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MESOSCOPIC LIMIT FOR NON-ISOTHERMAL PHASE TRANSITION

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Abstract. Motivated by the problem of modelling nucleation in non-isothermal systems, we consider the stochastic evolution of a coupled system of a lattice spin variable σ and a continuous variable e (corresponding to the phase and the energy density of a continuum system). The spin variables flip with rates depending both on a Kac-potential type interaction with the spins and on an intercation with the e-field, which plays the role of the external field in ferromagnetics but evolves by a diffusion equation with a forcing depending on the spins.

We analyse the mesoscopic limit, where space scales like the diverging interaction range of the Kac potential, γ^{-1}, while time is not rescaled. By writing σ as random time change of a family of *independent* spins, and thus reducing the problem to investigating integral equations parametrised by independent random variables, we show that as $\gamma \to 0$ the average of the spins over small cubes and the field e converge in probability to the solution of a system of nonlocal evolution equations which is similar to the phase field equations. In some cases the convergence holds until times of order $\log(\gamma^{-1})$.

1. Introduction

The phase field equations

$$\partial_t m(t,x) = \Delta m(t,x) - V'(m(t,x)) + \lambda\theta(t,x) \tag{1.1}$$

$$\partial_t\big(m(t,x)+\theta(t,x)\big) = \Delta\theta(t,x) \tag{1.2}$$

were introduced by Caginalp [3] to describe solidification in the presence of heat diffusion on a scale where regions of well defined phase form with a small transition layer between them. Here m is the order parameter, θ the temperature and V a double well potential whose wells correspond to the two phases. In this case, $e = m+\theta$ is the energy density. We refer to [12] for a large bibliography and an extensive discussion of deterministic models for moving phase boundaries coupled with diffusion.

After diffusive rescaling with a parameter of order of the coupling constant λ, one should expect in analogy to the behaviour under rescaling of (1.1) with *constant* θ the formation of a sharp interface moving by an evolution of the type $V = -\kappa+\theta$, where V is the velocity in direction of the outward normal and κ the mean curvature of the phase boundary.

Indeed, for classical solutions this was shown by Caginalp and Chen, [4]. Soner [11] showed that the rescaled solutions of a slightly modified version of the phase field equations converge on a subsequence to weak solutions of the Mullins-Sekerka problem with

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kinetic undercooling. The theorem holds beyond possible singularities of the (classical) limit equations.

It is important to choose the coupling constant and the rescaling parameter of the same order: The coupling should influence the (in unscaled time) slow motion of the interface, but should not interfere with the formation of a sharp interface. (Otherwise the Stefan problem is expected to be the limit problem, cf. [1], [4].)

Although we consider here only the mesoscopic limit, we were motivated by the problem of nucleation in the presence of heat diffusion. (For an introduction to the modelling of such problems, see for example [12].) One has to analyze a stochastic process approximating the phase field equations and its deviations from this deterministic limit, which allow the system to tunnel through a potential barrier and hence a droplet to nucleate.

In order to add stochasticity to a deterministic equation, one could in principle add noise to it. We chose however to model the phase change by spins $\sigma(x) \in \{-1, +1\}$, which flip at rates depending both on the average of their neighbours via a Kac-Potential and on the local average of the field. Thus we avoid the difficulties of adding noise to nonlinear equations in higher space dimensions, and we hope that our approach, though a caricature, is more natural because the mechanism behind phase change phenomena is actually collective behaviour of interacting atoms:

We think of a lattice where each point represents a more complicated cell which is characterized by two possible states, e.g. an inversion like in ferroelastica or a different type of symmetry breaking. In our simplified description this state is modelled by a $\{-1, +1\}$-field on the lattice, the spin field. Furthermore the thermal energy at a point of the lattice is characterized by a field which could be thought of as a field of phonons, for simplicity performing a random walk on the lattice. The energy is the sum of the potential energy and the thermal energy.

We assume that the evolution of the thermal energy for a given spin field is well approximated by a discretized heat equation. (This is for example the case if there is a large number of phonons on each lattice point which jump independently given the spin field.)

So in the following our model for the thermal field is a deterministic diffusion equation coupled to a spin flip model, where the interaction between the spins is described by a Kac-potential. In order to simplify the proofs, we work with the continuum heat kernel.

The choice of the Kac-potential was motivated by [6] and [10] who study spins on a lattice which interact with the average of their neighbours and a *constant* external field. However our methods are different.

Similar to the model studied in [1], our model has a quantity which is locally conserved during the flips: The sum of potential and thermal energy. This models the effect of latent heat and is responsible for recalescence, see [12]: As the total energy is conserved, a flip in a region can change the thermal energy locally in such a way that the spin has before and after the flip for some time the unfavourable sign with respect to the thermal field.

Note that in our model the spin-flip process is non-Markovian, the joint system (σ, e) however is a Markov process. We hope that the behaviour of a coupled system consisting of one fluctuating quantity and a deterministic equation for a locally conserved quantity will be interesting from a purely mathematical point of view, e.g. with respect to spinodal decomposition and large deviations.

For simplicity, we will assume from now on only *periodic boundary conditions.*

As in [6], we expect convergence to a *nonlocal* equation in the mesoscopic limit, where space scales like the diverging interaction range of the Kac potential, γ^{-1}, while time is not rescaled:

$$\begin{aligned} \partial_t m &= -m + b(\beta[J * m + \lambda\theta]) && (1.3)\\ \partial_t e(\theta, m) := \partial_t(\theta + m) &= \Delta\theta. && (1.4) \end{aligned}$$

Here $b \in C_b^1$ antisymmetric and monotone, e.g. $b = \tanh$, $*$ denotes convolution and λ is a small parameter. J is a C^1 kernel with compact support which we assume for simplicity to depend only on $|x-y|$, however we expect the results to hold for kernels $J(x,y)$ as long as they are smooth enough and supported in $|x-y| < 1$, say. A kernel $J(x,y) = \widetilde{J}(x, x-y)$ would also include the case of reflecting (Neumann) boundary conditions. Note that at this stage we do not require $J \geq 0$, i.e. not necessarily ferromagnetic interactions, but for the sharp interface limit, which is not studied in this paper, $J \geq 0$ is expected to be crucial.

For technical reasons we are forced to introduce a smoothing $K^\epsilon * m(t,x)$ which makes the part of the interaction of the spins which is done through their action on the field and the feedback of the field on the spins also of local mean field type. Here $K^\epsilon(x) := \epsilon^{-d}K(\epsilon^{-1}x)$ for a rapidly decreasing nonnegative $K \in C^2$ which has integral normalized to 1, such that $K^\epsilon * m$ approximates m for small ϵ. We will let $\epsilon \to 0$ slowly.

As ϵ is much larger than the lattice spacing, the kernel seems to be necessary for a discretized heat equation as well. We remark that because of $P(\epsilon) * P(t) = P(t+\epsilon)$, this corresponds to a cut-off at the singularity of the heat kernel.

Setting $\theta = e - K^\epsilon * m$, we arrive at the so-called mean-field equation for m :

$$\begin{aligned} \partial_t m(t) &= -m(t) + b(\beta[J * m(t) + \lambda K^\epsilon * (e - K^\epsilon * m)]) && (1.5)\\ \partial_t e(t) &= \Delta e(t) - \Delta(K^\epsilon * m)(t) && (1.6) \end{aligned}$$

From now on we will always express the field θ as a function of e and m.

In this paper we allow a more general version of equation (1.5), which we replace by

$$\partial_t m = -m\, c_+(J * m, \lambda K_1^\epsilon * m, \lambda K_2^\epsilon * e) + c_-(J * m, \lambda K_1^\epsilon * m, \lambda K_2^\epsilon * e). \qquad (1.7)$$

Here $0 \leq |c_-(x,y,z)| < c_+(x,y,z)$ are bounded and uniformly Lipschitz in all arguments, so that we can define a jump intensity by $c(\pm 1, x, y, z) := \frac{1}{2}(c_+(x,y,z) \pm c_-(x,y,z))$. J is as above, the kernels K_1, K_2 are C^1, symmetric, rapidly decreasing or with compact support.

Now we want to define the stochastic process on the lattice which converges on the mesoscopic scale to this equation (or rather to a space-discretized version with discretization parameter γ.) We require that this stochastic process takes values in $\{-1, 1\} \times \mathbb{R}$ and we construct it as *random time change* of a family of independent spins:

Let $N(\cdot, x)_{x \in \mathbb{Z}^d}$ be a family of independent Poisson processes with rate 1, and let $\mathcal{Z}^\gamma$ be a cube such that for a period n_γ of order $(\lambda\gamma)^{-1}$ or slightly larger $\mathbb{Z}^d = \mathcal{Z}^\gamma + n_\gamma \mathbb{Z}^d$ holds. For any lattice site $x \in \mathcal{Z}^\gamma$ we construct a strictly monotone function (random time change) $T^\gamma_{\sigma_0^\gamma, e_0}(t, x, N) : [0, \infty) \to [0, \infty)$, $T^\gamma_{\sigma_0^\gamma, e_0}(0, x) = 0$. We set $T^\gamma(t, x, N) := T^\gamma_{\sigma_0^\gamma, e_0}(t, x, N)$ and define

$$\sigma^\gamma[\sigma_0^\gamma, T^\gamma, N](t, x) := \sigma_0^\gamma(x)(-1)^{N\left(T^\gamma(t,x,N), x\right)}. \qquad (1.8)$$

This allows us to construct the spins for any γ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, defined by the paths of $N(\cdot, x)_{x \in \mathbb{Z}^d}$: Remember that a single site flip process $\sigma(t)$ with intensity c can be constructed as time changed rate 1 flip process, $\sigma(s)=\widetilde{\sigma}(T(s))$, where $T(s)$ is given by

$$T(t):=\int_0^t c(\sigma(s))ds=\int_0^t c(\widetilde{\sigma}(T(s)))ds.$$

In our case T is vector-valued, however the principle is the same. (See also [9] for multi-parameter time changes.) As $\mathcal{Z}^\gamma$ is finite, we need only finitely many time changes for a fixed value of γ. The others we set equal to the identity.

