{h}|F_2 - A|^2\Big) \\
\le\; &\Big(\nabla\omega(F_1) + \frac{2\delta}{h}(F_1 - A)\Big)(F_1 - F_2).
\end{aligned} \tag{14}$$

To prove the convexity of g we apply (13), which itself is a consequence of the Andrews-Ball condition (for a proof see e.g. [FD97]). By the convexity of g we get: $g(F_1)-g(F_2) \leq \nabla g(F_1)(F_1-F_2)$ and this gives (14). □

We apply this lemma twice: once with $F_1 := \nabla u^{j+1}$, $F_2 := \nabla u^j$, $A := \nabla u^j$ and $\omega(X) := \phi(X, c^j)$ and once with $F_1 := \nabla c^{j+1}$, $F_2 := \nabla c^j$, $A := \nabla c^j$ and $\omega(X) := \psi(X, u^j)$. Furthermore we use the global Lipschitz continuity of S and K in the second variable (with Lipschitz constant L) to derive:

$$\begin{aligned}\Delta E_j \leq & \int_\Omega \left(\nabla_1\phi(\nabla u^{j+1}, c^j) + \frac{2\delta}{h}(\nabla u^{j+1} - \nabla u^j)\right)(\nabla u^{j+1} - \nabla u^j) \\ & -\frac{\delta}{h}|\nabla u^{j+1} - \nabla u^j|^2 + L|c^j - c^{j-1}|^2 + \\ & \eta\left(\nabla_1\psi(\nabla c^{j+1}, u^j) + \frac{2\delta}{h}(\nabla c^{j+1} - \nabla c^j)\right)(\nabla c^{j+1} - \nabla c^j) \\ & -\frac{\delta\eta}{h}|\nabla c^{j+1} - \nabla c^j|^2 + L\eta|u^j - u^{j-1}|^2 \\ & +\frac{1}{2}\left(|v^{j+1}|^2 - |v^j|^2\right)\, dx.\end{aligned}$$

By rearranging the terms we get:

$$\begin{aligned}\Delta E_j \leq & \int_\Omega \left(\nabla_1\phi(\nabla u^{j+1}, c^j) + \frac{\varepsilon}{h}(\nabla u^{j+1} - \nabla u^j)\right)(\nabla u^{j+1} - \nabla u^j) \\ & -\frac{\varepsilon-\delta}{h}|\nabla u^{j+1} - \nabla u^j|^2 + L|c^j - c^{j-1}|^2 \\ & +\eta\left(\nabla_1\psi(\nabla c^{j+1}, u^j) + \frac{\varepsilon}{h}(\nabla c^{j+1} - \nabla c^j)\right)(\nabla c^{j+1} - \nabla c^j) \\ & -\eta\frac{\varepsilon-\delta}{h}|\nabla c^{j+1} - \nabla c^j|^2 + L\eta|u^j - u^{j-1}|^2 \\ & +\frac{1}{2}\left(|v^{j+1}|^2 - |v^j|^2\right)\, dx. \qquad (15)\end{aligned}$$

Before we continue with our estimate we now consider equation (11) with $\zeta := u^{j+1} - u^j$ (or to be precise a smooth sequence ζ_k converging to $u^{j+1} - u^j$ and considering the limit $k \to \infty$) which gives us the following expression:

$$\begin{aligned}& \int_\Omega S(\nabla u^{j+1}, c^j)(\nabla u^{j+1} - \nabla u^j)\, dx \\ = & \int_\Omega -(v^{j+1} - v^j)v^{j+1} - \frac{\varepsilon}{h}|\nabla u^{j+1} - \nabla u^j|^2\, dx.\end{aligned}$$

Using the same ideas for equation (12) we get:

$$\begin{aligned}& \int_\Omega K(\nabla c^{j+1}, u^j)(\nabla c^{j+1} - \nabla c^j)\, dx \\ = & \int_\Omega -\frac{1}{h}|c^{j+1} - c^j|^2 - \frac{\varepsilon}{h}|\nabla c^{j+1} - \nabla c^j|^2\, dx.\end{aligned}$$

