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Some fine properties of currents and applications to distributional Jacobians

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SOME FINE PROPERTIES OF CURRENTS AND APPLICATIONS TO DISTRIBUTIONAL JACOBIANS

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Abstract. We study fine properties of currents in the framework of geometric measure theory on metric spaces developed by Ambrosio and Kirchheim in [5] and we prove a rectifiability criterion for flat currents of finite mass. We apply these tools to study the structure of the distributional Jacobians of functions in the space BnV, defined by Jerrard and Soner in [9]. We define the subspace of *special functions of bounded higher variation* and we prove a closure theorem.

1. Introduction

In this paper we generalize some tools of geometric measure theory on metric spaces developed by Ambrosio and Kirchheim in [5] and we apply them to the space BnV. This space, which has been defined by Jerrard and Soner in [9], is composed, roughly speaking, by those functions such that their weak Jacobians are measures.

If $u \in C^1(\mathbf{R}^m, \mathbf{R}^n)$, with $m \geq n$, then the Jacobian of u can be seen as the differential form $\omega = du_1 \wedge \ldots \wedge du_n$. Of course this notion can be easily extended to functions $u \in W^{1,n}$ but the main idea for a broader extension is based on the fact that $\omega = d(u_1 du_2 \wedge \ldots \wedge du_n)$. Indeed we need less summability on the derivatives of u to handle the form $\nu = u_1 du_2 \wedge \ldots \wedge du_n$ and we can define the weak Jacobian of u as the exterior derivative of ν in the distributional sense. A lot of attention has been devoted to this notion in the last years and we refer to [9] for an account of its applications and of the main papers on the argument.

In this work we propose to think of $u_1 du_2 \wedge \ldots \wedge du_n$ as a current T via the natural action

$$T(dw_1 \wedge \ldots \wedge dw_{m-n+1}) = \int_{\mathbf{R}^m} u_1 \det(\nabla u_2, \ldots, \nabla u_n, \nabla w_1, \ldots, \nabla w_{m-n+1}) d\mathcal{L}^m.$$

Thus we can define the weak Jacobian $[Ju]$ as the boundary of T and the space BnV can be identified with those u such that $[Ju]$ is a normal current. Instead of working in the framework of classical geometric measure theory we prefer to use the "metric currents theory" of [5] because we think that it is much easier to use and provides more powerful tools for studying the

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structure of weak Jacobians. The main idea of this approach, suggested by De Giorgi in [6], is to replace the duality with differential forms with the duality with $(k+1)$-ples of Lipschitz functions. We hope to show that in this way we simplify notations and proofs. In the last section we define a new class of functions, called SBnV, that is a generalization of the space of special functions of bounded variations (see [1], [3]). We prove for SBnV a closure theorem which is a generalization of the closure theorem for SBV (see Theorem 5.5 and Theorem 5.7).

The definition of SBnV is induced, as a particular case, by a more general decomposition of flat currents of finite mass, which is proposed in section 3. Indeed we show that it is possible to decompose every k-dimensional flat metric current T of finite mass into two currents of finite mass T_l and T_u such that:

(a) T_l is concentrated on a $\mathcal{H}^k$- rectifiable set S;

(b) the mass of T_l is absolutely continuous with respect to $\mathcal{H}^k \llcorner S$;

(c) T_u neglects all $\mathcal{H}^k$ σ-finite sets.

One of the consequence of this decomposition is the following criterion of rectifiability for flat metric currents:

(A) a flat k-dimensional current T of finite mass on E is rectifiable if and only if for every Lipschitz function $\pi : E \to \mathbf{R}^k$ almost every slice of T with respect to π is composed of atoms (see Theorem 3.3).

This criterion has been already proved by Ambrosio and Kirchheim in [5] for normal metric currents, and by White in [13], with a different approach, for flat currents on Euclidean spaces with coefficients in normed groups.

The paper is organized as follows.

The next section contains the basic definitions and theorems (available in the first part of [5]) of geometric measure theory on metric spaces. We develop the main tools for proving criterion (A) and we introduce the notion of BV functions that take values in metric spaces (first defined by Ambrosio in [2]).

In the third section we define the decomposition of currents and we prove that the lower dimensional part of a flat current is rectifiable. In order to prove this fact we need a basic BV-estimate on the slicing of currents (first due to Jerrard and Soner in the Euclidean case and then developed by Ambrosio and Kirchheim).

In the fourth section we apply to BnV the tools just developed. Taken a function $u \in$ BnV we single out a "lower dimensional part" $[Ju]_l$ of the Jacobian and we prove that it is a rectifiable current. The remaining part of the Jacobian (namely $[Ju] - [Ju]_l$) can be split further into two currents: one that is absolutely continuous with respect to the Lebesgue measure and the other that is singular (which we call Cantor part, in analogy with the case of functions of bounded variation). Thanks to its flatness the lower dimensional part of $[Ju]$ can be represented as

$$\langle [Ju]_l, \omega\rangle = \int_{S_l} m(x)\langle \tau(x), \omega(x)\rangle d\mathcal{H}^{m-n}$$

where S_l is a $\mathcal{H}^k$-rectifiable set, $\tau(x)$ is its approximate tangent space in x and ω is any smooth $(m-n)$-form.

Then we analyze the structure of the absolutely continuous part of the Jacobian and extending a result of Müller (see [11]) we prove that it can be represented as

$$[Ju]_a = H(du_1 \wedge \ldots \wedge du_n)\mathcal{L}^m$$

where H is the Hodge star operator. Thus $[Ju]_l + [Ju]_a$ can be represented as $\nu d\mu$, where ν is a simple covector and μ is a measure. We conjecture that even the Cantor part has a similar structure but we are not able to prove it.

In the last section we define the functions of special bounded higher variation as those BnV functions whose Jacobian has zero Cantor part. Finally we prove that under suitable conditions (i.e. equiintegrability of the absolutely continuous part and equiboundedness of the Hausdorff measure of the singular supports) a closure property holds for SBnV.

2. Metric currents

Throughout the paper (E, d) is a complete metric space and $\mathrm{Lip}_b(E)$ is the space of Lipschitz and bounded real functions on E. We denote by $\mathcal{D}^k(E)$ the set of all $(k+1)$-ples $(f, g_1, \ldots, g_k)$ of functions such that $f, g_1, \ldots, g_k \in \mathrm{Lip}_b(E)$ and we refer to it as the space of k-dimensional differential forms (or simply k-forms). For every k-form $\omega = (f, g_1, \ldots, g_k)$ we define its exterior derivative as the $(k+1)$-form

$$d\omega = (1, f, g_1, \ldots, g_k). \tag{1}$$

If $\phi : F \to E$ is Lipschitz and bounded (and F is a complete metric space), we define the pull-back of ω as the k-form on F given by

$$\phi^{\#}\omega = (f\circ\phi, g_1\circ\phi, \ldots, g_k\circ\phi). \tag{2}$$

If $\omega_1 = (f, g_1, \ldots, g_n)$ and $\omega_2 = (w, h_1, \ldots, h_k)$ then their exterior product is the $(n+k)$-form

$$\omega_1 \wedge \omega_2 := (fw, g_1, \ldots, g_n, h_1, \ldots, h_k).$$

Let us fix $\omega = (f, g_1, \ldots, g_n) \in \mathcal{D}^n(E)$. For every i we define

$$\mathcal{C}_i := \{C \text{ open} |\ g_i \text{ is constant in every connected component of } C\}.$$

After setting $C_i := E\setminus (\bigcup\{U \in \mathcal{C}_i\})$ we define the closed set

$$\mathrm{supp}\,(\omega) = \mathrm{supp}\,(f) \cap \bigcap_{i=1}^{n} C_i \tag{3}$$

and we refer to it as *support* of ω.

Definition 2.1. *Let $k \in \mathbb{N}$. A k-dimensional current in E is a functional $T : \mathcal{D}^k(E) \to \mathbf{R}$ such that*

(a) $\lim_i T(f, g_1^i, \dots, g_k^i) = T(f, g_1, \dots, g_k)$ *if* $g_k^i \to g_k$ *pointwise and* $(Lip\,(g_k^i))$ *is bounded for every k;*

(b) T *is multilinear with respect to* $(f, g_1, \dots, g_k)$*;*

(c) $T(f, g_1, \dots, g_k) = 0$ *if* $\operatorname{supp}((f, g_1, \dots, g_k)) = \emptyset$.

We denote by $\mathcal{M}_k(E)$ the vector space of k-dimensional currents.

Remark 2.2. We could replace $\mathcal{D}^k(E)$ with $\mathcal{D}^k_c(E)$, namely the set of differential forms with compact support, and we could define as well a k-dimensional "local current" as a linear functional that satisfies conditions (b), (c) above and condition (a') below:

(a') $\lim_i T(f, g_1^i, \dots, g_k^i) = T(f, g_1, \dots, g_k)$ if $g_k^i \to g_k$ pointwise, $(\mathrm{Lip}\,(g_k^i))$ is bounded for every k and $\operatorname{supp}((f, g_1^i, \dots, g_k^i))$ is contained on a compact subset K for every i.

All the definitions and theorems of this paper work as well with slight modifications. Moreover in the applications to distributional Jacobians we will use local currents.

Definition 2.3. *Let T be a k-dimensional current. If there exists a σ-finite positive measure μ such that*

$$T(f, g_1, \dots, g_k) \leq \prod_{i=1}^{k} Lip\,(g_i) \int_{\mathbf{R}^n} |f| d\mu \tag{4}$$

then we say that T is of finite mass*. We call* mass *of the current T the minimal μ that satisfies (4) and we denote it with $\|T\|$. We say that T is concentrated on a Borel set B if $\|T\|(E \setminus B) = 0$.*

We denote by $\mathbf{M}_k(E)$ the vector space of k-dimensional currents of finite mass.

From now on, given a current T of finite mass we will denote by $\mathbf{M}(T)$ the total variation of $\|T\|$ in E. If T has not finite mass we set $\mathbf{M}(T) = \infty$.

Remark 2.4. We will always assume that $\|T\|$ is concentrated on a σ-compact set. However, as observed in [5], this fact can be proved if E is separable or if the cardinality of E is a Ulam number. The assumption that the cardinality of any set E is a Ulam number is consistent with the standard ZFC theory.