Set for $\sigma(x)\in\{-1,1\}$, $f,\ e\in L^\infty(\mathbb{R}^d)$ (or in $L^\infty(\mathbb{T}_\lambda)$, where $\mathbb{T}_\lambda$ is a cube of sidelength λ^{-1} with periodic boundary conditions)

$$c[\sigma,f,e](x):=c\left(\sigma(x),(J*f)(\gamma x),\lambda(K_1^\epsilon*e)(\gamma x),\lambda(K_2^\epsilon*f)(\gamma x)\right). \tag{1.9}$$

The intensity c is a positive function and $\lambda>0,\ \epsilon>0$ may tend to 0 slowly as $\gamma\to 0$.

In order to treat lattice functions as function defined on $\mathbb{T}_\lambda$ or $\mathbb{R}^d$, we define the operation

$$f\mapsto\overline{f}:\quad \overline{f}(s,r):=f(s,x)\ \text{for}\ -\frac{\gamma}{2}\le r_i-\gamma x_i<\frac{\gamma}{2},\ i=1,\dots,d. \tag{1.10}$$

If the spins interact by the Kac potential $J*\overline{\sigma^\gamma}$, this leads to the following integral equations for the time changes, where we set $T^\gamma(t,x):=T^\gamma_{\sigma_0^\gamma,e_0}(t,x,N)$ to simplify notation.

$$\begin{aligned} T^\gamma(t) &= \int_0^t c\Big[\sigma^\gamma[T^\gamma](s),\overline{\sigma^\gamma}[T^\gamma](s),e[\sigma^\gamma[T^\gamma]](s)\Big]ds \quad (1.11)\\ e[\sigma^\gamma[T^\gamma]](t) &= P(t)*e_0-\int_0^t P(t-s)*\Delta[K^\epsilon*\overline{\sigma^\gamma}[T^\gamma](s)]ds. \quad (1.12)\end{aligned}$$

Whenever necessary, we will extend functions defined originally on $\mathbb{T}_\lambda$ or $\mathcal{Z}^\gamma$ periodically to functions defined on the whole of $\mathbb{R}^d$ or $\mathbb{Z}^d$ without stating it explicitly.

(1.5) corresponds to the case where $K_1=K,\ K_2=K*K$, $c(+1,\cdot,\cdot,\cdot)+c(-1,\cdot,\cdot,\cdot)=1$ and $c(+1,x_1,x_2,x_3)-c(-1,x_1,x_2,x_3)=b(\beta x_1+\beta(x_2-x_3))$.

If one requires the spins to be independent and their mean to be exactly the solution m^γ of (1.7), then the corresponding $T^{m^\gamma}_{\sigma_0^\gamma,e_0}(t,x,N)$ solve for any path N of the family of Poisson processes and for any σ_0^γ,e_0 the following equations, where again $T^{m^\gamma}:=T^{m^\gamma}_{\sigma_0^\gamma,e_0}$.

$$\begin{aligned} T^{m^\gamma}(t) &= \int_0^t c\left[\sigma^\gamma[T^{m^\gamma}](s),\overline{m^\gamma}(s),e[m^\gamma](s)\right]ds \quad (1.13)\\ e[m^\gamma](t) &= P(t)*e_0-\int_0^t P(t-s)*\Delta[K^\epsilon*\overline{m^\gamma}(s)]\ ds \quad (1.14)\\ m^\gamma(t,x) &= \mathbb{E}[\sigma^\gamma[T^{m^\gamma}](t,x)]. \quad (1.15)\end{aligned}$$

The expectation is taken with respect to the product of the distributions of the independent Poisson processes N and of the independent $\sigma_0(x)$. The spins interact only with the expectation of their neighbours, not with the actual configuration. Note that (1.14) is purely deterministic.

With the help of the time changes, this auxiliary process (called Mean-Field process) can be constructed on the same probability space as the process defined by (1.11-1.12).

In chapter 4 we will show that on not too large time intervals, (1.13) to (1.15) approximate (1.11), (1.12) in probability as $\gamma\to 0$. The weak dependence of expressions like $J*\overline{\sigma^\gamma}$ on each single spin makes the convolution behave like its mean.

To show this, we make use of the fact that the inverse functions $(T^\gamma)^{-1} =: T_1$ and $(T^{m^\gamma})^{-1} =: T_2$ exist and solve a fixed point problem of the type

$$T_i(t) = \int_0^t F_i(T_i, N)(s)ds,$$

were N is a path of the family of independent Poisson processes and the dependence of F_i on $T_i(x)$ for a single lattice point x is weak. Formally, we write

$$T_1(t) - T_2(t) = \int_0^t [F_1(T_1, N)(s) - F_1(T_2, N)(s)]ds + \int_0^t [F_1(T_2, N)(s) - F_2(T_2, N)(s)]ds.$$

For small times, the first integral can be estimated by

$$\int_0^t [F_1(T_1, N)(s) - F_1(T_2, N)(s)]ds \leq \frac{1}{2} \sup_{[0,t]} |T_1(s) - T_2(s)|$$

on a set of large probability, using the fact that expressions as $\int_0^t J * \overline{\sigma^\gamma}[T](s)ds$ are by the law of large numbers "almost Lipschitz in T" in a sense to be specified in chapter 4.

The second integral is small on a set of large probability by the law of large numbers for the independent $\sigma^\gamma[T^{m^\gamma}]$. The convergence of the inverse time changes implies the convergence of the time changes for c bounded away from 0.

In chapter 5 we use the short time convergence of the time changes to show the convergence of the block spins

$$\mathcal{A}_{\gamma^{-\alpha}}(\sigma^\gamma[T](s, x)) := \gamma^{\alpha d} \sum_{|x-y|<\gamma^{-\alpha}} \sigma^\gamma[T](s, y) \tag{1.16}$$

to the solution of a system similar to (1.6, 1.7) on intervals of length of order $\log(\gamma^{-1})$.

Remarks Corresponding to the more general form of the order parameter equation (1.7), it would be desirable to replace (1.6) by a diffusive equation of the form $\partial_t e = \nabla(L(\theta)\nabla\theta)$ and a relation between e and θ involving the phase which is some regularization of

$$e(\theta, \sigma) = \frac{1+\sigma}{2}\Phi_1(\theta) + \frac{1-\sigma}{2}\Phi_2(\theta).$$

But although we need not really the precise form of the heat kernel but only the short and long time asymptotics and the averaging behaviour, the question whether our results extend to nonlinear diffusive equations is not clear.

It would also be desirable to get rid of the auxiliary kernels K^ϵ, which force the interaction to be of Kac-type everywhere. We expect however that the qualitative behaviour of the diffusively rescaled limit equations (1.5, 1.6) does not depend on the auxiliary kernel, because ϵ can be chosen much smaller than the transition layer thickness.

Our methods use only basic calculus and elementary probability, i.e. the weak law of large numbers. This can be formulated as a property of product measures. Thus we hope that that our approach might appeal to researchers interested in phase-change models but without probabilistic background.

Parts of this paper are based on a diploma thesis at the University of Bonn, [8].

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2. Definitions and results

2.1. **Definitions.** We give a precise definition of the periodic boundary conditions:

Definition 2.1. *Let $\gamma \in \mathbb{R}$, $\gamma \searrow 0$, let $0 < C^{-1} \le \lambda(\gamma)^{-1} \le C|\log(\gamma)|^{-1}$ and further let $0 < p_\lambda \lambda \le C < \infty$. Assume γ and $\lambda(\gamma)$ are such that $n_\gamma := \gamma^{-1} p_\lambda \in \mathbb{Z}$ and let*

$$\mathcal{Z}^\gamma := \{x \in \mathbb{Z}^d : 0 \le x_i < n_\gamma \text{ for } 1 \le i \le d\}.$$

$\mathbb{T}_\lambda := \mathbb{R}^d/p_\lambda$, $\mathbb{T} := \mathbb{T}_1$. We require functions on $\mathbb{R}$ to be p_λ-periodic, and so the discretized functions on the lattice are periodic with period n_γ. They are uniquely defined on $\mathbb{Z}^d$ by their values on $\mathcal{Z}^\gamma$.

Definition 2.2. *Define*

$$\begin{aligned}
C^0([t_1,t_2],\mathcal{Z}^\gamma) &:= \{f\colon [t_1,t_2]\times\mathbb{Z}^d \to \mathbb{R} \,|\, f(\cdot,x) \in C^0([t_1,t_2]),\ f(t,\cdot) \text{ is } n_\gamma\text{-periodic},\\
L^\infty([t_1,t_2],\mathcal{Z}^\gamma) &:= \{f\colon [t_1,t_2]\times\mathbb{Z}^d \to \mathbb{R} \,|\, f(\cdot,x) \in L^\infty([t_1,t_2]),\ f(t,\cdot) \text{ is } n_\gamma\text{-periodic},\\
\|f\|_{\infty,t_1,t_2} &:= \sup_{x\in\mathcal{Z}^\gamma,\, t\in[t_1,t_2]} |f(t,x)| \ (= \sup_{\mathbb{Z}^d\times[t_1,t_2]} |f(t,x)|, \text{ if } f \text{ is } n_\gamma\text{- periodic.})
\end{aligned}$$

Definition 2.3. *(Space of time changes)*

$$C^0_{M,a}([t_1,t_2],\mathcal{Z}^\gamma) := \{f \in C^0([t_1,t_2],\mathcal{Z}^\gamma) \,|\, f(t_1,x) = a \text{ and } (*) \textit{ holds for all } x \in \mathcal{Z}^\gamma\}.$$

$$\frac{1}{M} \le \frac{f(s)-f(t)}{s-t} \le M \textit{ for } s,t \in [t_1,t_2] \qquad (*)$$

Remark 2.4. *For any f, g such that $(*)$ holds, we have*

$$\frac{1}{M}\sup_{[0,t]} |f(s)-g(s)| \le \sup_{f([0,t])\cap g([0,t])} |f^{-1}(s) - g^{-1}(s)| \le M \sup_{[0,t]} |f(s)-g(s)|.$$

Remarks on Notation

We denote by $D([0,t]\times\mathcal{Z}^\gamma, \mathbb{Z}_0^+)$ the cad-lag-functions on $[0,t]\times\mathcal{Z}^\gamma$ with values in the nonnegative integers $\mathbb{Z}_0^+$, equipped with the product (over $x \in \mathcal{Z}^\gamma$) of the Skorohod-metrics and the corresponding Borel σ-Algebra. For the precise definition we refer to standard textbooks of probability theory such as [9].
Further we use $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$.