We insert these equations into (15) and use the Poincaré inequality for $u^{j+1} - u^j$ to get the following estimate:

$$\begin{aligned}
E_{j+1}-E_j &\le \int_\Omega -(v^{j+1}-v^j)v^{j+1}+\frac{1}{2}\left(|v^{j+1}|^2-|v^j|^2\right)\\
&\quad -\frac{\varepsilon-\delta}{h}|\nabla u^{j+1}-\nabla u^j|^2-\frac{\varepsilon}{h}|\nabla u^{j+1}-\nabla u^j|^2+\frac{\varepsilon}{h}|\nabla u^{j+1}-\nabla u^j|^2\\
&\quad -\frac{\varepsilon-\delta}{h}|\nabla c^{j+1}-\nabla c^j|^2-\frac{\varepsilon\eta}{h}|\nabla c^{j+1}-\nabla c^j|^2+\frac{\varepsilon\eta}{h}|\nabla c^{j+1}-\nabla c^j|^2\\
&\quad +L\eta|u^j-u^{j-1}|^2-\frac{\eta}{h}|c^{j+1}-c^j|^2+L|c^j-c^{j-1}|^2\,dx\\
&\le \int_\Omega -\frac{1}{2}|v^{j+1}+v^j|^2-\frac{\varepsilon-\delta}{2h}|\nabla u^{j+1}-\nabla u^j|^2-\eta\frac{\varepsilon-\delta}{h}|\nabla c^{j+1}-\nabla c^j|^2\\
&\quad -\left(\frac{\varepsilon-\delta}{2h}M-L\eta\right)|u^{j+1}-u^j|^2\\
&\quad -\frac{\eta}{h}|c^{j+1}-c^j|^2+L|c^j-c^{j-1}|^2\,dx.
\end{aligned}$$

If we choose $\eta \le \frac{\varepsilon-\delta}{2}\frac{M}{L}$, $h<\min\left(\frac{\eta}{L},1\right)$ and sum over all $j\ge 1$, then we get:

$$E_j-E_0 \le -\sum_{i=1}^{j}(\varepsilon-\delta)\frac{h}{2}||\nabla v^i||^2-\sum_{i=1}^{j}(\varepsilon-\delta)\frac{h}{2}\left|\left|\frac{\nabla c^i-\nabla c^{i-1}}{h}\right|\right|^2+M(c_0).$$

This gives the statement of the lemma. □

The following inequality is an easy corollary of Lemma 3.4:

Corollary 3.6 *For every $T=kh>0$ there exists a constant $M>0$ such that:*

$$\sup_j\left(||\nabla u^{h,j}||^2+||\nabla c^{h,j}||^2+||v^{h,j}||^2\right)+\sum_{j=1}^{\frac{T}{h}}h||\nabla v^{h,j}||^2+\sum_{j=1}^{\frac{T}{h}}h\left|\left|\frac{\nabla c^i-\nabla c^{i-1}}{h}\right|\right|^2\le M<\infty$$

For the proof one simply applies the growth conditions for ϕ and ψ and Lemma 3.4. □

We are now able to prove the existence of solutions $(u^{h,j},c^{h,j})$ of our time-discretized system. We first solve the time-step problem with the help of a variational ansatz, i.e. we consider the functional:

$$\begin{aligned}
W^{h,j}(u,c) &:= \int_\Omega \phi(\nabla u,c^{h,j-1})+\psi(\nabla c,u^{h,j-1})\\
&\quad +\frac{\varepsilon}{2h}|\nabla u-\nabla u^{h,j-1}|^2+\frac{1}{2h^2}|u-2u^{h,j-1}+u^{h,j-2}|^2\\
&\quad +\frac{\varepsilon}{2h}|\nabla c-\nabla c^{h,j-1}|^2+\frac{1}{2h}|c-c^{h,j-1}|^2\,dx.
\end{aligned}$$

The functional $W^{h,j}$ is weakly lower semicontinuous since its integrand is convex in $(\nabla u,\nabla c)$, which is true since the "critical" terms $\phi(\nabla u,c^{h,j-1})+\frac{1}{2h}|\nabla u|^2$ and $\psi(\nabla c,u^{h,j-1})+\frac{1}{2h}|\nabla c|^2$ are convex for sufficiently small $h>0$.

Since $W^{h,j}$ is also bounded from below by zero, there exists a (not neccessarily unique) minimizer (u,c). By a standard calculation one can show, that (u,c) solves the time-step problem. We define $(u^{h,j},c^{h,j}):=(u,c)$. By induction we get the existence of a time-discretized solution to the discrete problem.

In the next step we interpolate this discrete approximation $(u^{h,j},c^{h,j})$ in time. Here it is convinient to use two different approximation schemes, i.e. the piecewise constant and the piecewise affine interpolation.