Definition 2.5. *Given a sequence $(T_n) \subset \mathcal{M}_k(E)$, we say that*

$$T_n \rightharpoonup T \in \mathcal{M}_k(E) \tag{5}$$

if $T_n(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^k(E)$.

Sometimes we will write $\langle T, \omega \rangle$ for $T(\omega)$. As we can see in [5] from the assumptions of definition 2.1 it follows that a k-dimensional current is always alternating in $(g_1, \dots, g_k)$; hence we use for differential forms the usual notation

$$f dg_1 \wedge \dots \wedge dg_n.$$

Sometimes, for sake of simplicity, we will denote by g the n-tuple $(g_1, \dots, g_n)$ and we will write $f dg$ for $f dg_1 \wedge \dots \wedge dg_n$. A trivial computation shows that if $\omega \in \mathcal{D}^n(E)$, $\nu \in \mathcal{D}^k(E)$ and $T \in \mathcal{M}_{n+k}(E)$, then

$$T(d(\omega \wedge \nu)) = T(d\omega \wedge \nu) + (-1)^n T(\omega \wedge d\nu).$$

Moreover every current satisfies the usual chain rule

$$T(f\, dg_1 \wedge \dots \wedge dg_n) + T(g_1\, df \wedge \dots \wedge dg_n) = T(1\, d(fg_1) \wedge \dots \wedge dg_n).$$

If $T \in \mathcal{M}_k(\mathbf{R}^k)$, then for every $g \in C^1_c(\mathbf{R}^k, \mathbf{R}^k)$ and $f \in \mathrm{Lip}\,(\mathbf{R}^k)$ we have

$$T(f dg_1 \wedge \dots \wedge dg_n) = T(f \det(\nabla g)\, dx_1 \wedge \dots \wedge dx_n)$$

(with x_i we denote the projection on the i-th coordinate of the canonical system of $\mathbf{R}^k$).

We can define a boundary operator $\partial : \mathcal{M}_k \to \mathcal{M}_{k-1}$ with the duality relation $\partial T(\omega) := T(d\omega)$; it is not difficult to see that ∂T satisfies conditions (a), (b) and (c) of 2.1, but it can fail to be of finite mass, even if T itself has finite mass.

Definition 2.6. *If T and ∂T are currents of finite mass then we call T* normal. *We denote by $\mathcal{N}^k(E)$ the vector space of normal currents.*

Remark 2.7. Given $T \in \mathcal{N}^k(E)$ we can define

$$\|T\|_N := \|T\|(E) + \|\partial T\|(E).$$

It is easy to check that $\mathcal{N}^k(E)$ endowed with the norm $\|\cdot\|_N$ is a Banach space.

Definition 2.8. *Let T be a k-dimensional current on E. We define the flat norm $\mathbf{F}(T)$ as*

$$\inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(\partial S) | S \textit{ is a } (k+1)\textit{-dimensional current}\}.$$

Definition 2.9. *Let T be a k-dimensional current. We say that T is a flat current if there exists a sequence of normal currents (T_n) such that*

$$\lim_{n\to\infty} \mathbf{F}(T_n - T) = 0.$$

It is easy to see that a current T of finite mass is flat if and only if there exists a sequence of normal currents (T_n) such that $\mathbf{M}(T_n - T) \to 0$. Indeed one implication is trivial because, for every current S, we have $\mathbf{F}(S) \le \mathbf{M}(S)$. Moreover if T is a flat current of finite mass then for every n there exist $T_n \in \mathcal{N}_k(E)$ and $S_n \in \mathcal{M}_k(E)$ such that

$$\mathbf{M}(T - T_n - \partial S_n) + \mathbf{M}(\partial S_n) \le \frac{1}{n}.$$

So we have that $T_n' := T_n + \partial S_n$ is a normal current and $\mathbf{M}(T - T_n') \leq 1/n$. A useful consequence of the 1111last statement is that for every current T we can find a sequence of normal current T_n such that

$$\lim_{n\to\infty} \mathbf{M}\left(T - \sum_{i=1}^{n} T_n\right) = 0 \tag{6}$$

and

$$\sum_{i=1}^{\infty} \mathbf{M}(T_n) < \infty. \tag{7}$$

If (6) and (7) hold we simply write

$$T = \sum_{i=1}^{\infty} T_n.$$

Definition 2.10. *We say that a k-dimensional current T of finite mass is rectifiable if it is concentrated on a k-dimensional rectifiable set and $\|T\| << \mathcal{H}^k$.*

As for the notion of boundary, we can define by duality the push-forward of currents. Indeed, given a Lipschitz and bounded map $\phi : E \to F$ and a k-dimensional current T on E it is not difficult to check that $\phi_\# T$ defined by

$$\langle \phi_\# T, \omega\rangle := T(\phi^\# \omega) \tag{8}$$

is a k-dimensional current. Moreover if T is a current of finite mass, then $\phi_\# T$ has finite mass and

$$\|\phi_\#(T)\| \leq \phi_\# \|T\|.$$

(We recall that if μ is a measure then its push forward $\phi_\#\mu$ is defined by $\phi_\#\mu(U) = \mu(\phi^{-1}(U))$.)

From these definitions one can develop a self-contained theory of normal currents in E which is equivalent to the classical theory in the Euclidean case. Hereafter we study the aspects that are useful for our purposes. We begin with the definitions of restriction and slicing.

Definition 2.11. *Let $T \in \mathcal{M}_k(E)$ and $\omega \in \mathcal{D}^h(E)$, with $h \leq k$. We define the* restriction of T to ω *as the $(k-h)$-dimensional current given by*

$$T \llcorner \omega\,(\nu) := T(\omega \wedge \nu).$$

Remark 2.12. If T is a current of finite mass, then we can extend its action to the $(k+1)$-ples $(f, g_1, \ldots, g_k)$ such that $g_i \in \mathrm{Lip}_b(E)$ and f is bounded and Borel measurable. Indeed, $T \llcorner dg$ is a 0-dimensional current of finite mass and so there exists a finite measure μ_g such that

$$\langle T, w\, dg\rangle = \langle T \llcorner dg, w\rangle = \int_E w\, d\mu_g \qquad \text{for every } w \in Lip_b(E) \tag{9}$$

and

$$\|\mu_g\|_{\mathrm{var}} \leq \|T \llcorner dg\|(E) \leq \left(\prod_{i=1}^{k} \mathrm{Lip}(g_i) \right) \|T\|(E).$$

Using equation (9) the action of $T \llcorner dg$ can be easily extended to every Borel measurable and bounded function. Of course, if $f^k \to f$ uniformly, $g_i^k \to g_i$ pointwise and $(\mathrm{Lip}\,(g_i^k))$ is bounded for every i, then

$$\langle T, f^i dg_1^i \wedge \ldots \wedge dg_k^i \rangle \to \langle T, f dg_1 \wedge \ldots \wedge dg_k \rangle.$$

From this last last remark it follows that if T is a current of finite mass then for every Borel set A we can define the current $T \llcorner A$:

$$\langle T \llcorner A, f dg \rangle := \langle T \llcorner \chi_A, f dg \rangle = \langle T, f \chi_A \, dg \rangle.$$

Moreover $\|T \llcorner \chi_A\| \leq \|T\|$.

Theorem 2.13. *Let T be a k-dimensional normal current in E and π a Lipschitz function from E to $\mathbf{R}^h$, with $h \leq k$. Then there exist normal $(k-h)$-dimensional currents $\langle T, \pi, x \rangle$ such that:*

(i) *$\langle T, \pi, x \rangle$ and $\partial \langle T, \pi, x \rangle$ are concentrated on $E \cap \pi^{-1}(x)$;*

(ii) *for every $\psi \in C_c(\mathbf{R}^h)$,*

$$\int_{\mathbf{R}^h} \langle T, \pi, x \rangle \psi(x) d\mathcal{L}^h = T \llcorner (\psi \circ \pi) d\pi; \tag{10}$$

(iii)

$$\int_{\mathbf{R}^h} \|\langle T, \pi, x \rangle\| d\mathcal{L}^h = \|T \llcorner d\pi\|. \tag{11}$$

We refer to [5] for the proof. Such a map $\langle T, \pi, x \rangle$ is called *slicing of T with respect to π*. The previous theorem can be easily extended to flat currents.

Theorem 2.14. *Let T be a k-dimensional flat current of finite mass on E and $\pi : E \to \mathbf{R}^h$ a Lipschitz function (with $h \leq k$). Then there exist $(k-h)$-dimensional flat currents $\langle T, \pi, x \rangle$ of finite mass such that:*

(i) *$\langle T, \pi, x \rangle$ is concentrated on $E \cap \pi^{-1}(x)$;*

(ii) *for every $\psi \in C_c(\mathbf{R}^h)$,*

$$\int_{\mathbf{R}^h} \langle T, \pi, x \rangle \psi(x) d\mathcal{L}^h = T \llcorner (\psi \circ \pi) d\pi; \tag{12}$$

(iii)

$$\int_{\mathbf{R}^h} \|\langle T, \pi, x \rangle\| d\mathcal{L}^h = \|T \llcorner d\pi\|. \tag{13}$$

Proof Let T_n be a sequence of normal currents such that

$$T = \sum_{i=1}^{\infty} T_n.$$

From Theorem 2.13 we have that there exist normal $(k-h)$-dimensional currents $\langle T_n, \pi, x\rangle$ that verifies conditions (a), (b) and (c) above. Let us think of $\langle T_n, \pi, x\rangle$ as an L^1 function of x that takes values on the Banach space $\mathbf{M}_{k-h}(E)$ (endowed with the norm $\mathbf{M}$). Condition (c) and inequality (7) imply that

$$\sum_{i=1}^{\infty} \langle T_n, \pi, \cdot\rangle \tag{14}$$

is a totally convergent series in $L^1(\mathbf{R}^h, \mathbf{M}_{k-h}(E))$. We define $\langle T, \pi, \cdot\rangle$ as the sum of (14). It is easy to check that T verifies conditions (i), (ii) and (iii). Moreover we can extract a subsequence $T_{j(n)}$ such that for $\mathcal{L}^h$-a.e. $x \in \mathbf{R}^h$

$$\lim_{n\to\infty} \mathbf{M}\left(\langle T, \pi, x\rangle - \sum_{i=1}^{n} \langle T_{j(n)}, \pi, x\rangle\right) = 0.$$

We conclude that, for $\mathcal{L}^h$ a.e. x, $\langle T, \pi, x\rangle$ is a flat current of finite mass.