2.2. **Results.**

Lemma 2.5. *Existence*

For $c(\pm1,\cdot,\cdot,\cdot)$ uniformly Lipschitz-continuous, $\frac{1}{M} \le c \le M$, and N Cad-lag with values in $\mathbb{N}^{\mathcal{Z}^\gamma}$, $\{\lambda,\epsilon,\gamma\} \subset (0,1]$, there is a family of time changes $T^\gamma_{\sigma_0^\gamma,e_0}(t,x) \in C^0_{M,0}([0,t],\mathcal{Z}^\gamma)$ such that for $\sigma^\gamma[T^\gamma_{\sigma_0^\gamma,e_0}]$ as in (1.8) the system (1.11), (1.12) *is solved for any $\sigma^\gamma(0)$ and $e_0 \in L^\infty$. Moreover, let*

$$\sigma^\gamma_{(\sigma_0^\gamma,e_0)}(t,N) := \sigma^\gamma[\sigma_0^\gamma, T^\gamma_{\sigma_0^\gamma,e_0}(N), N](t) \tag{2.1}$$

$$e^\gamma_{(\sigma_0^\gamma,e_0)}(t) := e[\sigma^\gamma_{(\sigma_0^\gamma,e_0)}](t), \tag{2.2}$$

then $\sigma^\gamma_{(\sigma_0,e_0)}(t,N)$ is cadlag and we have with respect to its natural filtration $\mathcal{F}_t$ (which is not *the filtration generated by N):*

$$\mathbb{E}[\sigma^\gamma_{(\sigma_0^\gamma,e_0)}(t+s)|\mathcal{F}_t] = \sigma^\gamma_{\left(\sigma^\gamma_{(\sigma_0^\gamma,e_0)}(t),\, e^\gamma_{(\sigma_0^\gamma,e_0)}(t)\right)}(s). \tag{2.3}$$

As $e[\sigma^\gamma[T^\gamma]]|_{[t_1,t_2]}$ is a measurable function of $(\sigma^\gamma_{(\sigma_0^\gamma,e_0)}(t,N)|_{[t_1,t_2]}, e(t_1))$, standard methods from measure theory show that $(\sigma^\gamma(t), e^\gamma(t))$ is Markov.

Lemma 2.6. *Existence of the Mean-Field Process*
Under the same conditions, there is a solution of (1.13)-(1.15) *such that* $\sigma^\gamma[T^{m^\gamma}, N](\cdot, x)$ *are Markov and independent for different* x*. Moreover,* $m^\gamma(t,x)$, $x \in \mathcal{Z}^\gamma$*, solves for* $t > 0$ *the system*

$$\partial_t m^\gamma(s) = -m^\gamma c_+(m^\gamma, e[m^\gamma])(s) - c_-(m^\gamma, e[m^\gamma])(s) \tag{2.4}$$

$$e[m^\gamma](t) = P(t) * e_0 - \int_0^t P(t-s) * \Delta[K^\epsilon * \overline{m^\gamma}(s)]ds \tag{2.5}$$

$$m^\gamma(0,x) = m_0(x), \tag{2.6}$$

$$c_\pm(m,e,x)(s) := c[+1, \overline{m}, e] \pm c[-1, \overline{m}, e].$$

Theorem 2.7. *Convergence of the time changes on short time intervals.*
If the conditions of Lemma 2.5 *and Lemma* 2.6 *hold and if* T^γ *and* T^{m^γ} *as in Lemmas* 2.5 *and* 2.6 *are constructed from the same initial values* σ_0 *and* e_0*, then there are constants* C_1, C *depending only on the kernels* J, K_1, K_2*, on the bound* M *and the Lipschitz constant of the intensity* c *such that for all* $0 < \Delta t < \left(\frac{1}{4}C_1\left((1+\|e_0\|_{L^\infty})\lambda\epsilon^{-1}\right)^2\right)$ *and for all* $\delta > 0$

$$\mathbb{P}\left[\sup_{x\in\mathcal{Z}^\gamma, 0\le t\le\Delta t} |T^\gamma(t,x) - T^{m^\gamma}(t,x)| > \delta\right] \le C\frac{\Delta t}{\delta^{d+3}}\left(\frac{\gamma p^2}{\epsilon^3}\right)^d. \tag{2.7}$$

Theorem 2.8. *Block-spin convergence*
Under the assumptions of 2.7*, but now for* $0 < c(\cdot) \le M < \infty$*, let* $(m^\gamma, e[m^\gamma])$ *solve* (2.4, 2.5) *starting from* σ_0, e_0*. Let* $\mathcal{A}_{\gamma^{-\alpha}}(\cdot)$ *be as in (1.16). If* $C^{-1} < \epsilon\log(\gamma^{-1})$ *and* λ *is as in* 2.1*, then there is a* $0 < h_0 \ll 1$*, such that for all* $1 - h_0 < \alpha < 1$ *there are real constants* $a,\ \zeta,\ b,\ \gamma_0 \in (0,1)$ *such that for* $\gamma \le \gamma_0$

$$\mathbb{P}\left[\sup_{[0,a\log(\gamma^{-1})]\times\mathcal{Z}^\gamma} |\mathcal{A}_{\gamma^{-\alpha}}(m^\gamma(t)) - \mathcal{A}_{\gamma^{-\alpha}}(\sigma^\gamma[T^\gamma](t))| > \gamma^\zeta\right] \le \gamma^b \tag{2.8}$$

$$\mathbb{P}\left[\sup_{[0,a\log(\gamma^{-1})]} \|e[\sigma^\gamma[T^\gamma]](t) - e[m^\gamma](t)\|_{L^\infty(\mathbb{T}_\lambda)} > \gamma^\zeta\right] \le \gamma^b \tag{2.9}$$

The proofs allow to give explicit bounds for $h_0,\ \gamma_0,\ a,\ \zeta,\ b$.

Corollary 2.9. *Continuum limit*

1. *Let* $(m^{\lambda,\epsilon}, e^{\lambda,\epsilon})$ *solve* (1.6), (1.7) *on* $[0, a\log\gamma^{-1}] \times \mathbb{T}_\lambda$ *with initial value* $(m_0, e_0) \in C^1$*. Let* $\lambda^{-2} + \epsilon^{-1} \le C|\log\gamma|$*. Let* σ_0 *be distributed according to a product measure* μ_0 *such that for some* $\tilde{a},\ \tilde{b}\ \in (0,1)$

$$\mu_0(\|\mathcal{A}_{\gamma^{-\alpha}}(\sigma_0) - m_0(\gamma x)\|_{L^\infty} > \gamma^{\tilde{a}}) \le \gamma^{\tilde{b}}. \tag{2.10}$$

   Then there are ζ, b, γ_0 *such that* (2.8) *holds with* $m^{\lambda,\epsilon}(t,\gamma x)$ *replacing* $\mathcal{A}_{\gamma^{-\alpha}}(m^\gamma(t))$.
2. *Let* (m,θ) *solve* (1.3), (1.4) *on* $[0,T]\times\mathbb{T}$*. Construct the process* $(\sigma^{\gamma,\epsilon}, e^{\gamma,\epsilon})$ *from the time changes as in* (1.11) *for* σ_0 *independent with distribution* μ_0, $e^{\gamma,\epsilon}(0) = \theta_0 + K^\epsilon * m_0$, $K_2 = K * K$, $K_1 = K$*, and* $c(\sigma, x_1, x_2, x_3) = \frac{1}{2}[1 - b(\beta\sigma\cdot(x_1 + x_2 - x_3))]$*. If (2.10) holds and* $\epsilon(\gamma) \to 0$*, but* $\epsilon^{-1} \le C\log(\gamma^{-1})$*, then for all* $\delta > 0$

$$\mathbb{P}\left[\sup_{[0,T]\times\mathbb{T}} \left|\overline{\mathcal{A}_{\gamma^{-\alpha}}(\sigma^{\gamma,\epsilon})(t,r) - m(t,r)}\right| > \delta\right] \to 0 \text{ as } \gamma \to 0. \tag{2.11}$$

Remarks: Ad 1: The conditions on m_0 could be relaxed, we refer to [6] for a refined discussion of initial values for spin-flip processes with Kac potentials.

Ad 2: The convergence of the equations with auxiliary kernel K^ϵ to the equations (1.3), (1.4) on long time intervals $[0,\lambda^{-2}]$ for ϵ and λ of the same order could fail because there is an unstable equilibrium for (1.5).

3. Existence and properties of time changes

3.1. **Proof of Lemma 2.6.** We use the Banach fixed point theorem for short time existence and iterate the result. Let

$$\mathcal{B}([0,t_0]) := \Big\{ f \in C^0([0,t_0],\mathcal{Z}^\gamma) \Big| \ f(0,x) = m_0(x), \ \|f(t,x)\|_{\infty,0,t_0} \le 1, \\ \sup_{x,\ 0\le r<s\le t_0} \frac{|f(r,x)-f(s,x)|}{|r-s|} \le 2M \Big\}.$$

For $f \in \mathcal{B}([0,t_0])$, $c[\cdot,\cdot,\cdot]$ as in (1.9) and $e[f]$ as in (1.14) let $T(f)(t,x)$ be the solution of

$$T(f)(t,x) = \int_0^t c\big[\sigma_0^\gamma(\cdot)(-1)^{N\left(T(f)(s,\cdot),\cdot\right)}, \overline{f(s)}, e[f](s)\big](x)ds.$$

Now set $A(f)(t,x) := \mathbb{E}[\sigma^\gamma[T(f)](t,x)]$, where $\sigma^\gamma[T(f)](t,x)$ is defined in (1.8). We show $A(f) \in \mathcal{B}$ and A is a contraction for t_0 small enough. Set $\|\cdot\| := \|\cdot\|_{\infty,0,t_0}$.