We define for $h > 0$, $0 \le j < \frac{T}{h}$ and the characteristic function $\chi^{h,j} := \chi_{[hj,h(j+1)]}$:

- $w^h(t) := \sum_j \chi^{h,j}(t) \frac{v^{h,j+1} - v^{h,j}}{h}$ (step function approximation of u_{tt}),
  $\tilde{v}^h(t) := \sum_j \chi^{h,j}(t) \left(v^{h,j} + \frac{v^{h,j+1} - v^{h,j}}{h}(t - hj) \right)$ (its primitive),
- $v^h(t) := \sum_j \chi^{h,j}(t) v^{h,j+1}$ (step function approximation of u_t),
  $\tilde{u}^h(t) := \sum_j \chi^{h,j}(t) \left(u^{h,j} + v^{h,j+1}(t - hj) \right)$ (its primitive),
- $u^h(t) := \sum_j \chi^{h,j}(t) u^{h,j+1}$ (step function approximation of u),
- $d^h(t) := \sum_j \chi^{h,j}(t) \frac{c^{h,j+1} - c^{h,j}}{h}$ (step function approximation of c_t),
  $\tilde{c}^h(t) := \sum_j \chi^{h,j}(t) \left(c^{h,j} + \frac{c^{h,j+1} - c^{h,j}}{h}(t - hj) \right)$ (its primitive),
- $c^h(t) := \sum_j \chi^{h,j}(t) c^{h,j+1}$ (step function approximation of c).

We have choosen the notation in such a way that the step functions are each labeled with different characters (w, v, u resp. d and c) depending on the order of derivative they are approximating. Their primitives are denoted with the character of the corresponding lower order terms and an additional squiggle, e.g. the primitive of w^h is denoted as $\tilde{v}^h$. Later we will show that the interpolations denoted with and without a squiggle of the same character (i.e. terms of the same order) coincide in the limit $h \to 0$ and converge to our solution or its derivatives.
To prove convergence for these sequences we use Corollary 3.6 and the growth conditions (in the cases where the H^{-1}-norm is involved we also use the discretized partial differential equations) to prove the following bounds (uniformly in h) for fixed $T > 0$:

$$
\begin{aligned}
\sup_{0 \le t \le T} ||u^h(t)||^2_{H^1_0} &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||\tilde{u}^h(t)||^2_{H^1_0} &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||v^h(t)||^2 &\le M(u_0, z_0, c_0), \\
\int_0^T ||v^h(t)||^2_{H^1_0} \, dt &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||\tilde{v}^h(t)||^2 &\le M(u_0, z_0, c_0), \\
\int_0^T ||\tilde{v}^h(t)||^2_{H^1_0} \, dt &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||w^h(t)||^2_{H^{-1}} &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||S(\nabla u^h)||^2 &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} ||c^h(t)||^2_{H^1} &\le M(u_0, z_0, c_0), \\
\sup_{0 \le t \le T} \left(||\tilde{c}^h(t)||^2 + ||d^h(t)||^2_{H^{-1}} \right) &\le M(u_0, z_0, c_0), \\
\int_0^T ||d^h(t)||^2_{H^1} \, dt &\le M(u_0, z_0, c_0).
\end{aligned}
$$

From these bounds we get the following weak convergence results (again choosing subsequences):

$$
\begin{aligned}
u^h &\overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty\left((0,T), H^1(\Omega)\right), \\
\tilde{u}^h &\overset{\star}{\rightharpoonup} \tilde{u} \quad \text{in } L^\infty\left((0,T), H^1(\Omega)\right) \cap W^{1,\infty}\left((0,T), L^2(\Omega)\right) \cap W^{1,2}\left((0,T), H^1(\Omega)\right), \\
v^h &\overset{\star}{\rightharpoonup} v \quad \text{in } L^\infty\left((0,T), L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right), \\
\tilde{v}^h &\overset{\star}{\rightharpoonup} \tilde{v} \quad \text{in } L^\infty\left((0,T), L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right) \cap W^{1,\infty}\left((0,T), H^{-1}(\Omega)\right),
\end{aligned}
$$

$$\begin{aligned} w^h &\stackrel{\star}{\rightharpoonup} w \quad \text{in } L^2\left((0,T),H^{-1}(\Omega)\right), \\ c^h &\stackrel{\star}{\rightharpoonup} c \quad \text{in } L^\infty\left((0,T),H^{1}(\Omega)\right), \\ \tilde{c}^h &\stackrel{\star}{\rightharpoonup} \tilde{c} \quad \text{in } W^{1,\infty}\left((0,T),H^{-1}(\Omega)\right), \\ d^h &\stackrel{\star}{\rightharpoonup} d \quad \text{in } L^2\left((0,T),H^{1}(\Omega)\right). \end{aligned}$$

