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Abstract

The dimension of the kernels of the edth and edth-prime operators on closed, orientable spacelike 2-surfaces with arbitrary genus is calculated, and some of its mathematical and physical consequences are discussed.

1 Introduction

The "edth" operator $\eth$ made its first appearance in General Relativity in an article by Newman and Penrose [1] on the symmetry group of asymptotically flat spacetimes. It was introduced as a particular differential operator on the unit-sphere acting on spin-weighted functions, i.e., sections of certain complex line bundles over the sphere. In this paper and [2] the relationship of $\eth$ and its complex conjugate counterpart $\eth'$ with the representation theory of the Lorentz group resp. $SL(2,\mathbb{C})$ has been elaborated. In particular, the so called spin weighted spherical harmonics on the sphere were introduced. Since then, the edth operator has been intensely studied and it has found numerous applications in General Relativity because it provides a useful tool for all problems which lead to elliptic tensor or spinor equations on the sphere [3].

In [4] it was pointed out that $\eth$ and $\eth'$ are closely related to the ∂ and $\bar{\partial}$ operators of complex analysis. In fact, this analogy can be used to define $\eth$ and $\eth'$ on arbitrary compact Riemann surfaces, the sphere being merely the special case of a surface with genus $g = 0$. The natural framework for this definition of $\eth$ and $\eth'$ is the theory of complex line bundles over Riemann surfaces.

Riemann surfaces with higher genus have many uses in conformal field theories and string theory (see e.g. [5, 6] for an introduction). In particular, the discussion of higher genus black holes (e.g., [7, 8]) has recently gained importance due to the adS/CFT conjecture which asserts that supergravity in an asymptotically adS (anti-de Sitter) spacetime corresponds to a conformal field theory on the conformal boundary of this spacetime in a certain limit. Such spacetimes admit black holes with non-trivial topology which is accordingly reflected in the topology of their conformal boundary.

But also in the classical theory of gravity do they make an appearance. E.g., when the Einstein equations have been reduced by symmetry assumptions to the Ernst equation, a completely integrable system, then one can employ the methods known from soliton theory to produce exact solutions of the field equations which are given in terms of higher genus Riemann surfaces and their associated theta-functions (see e.g. [9, 10, 11] for some applications).

Our immediate motivation for the present paper arose from two different sources. Schmidt [12] has presented a solution of the vacuum Einstein equations which has a conformal boundary with toroidal cross sections. This solution has been very useful for tests of numerical codes solving Friedrich's conformal field equations [13, 14, 15]. The discussion of the asymptotic properties like e.g., their Bondi 4-momentum, radiation field and asymptotic symmetry group, of this and more general such spacetimes necessarily leads to the examination of $\eth$ equations on higher genus Riemann surfaces [16]. One particular problem which arises in this context is the question as to whether the Weyl tensor also necessarily vanishes on the conformal boundary of these spacetimes. Recall [17], that this question can be answered to the affirmative in the spherical case because a certain $\eth$ equation has no non-trivial solution. Or, to put it into more geometric language, there are no symmetric and trace-free second-rank tensorfields on the sphere (see [18] for a proof of the above statement and more information on the relationship between $\eth$ and trace-free symmetric tensorfields).

Another motivation comes from the construction of two-surface observables (e.g. quasi-local energy-momentum and angular momentum) associated with spacelike 2-surfaces embedded in a spacetime [19]. While the quasi-local constructions have been mainly carried out for the spherical case, there is no reason to believe that this should be the only possibility. Again, one is lead to solve equations of the type $\eth\phi = \omega$ for given ω on a closed spacelike 2-surface $\mathcal{S}$. However, the structure of the solutions depends on the kernel and cokernel of the operator $\eth$, which in turn depends on the topology of the line bundle E whose cross section is ϕ. Since the topology of E depends on the topology of $\mathcal{S}$, to solve certain physical problems one should know the kernel and cokernel of the edth operators on complex line bundles E over 2-surfaces that have more complicated topology than that of S^2.

In this paper, we do not intend to discuss the general theory of such operators on complex line bundles. These issues are discussed in the basic monograph [20] in general. Rather, we concentrate on the applicability of our results to General Relativity, motivated by the problems above. Thus, the line

bundles that we consider are connected with the spacetime structure. They arise as the bundles associated with the specific bundle of the spacetime spin frames over the 2-surface $\mathcal{S}$. We regard them and various structures on them as being derived from the four-dimensional geometry.

The necessary background on the geometry of spacelike 2-surfaces and, in particular, the spinor geometry of $\mathcal{S}$ can be found in [21, 22, 23], which also contain the calculation of $\dim\ker \eth_{(p,q)}$ and $\dim\ker \eth'_{(p,q)}$ for $\mathcal{S} \approx S^2$. Essentially that technique is used here to calculate these kernels also for higher-genus 2-surfaces.

Our conventions are those of [17] while our use of global differential geometry is based on [24, 25, 26].