First note that for any x the inverse time change $T(f)^{-1}(t,x)$ solves

$$T(f)^{-1}(t,x) = \int_0^t \frac{ds}{c\left[\sigma_0^\gamma(-1)^{N(s)}, f\big(T(f)^{-1}(s,x)\big)e[f]\big(T(f)^{-1}(s,x)\big)\right](x)}. \tag{3.1}$$

This shows that $T(f)^{-1}(t,x)$ is adapted to the σ-algebra $\mathcal{F}_t^x := \sigma(N(s,x), s \le t, \sigma_0(x))$, hence the inverse $T(f)(t,x)$ is a stopping time for $\mathcal{F}_t^x$. The bound $c \le M$ and the strong Markov property of the Poisson process N with the stopping time $T(f)$ imply $A(f) \in \mathcal{B}$:

$$\begin{aligned} |A(f)(t)-A(f)(s)| &\le 2\mathbb{P}\left[N\big(T(f)(s)+M|t-s|\big)-N(T(f)(s))>0\right] \\ &= 2\left(1-e^{-M|t-s|}\right) \le 2M|t-s|. \end{aligned}$$

Now we show that $\|A(f)-A(g)\| \le \frac{1}{2}\|f-g\|$ for t_0 small enough. We will show that for t_0 small enough

$$\|T(f)-T(g)\| \le \frac{1}{4}\|f-g\|. \tag{3.2}$$

From (3.2) we get, using the strong Markov property of the Poisson process $N(\cdot,x)$ with the stopping time $T(f)(t,x)\wedge T(g)(t,x)$ and the independence of σ_0 and N

$$\begin{aligned} &|A(f)(t,x)-A(g)(t,x)| \le 2\mathbb{P}\Big[\big|N\big(T(f)(t,x),x\big)-N\big(T(g)(t,x),x\big)\big|>0\Big] \\ &= 2\mathbb{P}\Big[N\big(T(f)(t,x)\vee T(g)(t,x),x\big)-N\big(T(f)(t,x)\wedge T(g)(t,x),x\big)>0\Big] \\ &\le 2\mathbb{P}\Big[N\Big(T(f)(t,x)\wedge T(g)(t,x)+\frac{1}{4}\|f-g\|,x\Big)-N\big(T(f)(t,x)\wedge T(g)(t,x),x\big)>0\Big] \\ &\le \frac{1}{2}\|f-g\|. \end{aligned}$$

Proof of (3.2):
As $T(f),\ T(g) \in C^0_{M,0}([0,t_0],\mathcal{Z}^\gamma)$, remark 2.4 shows that it is enough to give an estimate for the *inverse* time changes. As c is Lipschitz and bounded away from 0, $\frac{1}{c}$ is Lipschitz, say with Lipschitz constant L. So we get from (3.1) for all x :

$$
\begin{aligned}
\left|T(f)^{-1}(t)-T(g)^{-1}(t)\right| \;\le\; & L \int_0^t \left|J*\overline{f}\big(T(f)^{-1}(s)\big)-J*\overline{g}\big(T(g)^{-1}(s)\big)\right| ds \\
& +\lambda L \int_0^t \left|K_2^\epsilon*\overline{f}\big(T(f)^{-1}(s)\big)-K_2^\epsilon*\overline{g}\big(T(g)^{-1}(s)\big)\right| ds \\
& +\lambda L \int_0^t \left|K_1^\epsilon*e[f]\big(T(f)^{-1}(s)\big)-K_1^\epsilon*e[g]\big(T(g)^{-1}(s)\big)\right| ds.
\end{aligned}
$$

We denote the first integral by $I_{1.1}$, the second by $I_{1.2}$, and the third by I_2.
Further we have for any $y \in \mathcal{Z}^\gamma$, $s \in T(f)([0,t_0],x) \cap T(g)([0,t_0],x)$, $f,g \in \mathcal{B}$ by adding and subtracting $f(T(g)^{-1}(s,x),y)$

$$
\begin{aligned}
|f(T(f)^{-1}(s,x),y)-g(T(g)^{-1}(s,x),y)| \;\le\; & 2M^2 \sup_{[0,t_0]} |T(f)(t,x)-T(g)(t,x)| \\
& + \sup_{[0,t_0]} |f(t,y)-g(t,y)|.
\end{aligned}
$$

This implies $I_{1.1}+I_{1.2} \le t_0 C(J,M,K_2)\left(\|T(f)(s)-T(g)(s)\|+\|f-g\|\right)$. Further

$$
\begin{aligned}
I_2 \;\le\; & \int_0^t \left|K_1^\epsilon*e[f]\big(T(f)^{-1}(s)\big)-K_1^\epsilon*e[g]\big(T(f)^{-1}(s)\big)\right| ds \\
+ & \int_0^t \left|K_1^\epsilon*e[g]\big(T(f)^{-1}(s)\big)\right| - K_1^\epsilon*e[g]\big(T(g)^{-1}(s)\big)\Big| ds.
\end{aligned}
$$

Let the first integral be I_3 and the second I_4. We claim:

$$
I_4 \le \sqrt{t_0}\,\epsilon^{-1} C(K,K_1,M)\|e_0\|_{L^\infty}\|T(f)(s)-T(g)(s)\| : \tag{3.3}
$$

As $T(f)^{-1}(s) \wedge T(g)^{-1}(s) \ge M^{-1}s$, we get

$$
I_4 \le M\|T(f)-T(g)\| \int_0^{t_0} \|K_1^\epsilon*e[g](r)\|_{C^{0,1}([M^{-1}s,Mt_0];L^\infty)}\ ds,
$$

where $C^{0,1}([a,b];L^\infty)$ means Lipschitz in time uniformly in space. For $r_1<r_2$ we get from (1.14) (with m^γ replaced by g) by a change of variables in time

$$
\begin{aligned}
e[g](r_2)-e[g](r_1) \;=\; & \big(P(r_2)-P(r_1)\big)*e_0+\int_{r_1}^{r_2} P(s)*[\Delta(K^\epsilon*\overline{g})(r_2-s)]ds \\
& +\int_0^{r_1} P(s)*[(\Delta K^\epsilon)*(\overline{g}(r_2-s)-\overline{g}(r_1-s))]ds.
\end{aligned}
$$

From standard estimates on the heat kernel (use $K*\partial_t P = \nabla K*\nabla P$) and the Lipschitz continuity of g we get

$$
\|K_1^\epsilon*e[g](r)\|_{C^{0,1}([M^{-1}s,Mt_0];L^\infty)} \le C(M)s^{-\frac{1}{2}}\epsilon^{-1}.
$$

(For the precise calculations and justification of the change of variables see also [8].)

To estimate I_3, note that the inverse time changes are piecewise differentiable with derivative bounded by M. Hence by the change of variables $T(f)^{-1}(s) = r$ and estimates on the heat kernel we get

$$I_3 \leq M \int_0^{t_0}\int_0^t \left|\nabla P(t-s) * (\nabla K^\epsilon)[\overline{f}(s) - \overline{g}(s)]\right| dsdt \leq \epsilon^{-1} t_0^{\frac{3}{2}} C(K,M)\|f-g\|.$$

We can iterate this short time existence with the new independent initial distribution $\sigma_0 := \sigma^\gamma[T](t_0)$, as long as $\|e\|_\infty$ does not explode. This cannot happen on compact time intervals. The Markov property follows directly from the strong Markov property of the independent N with stopping time T^{m^γ}. $\square$

Proof of the differential equations (2.4) (2.6) (Mean field equations):
Using the fact that the first jump time of a Poisson process is exponentially distributed, one gets from the representation (1.15) and the strong Markov property the following estimates:

$$\frac{m^\gamma(t+h) - m^\gamma(t)}{h} = (1 - m^\gamma(t))c\big[-1, m^\gamma, e[m^\gamma]\big](t) - (1 + m^\gamma(t))c\big[+1, m^\gamma, e[m^\gamma]\big](t) + o(1)$$

$$\frac{m^\gamma(t) - m^\gamma(t-h)}{h} = (1 - m^\gamma(t-h))c\big[-1, m^\gamma, e[m^\gamma]\big](t-h) - (1 + m^\gamma(t-h))c\big[+1, m^\gamma, e[m^\gamma]\big](t-h) + o(1).$$

Letting $h \to 0$, the claim follows from the uniform continuity in time of m^γ and $e[m^\gamma]$.

3.2. *Proof* of Lemma 2.5
As $\mathcal{Z}^\gamma$ contains only finitely many sites, only finitely many spins flip in any compact time interval, so we can construct the time changes piecewise, thus the existence is clear.

Lemma 3.1. Measurability properties
For the definition of $D([0, Mt] \times \mathcal{Z}^\gamma, \mathbb{Z}_0^+)$ *see the remarks on notation in section 2.*

1. *The Map* $\{-1,1\}^{\mathcal{Z}^\gamma} \times L^\infty(\mathbb{T}_\lambda) \times D([0, Mt] \times \mathcal{Z}^\gamma, \mathbb{Z}_0^+) \to C^0_{M,0}([0,t], \mathcal{Z}^\gamma)$:
$(\sigma_0, e_0, N) \to T^\gamma_{\sigma_0,e_0}(N)$ *is continuous.*
2. *The Map from* $\{-1,1\}^{\mathcal{Z}^\gamma} \times L^\infty(\mathbb{T}_\lambda) \times D([0, Mt] \times \mathcal{Z}^\gamma, \mathbb{Z}_0^+)$ *to* $\{-1,1\}^{\mathcal{Z}^\gamma}$:
$(\sigma_0, e_0, N) \to \sigma^\gamma[\sigma_0, T^\gamma_{\sigma_0,e_0}(N), N](t,x)$ *is Borel-measurable.*

Proof Claim 2. is a consequence of 1., as for large integers n the map

$$N \mapsto n \int_t^{t+\frac{1}{n}} \sigma_0^\gamma(x)(-1)^{N\left(T^\gamma(s,x,N),x\right)} ds$$

is continuous because of the continuity from the right of N and 1., moreover because of the cadlag property, this map converges to $\sigma_0^\gamma(x)(-1)^{N\left(T^\gamma(t,x),x\right)}$.
1. follows directly from

Lemma 3.2. *Fix* $N \in D([0, Mt_0] \times \mathcal{Z}^\gamma, \mathbb{Z}_0^+)$ *and let* $e_0, \widetilde{e}_0 \in L^\infty(\mathbb{T}_\lambda)$, $t_0 > 0$ *and* $0 < \gamma < 1$. *Let* d *denote the Skorohod-metric. Then there is for any such* N *a* $\delta_0(N) > 0$, *such that for all* $0 < \delta < \delta_0(N)$ *the inequality* $\|\widetilde{e}_0 - e_0\|_{L^\infty(\mathbb{T}_\lambda)} + \sup_{\mathcal{Z}^\gamma} d(N(\cdot,x), \widetilde{N}(\cdot,x)) < \delta$ *implies*

$$\sup_{[0,t_0]\times\mathcal{Z}^\gamma} |T_{\sigma_0,e_0}(t,x,N) - T_{\sigma_0,\widetilde{e}_0}(t,x,\widetilde{N})| \leq C(N)\delta.$$

Sketch of the proof of 3.2:

We give here a sketch of the proof and refer to [8] for the calculations. The proof is done by induction on the number of jumps of N. If two paths are close in the Skorohod-metric, then their respective jump times are close. From (1.11) we see: As long as both time changes are on each site x between jump number i_x and jump number i_x+1, the change of their difference depends only on the error in e, which in turn depends on $\|\widetilde{e}_0 - e_0\|$ and the difference of the time changes.