Additionally we deduce by applying Corollary 3.6 and the growth conditions on S and K, that there exists $\tilde{S}$ and $\tilde{K}$, such that for $c \in L^\infty((0,T),L^2(\Omega,\mathbb{R}^d))$, $u \in L^\infty((0,T),L^2(\Omega,\mathbb{R}^m))$:

$$\begin{aligned} S(\nabla u^h, c) &\stackrel{\star}{\rightharpoonup} \tilde{S}_c \quad \text{in } L^\infty\left((0,T),L^2(\Omega)\right), \\ K(\nabla c^h, u) &\stackrel{\star}{\rightharpoonup} \tilde{K}_u \quad \text{in } L^\infty\left((0,T),L^2(\Omega)\right). \end{aligned}$$

We now have to make sure, that the different interpolations we have choosen converge to the same limit. For this we use a standard lemma (see e.g. [KP92]):

Lemma 3.7 *Suppose that $(f^{h,j})_{h,j}$ is bounded in $L^2(\Omega)$, that $f^h(t)$ is its step function interpolation, $g^h(t)$ its continuous and piecewise affine interpolation. Assume furthermore that $f^h \rightharpoonup f$ and $g^h \rightharpoonup g$ in $L^2_{loc}(\Omega \times \mathbb{R}^+)$. Then we have: $f = g$.*

Sketch of the proof: We show the equivalence after testing with a smooth function. Therefore we only need to consider test functions of the 'separated' form $w(x)z(t)$. Let $\xi^h(t)$ be the step function approximation of $z(t)$ and $\zeta^h(t)$ be the piecewise affine approximation of $z(t)$. Then $w(x)\xi^h(t)$ and $w(x)\zeta^h(t)$ converge strongly to $w(x)z(t)$. If we now test $f^h(t)$ with $w(x)\xi^h(t)$ and $g^h(t)$ with $w(x)\zeta^h(t)$ we get the same result, and this equation holds also for $h \to 0$. (See [KP92] for the complete proof.) □

We can apply this lemma to deduce $u = \tilde{u}$, $v = \tilde{v}$ and $c = \tilde{c}$. This is nearly enough to consider the limit $h \to 0$ in our equation, but the nonlinearities S and K cannot be handled in this way, since weak convergence of ∇u^h to ∇u is not enough to get weak convergence of $S(\nabla u^h, c)$ to $S(\nabla u, c)$. (And the analogous statement holds for K.) Fortunately we can prove strong convergence of ∇u^h, $\nabla \tilde{u}^h$, ∇c^h and $\nabla \tilde{c}^h$ in $L^2\left((0,T),L^2(\Omega)\right)$ as $h \to 0$.

We first need some lemmata, where we state only simplified counterparts of the corresponding lemmata in [FD97]. The proofs can also be found there.

Lemma 3.8 (Aubin type result) *Let $X_s := W^{1,2}(\Omega)$, $X := L^2(\Omega)$ and $X_w := W^{-1,2}(\Omega)$. Then the imbedding of $L^2\left((0,T),X_s\right) \cap W^{1,2}\left((0,T),X_w\right)$ equipped with the natural norm $||\cdot||_{L^2(X_s)}$ + $||\partial_t \cdot||_{L^2(X_w)}$ into $L^2\left((0,T),X\right)$ is compact.*

The next lemma is giving a closer connection between the two kinds of interpolations we have used:

Lemma 3.9 *Let X be a Banach space and $\{f^{h,j}\}_{j\geq 1, h>0}$ a collection of elements in X. Let f^h be the piecewise constant and $\tilde{f}^h$ be the piecewise linear interpolation of $\{f^{h,j}\}$ defined (as usual) by:*

$$\begin{aligned} f^h(t) &:= \sum_j \chi_j(t) f^{h,j}, \\ \tilde{f}^h(t) &:= \sum_j \chi_j(t) \left(\left(j - \frac{t}{h} \right) f^{h,j-1} + \left(\frac{t}{h} - (j-1) \right) f^{h,j} \right), \end{aligned}$$

where χ_j is the characteristic function of $(jh,(j+1)h)$.
Assume that $\sup_j ||f^{h,j}||^2 \leq M_1$ and for some $\alpha > 0$:

$$\sum_{j=1}^{\frac{T}{h}} h \left| \left| \frac{f^{h,j} - f^{h,j-1}}{h^\alpha} \right| \right|^2 \leq M_2.$$

Then for all $f \in L^2((0,T),X)$ *with* $\sup_t ||f(t)||^2 \le M_1$ *we have the following estimate:*

$$\int_0^T ||f^h - f||^2\,dt \;\le\; 2\int_0^T ||\tilde{f}^h - f||^2\,dt + 4hM_1 + \frac{2}{3}h^{2\alpha}M_2.$$

We also use the following fact following from the definition of weak convergence and compactness:

Lemma 3.10 *Let* $G \subset I\!R^N$ *be open,* $\{f^h\}_h \subset L^2(\Omega)$, $f^h \rightharpoonup 0$ *in* $L^2(\Omega)$ *as* $h \to 0$ *and let* K *be a compact subset of* $L^2(\Omega)$*, then:*

$$\sup_{\xi\in K}\left|\int_G f^h\xi\,dx\right| \to 0 \text{ as } h \to 0.$$

Now we have collected all ingredients for the proof of the strong convergence of ∇u^h, $\nabla\tilde{u}^h$, ∇c^h and $\nabla\tilde{c}^h$. First we consider ∇u^h and $\nabla\tilde{u}^h$, later we will apply the methods introduced there to prove the strong convergence of ∇c^h and $\nabla\tilde{c}^h$.
We start with the following time-integrated version of our elasticity equation, having the property that the test function ζ does not need to be differentiable in time:

$$\begin{aligned}\int_0^T\int_\Omega \left(S(\nabla u^h, c^h(\cdot - h)) + \varepsilon\nabla v^h\right)\nabla\zeta - v^h\frac{\zeta(\cdot+h)-\zeta}{h}\,dx\,dt&\\ +\frac{1}{h}\int_{T-h}^T\int_\Omega v^h\zeta(\cdot+h)\,dx\,dt - \frac{1}{h}\int_{-h}^0\int_\Omega v_0\zeta(\cdot+h)\,dx\,dt &= 0. \qquad (16)\end{aligned}$$

We consider the limit $h \to 0$ in equation (16), where we use that $c^h(\cdot{-}h) \to c$ in $L^\infty((0,T),L^2(\Omega))$. Using the definition of $\tilde{S}_c$ we get:

$$\begin{aligned}\int_0^T\int_\Omega \tilde{S}_c\nabla\zeta + \varepsilon\nabla u_t\nabla\zeta\,dx\,dt&\\ +\int_{T-h}^T\int_\Omega u_t\zeta\,dx\,dt &= 0. \qquad (17)\end{aligned}$$

We insert $\zeta := u^h - u$ in (16) and $\zeta := \tilde{u}^h - u$ in (17) and substract the resulting equations. (To be exact we have to approximate $u^h - u^h(\cdot - h)$ and $\tilde{u}^h - u$ by sequences of smooth functions.) This gives for $t \in (0,T)$:

$$\begin{aligned}0 \;=\; &\underbrace{\int_0^t\int_\Omega S(\nabla u^h, c^h(\cdot-h))(\nabla u^h - \nabla u) - \tilde{S}_c(\nabla\tilde{u}^h - \nabla u)\,dx\,dt}_{=:T_1}\\
&+\varepsilon\underbrace{\int_0^t\int_\Omega \nabla v^h(\nabla u^h - \nabla u) - \nabla u_t(\nabla\tilde{u}^h - \nabla u)\,dx\,dt}_{=:T_2}\\
&-\underbrace{\int_0^t\int_\Omega v^h\left(v^h(\cdot+h) - \frac{\tilde{u}(\cdot+h)-u}{h}\right) - u_t\left((\tilde{u}^h)_t - u_t\right)\,dx\,dt}_{=:T_3}\\
&+\underbrace{\int_\Omega\int_{t-h}^t v^h(u^h(\cdot+h) - u(\cdot+h)) - u_t\left(\tilde{u}^h - u\right)\,dt\,dx}_{=:T_4}\\
&-\underbrace{\int_\Omega v_0\frac{1}{h}\int_{-h}^0 u^h(\cdot+h) - u(\cdot+h)\,dt\,dx,}_{=:T_5}\end{aligned}$$

where we have defined the terms T_1,...,T_5 which we will estimate in the following calculation. To simplify notation we denote all terms converging to zero as $h \to 0$ (uniformly in t) by $\alpha(h)$. We start by estimating T_1, where we use the global Lipschitz continuity of S giving us for a certain $M > 0$ and every $F_1, F_2 \in \mathbb{R}^{m\times n}$ and $\hat{c} \in \mathbb{R}^d$ the inequality:

$$(S(F_1, \hat{c}) - S(F_2, \hat{c})(F_1 - F_2)) \geq -M|F_1 - F_2|^2.$$

(This corresponds to condition (4) in the last section.)