$\square$

As we will see at the end of this section the slicing map of a normal current has a remarkable property. In order to state it we need the definition of map of bounded variation from an open set of $\mathbf{R}^n$ to a weakly separable metric space (M, d) (see [5] and [2]).

Definition 2.15. *We say the metric space (M, d) is weakly separable if there exists a countable family $\mathcal{F} \subset Lip_b(M)$ such that*

$$d(x, y) = \sup_{\varphi\in\mathcal{F}} |\varphi(x) - \varphi(y)| \qquad \textit{for every } x, y \in M. \tag{15}$$

Definition 2.16. *Let $\{\mu_i\}_{i\in I}$ be a family of positive measures μ on E. Then for every Borel subset of E we define*

$$\bigvee_{i\in I} \mu_i(B) := \sup\left\{\sum_{i\in J} \mu_i(B_i) |\ B_i \textit{ are pairwise disjoint and } \bigcup_{i\in J} B_i = B\right\}$$

where J runs through all countable subsets of I.

Definition 2.17. *Let $U \subset \mathbf{R}^k$ be an open set, (M, d) a weakly separable metric space and $u : U \to M$. We say that u is of metric bounded variation if*

(a) *$\varphi \circ u$ is of locally bounded variation for every $\varphi \in \mathcal{F}$;*
(b)

$$\|Du\|_{MBV} := \bigvee_{\varphi\in\mathcal{F}} |D(\varphi \circ u)|(\Omega) < \infty.$$

We remark that this definition does not depend on the choice of $\mathcal{F}$ and that

$$\|Du\|_{MBV} = \bigvee_{\varphi \in \mathrm{Lip}_b(M)} |D(\varphi \circ u)|(\Omega)$$

(see [5] for the proofs). From now on we will denote by $\|Du\|$ the measure $\bigvee |D(\varphi \circ u)|$.

The key of the proof of Theorem 3.2 in the next section is the fact that the slicing map of a k-dimensional normal current T with respect to $\pi \in \mathrm{Lip}_b(E, \mathbf{R}^k)$ is a map of metric bounded variation if we endow $\mathbf{M}_0(E)$ with the flat norm

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in \mathrm{Lip}_b(E),\ \mathrm{Lip}(\phi) \leq 1\}.$$

This observation, due to Jerrard and Soner in the case of weak Jacobians ([9]), has been developed by Ambrosio and Kirchheim ([5]) in the framework of normal currents. (With a little effort one can see that the last definition of flat norm coincide with that given in Definition 2.9 when $E = \mathbf{R}^n$.)

Theorem 2.18. *Let E be a weak separable metric space, T a normal n-dimensional current in E and $\pi : E \to \mathbf{R}^n$ a Lipschitz map. Then the slicing map*

$$\mathcal{S} : \mathbf{R}^n \ni x \to \langle T, \pi, x\rangle \in \mathbf{M}_0(E)$$

is metric bounded variation if we endow $\mathbf{M}_0(E)$ with the flat norm. Moreover the MBV semi-norm of $\langle T, \pi, x\rangle$ is bounded by the norm of T in $\mathcal{N}_k(E)$.

Proof. With little effort one can see that there is a countable family $\mathcal{F} \subset \mathrm{Lip}_b(E)$ such that

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in \mathcal{F}\}$$

and $\mathrm{Lip}(\phi) \leq 1$ for every $\phi \in \mathcal{F}$. We can think of $\phi \in \mathcal{F}$ as a Lipschitz real function defined on $\mathbf{M}_0$. Then (recall Definition 2.17) we will show that

(a) for every such ϕ, $\phi \circ \mathcal{S}(x) = \langle \mathcal{S}(x), \phi\rangle$ is a function of locally bounded variation (as real-valued function of x);

(b)
$$\bigvee_{\phi \in \mathcal{F}} |D(\mathcal{S} \circ \phi)| \leq n\pi_\#\|T\| + n\pi_\#\|\partial T\|.$$

Indeed, fix a bounded ϕ such that $\mathrm{Lip}(\phi) \leq 1$. If we consider a test function $\psi \in C^1_c(\mathbf{R}^n)$, then

$$\begin{aligned}
(-1)^{i-1} \int_{\mathbf{R}^n} \mathcal{S}(\phi(x))\partial_i\psi(x)dx &= (-1)^{i-1} T \llcorner d\pi(\phi\, \partial_i\psi \circ \pi) \\
&= T(\phi\, d(\psi \circ \pi) \wedge d\tilde{\pi}_i) \\
&= \partial T(\phi(\psi \circ \pi)d\tilde{\pi}_i) - T(\psi \circ \pi d\phi \wedge d\tilde{\pi}_i) \\
&\leq \|\partial T\|(\psi \circ \pi) + \|T\|(\psi \circ \pi),
\end{aligned}$$

where $d\tilde{\pi} = d\pi_1 \wedge \ldots \wedge d\pi_{i-1} \wedge d\pi_{i+1} \wedge \ldots \wedge d\pi_{m-n}$. Then $\phi \circ S$ is a function of locally bounded variation and

$$|D(\phi \circ \mathcal{S})| \leq n\pi_{\#}\|T\| + n\pi_{\#}\|\partial T\|.$$

□

3. Decomposition of currents and rectifiability theorem

Given a k-dimensional current T of finite mass we can find a $\mathcal{H}^k$ σ-finite set L_T such that:

$\|T\|(F) = 0$ whenever $\mathcal{H}^k(F) < \infty$ and $E \cap F = \emptyset$.
We construct this set a follows. Let us consider

$$K = \sup\{\|T\|(L)|L \text{ is } \mathcal{H}^k \ \sigma\text{-finite}\}.$$

We choose a sequence (L_n) of $\mathcal{H}^k$ σ-finite sets such that $\|T\|(L_n) \uparrow K$ and we put $L_T = \cup L_n$. Then we have that L_T is $\mathcal{H}^k$ σ-finite and $\|T\|(L_T) = K$: hence L_T has the desired properties.

Definition 3.1. *Let T be a k-dimensional current of finite mass and L_T be defined as above. Then we define*

$$\begin{cases} T_u := T \llcorner (E \backslash L_T) \\ T_l := T \llcorner (L_T) \end{cases} \tag{16}$$

and we refer to T_l as lower dimensional part of T.

Of course $\|T_u\|$ and $\|T_l\|$ are mutually singular and $T_u + T_l = T$. Moreover $\|T_u\| \leq \|T\|$ and $\|T_l\| \leq \|T\|$. If E is $\mathcal{H}^p$ σ-finite for some $p > k$ then we define $\|T\|_a$ as the absolutely continuous part of $\|T\|$ with respect to $\mathcal{H}^p$. Of course $\|T\|_a$ and $\|T_l\|$ are mutually singular and so there exists a Borel set A_T disjoint from L_T such that $\|T\| \llcorner A_T = \|T\|_a$. Therefore we can define

$$\begin{aligned} T_a &= T_u \llcorner A_T \\ T_c &= T_u \llcorner (E \backslash A_T) \end{aligned} \tag{17}$$

and we refer to T_a and T_c as, respectively, *absolutely continuous part* and *cantor part* of T. Notice that $T_a + T_c + T_l = T$.

When T is a flat current of finite mass it is easy to see that there is a Borel set R_T such that $\|T_l\|$ is absolutely continuous with respect to the measure $\mathcal{H}^k \llcorner R_T$. The main result of this section is that in this case T_l is a rectifiable current: for proving it we need only check that R_T is rectifiable.

Theorem 3.2. *If E is separable and T is a k-dimensional flat current of finite mass on E, then T_l is a rectifiable current.*

A consequence of this fact is the following criterion of rectifiability, obtained in another framework by White in [13].

Theorem 3.3. *Let T be a k-dimensional flat current of finite mass on a separable metric space E. Then T is rectifiable if and only if for every Lipschitz function $\pi : E \to \mathbf{R}^k$ and for $\mathcal{L}^k$ a.e. $x \in \mathbf{R}^k$ the sliced current $\langle T, \pi, x\rangle$ is supported on a finite number of points.*

We remark that one implication is trivial. Indeed if T is rectifiable and $\pi : E \to \mathbf{R}^k$ is Lipschitz, then $\langle T, \pi, x\rangle$ is concentrated on $\pi^{-1}(\{x\})$, which for almost every x consists of a finite number of points.

Before proving Theorem 3.2 and the other implication of Theorem 3.3 we need some tools.

Theorem 3.4. *Let E be a separable metric space and let us endow $\mathbf{M}_0(E)$ with the norm*

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in Lip_b(E),\ Lip(\phi) \leq 1\}.$$

If $\mathcal{S} \in MBV(\mathbf{R}^k, E)$ and $K \subset E$ is a compact set then there exists an $\mathcal{L}^k$-negligible set $A \in \mathbf{R}^k$ such that

$$S := \{y \in K | \|\mathcal{S}(x)\|(\{y\}) > 0 \quad \textit{for some } x \in \mathbf{R}^k \backslash A\}$$

is countably $\mathcal{H}^k$-rectifiable.

Before proving this theorem we introduce the notion of maximal function for MBV mappings. Given a function $u \in MBV(\mathbf{R}^k, M)$, where M is a weakly separable metric space, we set

$$MDu(x) := \sup_{\rho>0} \frac{\|Du\|(B_\rho(x))}{\omega_k \rho^k}$$

(MDu is known in the literature as the *maximal function* of the measure $\|Du\|$). It is not difficult to see that this function is finite for almost every x: in fact we can estimate $\mathcal{L}^k(\{MDu > \lambda\}$ from above with a constant times $\|Du\|(\mathbf{R}^k)/\lambda$. As it happens for classical real-valued functions of bounded variation, MDu provides a Lipschitz property for u.

Lemma 3.5. *Let (M,d) be weakly separable and $\mathcal{S} : \mathbf{R}^k \to M$ a map of metric bounded variation. Then there exists $N \subset \mathbf{R}^k$ of measure zero such that*

$$d(\mathcal{S}(x), \mathcal{S}(y)) \leq c(MD\mathcal{S}(x) + MD\mathcal{S}(y))|x-y|, \qquad \forall x, y \in \mathbf{R}^k \backslash N \tag{18}$$

where c depends only on k

Proof Let us choose a family $\mathcal{F}$ of weakly dense Lipschitz functions. Then for every $\varphi \in \mathcal{F}$ we define L_φ as the set of Lebesgue points of $\varphi \circ \mathcal{S}$ (which is a real function on $\mathbf{R}^k$). For every $x, y \in L_\varphi$ we claim that inequality (18) holds with $w = \varphi \circ \mathcal{S}$ in place of $\mathcal{S}$.