As the times at which jump number i_x+1 occurs for N and $\widetilde{N}$ are less than δ apart and the intensities are bounded, we can estimate the contribution to the distance of the time changes coming from the neighbourhood of a jump: From the distance of the time changes before the jump, we can derive an upper bound for the time interval in which one time change has reached the jump time and the other has not. One has to be aware of the possibility that one time change could be several jumps ahead of the other.

In order to prove property (2.3), we need the following lemma:

Lemma 3.3. *Let $\mathbb{P}$ be a probability such that $N(t,x),\ x \in \mathbb{Z}^d$ is a family of independent Poisson processes and T^γ as in (1.11). Define for $u := (u_x)_{\ x\in\mathcal{Z}^\gamma},\ u_x \in [0,\infty)$ a σ-algebra by*

$$\mathcal{F}_u := \sigma(\sigma_0(x), N(s_x,x),\ s_x \le u_x,\ x \in \mathcal{Z}^\gamma),$$

and for $t>0$

$$\mathcal{F}_{T^\gamma(t)} := \left\{ A \in \bigcup_{u\in(0,\infty)^{\mathcal{Z}^\gamma}} \mathcal{F}_u :\ A \bigcap_{\mathcal{Z}^\gamma} [T^\gamma(t,x) \le v_x] \in \mathcal{F}_v \text{ for all } v \in (0,\infty)^{\mathcal{Z}^\gamma} \right\}.$$

Then $\widetilde{N}(t,x) := N\big(T^\gamma(t_0,x)+t,x\big) - N\big(T^\gamma(t_0,x),x\big),\ x \in \mathcal{Z}^\gamma$, is again a family of independent Poisson processes (with rate 1) and independent of $\mathcal{F}_{T^\gamma(t_0)}$.

Remark: T^γ is a multi-parameter stopping time in the sense of [9]. The theorem is true for multi-parameter stopping times in general. It is a generalisation of the strong Markov property for the Poisson process.

Proof (Sketch) The proof mimics the proof of the strong Markov property for the Poisson process. For $x_i \in \mathcal{Z}^\gamma$, $1 \le i \le m$, $0 \le s_{i,1} \le t_{i,1} \ldots \le s_{i,m_i} \le t_{i,m_i}$, $A \in \mathcal{F}_{T^\gamma(t)}$, $z_{i,j} \in \mathbb{Z}$ and $S := A \bigcap_{i=1}^m \bigcap_{j=1}^{m_i} \{\ \widetilde{N}(t_{i,j},x_i) - \widetilde{N}(s_{i,j},x_i) = z_{i,j}\}$, we show that

$$\mathbb{P}(S) \ = \ \prod_{i=1}^{m}\prod_{j=1}^{m_i} \mathbb{P}(\{N(t_{i,j}-s_{i,j}) = z_{i,j}\})\mathbb{P}(A)$$

by approximating T from above by functions with finitely many values and using the independence of the increments of the Poisson process and the independence of the family $(N(\cdot,x))_{x\in\mathcal{Z}^\gamma}$. □

Now we can continue the proof of property (2.3): As in the case of one-parameter stopping times, $\sigma_0^\gamma(x)(-1)^{N\left(T^\gamma_{\sigma_0,e_0}(t,x,N),x\right)}$ and hence $e[\sigma^\gamma[T^\gamma]](t)$ are $\mathcal{F}_{T^\gamma}(t)$- measurable and thus independent of $\widetilde{N}(s,x)$. From (1.11) and (1.12) we see that

$$T^\gamma_{\sigma_0,e_0}(t_0+s,N) - T^\gamma_{\sigma_0,e_0}(t_0,N) = T^\gamma_{\sigma_0^\gamma(x)(-1)^{N(T^\gamma_{\sigma_0,e_0}(t_0))},e[\sigma^\gamma[T^\gamma]](t_0)}(s,\widetilde{N}) =: T^\gamma[\widetilde{N}|t_0](s)$$

so for $\sigma^\gamma_{(\sigma_0,e_0)}$, $e^\gamma_{(\sigma_0,e_0)}$ as in (2.1, 2.2)

$$\sigma^\gamma_{(\sigma_0,e_0)}(t_0+s,x)=\sigma^\gamma_{(\sigma_0,e_0)}(t_0)(-1)^{\widetilde{N}(T^\gamma[\widetilde{N}|t_0](s),x)}=F(\sigma^\gamma_{(\sigma_0,e_0)}(t_0),e^\gamma_{(\sigma_0,e_0)}(t_0),\widetilde{N}),$$

F measurable. As $\widetilde{N}$ is in distribution a family of independent Poisson processes, (2.3) is a consequence of the theorem of Fubini.

4. Convergence of time changes

In this section, we will proof Theorem 2.7. Fix e_0, σ_0^γ and define a function $F^\gamma[T](t,x,N)$ for $T(x,t)\in C^0_{M,0}$, $\sigma^\gamma[\sigma_0^\gamma,T,N](t,x)$ as in (1.8), $e[\sigma^\gamma[T]]$ as in (1.12), T^{-1} denoting the inverse function and $c[\cdot,\cdot,\cdot]$ as in (1.9) by

$$F^\gamma[T](t,x):=\frac{1}{c\Big[\sigma_0^\gamma(-1)^{N(t,x)},\overline{\sigma^\gamma}[T](T^{-1}(t,x)),e[\sigma^\gamma[T]](T^{-1}(t,x))\Big](x)}.$$

Obviously, $(T^\gamma)^{-1}(t,x)=\int_0^t F^\gamma[T^\gamma](s,x)ds$. In the same way, we set

$$F^{m^\gamma}[T](t,x):=\frac{1}{c\Big[\sigma_0^\gamma(-1)^{N(t,x)},\overline{m^\gamma}(T^{-1}(t,x)),e[m^\gamma](T^{-1}(t,x))\Big](x)}$$

and have $(T^{m^\gamma})^{-1}(t,x)=\int_0^t F^{m^\gamma}[T^{m^\gamma}](s,x)ds$.

Set $I^\gamma(x,t_0):=T^\gamma([0,t_0],x)\cap T^{m^\gamma}([0,t_0],x)$, then it is an interval and

$$\sup_{[0,t_0]}|T^\gamma(t,x)-T^{m^\gamma}(t,x)|\le M\sup_{I^\gamma(x,t_0)}|(T^\gamma)^{-1}(t,x)-(T^{m^\gamma})^{-1}(t,x)|,$$

as the time changes are in $C^0_{M,0}$, so we have

$$\begin{aligned}\sup_{[0,t_0]}|T^\gamma(t,x)-T^{m^\gamma}(t,x)| &\le M\Bigg(\sup_{I^\gamma(x,t_0)}\left|\int_0^t\big(F^\gamma[T^\gamma](s,x)-F^\gamma[T^{m^\gamma}](s,x)\big)\,ds\right|\\ &\quad+\sup_{I^\gamma(x,t_0)}\left|\int_0^t\big(F^\gamma[T^{m^\gamma}](s,x)-F^{m^\gamma}[T^{m^\gamma}](s,x)\big)\,ds\right|\Bigg)\end{aligned}$$

The local convergence of the time changes is a consequence of two lemmas:

Lemma 4.1. *There is for any $t_0,\lambda\in(0,1]$, $[\epsilon(\gamma)+p_{\lambda(\gamma)}]\gamma^{-1}\to\infty$ a set G_0 and a constant C such that*

$$\mathbb{P}(\Omega\setminus G_0)\le C{t_0}^{-1}\left(\gamma p_\lambda\epsilon^{-2}\right)^d,$$

and such that on all paths in G_0 for all $T_1,T_2\in C^0_{M,0}$ the following holds:

$$\sup_{I^\gamma_x(t_0)}\left|\int_0^t(F^\gamma[T_1](s,x)-F^\gamma[T_2](s,x))\,ds\right|\le C\sqrt{t_0}\left(1+\frac{\lambda}{\epsilon}(1+\|e_0\|_{L^\infty})\right)\|T_1-T_2\|_{0,t_0,\infty}.$$

Lemma 4.2. *If $t_0, \lambda, \in (0,1]$, $[\epsilon(\gamma)+p_{\lambda(\gamma)}]\gamma^{-1} \to \infty$ then there is for any $\delta > 0$ a set $G_1(\delta)$ and a constant C such that*

$$\mathbb{P}[\Omega \setminus G_1(\delta)] \le Ct_0 \left(\delta^{-(d+3)} \left(\frac{p^2_{\lambda(\gamma)}\gamma}{\epsilon^3} \right)^d + \frac{t_0}{\delta^2} \left(\frac{\gamma p_{\lambda(\gamma)}}{\epsilon^2} \right)^d \right),$$

and for all paths in $G_1(\delta)$ the following estimate holds for all $x \in \mathcal{Z}^\gamma$:

$$\sup_{I^\gamma(x,t_0)} \left| \int_0^t \left(F^\gamma[T^{m^\gamma}](s,x) - F^{m^\gamma}[T^{m^\gamma}](s,x) \right) ds \right| \le \delta\sqrt{t_0}C(\sqrt{t_0}+\lambda\epsilon^{-1}).$$

Theorem 2.7 follows by taking t_0 so small that $C\sqrt{t_0}\left(1+\frac{\lambda}{\epsilon}(1+\|e_0\|_{L^\infty})\right) \le \frac{1}{2M}$.