$$\begin{aligned} T_1 &= \int_0^t \int_\Omega \left(S(\nabla u^h, c^h(\cdot - h)) - S(\nabla u, c^h(\cdot - h))\right)(\nabla u^h - \nabla u) + S(\nabla u, c^h(\cdot - h))(\nabla u^h - \nabla u) \\ &\quad -\tilde{S}_c(\nabla \tilde{u}^h - \nabla u)\, dx\, dt \\ &\geq -M \int_0^t \int_\Omega |\nabla u^h - \nabla u|^2\, dx\, dt - \sup_{t\in(0,T)} \left| \int_0^T \int_\Omega \left(\chi_{\Omega\times(0,t)} \tilde{S}_c\right)(\nabla \tilde{u}^h - \nabla u)\, dx\, dt \right|^2. \end{aligned}$$

Applying Lemma 3.10 we can show that the last three terms converge to zero for $h \to 0$, i.e.:

$$T_1 \geq -M \int_0^t \int_\Omega |\nabla u^h - \nabla u|^2 + \alpha(h).$$

Applying Lemma 3.9 we finally get:

$$T_1 \geq -2M \int_0^t \int_\Omega |\nabla \tilde{u}^h - \nabla u|^2 + \alpha(h).$$

We can use the same calculations as in the purely viscoelastic case (see [FD97] [1]) to derive:

$$-T_2 \geq -\frac{1}{2}\int_\Omega |\nabla \tilde{u}^h(t) - \nabla u(t)|^2\, dx + \frac{1}{2}\int_\Omega \underbrace{|\nabla \tilde{u}^h(0) - \nabla u(0)|^2}_{=0}\, dx + \alpha(h),$$

where the discrete energy estimate proved above is used. This (together with the estimate for T_1) is the key step to the desired strong convergence result, since at the end we want to apply the Gronwall Lemma to the inequality we get by estimating these terms. Therefore we need that the terms T_3-T_5 are "well behaved", i.e. that they are simply $\alpha(h)$.
In fact by applying Lemma 3.8 combined with Lemma 3.9 we can prove this:

$$T_3 = \alpha(h), \quad T_4 = \alpha(h), \quad T_5 = \alpha(h).$$

Taking everything together we have the inequality:

$$\partial_t \int_0^t \int_\Omega |\nabla \tilde{u}^h - \nabla u|^2\, dx\, dt \leq \frac{4M}{\varepsilon} \int_0^t \int_\Omega |\nabla \tilde{u}^h - \nabla u|^2\, dx\, dt + \alpha(h).$$

Now we can apply the Gronwall Lemma to get:

$$\int_0^T \int_\Omega |\nabla \tilde{u}^h - \nabla u|^2\, dx\, dt \leq \alpha(h) \frac{\varepsilon}{4M} e^{\frac{4MT}{\varepsilon}},$$

and this converges to zero for $h \to 0$, hence:

$$\nabla \tilde{u}^h \to \nabla u \text{ in } L^2\left((0,T), L^2(\Omega)\right).$$

[1] Remember the slightly different notation in their article.

Due to Lemma 3.9 the same convergence result holds for ∇u^h. This ensures that $\tilde{S}_{\hat{c}} = S(\nabla u, \hat{c})$. Now we are ready to apply the same methods to prove $\tilde{K}_{\hat{u}} = K(\nabla c, \hat{u})$. First we consider the following weak formulation of (12):

$$\int_0^T \int_\Omega K(\nabla c^h, u^h(\cdot - h))\nabla\zeta + \varepsilon \frac{\nabla c^h - \nabla c^h(\cdot - h)}{h}\nabla\zeta + \frac{c^h - c^h(\cdot - h)}{h}\zeta \, dx \, dt = 0. \tag{18}$$

Then we consider the limit $h \to 0$ in equation (18) to get:

$$\int_0^T \int_\Omega \tilde{K}_u \nabla\zeta + \varepsilon\nabla c_t \nabla\zeta + c_t\zeta \, dx \, dt = 0. \tag{19}$$

We insert $\zeta := c^h - c$ in (18) and $\zeta := \tilde{c}^h - c$ in (19) and substract the resulting equations. (To be exact we have to approximate $c^h - c^h(\cdot - h)$ and $\tilde{c}^h - c$ by sequences of smooth functions.) This gives:

$$\begin{aligned} 0 \quad = \quad & \int_0^T \int_\Omega K(\nabla c^h, u^h(\cdot - h))\nabla(c^h - c) - \tilde{K}_u \nabla(\tilde{c}^h - c) \\ & + \varepsilon\frac{\nabla c^h - \nabla c^h(\cdot - h)}{h}\nabla(c^h - c) - \varepsilon\nabla c_t \nabla(\tilde{c}^h - c) \\ & + \frac{c^h - c^h(\cdot - h)}{h}(c^h - c) - c_t(\tilde{c}^h - c)\, dx\, dt. \end{aligned} \tag{20}$$