Indeed, let us choose a ball B of radius $R = |x-y|/2$ centered at $(x-y)/2$; we obtain

$$\begin{aligned}
|w(x) - w(y)| &= \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(y)|}{|x-y|} dz \\
&\leq \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(z)|}{|x-z|} dz + \frac{1}{\omega_k R^k} \int_B \frac{|w(z) - w(y)|}{|z-y|} dz \\
&\leq \frac{1}{\omega_k R^k} \left(\int_{B_{2R}(x)} \frac{|w(x) - w(z)|}{|x-z|} dz + \int_{B_{2R}(y)} \frac{|w(y) - w(z)|}{|y-z|} dz \right)
\end{aligned}$$

Moreover we have

$$\frac{1}{\omega_k(2R)^k}\int_{B_{2R}(x)}\frac{|w(x)-w(z)|}{|x-z|}dz \le \int_0^1 \frac{|Dw|(B_{tR}(x))}{\omega_k(tR)^k)}dt \le MDw(x),$$

and the claim easily follows.

Now, if we consider $\bigcap_{\varphi\in\mathcal{F}} L_{\varphi\circ\mathcal{S}}$, recalling that

$$d(\mathcal{S}(x),\mathcal{S}(y)) = \sup_{\varphi\in\mathcal{F}}|\varphi\circ\mathcal{S}(x)-\varphi\circ\mathcal{S}(y)|$$

and

$$\begin{aligned} MD\mathcal{S}(x) &= \sup_{\rho>0}\frac{\|D\mathcal{S}\|(B_\rho(x))}{\omega_k\rho^k} \\ &\ge \sup_{\rho>0}\frac{\|D(\varphi\circ\mathcal{S})\|(B_\rho(x))}{\omega_k\rho^k} = MD(\varphi\circ\mathcal{S})(x), \end{aligned}$$

we obtain (18).

□

Proof of Theorem 3.4 First of all we set

$$A := N_1 \cup \{x\in\mathbf{R}^k | MD\mathcal{S}(x)=\infty\},$$

where N_1 is the set of measure zero that plays the role of N in Lemma 3.5. Of course $\mathcal{H}^k(A)=0$.

Following [5] we define $Z_{\varepsilon,\delta}$ as the set of points $z\in\mathbf{R}^k\backslash A$ such that

(a) $MD\mathcal{S}(z)\le 1/(2\varepsilon)$;

(b) for every $x\in K$ such that $\|\mathcal{S}(z)\|(\{x\})\ge\varepsilon$ there holds $\|\mathcal{S}(z)\|(B_{3\delta}(x)\backslash\{x\})\le\varepsilon/3$.

Next we define $R_{\varepsilon,\delta} := \{x\in E \,|\, \|\mathcal{S}(z)\|(\{x\})\ge\varepsilon \text{ for any } z\in Z_{\varepsilon,\delta}\}$. Observing that

$$S = \bigcup_{\varepsilon,\delta} R_{\varepsilon,\delta},$$

we will prove that for each ε,δ the set $R_{\varepsilon,\delta}$ is $\mathcal{H}^k$-rectifiable. Indeed, for every $x,x'\in R_{\varepsilon.\delta}$ and every $z,z'\in Z_{\varepsilon,\delta}$ such that

(i) $\|\mathcal{S}(z)\|(\{x\})\ge\varepsilon$, $\|\mathcal{S}(z)\|(\{x'\})\ge\varepsilon$,

(ii) $d(x,x')\le\delta$,

it holds

$$d(x,x') \le \frac{3c(\delta+1)}{\varepsilon^2}|z-z'|. \tag{19}$$

Before proving this estimate we remark it implies that $R_{\varepsilon,\delta}\cap B$ is the image of a Lipschitz function whenever $\operatorname{diam}(B)\le\delta$: indeed for any $z\in Z_{\varepsilon,\delta}\cap B$ there is an only one $x=f(z)\in R_{\varepsilon,\delta}$ such that $\|\mathcal{S}\|(x)\ge\varepsilon$; moreover f is Lipschitz and, if D is the domain of f, $f(D)=B\cap R_{\varepsilon,\delta}$.

Now we complete the proof by showing that (19) holds. Let us set $d=d(x,x')$ and consider a Lipschitz function $\phi: E\to\mathbf{R}$ such that (a') $\phi(y) = d(y,x)$ for every $y \in B_d(x)$;

(b') $\phi \equiv 0$ in $\mathbf{R}^m \backslash B_{2\delta}(x)$;

(c') $\sup|\phi| = d$ and $\mathrm{Lip}(\phi) \leq 1$.

We have $|(\mathcal{S}(z))(\phi)| \leq \varepsilon\delta/3$ and $|(\mathcal{S}(z'))(\phi)| \geq \varepsilon\delta - \varepsilon\delta/3$, so we get

$$\begin{aligned}\frac{\varepsilon}{3}d(x,x')| &\leq |(\mathcal{S}(z'))(\phi) - (\mathcal{S}(Z))(\phi)| \leq (\delta+1)\mathbf{F}(\mathcal{S}(z) - \mathcal{S}(z')) \\ &\leq c(\delta+1)(MD\mathcal{S}(z) + MD\mathcal{S}(z'))|z - z'|.\end{aligned}$$

Recalling that $MD\mathcal{S}(z) \leq 1/(2\varepsilon)$, we obtain the desired estimate.

□

Proof of Theorem 3.2 We need only prove that T_l is concentrated on a $\mathcal{H}^k$ σ-rectifiable set. First let us fix a Lipschitz function $\pi : E \to \mathbf{R}^k$. We want to prove that $T_l \llcorner d\pi$ is concentrated on a rectifiable set. We set $\mathcal{S}_\pi(x) = \langle T, d\pi, x\rangle$ and we claim that

(S) there exists a set $N \subset \mathbf{R}^k$ such that $\mathcal{L}^k(N) = 0$ and

$$S_\pi := \{y \in E | \|\mathcal{S}_\pi(x)\|(\{y\}) > 0 \quad \text{for some } x \in \mathbf{R}^k \backslash N\}$$

is countably rectifiable.

For proving it let us choose a sequence T_n of normal currents such that

(i) $\mathbf{M}(T - T_n) \to 0$;

(ii) there exists a set $N^\infty \subset \mathbf{R}^k$ such that $\mathcal{L}^k(N^\infty) = 0$ and

$$\lim_{n\to\infty} \mathbf{M}(\mathcal{S}(x) - \langle T_n, \pi, x\rangle) = 0$$

for every $x \in \mathbf{R}^k \backslash N^\infty$.

To simplify the notation we write $\mathcal{S}_n(x) = \langle T_n, \pi, x\rangle$. We remark that if (μ_n) is a sequence of finite measures and there exists a measure μ such that $\|\mu_n - \mu\|_{var} \to 0$, then the set of atoms of μ is contained in the union of the sets of atoms of μ_n. Recalling that $\mathbf{M}_0(E)$ can be represented as the space of finite measures on E, we conclude that for almost every $x \in \mathbf{R}^k \backslash N^\infty$

$$\{z | \|\mathcal{S}_\pi(x)\|(z) > 0\} \subset \bigcup_n \{z | \|\mathcal{S}_n(x)\|(z) > 0\}.$$

Using Theorem 3.4 we infer that for every i there exists a set $N^i \subset \mathbf{R}^k$ of measure zero such that

$$S^i := \{z \in E | \|\mathcal{S}_i(x)\|(z) > 0 \quad \text{for some } x \in \mathbf{R}^k \backslash N_i\}$$

is countably rectifiable. If in statement (A) we set

$$N = N^\infty \cup \bigcup_i N_i$$

then we have

$$S_\pi \subset \bigcup_{i=1}^{\infty} S^i.$$

We conclude that S_π is countably rectifiable.

Now, let us prove that

$$\|T_l \llcorner d\pi\|(E \backslash S_\pi) = 0. \tag{20}$$

Recalling Definition 3.1 we must only check that $\|T \llcorner d\pi\|(A) = 0$ for every $\mathcal{H}^k$ σ-finite set A such that $A \cap S_\pi = \emptyset$.

Since A is $\mathcal{H}^k$ σ-finite, for a.e. $x \in \mathbf{R}^k$, $(\pi^{-1}\{x\}) \cap A$ contains at most a countable number of points; this fact combined with $A \cap S_\pi = \emptyset$ implies that, for a.e. $x \in \mathbf{R}^k$, $\|\mathcal{S}_\pi(x)\|(A) = 0$. Then we have

$$\|T \llcorner d\pi\|(A) = \int_{\mathbf{R}^n} \|\mathcal{S}_\pi(x)\|(A) d\mathcal{L}^k(x) = 0$$

and this proves (20).

Now we recall that $\|T\|$ is concentrated on a σ-compact set; because of this fact it can be proved (see [5], Lemma 5.4) that there exists a countable set $D \subset \mathrm{Lip}_1(E) \cap \mathrm{Lip}_b(E)$ such that

$$\|T\| = \bigvee \{ \|T \llcorner d\pi\| | \pi_1, \dots, \pi_k \in D \}. \tag{21}$$

If we take the countably rectifiable set

$$S := \bigcup \{ S_\pi | \pi_1, \dots, \pi_k \in D \},$$

then from (21) it follows that $\|T_l\|(E/S) = 0$.

□

Notice that if we are in the hypotheses of Theorem 3.3, then we can reason as in the previous case. In fact we have that for every π, $T \llcorner d\pi$ is concentrated on a $\mathcal{H}^k$ rectifiable set. Then it follows that T is concentrated on a $\mathcal{H}^k$ rectifiable set and coincides with T_l.

Remark 3.6. Assuming that every set has a cardinality that is a Ulam number, we can drop the assumption that E is separable (see Remark 2.4).

4. Distributional Jacobians and BnV functions

In this section we are going to transpose some definitions and concepts from [9] in the language introduced above. We will work with differential forms with compact support and local currents, but this does not create any problem, as observed in Remark 2.2. Finally we recall that what we call differential forms in this paper are not the usual Lipschitz differential forms: in terms of the classical theory $\mathcal{D}^k_c(\mathbf{R}^n)$ is the set of Lipschitz *simple* differential forms with compact support.