Proof of Lemma 4.1:

Set $e[T] := e[\sigma^\gamma[T]]$ and $\|\cdot\| := \|\cdot\|_{\infty,0,t_0}$. We use the Lipschitz continuity of $\frac{1}{c}$ and split the integral up as in the proof of (3.2). It is enough to estimate integrals of the following type uniformly in x :

$$\int_0^t \left| K^\epsilon * \left(\overline{\sigma^\gamma}[T_1](T_1^{-1}(s,x)) - \overline{\sigma^\gamma}[T_2](T_2^{-1}(s,x)) \right) \right| ds \tag{4.1}$$

$$\int_0^t \left| K^\epsilon * \left(e[T_1](T_1^{-1}(s)) - e[T_1](T_2^{-1}(s)) \right) \right| ds \tag{4.2}$$

$$\int_0^t \left| K^\epsilon * \left(e[T_1](T_2^{-1}(s)) - e[T_2](T_2^{-1}(s)) \right) \right| ds \tag{4.3}$$

In order to estimate (4.3), we perform a change of variables $T_2^{-1}(s) = s'$, put one derivative from the Laplacian on the heat kernel and interchange the order of integration of the time integrals to get

$$(4.3) \le C\sqrt{t_0} \left\| \int_0^{t_0} |\nabla K^\epsilon| * |\overline{\sigma^\gamma}[T_1](s) - \overline{\sigma^\gamma}[T_2](s)| ds \right\|_{L^\infty}. \tag{4.4}$$

For (4.2) we set $T^s := T_1^{-1} \vee T_2^{-1}$, $T_i := T_1^{-1} \wedge T_2^{-1}$ and get:

$$\begin{aligned}(4.2) \le \int_0^t K^\epsilon * \Bigg(& \left| \left[P(T_1^{-1}(r)) - P(T_2^{-1}(r)) \right] * e_0 \right| + \left| \int_{T_i(r)}^{T^s(r)} P(s) * \left[\Delta K^\epsilon * \overline{\sigma^\gamma}[T_1](T^s(r)-s) \right] ds \right| \\ & + \left| \int_0^{T_i(r)} P(s) * \Delta K^\epsilon * \left(\overline{\sigma^\gamma}[T_1](T_i(r)-s) - \overline{\sigma^\gamma}[T_1](T^s(r)-s) \right) ds \right| \Bigg) dr.\end{aligned}$$

We call the three integrals on the right hand side (over ds and dr) I_1, I_2 and I_3. From standard estimates for the heat kernel (put one space derivative on K^ϵ and one on $P(s)$, then use $T_i(s) \ge Ms^{-1}$), we get:

$$\begin{aligned} I_1 &\le \left| \int_0^t K^\epsilon * \left(P(T_1^{-1}(x,s)) - P(T_2^{-1}(x,s)) \right) * e_0 \right| \le \sqrt{t_0}\epsilon^{-1}C\|e_0\|_{L^\infty}\|T_1 - T_2\|, \\ I_2 &\le C\epsilon^{-1}\sqrt{t_0}\|T_1 - T_2\| \end{aligned}$$

For I_3, we put one derivative on the heat kernel and interchange the order of integration. We get

$$I_3 \leq \int_0^{Mt} |\nabla P(s)| * |\nabla K^\epsilon| * \left[\int_{T_i^{-1}(s)}^{Mt} \left|\overline{\sigma^\gamma}[T_1](T^s(r)-s) - \overline{\sigma^\gamma}[T_1](T_i(r)-s)\right| dr \right] ds$$

We have for functions f, g such that $(*)$ in 2.3 holds and functions $\sigma(s)$ taking only the values ± 1 the inequality

$$\int_{t_1}^{t_2} |\sigma(f(s)) - \sigma(g(s))| \leq 4M\, N(t_1, t_2) \sup_{[t_1,t_2]} |f(s) - g(s)|,$$

where $N(t_1, t_2)$ is the number of jumps of σ in $[f(t_1) \wedge g(t_1), f(t_2) \vee g(t_2)]$. If we apply this to $\sigma_s(r) := (-1)^{N\left(T_1(r-s), x\right)}$ and $T_i(s, y)$, $T^s(s, y)$, it remains to bound

$$\sup_{u \in \mathbb{T}_\lambda} \sum_{y \in \mathcal{Z}^\gamma} N(Mt_0, y) \int_{|v - \gamma y| \leq \frac{\gamma}{2}} \left(\sum_{z \in \mathbb{Z}^d} |(\nabla K)^\epsilon (u - v + zp_\lambda)| \right). \tag{4.5}$$

on a set of large probability. If we can do this, the proof of the lemma is finished, because we can estimate (4.1) and (4.4) in the same way. Note that for $|\mathcal{Z}^\gamma| = O(\gamma^{-1})$, $\epsilon = O(1)$, it is simply a law of large numbers.

We can divide $\mathbb{T}_\lambda$ in k_λ cubes W_k, $[p_\lambda \epsilon^{-1}] \leq k_\lambda \leq 2[p_\lambda \epsilon^{-1}]$, each of them of side length $\widehat{\epsilon}$, $\epsilon \leq \widehat{\epsilon} \leq 2\epsilon$. (To estimate expressions concerning the kernel J, simply put $\epsilon = 1$.) Note that for K with compact support or rapidly decreasing,

$$\sum_k \left\| \sum_{\mathbb{Z}^d} |(\nabla K)^\epsilon (u - v + zp)| \right\|_{L^\infty(W_k)} \leq \epsilon^{-d} C(K).$$

Thus

$$(4.5) \leq C(K) \sup_{1 \leq k \leq k_\lambda} \sum_{y:\ dist(\gamma y, W_k) \leq \frac{\gamma}{2}} \left(\frac{\gamma}{\epsilon}\right)^d N(Mt_0, y).$$

If we apply the weak law of large numbers for the random variables $N(Mt_0, x) - Mt_0$ which are independent with mean 0 and variance Mt_0, we get

$$\mathbb{P}\left[\sum_{y:\ dist(\gamma y, W_k) \leq \frac{\gamma}{2}} \left(\frac{\gamma}{\epsilon}\right)^d N(Mt_0, y) \geq 2Mt_0 \right] \leq \frac{CM}{t_0} \left(\frac{\gamma}{\epsilon}\right)^d.$$

(The constant accounts also for the fact that there are not exactly $\left(\frac{\gamma}{\epsilon}\right)^{-d}$ grid points in each W_k.) Now sum over k. $\square$

Proof of lemma 4.2: By a change of variables $(T^{m^\gamma})^{-1}(s, x) = s'$ and as $\frac{1}{c} \leq M$ we see that it is sufficient to estimate uniformly on $[0, t]$

$$\int_0^t \left| c\left[\sigma^\gamma[T^{m^\gamma}](s), \overline{\sigma^\gamma}[T^{m^\gamma}](s), e[\sigma^\gamma[T^{m^\gamma}]](s)\right] - c\left[\sigma^\gamma[T^{m^\gamma}](s), \overline{m^\gamma}(s), e[m^\gamma](s)\right] \right| ds.$$

As c is Lipschitz continuous, we see from (1.15) that it is enough to estimate expressions of the type

$$I(u,t) := \sum_{y \in \mathcal{Z}^\gamma} \int_{|v-\gamma y| \le \frac{\gamma}{2}} \left(\sum_{z \in \mathbb{Z}^d} K^\epsilon(u-v+zp_\lambda) \right) dv \Big(\sigma^\gamma[T^{m^\gamma}](t,y) - \mathbb{E}\big[\sigma^\gamma[T^{m^\gamma}](t,y)\big] \Big)$$

(Also the e-dependent difference can be written in such a way.)

In principle, the estimate we need follows from the weak law of large numbers for the independent $\sigma^\gamma[T^{m^\gamma}](t,y)$. But we need a uniform estimate for $(t,u) \in Mt_0 \times \mathcal{Z}^\gamma$. So we divide $\mathcal{Z}^\gamma$ as in the proof of the previous lemma in cubes $W_k(\delta)$ of side length of order $\epsilon\delta$. Within such a cube, $I(u,t)$ varies at most by $C(d)\delta$, because the σ^γ are bounded.

Then we divide the time interval in subintervals I_j of length δ. The m^γ are Lipschitz-continuous, so they do not vary by more than $2M\delta$ in one I_k.

By the law of large numbers for the independent random variables $N\big(T^{m^\gamma}(t,x)+M\delta,x\big)$, we can bound the number of jumps of the $\sigma^\gamma[T^{m^\gamma}](t)$ in I_k on a set of large probability in the following way: For $|s-t| \le \delta$

$$\mathbb{P}\left[\sum_{y \in W_k} \big(\sigma^\gamma[T^{m^\gamma}](t,y) - \sigma^\gamma[T^{m^\gamma}](s,y) \big) > 2M\delta \right] \le C(M\delta)^{-2} M\delta \left(\frac{\gamma}{\epsilon} \right)^d$$

To conclude, we apply the law of large numbers for a family of representatives $I(u_k, t_j)$, $u_k \in W_k(\delta)$, $t_j \in I_j$.