Now we consider the three terms in (20). We start with the third one. We want to prove that:

$$\int_0^T \int_\Omega \frac{c^h - c^h(\cdot - h)}{h}(c^h - c) - c_t(\tilde{c}^h - c)\, dx\, dt \to 0 \quad \text{as } h \to 0.$$

But this is true for the first part, since $\left(\tilde{c}^h\right)_t$ is bounded in $L^\infty((0,T), H^{-1}(\Omega))$ and $c^h \overset{\star}{\rightharpoonup} c$ in $L^\infty((0,T), H^1(\Omega))$, and it is true for the second part, since $c_t \in L^2((0,T), H^1(\Omega))$ and $\tilde{c}^h \overset{\star}{\rightharpoonup} c$ in $W^{1,\infty}(H^{-1}(\Omega))$.

We now rewrite the first term in (20) denoted by T_6 and estimate it as follows:

$$\begin{aligned} T_6 \quad := \quad & \int_0^t \int_\Omega \Big(K(\nabla c^h, u^h(\cdot - h)) - K(\nabla c, u^h(\cdot - h))\Big)\nabla(c^h - c) \\ & + K(\nabla c, u^h(\cdot - h))\nabla(c^h - c) \\ & - \tilde{K}_{u^h(\cdot - h)}\nabla(\tilde{c}^h - c) + \left(\tilde{K}_{u^h(\cdot - h)} - \tilde{K}_u\right)\nabla(\tilde{c}^h - c)\, dx\, dt \\ \geq \quad & -M\int_0^t \int_\Omega |\nabla\tilde{c}^h - \nabla c|^2\, dx\, dt + \alpha(h). \end{aligned}$$

So we get:

$$T_6 \quad \geq \quad -M\int_0^t \int_\Omega |\nabla\tilde{c}^h - \nabla c|^2\, dx\, dt + \alpha(h). \tag{21}$$

It remains to estimate the second term in (20). Here we apply the methods we had used to estimate T_2. This gives the following inequality:

$$\begin{aligned} & -\varepsilon\int_0^T \int_\Omega \frac{\nabla c^h - \nabla c^h(\cdot - h)}{h}(\nabla c^h - \nabla c) - \nabla c_t(\nabla\tilde{c}^h - \nabla c) \\ \geq \quad & -\frac{\varepsilon}{2}\int_0^T \int_\Omega |\nabla\tilde{c}^h(t) - \nabla u(t)|^2\, dx\, dt + \alpha(h). \end{aligned} \tag{22}$$

If we insert (21) and (22) into (20) and apply the Gronwall Lemma in the same way as before we derive:

$$\int_0^T \int_\Omega |\nabla \tilde{c}^h - \nabla c|^2 \, dx \, dt \quad \leq \quad \alpha(h) \frac{\varepsilon}{4M} e^{\frac{4MT}{\varepsilon}},$$

and this is converging to zero for $h \to 0$. Therefore $\nabla \tilde{c}^h$ is converging to ∇c strongly in $L^2((0,T), L^2(\Omega))$. And due to Lemma 3.9 this also holds for ∇c^h. Hence for $h \to 0$ the nonlinear term $K(\nabla c^h, u^h)$ converges to $K(\nabla c, u)$.
Taking everything together we have proved that the solutions of the time-discretized equations converge to solutions of the hyperbolic-parabolic system (7).
To prove Theorem 3.3 it only remains to prove the energy inequality:

$$E(t) \quad \leq \quad M(u_0, v_0, c_0) - \varepsilon \int_0^t \int_\Omega |\nabla u_t|^2 + |\nabla c_t|^2 \, dx \, dt,$$

where $E(t) := \int_\Omega \phi(\nabla u(t), c(t)) + \psi(\nabla c(t), u(t)) + \frac{1}{2}|u_t(t)|^2 \, dx$ and $M(u_0, v_0, c_0)$ is a constant depending only on the initial values u_0, v_0, c_0.
To prove this we start from the discrete energy inequality (Lemma 3.4), telling us that for $\eta > 0$ sufficiently small, $h < \min\left(1, \frac{\eta}{L}\right)$ and $\delta \in (0,1)$ the following inequality holds for every $t \in (0,T)$:

$$\begin{aligned} &\int_\Omega \phi(\nabla u^h, c^h(\cdot - h)) \, dx + \eta \int_\Omega \psi(\nabla c^h, u^h(\cdot - h)) \, dx \\ &+\frac{1}{2} \int_\Omega |v^h|^2 \, dx + (\varepsilon - \delta) \int_0^t \int_\Omega |\nabla v^h|^2 + |\nabla d^h|^2 \, dx \, dt \quad \leq \quad M(u_0, v_0, c_0). \end{aligned}$$