Definition 4.1. *We define the k-dimensional local current H_k in $\mathbf{R}^k$ as*

$$H_k(f dg) = \int_{\mathbf{R}^k} f \det(\nabla g_1, ..., \nabla g_k) d\mathcal{L}^k.$$

The continuity axiom (Definition 2.1, condition (c)) is satisfied because the Jacobian determinant is weakly* continuous in $W^{1,\infty}$. We remark that the classical Hodge star-operator assigns to every $\omega \in \mathcal{D}^k_c(\mathbf{R}^n)$ the local $(n-k)$-dimensional current given by $H_n \llcorner \omega$.

In the definition of $H_n \llcorner \omega$ the regularity assumptions on ω can be weakened. In particular, let us suppose that $\omega = f dg$ satisfies

1. $f \in L^p$;
2. $g \in W^{1,q}(\mathbf{R}^n, \mathbf{R}^k)$;
3. $1/p + k/q = 1/r < 1$.

Then it is well known (see for example [10] or [8]) that the map $F : W^{1,\infty}(\mathbf{R}^n, \mathbf{R}^{n-k}) \to L^r(\mathbf{R}^n)$ given by

$$F(u) = f \det(\nabla g_1, \ldots, \nabla g_k, \nabla u_1, \ldots, \nabla u_{n-k})$$

is continuous if we endow $L^r(\mathbf{R}^n)$ with the weak topology and $W^{1,\infty}(\mathbf{R}^n, \mathbf{R}^{n-k})$ with the weak* one. Even in this case, with a slight abuse of notation, we define $H_n \llcorner \omega$ as the k-dimensional local current T given by

$$T(f dg) = \int_{\mathbf{R}^n} f \det(\nabla g_1, \ldots, \nabla g_k, \nabla u_1, \ldots, \nabla u_{n-k}) d\mathcal{L}^n.$$

We will see below that this is a crucial point in the definition of weak Jacobians. In the rest of this section U will denote an open set.

Definition 4.2. *Let $u \in W^{1,p}_{loc}(U, \mathbf{R}^n) \cap L^\infty$ with $U \subset \mathbf{R}^m$, $p \geq n-1$ and $m > n$ (or $u \in W^{1,mn/(m+1)}$). We define $j(u)$ as the $(m-n+1)$-dimensional local current $(-1)^n H_m \llcorner (u^1 \wedge du_2 \wedge \ldots \wedge du_n)$.*

Definition 4.3. *Let u be as in the previous definition. Then we define*

$$[Ju] := \partial j(u).$$

We say that $u \in \mathrm{BnV}(U, \mathbf{R}^n)$ if $j(u)$ is a normal local current.

The last definition is motivated as follows. Let us put $\nu = u_1 du_2 \wedge \ldots \wedge du_n$ and suppose that u is sufficiently regular (i.e. Lipschitz). Then we have

$$\begin{aligned} \langle [Ju], \omega \rangle &= \langle j(u), d\omega \rangle = (-1)^n H_m(\nu \wedge d\omega) \\ &= (-1)^n H_m((-1)^{n-1}(d(\nu \wedge \omega) - d\nu \wedge \omega))) \\ &= -\partial H_m(\nu \wedge \omega) + H_m(d\nu \wedge \omega) \\ &= H_m \llcorner d\nu(\omega). \end{aligned}$$

We remark that in view of this fact we could have defined $j(u)$ as

$$\mathrm{sign}\,(\pi) H_m \llcorner (u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}),$$

where π is any permutation of the set $\{1, \ldots, n\}$. Indeed , if u is a smooth function, then

$$(-1)^n \partial (H_m \llcorner u_1 du_2 \wedge \ldots \wedge du_n) = H_m \llcorner d(u_1 du_2 \wedge \ldots \wedge du_n)$$

$$
\begin{aligned}
&= H_m \llcorner (du_1 \wedge du_2 \wedge \ldots \wedge du_n) \\
&= H_m \llcorner \operatorname{sign}(\pi)(du_{\pi(1)} \wedge du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}) \\
&= (-1)^n \operatorname{sign}(\pi) H_m \llcorner d(u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}) \\
&= (-1)^n) \operatorname{sign}(\pi) \partial (H_m \llcorner u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}). \qquad (22)
\end{aligned}
$$

These equalities follows from the fact that, if T is a k-dimensional local current and ω is an h-dimensional form with $k \leq h-1$, then

$$\partial T \llcorner \omega = (\partial T) \llcorner \omega - T \llcorner d\omega.$$

Now, approximating every $u \in \mathrm{BnV}$ by convolutions with standard mollifiers, we obtain the identity (22) in its full generality. Indeed it is easy to see that if $u_n \to u$ in the strong Sobolev topology and $\|u_n\|_\infty \leq c$, then $j(u_n)$ converges to $j(u)$ as local current.

Actually $j(u)$ satisfies a stronger continuity result: appropriate weak convergence of the functions induces weak convergence on the Jacobians. More precisely

Theorem 4.4. *Suppose that $(u_n), u$ satisfy the conditions of Definition 4.2 and*

(a) *$u_k \rightharpoonup u$ weakly in W^{1,p_1}_{loc};*

(b) *$u_k \to u$ strongly in $L^{p_2}_{loc}$;*

(c) *$(n-1)/p_1 + 1/p_2 < 1$*

(or $u_k \rightharpoonup u$ in $W^{1,n-1}_{loc}$, $u_k \in C(U)$ and $u_k \to u$ uniformly on compact sets).

Then $j(u_n) \rightharpoonup j(u)$ as local current.

(For the proof of this theorem we refer to the weak continuity of Jacobian determinant maps, [10], [8]). If the hypotheses of the previous theorem hold then we have

$$\langle [Ju_n], \omega \rangle = \langle \partial j(u_n), \omega \rangle = \langle j(u_n), d\omega \rangle \to \langle j(u), d\omega \rangle = \langle [Ju], \omega \rangle.$$

Now let us see how the local current $[Ju]$ behaves with respect to slicing when $u \in \mathrm{BnV}$. Let us consider a projection π of $\mathbf{R}^m$ onto a subspace of dimension $m-k \leq m-n$. For the sake of simplicity we choose a system of coordinates and we suppose that π is the projection on the first $m-k$ coordinates: we will adopt the notation $\mathbf{R}^m \ni z = (x,y) \in \mathbf{R}^{m-k} \times \mathbf{R}^k$.

We notice that, for a.e. $x \in \mathbf{R}^{m-k}$, $j(u(x,\cdot))$ is a $(x1k-n+1)$ dimensional local current in $\mathbf{R}^k$: indeed, because of the Fubini-Tonelli Theorem, for a.e. x the map $u(x,\cdot)$ belongs to the appropriate Sobolev space that allows us to define

$$j(u(x,\cdot)) := (-1)^n H_k \llcorner u_1(x,\cdot) d_y \tilde{u}(x,\cdot).$$

Definition 4.5. *We denote by i^x the natural identification between $\mathbf{R}^k$ and the affine subspace $\{x\} \times \mathbf{R}^k$ of $\mathbf{R}^m$.*

Theorem 4.6. *Let u be as in Definition 4.2 and $\pi : \mathbf{R}^{m-k} \times \mathbf{R}^k \to \mathbf{R}^{m-k}$ a projection, with $k \geq n$. Then we have*

$$\langle j(u), d\pi, x\rangle = (-1)^l (i^x)_\# j(u(x,\cdot)), \tag{23}$$

$$\langle [Ju], d\pi, x\rangle = (-1)^r (i^x)_\# [Ju(x,\cdot)]. \tag{24}$$

with $l = (m-k)(n-1)$ and $r = (m-k)n$.

Proof We use the notations of the previous paragraph to simplify the calculations. We observe that

$$j(u) \llcorner d\pi = H_m \llcorner (u_1 du_2 \wedge \ldots \wedge du_n \wedge dx_1 \ldots \wedge dx_{m-k}).$$

So, for every $f dg \in \mathcal{D}_c^{k-n+1}$, we have

$$H_m \llcorner (u_1\, du_2 \wedge \ldots \wedge du_n \wedge d\pi)(f dg)$$

$$\begin{aligned} &= \int_{\mathbf{R}^m} f u_1 \det(\nabla u_2, \ldots, \nabla u_n, e_1, \ldots, e_{m-k} \nabla g_1, \ldots, \nabla g_{k-n+1}, dz \qquad (25)\\ &= (-1)^{(m-k)(n-1)} \int_{\mathbf{R}^m} f u_1 \det(e_1, \ldots, e_{m-k}, \nabla \tilde{u}, \nabla g)\, dz, \end{aligned}$$

where $e_1, \ldots, e_k$ are the first $m-k$ vectors of the canonical basis and $\tilde{u}$ denotes the vector $(u_2, \ldots, u_n)$.

We remark that the matrix $(e_1, \ldots, e_{m-k}, \nabla\tilde{u}, \nabla g)$ can be written as

$$\begin{pmatrix} Id & \nabla_x \tilde{u} & \nabla_x g \\ 0 & \nabla_y \tilde{u} & \nabla_y g \end{pmatrix}$$

(where Id is the identical $k \times k$ matrix, 0 is the $(m-k) \times k$ null matrix). Therefore $(\nabla_y \tilde{u}, \nabla_y g)$ is a $(k \times k)$ matrix and

$$\det(e_1, \ldots, e_{m-k}, \nabla\tilde{u}, \nabla g) = \det(\nabla_y \tilde{u}, \nabla_y g);$$

this means that (25) is equal to

$$(-1)^l \int_{\mathbf{R}^{m-k}} \int_{\mathbf{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\, dy\, dx. \tag{26}$$

Then the expression

$$\int_{\mathbf{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\, dy$$

can be read as

$$\langle (i^x)_\# j(u(x,\cdot)), f\, dg\rangle.$$

We conclude from (26) that

$$\mathbf{R}^k \ni x \to \mathcal{S}(x) = (-1)^l (i^x)_\# j(u(x,\cdot))$$

is the slicing for $j(u)$ with respect to $d\pi$.

Notice that

$$[Ju] \llcorner d\pi = (-1)^{m-k}(\partial(j(u) \llcorner d\pi) - j(u) \llcorner d(d\pi)) = (-1)^{m-k} \partial(j(u) \llcorner d\pi);$$

hence $(-1)^r (i^x)_\# [Ju(x,\cdot)]$ is a slicing map for $[Ju]$.