5. Convergence of the Block Spins

In this section we prove theorem 2.8. For $\mathcal{A}_{\gamma^{-\alpha}}$ as in (1.16), we use the abbreviation $\|f-g\|_\alpha := \|\mathcal{A}_{\gamma^{-\alpha}}(f-g)\|_{L^\infty}$ and show as a first step:

Lemma 5.1. *For Δt as in Theorem 2.7 and m^γ and $\sigma^\gamma[T^\gamma]$ starting from the same e_0, σ_0^γ, we have for small γ*

$$\mathbb{P}\left[\sup_{[0,\Delta t]} \|\sigma^\gamma[T^\gamma](t) - m^\gamma(t)\|_\alpha > \delta \right] \le \frac{C}{\Delta t \delta^{d+3}} \left(p_\lambda \gamma^{\alpha-1} \right)^d \left(\Delta t \left(\frac{\gamma p_\lambda^2}{\epsilon^3} \right)^d + \gamma^{\alpha d} \right).$$

Proof We write

$$G^\alpha(\delta,t,x) := \left\{ \left| \mathcal{A}_{\gamma^{-\alpha}} \Big(\sigma^\gamma[T^\gamma](t) - m^\gamma(t) \Big)(x) \right| > \delta \right\}$$

and have

$$\begin{aligned} G^\alpha(\delta,t,x) \quad &\subseteq \quad \left\{ \left| \mathcal{A}_{\gamma^{-\alpha}} \Big(\sigma^\gamma[T^{m^\gamma}](t) - m^\gamma(t) \Big)(x) \right| > \frac{\delta}{2} \right\} \\ &\qquad \cup \left\{ \left| \mathcal{A}_{\gamma^{-\alpha}} \Big(\sigma^\gamma[T^\gamma](t) - \sigma^\gamma[T^{m^\gamma}](t) \Big)(x) \right| > \frac{\delta}{2} \right\}. \end{aligned}$$

Call the first expression on the right hand side $G_1^\alpha(\delta,x,t)$ and the second one $G_2^\alpha(\delta,x,t)$. We have for $t_\delta := t - \frac{\delta}{16M}$

$$\begin{aligned} G_2^\alpha(\delta,x,t) &\subseteq \left\{ \|T^\gamma - T^{m^\gamma}\|_{\infty,0,\Delta t} > \frac{\delta}{16M^2} \right\} \\ &\cup \left\{ \mathcal{A}_{\gamma^{-\alpha}} \left(\left| N\left(T^{m^\gamma}(t_\delta,y) + \frac{\delta}{8}, y\right) - N\left(T^{m^\gamma}(t_\delta,y),y\right) \right| \wedge 1 \right)(x) > \frac{\delta}{4} \right\} \\ &= G_3(\delta) + G_4(\delta,t,x), \end{aligned}$$

observe that $\{T^\gamma(t), T^{m^\gamma}(t)\} \subset [T^{m^\gamma}(t-\frac{\delta}{16M}), T^{m^\gamma}(t-\frac{\delta}{16M}) + \frac{\delta}{16}(1+M^{-2})]$ follows from $|T^\gamma(t) - T^{m^\gamma}(t)| \le \frac{\delta}{16M^2}$, and G_4 contains paths where the $\mathcal{A}_{\gamma^{-\alpha}}$- average of jumps in this time interval exceeds $\delta/4$.

The probability of the set $G_3(\delta)$ is bounded by lemma 2.7. For each fixed (t,x) we can estimate $\mathbb{P}(G_1^\alpha(\delta,t,x))$ by the Chebyshev-inequality for the independent $\sigma^\gamma[T^{m^\gamma}]$.

As $N(T^{m^\gamma}(t,y)+\cdot,y)$ are distributed as independent Poisson processes, the probability of $G_4(\delta,t,x)$ for a *fixed* x can be estimated by the Chebyshev inequality as well. To estimate the probability of their union, we proceed as in the convergence proof for the time changes: We divide $[0,\Delta t]$ in subintervals of length of order δ and $\mathbb{T}_\lambda$ in cubes of side length of order $\gamma^{1-\alpha}\delta$. Then $\mathcal{A}_{\gamma^{-\alpha}}(f(t))(x)$ varies on these subintervals and cubes by less than δ, if f is either $\sigma^\gamma[T^\gamma](t) - \sigma^\gamma[T^{m^\gamma}](t)$ or $|N(T^{m^\gamma}(t_\delta)+\frac{\delta}{8}) - N(T^{m^\gamma}(t_\delta))| \wedge 1$. So we choose again representatives (t_i,x_k) in each cube and have

$$\bigcup_{t,x} \left(G_1(\delta,t,x) \cup G_4(\delta,t,x)\right) \subseteq \bigcup_{k,i} \left(G_1(C^{-1}\delta,t_i,x_k) \cup G_4(C^{-1}\delta,t_i,x_k)\right).$$

Now we use the results obtained on intervals of lenght Δt to obtain convergence on long time intervals. This will be done by a method adapted from [6]: We define a piecewise deterministic process, called *quasi-solution* of the Mean field equation on an interval $[0,t(\lambda(\gamma))]$ for $t(\lambda(\gamma)) \le C\log(\gamma^{-1})$:

Let $(m^\gamma_{e_0,\sigma_0}(t), e_{e_0,\sigma_0}(t))$ be the solution of (2.4), (2.5) starting from the initial values e_0, σ_0. The quasi-solution $(m^\gamma_{\Delta t}(N), e^\gamma_{\Delta t}(N))$ for the interval size Δt and the path N is defined as follows: Let $(e^\gamma(t,N), \sigma^\gamma(t,N)) := (e[\sigma^\gamma[T^\gamma(N),N](t), \sigma^\gamma[T^\gamma(N),N](t))$, then

$$(m^\gamma_{\Delta t}, e^\gamma_{\Delta t})(t) := (m_{e^\gamma(k\Delta t,N),\sigma^\gamma(k\Delta t,N)}, e_{e^\gamma(k\Delta t,N),\sigma^\gamma(k\Delta t,N)})(t-k\Delta t) \text{ for } t \in [k\Delta t,(k+1)\Delta t).$$

This means that $(m^\gamma_{\Delta t}(N), e^\gamma_{\Delta t}(N))$ is updated at the end of each subinterval of length Δt by the true value of the stochastic process $(\sigma^\gamma[T^\gamma], e[\sigma^\gamma[T^\gamma]])$.

Note that a rough a priori estimate gives $\|e[T^\gamma]\|_{L^\infty} \le C + t(\lambda) + \epsilon^{-2}$, hence we can find a $\Delta t(\gamma)$ such that Theorem 2.7 can be applied for $e_0 := e[T^\gamma](k\Delta t)$, $k \le t(\gamma)\Delta t^{-1}$.

Now we condition stepwise on the σ-algebra generated by $\sigma^\gamma[T^\gamma](k\Delta t)$ and we derive from property (2.3) and Lemma 5.1: The probability of $\sigma^\gamma[T^\gamma](t)$ to deviate in the α-norm from the quasi solution on $[0,t(\lambda)]$ by more than δ is $t(\lambda)(\Delta t)^{-1}$ times the probability estimated in Lemma 5.1.

Now we choose $\delta := \gamma^\zeta$ for a ζ small enough. As $\epsilon^{-1}, (\Delta t)^{-1}, p_{\lambda(\gamma)}$ and $t(\gamma)$ are all of at most logarithmic growth in γ, we have the desired estimate (2.8) with the quasi-solution $m^\gamma_{\Delta t}$ in place of m^γ.

As ϵ is much larger than $\gamma^{-\alpha}$, the convergence of $\mathcal{A}_{\gamma^{-\alpha}}(\sigma^\gamma[T^\gamma])$ implies the L^∞-convergence of $e[\sigma^\gamma[T^\gamma]]$ in probability to the quasi-solution, this means we have (2.9) with another ζ and $e^\gamma_{\Delta t}$ in place of $e[m^\gamma]$.

An estimate for the distance of the quasi- solution and the solution of the mean-field equation will conclude the proof. For this, we have to estimate the distance of two solutions of (2.4), (2.5) starting from different initial values.

Denote for simplicity the solutions by (m, e) and $(\widehat{m}, \widehat{e})$ respectively. As K^ϵ is C^2, $\theta := e - K^\epsilon * m$ solves $\partial_t\theta = \Delta\theta - K^\epsilon * \partial_t m$, so e solves

$$e(t) = P(t) * \big[e(0) - K^\epsilon * m(0)\big] + K^\epsilon * m(t) - \int_0^t P(t-s) * \big[K^\epsilon * \partial_t m\big] \tag{5.1}$$

The operator $\mathcal{A}_{\gamma^{-\alpha}}$ commutes with convolutions, so we have

$$\begin{aligned} \|e - \widehat{e}\|_\alpha \ \le\ & \|e(0) - \widehat{e}(0)\|_\alpha + C\|P(1)\|_{L^1}\|m(0) - \widehat{m}(0)\|_\alpha \\ & + C\left(\|m(t) - \widehat{m}(t)\|_\alpha + \|P(1)\|_{L^1}\int_0^t \|\partial_t(m(s) - \widehat{m}(s))\|_\alpha ds\right). \end{aligned}$$

For $\partial_t(m(s) - \widehat{m}(s))$ we substitute equation (2.4). We now make use of the fact that for Lipschitz-functions c

$$|c(K^\epsilon * g_1) - c(K^\epsilon * g_2)| \le LC\|g_1 - g_2\|_\alpha + LC\epsilon^{-1}\gamma^{1-\alpha}. \tag{5.2}$$

Then we get

$$\|e(t) - \widehat{e}(t)\|_\alpha \ \le\ f_\alpha((e - \widehat{e})(0), m - \widehat{m}, t) + C\int_0^t \lambda\|e - \widehat{e}\|_\alpha(s)ds + C\epsilon^{-1}\gamma^{1-\alpha},$$

$$f_\alpha((e-\widehat{e})(0), m-\widehat{m}, t) \ =\ \|(e-\widehat{e})(0)\|_\alpha + C\left(\|m(0)-\widehat{m}(0)\|_\alpha + \|m(t)-\widehat{m}(t)\|_\alpha + \int_0^t \|m(s)-\widehat{m}(s)\|_\alpha ds\right).$$

The Gronwall lemma gives:

$$\|e(t) - \widehat{e}(t)\|_\alpha \le e^{C\lambda t}\left((f_\alpha(e(0) - \widehat{e}(0), m - \widehat{m}, t) + \epsilon^{-1}\gamma^{1-\alpha}\right) \tag{5.3}$$