Now we notice that we can apply these convergence results:

$$\begin{aligned} v^h(t) &\to u_t(t) \text{ in } L^2(\Omega) \text{ for a.e. } t \in (0,T), \\ \nabla v^h &\rightharpoonup \nabla u_t \text{ in } L^2((0,T) \times \Omega), \\ \nabla d^h &\rightharpoonup \nabla c_t \text{ in } L^2((0,T) \times \Omega). \end{aligned}$$

By the weakly lower semicontinuity of the $L^2((0,T) \times \Omega)$-norm we get for a.e. $t \in (0,T)$:

$$\begin{aligned} &\limsup_{h \to 0} \int_\Omega \phi(\nabla u^h, c^h(\cdot - h)) \, dx + \eta \limsup_{h \to 0} \int_\Omega \psi(\nabla c^h, u^h(\cdot - h)) \, dx \\ &\qquad +\frac{1}{2} \int_\Omega |u_t|^2 \, dx + (\varepsilon - \delta) \int_0^t \int_\Omega |\nabla u_t|^2 + |\nabla c_t|^2 \, dx \, dt \quad \leq \quad M(u_0, v_0, c_0). \quad (23) \end{aligned}$$

Now we apply the strong convergence of $\nabla u^h(t)$ to estimate:

$$\begin{aligned} \int_\Omega \phi(\nabla u, c) \, dx &= \int_\Omega \left(\phi(\nabla u, c) + \frac{M}{2} |\nabla u|^2 \right) dx - \int_\Omega \frac{M}{2} |\nabla u|^2 \, dx \\ &\leq \limsup_{h \to 0} \int_\Omega \left(\phi(\nabla u^h, c^h(\cdot - h)) + \frac{M}{2} |\nabla u^h|^2 \right) dx - \int_\Omega \frac{M}{2} |\nabla u|^2 \, dx \\ &\leq \limsup_{h \to 0} \int_\Omega \phi(\nabla u^h, c^h(\cdot - h)) \, dx + \limsup_{h \to 0} \int_\Omega \frac{M}{2} |\nabla u^h|^2 \, dx - \int_\Omega \frac{M}{2} |\nabla u|^2 \, dx \\ &= \limsup_{h \to 0} \int_\Omega \phi(\nabla u^h, c^h(\cdot - h)) \, dx. \end{aligned}$$

Similiar we get applying the strong convergence of $\nabla c^h(t)$:

$$\int_\Omega \psi(\nabla c, u) \, dx \quad = \quad \limsup_{h \to 0} \int_\Omega \psi(\nabla c^h, u^h(\cdot - h)) \, dx.$$

If we insert these estimates into (23) and take the limit $\delta \to 0$, then we get:

$$\int_\Omega \phi(\nabla u, c)\,dx + \eta \int_\Omega \psi(\nabla c, u)\,dx + \frac{1}{2}\int_\Omega |u_t|^2\,dx$$
$$+\varepsilon \int_0^t \int_\Omega |\nabla u_t|^2 + |\nabla c_t|^2\,dx\,dt \quad \le \quad M(u_0, v_0, c_0),$$

for a.e. $t \in (0,T)$.
By adjusting the constant M we get the desired estimate (10). This completes the proof of Theorem 3.3. □
No we apply this to prove Theorem 3.2 by considering $\varepsilon \to 0$ in the same spirit as in the previous section: First the energy inequality (10) gives the following weak convergence results (for subsequences) as $\varepsilon \to 0$:

$$\begin{aligned} u^\varepsilon &\overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty((0,T), H_0^1(\Omega)),\\ c^\varepsilon &\overset{\star}{\rightharpoonup} c \quad \text{in } L^\infty((0,T), H_0^1(\Omega)),\\ u^\varepsilon &\overset{\star}{\rightharpoonup} u \quad \text{in } W^{1,\infty}((0,T), L^2(\Omega)). \end{aligned}$$

Furthermore for a.e. $t \in (0,T)$ the sequence $\nabla u^\varepsilon(t)$ generates the Young measure $\nu_{\cdot,t}$ and $\nabla c^\varepsilon(t)$ generates the Young measure $\mu_{\cdot,t}$.
For a subsequence we can consider the limit of the viscoelastic equations for $\varepsilon \to 0$ by using the growth conditions on S and K and the strong convergence of u^ε and c^ε in L^2. The Neumann boundary condition on c^ε is converging to the Neumann boundary condition on c. This calculation concludes the proof of the existence Theorem 3.2. □

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