$\square$

For the sake of simplicity from now on we will identify the local current $[Ju(x,\cdot)]$ and its push-forward via i^x.

Using the decomposition defined in the previous section, when $u \in \mathrm{BnV}$ we can consider $[Ju]_a$, $[Ju]_c$ and $[Ju]_l$; moreover from theorem (3.2) it follows that $[Ju]_l$ is a rectifiable local current. From now on we define S_u as the set on which $[Ju]_l$ is concentrated.

From the classical theory we know that there exists a Borel function ν from $\mathbf{R}^m$ to the linear space of $m-n$ covectors $\Lambda_{m-n}(\mathbf{R}^m)$ such that

$$\langle [Ju], \omega\rangle = \int_{\mathbf{R}^m} \langle \nu(x), \omega(x)\rangle d\|Ju\| \tag{27}$$

(where $\|Ju\|$ is the mass of $[Ju]$). Of course similar representations hold for the three parts of $[Ju]$:

$$\langle [Ju]_a, \omega\rangle = \int_{\mathbf{R}^m} \langle \nu_a(x), \omega(x)\rangle d\|Ju\|_a$$

$$\langle [Ju]_c, \omega\rangle = \int_{\mathbf{R}^m} \langle \nu_c(x), \omega(x)\rangle d\|Ju\|_c$$

$$\langle [Ju]_l, \omega\rangle = \int_{S_u} \langle \nu_l(x), \omega(x)\rangle d\mathcal{H}^{m-n}.$$

In fact we can say a little more: from the fact that $[Ju]$ is a flat current it follows that

$$\nu_l(x) = m(x)\tau(x) \qquad \text{for } \mathcal{H}^{m-n} \text{ a.e. } x,$$

where $\tau(x)$ is the approximate tangent plane to S_u in x and m is a Borel measurable real-valued function.

Even for the absolutely continuous part we have a similar property. Indeed from a result of Müller ([11]) we know that if $u \in \mathrm{BnV}(U, \mathbf{R}^n)$ with $U \subset \mathbf{R}^n$, then $\det(\nabla u) \in L^1_{loc}$ and $[Ju]_a = \det(\nabla u)\mathcal{L}^n$. In theorem 4.8 we will prove a slight generalization of this result, namely that for a general BnV function u we can represent $[Ju]_a$ as

$$[Ju]_a = H_m \llcorner (du_1 \wedge \ldots \wedge du_n). \tag{28}$$

Lemma 4.7. *Let* $u \in \mathrm{BnV}(U, \mathbf{R}^n)$*, with* $U \subset \mathbf{R}^n$*, and let us choose a pointwise representative* $\tilde{u}$ *of* u*. Then* $\det(\nabla u)$ *is summable and*

$$\langle [Ju]_a, \omega\rangle = \int_{\mathbf{R}^m} \omega(x) \det(\nabla u(x)) d\mathcal{L}^m \tag{29}$$

Proof We follow the proof of Müller in [11] and to simplify notations we identify u and $\tilde{u}$.

We know that $[Ju]_a$ acts on 0 dimensional forms, i.e. on bounded measurable functions (recall Remark 2.12 and the fact that $[Ju]_a$ has finite mass). So we can write

$$\langle [Ju]_a, f\rangle = \int_{\mathbf{R}^n} \nu(x) f(x) d\mathcal{L}^n$$

where $\nu \in L^1_{loc}(\mathbf{R}^n)$: we only have to check

$$\nu(x) = \det(\nabla u(x)) \qquad \text{for a.e. } x.$$

Therefore let us fix $x_0 \in \mathbf{R}^n$ such that

(a) x_0 is a Lebesgue point for ν, $|\nabla u|^p$ and u (where p depends on the Sobolev space chosen in Definition 4.2);

(b)
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} \|Ju\|_s(B_\varepsilon(x_0)) = 0$$

(we recall that $\|Ju\|_s$ is the singular part of $\|Ju\|$).

Without loss of generality we can suppose that $x_0 = 0$ and $u(x_0) = 0$ and we define the rescaled functions

$$u^\varepsilon := \frac{1}{\varepsilon} u(\varepsilon x).$$

We observe that they are BnV and they converge strongly (in the appropriate Sobolev space) to the linear function given by $\nabla u(0)$, which we denote by u^∞. So we have that $[Ju^\varepsilon]$ converges to $[Ju^\infty]$ as local current and this implies that

$$\lim_{\varepsilon\to 0} \langle [Ju^\varepsilon], \omega\rangle = \langle [Ju^\infty], \omega\rangle = \int_{\mathbf{R}^m} \omega \det(\nabla u(0)) x1 \, d\mathcal{L}^m \tag{30}$$

for every Lipschitz function ω with compact support.

On the other hand, for every $f dg = \omega \in \mathcal{D}^1_c$, we also have

$$\langle j(u^\varepsilon), f dg\rangle = \int_{\mathbf{R}^m} \frac{1}{\varepsilon} f(x) u_1(\varepsilon x) \det(\nabla u_2(\varepsilon x), \dots, \nabla u_n(\varepsilon x), \nabla g(x)) d\mathcal{L}^m$$

and, by change of variables,

$$= \frac{1}{\varepsilon^n} \int_{\mathbf{R}^m} u_1(y) f(y/\varepsilon) \det(\nabla u_2(y), \dots, \nabla u_n(y), \frac{1}{\epsilon} \nabla g(y/\varepsilon) dy.$$

Thus we have obtained

$$\langle j(u^\varepsilon), \omega\rangle = \frac{1}{\varepsilon^n} \langle j(u), \omega(y/\varepsilon)\rangle$$

from which it follows that

$$\begin{aligned} \langle [Ju^\varepsilon], \omega\rangle &= \langle j(u^\varepsilon), d\omega\rangle \\ &= \frac{1}{\varepsilon^n} \langle j(u), (d\omega)(y/\varepsilon)\rangle \\ = \frac{1}{\varepsilon^n} \langle j(u), d(\omega(y/\varepsilon))\rangle &= \frac{1}{\varepsilon^n} \langle [Ju], \omega(y/\varepsilon)\rangle. \end{aligned}$$

From condition (b) we have

$$\lim_{\varepsilon\to 0} \left| \frac{1}{\varepsilon^n} \langle [Ju]_s, \omega(y/\varepsilon)\rangle \right| \le \lim_{\varepsilon\to 0} \frac{\mathrm{Lip}\,(\omega)}{\varepsilon^n} \|Ju\|_s(\varepsilon \,\mathrm{supp}\,(\omega)) = 0.$$

So we can write

$$\lim_{\varepsilon\to 0} \langle [Ju^\varepsilon], \omega\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} \langle [Ju]_a, \omega(y/\varepsilon)\rangle$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} \int_{\mathbf{R}^n} \nu(y)\,\omega(y/\varepsilon)\,dy$$
$$= \lim_{\varepsilon\to 0} \int_{\mathbf{R}^n} \nu(\varepsilon x)\,\omega(x)\,dx.$$

Since 0 is a Lebesgue point for ν, $\nu(\varepsilon y)$ converges in L^1_{loc} to the function $\nu(0)$. Recalling (30) we have

$$\int_{\mathbf{R}^n} \nu(0)\,\omega\,dx = \int_{\mathbf{R}^n} \det(\nabla u(0))\omega\,dx$$

for every Lipschitz function ω with compact support. Then we conclude that $\nu(0) = \det(\nabla u(0))$.

Now we remark that $\mathcal{L}^n$ a.e. x satisfies (a) and (b) and this completes the proof.

□

Theorem 4.8. *Let u be a* BnV *function and let us choose a pointwise representative $\tilde{u}$ of u. Then there exists a Borel function $\nu_a : \mathbf{R}^m \to \Lambda_{m-n}(\mathbf{R}^m)$ such that*

$$\langle [Ju]_a, \omega\rangle = \int_{\mathbf{R}^m} \langle \nu_a(x), \omega(x)\rangle d\mathcal{L}^m \tag{31}$$

and

$$\nu_a(x) = d\tilde{u}_1(x) \wedge \ldots \wedge d\tilde{u}_n \qquad \textit{for } \mathcal{L}^k \textit{ a.e. } x \in \mathbf{R}^m. \tag{32}$$

Before proving the theorem we put on the space of covectors $\Lambda_{m-n}(\mathbf{R}^m)$ the norm

$$|\nu| = \sup\{\langle \nu, f_1 \wedge \ldots \wedge f_n\rangle \,|\, f_i \in \mathbf{R}^m \text{ and } |f_i| \leq 1\}.$$

From the classical theory of currents we know that $|\nu_a(x)| \in L^1(\mathbf{R}^m)$. This fact and equation (32) imply that

$$\det(\nabla u_1, \ldots, \nabla u_n, \nabla g_1, \ldots, \nabla g_{m-n}) \in L^1(\mathbf{R}^m)$$

for every $(m-n)$-tuple of Lipschitz functions $(g_1, \ldots, g_{m-n})$.

Proof First we choose a Borel function ν'_a such that

$$\langle [Ju]_a, \omega\rangle = \int_{\mathbf{R}^m} \langle \nu'_a(x), \omega(x)\rangle d\|Ju\|_a$$

Recalling that $\|Ju\|_a$ is absolutely continuous with respect to $\mathcal{L}^m$ we set

$$\nu_a := \frac{d\|Ju\|_a}{d\mathcal{L}^m}\nu'_a.$$

Using Lemma 4.7 and the slicing techniques introduced abovex1 we will prove that ν_a satisfies equation (32). To simplify notations we will identify u and $\tilde{u}$ and we put $m = n + k$.