We observe that for $\frac{1}{M} \le c(\cdot)$, c Lipschitz

$$\left|e^{-\int_s^t c(f_1)(s)ds} - e^{-\int_s^t c(f_2)(s)ds}\right| \le e^{-\frac{t-s}{M}} L \int_s^t |f_1(s) - f_2(s)|ds. \tag{5.4}$$

Using the variation of constants formula for (2.4), which is

$$m(t,x) = e^{-\int_0^t c_+(m(s), e[m](s))ds} m(0) + \int_0^t e^{-\int_s^t c_+(m(s), e[m](s))dr} c_-(m(s), e[m](s))ds, \tag{5.5}$$

we get from (5.4), (5.3) and (5.2) for $0 < t < \Delta t < 1$:

$$\sup_{[0,t]} \|m - \widehat{m}\|_\alpha \le C\left(\epsilon^{-1}\gamma^{1-\alpha} + \|m(0) - \widehat{m}(0)\|_\alpha + \lambda\|e(0) - \widehat{e}(0)\|_\alpha\right) + C\int_0^t \sup_{[0,s]} \|m - \widehat{m}\|_\alpha ds$$

which allows to apply Gronwall's lemma to $\sup_{[0,s]} \|m(r) - \widehat{m}(r)\|_\alpha$, which implies

$$\|m(s) - \widehat{m}(s)\|_\alpha \le e^{C'\Delta t}\left(\frac{\gamma^{1-\alpha}}{\epsilon} + \|m(0) - \widehat{m}(0)\|_\alpha + \|e(0) - \widehat{e}(0)\|_\alpha\right). \tag{5.6}$$

The $\|\cdot\|_\alpha$-convegence of the $\sigma^\gamma[T^\gamma]$ implies the L^∞-convergence of the energy e, as ϵ is large compared to any power of γ.

Because $(\Delta t)^{-1}$ and $t(\lambda)$ are of logarithmic order in γ, this implies the convergence theorem for the $\sigma^\gamma[T^\gamma]$: We iterate the estimates for the different time steps of size Δt, using at each end point the estimate for the quasi solutions and (5.6) to get the error at the beginning of the next interval.

To conclude the proof of Theorem 2.8, we have to free ourselves from the restriction $\frac{1}{M} \leq c(\cdot)$. We first show that the solution e of (2.5) on $[0, t(\gamma)] \times \mathbb{T}_\lambda$ is uniformly bounded by $C(\log(\lambda^{-1}) + C)$. (Note that on a *fixed* time interval $[0, T]$, a uniform bound follows immediately from (5.1) and the boundedness of $\partial_t m$.) First note that $\partial_t e = \Delta(e - K^\epsilon * m)$ implies

$$\overline{e} := \int_{\mathbb{T}_\lambda} e_0(r)dr = \int_{\mathbb{T}_\lambda} e(t,r)dr \text{ for all } t > 0.$$

By considering $e - \overline{e}$ we can assume $\overline{e} = 0$. Then we have for $f^+ := f \vee 0$, $f^- := -(f \wedge 0)$: $\int_{\mathbb{T}_\lambda} f^+ = \int_{\mathbb{T}_\lambda} f^- \leq |\mathbb{T}_\lambda| \|f\|_{L^\infty}$.

Further define

$$P_{\mathbb{T}_\lambda}(t,r) := \sum_{z \in \mathbb{Z}^d} P(t, r + zp_\lambda), \ r \in \mathbb{T}_\lambda, \ t > 0$$

where P is the euclidean heat kernel on $\mathbb{R}^d$.

For $t \to \infty$, the kernel $P_{\mathbb{T}_1}(t,r) \to |\mathbb{T}_1|^{-1}$ uniformly in r. From this and the scaling relation $|\mathbb{T}_\lambda| P_{\mathbb{T}_\lambda}(\lambda^{-2}t, \lambda^{-1}r) = |\mathbb{T}| P_{\mathbb{T}}(t,r)$ we find a t^* such that $\frac{3}{4} \leq |\mathbb{T}_\lambda| P_{\mathbb{T}_\lambda}(\lambda^{-2}t^*, r) \leq \frac{5}{4}$. If we apply the estimate to $P_{\mathbb{T}_\lambda}(\lambda^{-2}t^*) * (e_0^+ - e_0^-)$, we get $\|P(\lambda^{-2}t^*) * e_0\|_{L^\infty} \leq \frac{1}{2}\|e_0\|_{L^\infty}$. Now we estimate the contribution of the $\partial_t m$-integral in (5.1). We assume $\lambda^{-2}t^* > 1$ and have by integrating by parts for $t^*_\lambda := t^* \lambda^{-2}$

$$\begin{aligned}
\int_0^{t^*_\lambda} P(t^*_\lambda - s) * \partial_t(K^\epsilon * m(s))ds \;&=\; \int_{t^*_\lambda - 1}^{t^*_\lambda} P(t^*_\lambda - s) * \partial_t(K^\epsilon * m)ds + \left[P(s) * (K^\epsilon * m(t^*_\lambda - s))\right]_1^{t^*_\lambda} \\
&\quad - \int_1^{t^*_\lambda} \partial_t P(s) * (K^\epsilon * m(t^*_\lambda - s))ds.
\end{aligned}$$

This implies (as $\int_{\mathbb{R}^d} \partial_t P(s) = Ct^{-1}$)

$$\left| \int_0^{\lambda^{-2}t^*} P(\lambda^{-2}t^* - s) * \partial_t(K^\epsilon * m(s)) \right| \leq C(\log(\lambda^{-1}) + C).$$

By dividing $[0, t(\gamma)]$ in subintervals of length $\lambda^{-2}t^*$, we find:

$$\sup_{[0,t(\gamma)] \times \mathbb{T}_\lambda} |e(t,x)| \leq C(e_0)(1 + \log(\lambda^{-1})). \tag{5.7}$$

So for e_0 fixed $\lambda e(t)$ is uniformly bounded by some $\lambda \log(\lambda^{-1}) C(e_0)$. (Note that without space rescaling, the same proof yields the boundedness of e_0 itself)

Now replace c by $c^A := c(\cdot, \cdot, \cdot, \chi(\cdot))$, where $\chi \in C_0^\infty(\mathbb{R})$ is such that $\chi(s) = s$ on $[-A-1, A+1]$. The random variables $e[T_A^\gamma]$ as in (1.14), where in (1.13) the intensity is c^A instead of c, converge by the convergence theorems for the case $c^{-1} < M$, which are already shown, in probability to the solution of (2.5) with c^A instead of c. (Remember that due to the scaling relation between γ and ϵ, the convergence of the spins in the α-norm implies the convergence of e in the L^∞-norm.) But from the bound on e we see that this is (for A large enough) equal to the solution for c, because e does not reach the cut-off. $e[T_A^\gamma]$ converges in probability to e, so they do not see the cutoff on a set of large probability. But on this set, T_A^γ are equal to the time changes defined without cut-off.

6. Continuum limit: Proof of Corollary 2.9

For the first part of the corollary, it remains to bound $\|\mathcal{A}_{\gamma^{-\alpha}}(m^\gamma)(t,\gamma x) - m(t,\gamma x)\|_\infty$, where m^γ solves (2.4) and m solves (1.7) for a sequence $|lambda(\gamma)$, $\epsilon(\gamma)$ with the usual logarithmic growth restrictions.

We use the integrated version of eqn. (2.4, 2.5) and (1.7, 1.6). (See (5.5), which gives the integrated version of (2.4), the space continous eqn. (1.7) is treated in the same way.)

We use the following facts:

- c_+, c_- are bounded and Lipschitz in their arguments, $0 < c_+$.
- Because of (5.7), we may still use (5.4).
- $\sup_{x\in\mathbb{Z}} |\mathcal{A}_{\gamma^{-\alpha}}(f) - f| \le L\gamma^{1-\alpha}$ if f is uniformly Lipschitz.

Setting

$$l(t) := \| |\overline{m^\gamma}(t,r) - m(t,r)| + \lambda |K^\epsilon * (e^\gamma - e)(t,r)| \|_{L^\infty(\mathbb{T}_\lambda)}$$

we get directly from the facts listed above:

$$l(t) \le l(0) + C\int_0^t l(s)ds + Ct\frac{\gamma^{1-\alpha}}{\epsilon}$$

The claim follows now from the Gronwall lemma together with the convergence of the initial conditions in the α-norm in probability and the logarithmic dependence of ϵ and λ on γ.

In order to prove the second part of the corollary, we only have to show the convergence of solutions of (1.5, 1.6) to those of (1.3, 1.4). For simplicity we will use that $K^\epsilon(y) = 0$ for $|y| > \epsilon$ and $\int_{\mathbb{R}^d} K^\epsilon(y)dy = 1$, but the result holds for rapidly decreasing kernels as well. Note that in this case in the integral formulation of (1.4) the part coming from the variation of constants formula is $\int_0^t P(t-s) * \partial_t m$ and $|\partial_t m| \le C$ by (1.5). We start with the following observation:

$$\begin{aligned}
&\left\| \int_0^t P(t-s) * (K^\epsilon * f - f) \right\|_{L^\infty(\mathbb{T})} \\
&\le \int_0^{t-\epsilon} \|P(t-s) * K^\epsilon - P(t-s)\|_{L^1(\mathbb{T})} \|f\|_{L^\infty(\mathbb{T})} + 2\epsilon \|f\|_{L^\infty(\mathbb{T})}.
\end{aligned}$$

Set $I(s) := \int_{\mathbb{R}^d} |(P(t-s) * K^\epsilon - P(t-s))(x)|dx$, then

$$I(s) \le \int_{\mathbb{R}^d}\int_{|y|\le\epsilon}\int_0^1 |\nabla P(t-s)_{|(x-ry)}|dr|x-y||K^\epsilon(y)|dydx \le C\epsilon I_1(s), \text{ where}$$

$$I_1(s) \;=\; \iint_{|y|\le\epsilon}\int_0^1 |\nabla P(t-s)_{|(x-ry)}|dr|K^\epsilon(y)|dydx \le \frac{C}{\sqrt{t-s}}$$

So $\int_0^{t-\epsilon} I(s)ds \le C(t)$. Extend periodically to get the result for $\mathbb{T}$, and the corollary follows from Gronwall type estimates as in the previous proofs.

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