First we choose an orthogonal system $x_1, \ldots, x_{n+k}$ and a particular partition of $\{1, \ldots, n+k\}$ into two disjoint sets $I = \{i_1, \ldots, i_k\}$ and $J = \{i_{k+1}, \ldots, i_{k+n}\}$. We call $y_1, \ldots, y_k$ the k coordinates $x_{i_1}, \ldots, x_{i_k}$ and $z_1, \ldots, z_n$ the remaining n; moreover we denote by π_I be the projection on the coordinates $y_1, \dots, y_k$. From Lemma 4.7 we know that equation (32) holds when $m = n$; hence for a.e. $y \in \mathbf{R}^k$ we have

$$\langle [Ju(y,\cdot)]_a, f(y,\cdot)\rangle = \int_{\mathbf{R}^n} \det(\nabla_z u(y,z)) f(y,z) d\mathcal{L}^n(z). \tag{33}$$

Then, from the slicing property of the Jacobians applied to π_I, it follows that

$$\begin{aligned}
\langle [Ju]_a \llcorner d\pi_I, f\rangle &= (-1)^{nk} \int_{\mathbf{R}^k} \langle [Ju(y,\cdot)]_a, f(y,\cdot)\rangle d\mathcal{L}^k(y) \\
&= \int_{\mathbf{R}^k} \int_{\mathbf{R}^n} (-1)^{nk} \det(\nabla_z u(y,z)) f(y,z) d\mathcal{L}^n(z)\, d\mathcal{L}^k(y) \\
&= \int_{\mathbf{R}^{n+k}} f \, \det(\nabla u_1, \dots, \nabla u_n, e_{i_1}, \dots e_{i_k}) d\mathcal{L}^{n+k}.
\end{aligned}$$

Of course this means that $\det(\nabla u_1, \dots, \nabla u_n, e_{i_1}, \dots e_{i_k})$ is an L^1 function. Moreover this fact is true for every choice of I and, from the multilinearity of the determinant, we argue that

$$\det(\nabla u_1, \dots, \nabla u_n, \nabla g_1, \dots, \nabla g_k)$$

is summable for every Lipschitz and bounded k-tuple $(g_1, \dots, g_k)$. The continuity of $[Ju]_a \llcorner d\pi_I$ for every choice of I and the multilinearity of the determinant give the continuity (as k-dimensional local current) of $H_{n+k} \llcorner du$, which is defined by

$$\langle H_{n+k} \llcorner du, f dg\rangle = \int_{\mathbf{R}^{n+k}} f \det(\nabla u_1, \dots, \nabla u_n, \nabla g_1, \dots, \nabla g_k) d\mathcal{L}^{n+k}.$$

Of course $H_{n+k} \llcorner du$ is the same local current as $[Ju]_a$.

□

Unfortunately we are not able to prove that something similar holds for the Cantor part, i.e. that $\nu_c(x)$ is a simple covector for $\|Ju\|_c$ a.e. x.

5. SBnV

In analogy with the case of SBV functions (see [1], [3]) we can define the space SBnV of special functions of bounded higher variation.

Definition 5.1. *We say that a map $u \in \mathrm{BnV}(U, \mathbf{R}^n)$ is a "special function of bounded higher variation" if $[Ju]_c = 0$.*

The next proposition provides an equivalent definition of $SBnV$ functions.

Proposition 5.2. *Let* Ind *be the collection of all subsets I of $\{1, \dots, m\}$ such that the cardinality of I is $m - n$. For every $I \in \mathrm{Ind}$ we denote by π_I the projection on the coordinates $\{x_i | i \in I\}$. A function $u \in \mathrm{BnV}(U, \mathbf{R}^n)$ is in* SBnV *if and only if:*

(A) for every set of indices $I := \{i_1, \dots, i_{m-n}\} \in \mathrm{Ind}$ and for $\mathcal{L}^{m-n}$ a.e. x, $u(x_{i_1}, \dots, x_{i_{m-n}}, y)$ is a SBnV *function of y.*

Proof The "only if" part is an easy consequence of the slicing of currents. So let us suppose that (A) holds. Then for every $I \in \mathrm{Ind}$ we have $[Ju]_c \llcorner d\pi = 0$. If ω is an n-form, we can write

$$\omega = \sum_{I \in \mathrm{Ind}} g_I d\pi_I$$

and so we obtain $[Ju]_c \llcorner \omega = 0$. We conclude that $[Ju]_c = 0$ □

An interesting fact is that the special functions of bounded higher variation satisfy a closure theorem similar to that proved in [1] for SBV.

Remark 5.3. From now on when ν is a k-covector we denote by $|\nu|$ the standard norm induced by its action on k-vectors (see the proof of Theorem 4.6).

First of all we prove the closure theorem in a particular case.

Theorem 5.4. *Let us consider* $(u_k) \subset \mathrm{BnV}(U, \mathbf{R}^n)$ *and* $u \in \mathrm{BnV}(U, \mathbf{R}^n)$, *with* $U \subset \mathbf{R}^n$. *Let us suppose that:*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc}, $u_k \to u$ *strongly in* $L^{p_2}_{loc}$, *and*

$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$

(or $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{loc}$, $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets);*

(b) *if we write*

$$[Ju_k] = m_k(x)\mathcal{H}^0 \llcorner E_k + H_n \llcorner \nu_k(x),$$

then $|\nu_k|$ *are equiintegrable and* $\mathcal{H}^0(E_k) \leq C < \infty$.

Then $u \in \mathrm{SBnV}(U, \mathbf{R}^n)$ *and*

$$[Ju_k]_a \rightharpoonup [Ju]_a$$

$$[Ju_k]_l \rightharpoonup [Ju]_l.$$

Proof We follow the ideas of the proof of SBV closure in [1]. First we notice that in this particular case the weak Jacobians $[Ju_n]$ are distributions. From the fact that the functions ν_k are equiintegrable, we can find a subsequence that converges weakly in L^1 to a function ν. To simplify the notation we will suppose that the whole sequence (ν_k) converges to ν.

We recall that, from the continuity of the Jacobians, $[Ju_k] \rightharpoonup [Ju]$ (which means

$$\lim_{k\to\infty} \langle [Ju_k], \omega \rangle = \langle [Ju], \omega \rangle$$

for every Lipschitz function ω with compact support). We notice that

$$[Ju_k]_l = [Ju_k] - [Ju_k]_a \rightharpoonup [Ju] - \nu\mathcal{L}^n$$

in the sense of distributions. Moreover we can write, for some integer N,

$$[Ju_k]_l = \sum_{i=1}^{N} a_N^k \delta_{x_N^k}.$$

Then we can find a subsequence $u_{k(r)}$ such that (possibly reordering each set $\{x_1^{k(r)}, \dots , x_N^{k(r)}\}$ in a proper way):

(B) for every $j \in \{1, \dots , N\}$ either $x_j^{k(r)}$ converges to $x_j \in \overline{U}$ or $|x_j^{k(r)}|$ tends to infinity.

Recalling that $[Ju] - \nu\mathcal{L}^n$ is the limit of $[Ju_{k(r)}]_l$ we obtain that its support is a finite number of points. But we know that $[Ju] - \nu\mathcal{L}^n$ is a measure: so it is the sum of a finite number of Dirac masses. We can conclude that $[Ju]$ is the sum of an absolutely continuous measure and a finite number of Dirac masses.

Moreover we have that

$$[Ju_k]_a \rightharpoonup [Ju]_a \tag{34}$$

and

$$[Ju_k]_l = [Ju_k]_s \rightharpoonup [Ju]_s = [Ju]_l. \tag{35}$$

(Actually we have proved these last statements only for a subsequence. However we notice that from every subsequence of u_k we can choose another subsequence such that (34) and (35) hold. Then (34) and (35) hold for the whole sequence (u_k)).

□

From the slicing property of $[Ju]$ we are now able to prove the next theorem.

Theorem 5.5. *Let us consider* $(u_k) \subset \mathrm{BnV}(U, \mathbf{R}^n)$ *and* $u \in BnV(U, \mathbf{R}^n)$, *with* $U \subset \mathbf{R}^m$. *Moreover suppose that*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc}, $u_k \to u$ *strongly in* $L^{p_2}_{loc}$, *and*

$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$

(or $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{loc}$, $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets);*

(b) *if we write*

$$[Ju_k] = m_k(x)\tau_k(x)\mathcal{H}^{m-n} \llcorner E_k + H_m \llcorner \nu_k(x),$$

then $|\nu_k|$ *are equiintegrable and* $\mathcal{H}^{m-n}(E_k) \leq C < \infty$.

Then u *is of special higher bounded variation.*

During the proof we will use the representations of the previous section. So the restriction of $[Ju]_a$ to dg becomes

$$\langle [Ju]_a \llcorner dg, \omega \rangle = \int_{\mathbf{R}^m} \langle \nu_a(z), \omega(w) \wedge dg(w) \rangle \, d\mathcal{L}^m(w)$$

and the slicing map with respect to the projection π on the first $m-n$ coordinates is given by

$$\langle \mathcal{S}(x), \omega \rangle = \int_{\{x\}\times\mathbf{R}^n} \langle \nu_a(x,y), \omega(x,y) \wedge d\pi(x,y) \rangle \, d\mathcal{L}^n(y).$$

From the slicing property of Jacobians we argue that for a.e. x we can find a 0-covector valued function $\xi(x,\cdot)$ (i.e. a *real function*) such that

$$\langle [J(u(x,\,\cdot))]_a, \omega \rangle = \int_{\mathbf{R}^n} \langle \xi(x,y), \omega(y) \rangle d\mathcal{L}^n(y).$$

We denote $\xi(x,y)$ by $\nu_a(x,y) \llcorner d\pi$ and we reamrk that $|\xi(x,y)| \leq |\nu(x,y)|$ for a.e. (x,y)

Proof We will prove that statement (A) in 5.2 holds.

Let us fix $I \in \mathrm{Ind}$: without loss of generality we can suppose that

$$I = \{1, \ldots, m-n\}.$$

We denote by z the projection on the first $m-n$ coordinates. Then we can write, for a.e. $x \in \mathbf{R}^{m-n}$,

$$[J(u_k(x,\cdot))]_a = (\nu_k(x,\cdot) \llcorner dz) \mathcal{H}^n \llcorner x \times \mathbf{R}^n \tag{36}$$

$$[J(u_k(x,\cdot))]_l = m_k(x,\cdot) \mathcal{H}^0 \llcorner (S_u \cap z^{-1}\{x\}) \tag{37}$$

and of course $[Ju_k(x,\cdot)]_c = 0$.

We split the proof into several steps.

Step 1 First we suppose that $u_k \to u$ strongly in $L^{p_2}_{loc}$ and weakly in W^{1,p_1}_{loc}. Let us fix an open set $V \subset\subset U$ and set $V_x = V \cap z^{-1}\{x\}$. Let us extract a subsequence u_k of u_n such that

$$\sum_{k=1}^{\infty} \int_{\mathbf{R}^{m-n}} \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)} dx < \infty.$$

From the Monotone convergence Theorem we infer that for a.e. x

$$\sum_k \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)}$$

is a convergent series: this implies that $u_k(x,\cdot) \to u(x,\cdot)$ in $L^{p_2}(V_x)$ for a.e. x. Let us choose a family of open sets $V_n \uparrow V$ such that $V_n \subset\subset V$. We reason as above for every V_n and we apply a diagonalization argument to conclude that there is a subsequence (u_k) such that $u_k(x,\cdot) \to u(x,\cdot)$ strongly in $L^{p_2}_{loc}$ for a.e. x.

Step 2 From Fatou Lemma we have that

$$\int_{\mathbf{R}^{m-n}} \liminf_{k\to\infty} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} dx \leq \liminf_{j\to\infty} \int_{\mathbf{R}^{m-n}} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty.$$

We conclude that for a.e. x we can extract a further subsequence (u_l) (possibly depending on x) such that $\|u_l(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty$ for every open set $V \subset\subset U$. Then, recalling that for a.e. x $(u_k(x,\cdot))$ converges strongly in $L^{p_2}_{loc}$ to $u(x,\cdot)$, we have that $u_l(x,\cdot)$ converges weakly in W^{1,p_1}_{loc} to $u(x,\cdot)$.

Summarizing we have proved that for a.e. $x \in \mathbf{R}^n$ we can extract a subsequence (u_k) (possibly depending on x) such that

- $u_k(x,\cdot) \rightharpoonup u(x,\cdot)$ in W^{1,p_1}_{loc} and $u_k(x,\cdot) \to u(x,\cdot)$ strongly in $L^{p_2}_{loc}$, with
$$(n-1)/p_1 + 1/p_2 < 1.$$

In a similar way we can treat the case in which $u_k(x,\cdot) \rightharpoonup u(x,\cdot)$ in $W^{1,n-1}_{loc}$ $u_k(x,\cdot) \to u(x,\cdot)$ uniformly on compact sets as continuous functions.

Step 3 From the Dunford-Pettis Theorem on L^1 weakly compact sequences (see for example [4]) we know that $|\nu_k|$ belong to some Orlicz space. So there exists a real convex function ϕ with superlinear growth such that

$$\int_{\mathbf{R}^m} \phi(|\nu_k|) \leq K < \infty.$$

Then we have

$$\begin{aligned} K &\geq \limsup_{k\to\infty} \int_{\mathbf{R}^m} \phi(|\nu_k \llcorner dz|) \\ &\geq \limsup_{k\to\infty} \int_{\mathbf{R}^{m-n}} \int_{\mathbf{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)\, dx\, dy \\ &\geq \int_{\mathbf{R}^{m-n}} \liminf_{k\to\infty} \int_{\mathbf{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)\, dx\, dy. \end{aligned}$$

This implies that for a.e. x we can find a subsequence $k(r)$ such that

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \phi(|\nu_{k(r)}(x,y) \llcorner dz|)\, dx < \infty,$$

which means that $\nu_{k(r)}(x,\cdot) \llcorner dz$ are equiintegrable (we remark that the chosen subsequence depends on x).

Step 4 Reasoning as in the previous cases we have

$$\int_{\mathbf{R}^{m-n}} \liminf_{k\to\infty} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}))dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^n} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}))$$

Then for a.e. x we can extract a subsequence (u_l) (possibly depending on x) such that

$$(\mathcal{H}^0(S_{u_l} \cap z^{-1}\{x\}))$$

is bounded.

Step 5 Now we want to put together all the informations of the previous steps. We notice that the subsequence extracted on the first step does not depend on x, whereas the choices of the other steps depend on x. However for a.e. x we can extract a subsequence that fulfills all the conditions. Indeed let us define

$$f_k(x) := \mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}) + \int_{\mathbf{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)dy + \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)};$$

then we have

$$\int_{\mathbf{R}^{m-n}} \liminf_{k\to\infty} f_k(x)dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^{m-n}} f_k(x)dx$$

$$\leq \liminf_{k\to\infty} \left(\mathcal{H}^{m-n}(S_{u_k}) + \|u_k\|_{W^{1,p_1}} + \int_{\mathbf{R}^m} \phi(\nu_k) \right) < \infty. \tag{38}$$

We conclude that for a.e. x we can choose a subsequence u_r such that $(u_r(x,\cdot))$ and $u(x,\cdot)$ satisfy all the hypotheses of Theorem 5.4. Then for a.e. x the Cantor part of $[J(u(x,\cdot))]$ is zero and from statement (A) it follows that u has no Cantor part.

□

We end this section by proving that, in the same hypotheses of Theorem 5.5 we have

$$[Ju_k]_l \rightharpoonup [Ju]_l$$
$$[Ju_k]_a \rightharpoonup [Ju]_a.$$

For doing it we need the next Lemma.

Lemma 5.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$ and (ν_k) a weakly compact sequence in $L^1(\Omega,\mu)$. Then $\nu_k \rightharpoonup \nu$ if and only if*

$$\int_\Omega |w+\nu|d\mu \leq \liminf_{k\to\infty} \int_\Omega |w+\nu_k|d\mu \qquad \forall\, w \in L^1(\Omega). \tag{39}$$

We refer to [1] for the proof.

Theorem 5.7. *Let us consider $(u_k) \subset \mathrm{BnV}(U,\mathbf{R}^n)$ and $ux \in BnV(U,\mathbf{R}^n)$, with $U \subset \mathbf{R}^m$. If conditions (a) and (b) of Theorem 5.5 hold then*

$$\begin{aligned} &[Ju_k]_l \rightharpoonup [Ju]_l \\ &[Ju_k]_a \rightharpoonup [Ju]_a \end{aligned} \tag{40}$$

Proof We use the notations of Theorem 5.5 and we reduce to prove

$$[Ju_k]_a \llcorner d\pi_I \rightharpoonup [Ju]_a \llcorner d\pi_I \tag{41}$$

for every $I \in$ Ind (we recall that Ind is the collection of all subsets of $\{1,\dots,n\}$ that have cardinality $(m-n)$). Indeed from this fact we could conclude that $[Ju_k]_a \rightharpoonup [Ju]_a$ and

$$[Ju_k]_l = [Ju_k] - [Ju_k]_a \rightharpoonup [Ju] - [Ju]_a = [Ju]_l.$$

Therefore we suppose that $I = \{1,\dots,m-n\}$ and we split the proof into several steps. To simplify the notation we suppose that $U = \mathbf{R}^n$ and that global convergence hold on (u_k): the proof can been adapted easily to the local case.

Step 1 Let us fix a convex real function ϕ with superlinear growth and a real number z such that

$$\liminf_{k\to\infty}\int_{\mathbf{R}^m}\phi(|z+\nu_k \llcorner d\pi|)d\mathcal{L}^m<\infty.$$

For every $x\in\mathbf{R}^{m-n}$ let us put

$$\begin{aligned}
J_1^x(u) &= \int_{\mathbf{R}^n}\phi(|z+\nu\llcorner d\pi(x,y)|)dy\\
J_2^x(u) &= \|u(x,\cdot)\|_{W^{1,p_1}}\\
J_3^x(u) &= \mathcal{H}^0(\pi^{-1}\{x\}\cap S_{u_k})
\end{aligned}$$

and fix a positive real number t. From Fatou Lemma we have that

$$\liminf_{k\to\infty} J_1^x(u_k)+tJ_2^x(u_k)+tJ_3^x(u_k)=K(x)<\infty$$

for almost every x and reasoning as in the last step of Theorem 5.5 we infer that

$$J_1^x(u)\le J_1^x(u)+tJ_2^x(u)+tJ_3^x(u)\le K(x).$$

Integrating this inequality with respect to x we obtain

$$\int_{\mathbf{R}^m}\phi(|z+\nu\llcorner d\pi|)d\mathcal{L}^m\le$$

$$\liminf_{k\to\infty}\left(\int_{\mathbf{R}^m}\phi(|z+\nu_k\llcorner d\pi|)d\mathcal{L}^m+t\|u_k\|_{W^{1,p_1}}+t\mathcal{H}^k(S_{u_k})\right).$$

Letting $t\downarrow 0$ we obtain

$$\int_{\mathbf{R}^m}\phi(|z+\nu\llcorner d\pi|)d\mathcal{L}^m\le\liminf_{k\to\infty}\int_{\mathbf{R}^m}\phi(|z+\nu_k\llcorner d\pi|)d\mathcal{L}^m. \tag{42}$$

We notice that the same arguments work if we replace $\mathbf{R}^m$ with an open set. Let us denote by $\mathcal{C}$ the class of functions $w\in L^1(\mathbf{R}^m)$ that can be written as

$$w=\sum_{i=1}^h\alpha_i\chi_{A_i},$$

for some open sets $A_1,\dots A_h$. Hence, equation (42) holds for every function $z\in\mathcal{C}$.

Step 2 We know that there exists a convex real function ψ with superlinear growth such that

$$\liminf_{k\to\infty}\int_{\mathbf{R}^m}\psi(|\nu_k\llcorner d\pi|)<\infty.$$

Let us take a convex real function ϕ with superlinear growth such that $\phi(0)=0$

$$\lim_{t\to\infty}\frac{\psi(t)}{\phi(t)}=+\infty.$$

We can easily conclude that the sequence $\phi(|\nu_k \llcorner d\pi|)$ is equiintegrable. Let us put

$$\phi_n(t) = \left(\frac{1}{n}\phi(t)\right) \vee t.$$

The equiintegrability of $\psi(|\nu_k \llcorner d\pi|)$ and the fact that $\phi_n(t) \downarrow t$ imply that

$$\liminf_{k\to\infty} \int_{\mathbf{R}^m} |\nu_k \llcorner d\pi| = \lim_{n\to\infty} \liminf_{k\to\infty} \int_{\mathbf{R}^m} \phi_n(|\nu_k \llcorner d\pi|).$$

From the previous step we easily conclude that

$$\int_{\mathbf{R}^m} |z + \nu \llcorner d\pi| \leq \int_{\mathbf{R}^m} \liminf_{k\to\infty} |z + \nu_k \llcorner d\pi| \tag{43}$$

for every $z \in \mathcal{C}$.

Step 3 By a standard approximation argument we have that (43) holds for every $z \in L^1(\mathbf{R}^n)$. Then applying Lemma 5.6 we conclude that

$$\nu_k \llcorner d\pi \rightharpoonup \nu \llcorner d\pi.$$

$\square$

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