2 The geometry of closed spacelike 2-surfaces

It is known that for any connected, closed, orientable two dimensional smooth manifold $\mathcal{S}$ the de Rham cohomology spaces are $H^0(\mathcal{S}) = H^2(\mathcal{S}) = \mathbb{R}$ and $H^1(\mathcal{S}) = \mathbb{R}^{2g}$, where g is the genus of $\mathcal{S}$. Let $\{a_i, b_i\}$, $i = 1, \dots, g$, be a canonical homology basis on $\mathcal{S}$. These are closed non-contractible curves on $\mathcal{S}$, and for the fundamental group of $\mathcal{S}$ we have

$$\pi_1(\mathcal{S}) = \langle a_i, b_i |\, i = 1, \dots, g;\ \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \,\rangle,$$

i.e. $\pi_1(\mathcal{S})$ is generated by the $2g$ elements a_i, b_i with the only relation that the product of all the commutants $a_i b_i a_i^{-1} b_i^{-1}$ is homotopically trivial [27]. Let q_{ab} be a (negative definite) metric on $\mathcal{S}$, ε_{ab} the corresponding volume 2-form and δ_e the Levi–Civita covariant derivative. By the Hodge decomposition every 1-form α_a is the sum of a closed, a harmonic and an exact 1-form: $\alpha_a = \beta_a + \omega_a + \delta_a f$, where $\delta_{[a}\beta_{b]} = 0$, $\delta_{[a}\omega_{b]} = 0$ and $\delta_a \omega^a = 0$ hold, and $f : \mathcal{S} \to \mathbb{R}$ is some function. Furthermore, this decomposition is unique, and hence, there are $2g$ linearly independent harmonic 1-forms on $\mathcal{S}$ [24]. Any basis $\{\alpha^i_a, \beta^i_a\}$, $i = 1, ..., g$, in the space of the (real) Harmonic 1-forms can be uniquely characterized by the four $g \times g$ matrices $M^i{}_j := \oint_{a_j} \alpha^i_a$, $M^{g+i}{}_j := \oint_{a_j} \beta^i_a$, $M^i{}_{g+j} := \oint_{b_j} \alpha^i_a$ and $M^{g+i}{}_{g+j} := \oint_{b_j} \beta^i_a$, and the $2g \times 2g$ matrix $M^I{}_J$, $I, J = 1, \dots, 2g$, is nonsingular[1]. Another set of 1-forms, associated naturally with $(\mathcal{S}, q_{ab})$ are the conformal Killing 1-forms. They are defined by the vanishing of the symmetric trace-free part of their covariant derivative: $\delta_{(a}\xi_{b)} - \frac{1}{2} q_{ab} \delta_e \xi^e = 0$.

Next, suppose that $\mathcal{S}$ is embedded as a smooth spacelike submanifold in M (for a detailed discussion of what follows, see [21, 22, 23]). If t^a and v^a are future timelike and spacelike unit normals to $\mathcal{S}$ satisfying $t^a v_a = 0$, respectively, then $\Pi^a_b := \delta^a_b - t^a t_b + v^a v_b$ is the orthogonal projection to $\mathcal{S}$, and $O^a_b := \delta^a_b - \Pi^a_b$ is the projection to the timelike 2-planes orthogonal to $\mathcal{S}$. Such vector fields are globally well defined if $\mathcal{S}$ is orientable and at least an open neighbourhood of $\mathcal{S}$ in M is time and space orientable. The induced metric on $\mathcal{S}$ is $q_{ab} := \Pi^c_a \Pi^d_b g_{cd}$, and if ε_{abcd} is the natural volume form on M, then the induced area 2-form and area element on $\mathcal{S}$ is $\varepsilon_{cd} := t^a v^b \varepsilon_{abcd}$ and $\mathrm{d}\mathcal{S} := \frac{1}{2}\varepsilon_{cd}$, respectively. The area 2-form on the orthogonal 2-planes is ${}^{\perp}\varepsilon_{ab} := t_a v_b - t_b v_a$. Then any four-vector X^a at the points of $\mathcal{S}$ can be decomposed in a unique way into the sum of its tangential and normal parts as $X^a = \Pi^a_b X^b + O^a_b X^b$. This implies that $\mathbf{V}^a(\mathcal{S})$, the restriction to $\mathcal{S}$ of the tangent bundle TM of M, has the g_{ab}-orthogonal decomposition $\mathbf{V}^a(\mathcal{S}) = T\mathcal{S} \oplus N\mathcal{S}$ into the sum of the tangent bundle $T\mathcal{S}$ and the (globally trivializable) normal bundle $N\mathcal{S}$ of $\mathcal{S}$. By the orientability of $\mathcal{S}$ the bundle of q_{ab}-orthonormal frames in $T\mathcal{S}$ is reducible to a $B(\mathcal{S}, SO(2))$ principal bundle, while the time and space orientability of (M, g_{ab}) implies that the bundle of orthonormal frames in the normal bundle $N\mathcal{S}$ is reducible to a $B(\mathcal{S}, SO(1,1))$ principal bundle. While the latter is always trivial, the former is not. In particular, if $g = 0$, i.e. $\mathcal{S} \approx S^2$, then $B(\mathcal{S}, SO(2)) \approx \mathbb{R}P^3$, but for $g = 1$, i.e. for $\mathcal{S} \approx S^1 \times S^1$, $B(\mathcal{S}, SO(2)) \approx \mathcal{S} \times SO(2) \approx S^1 \times S^1 \times S^1$. Therefore, the bundle of g_{ab}-orthonormal

[1] The $2g$ real harmonic forms can be combined to form g holomorphic 1-forms, the so called Abelian differentials of the first kind. The real matrix M then corresponds to the complex $g \times 2g$-matrix of periods of the holomorphic differentials. This matrix can always be put into the form $(1_g|P)$, where P is a symmetric complex matrix with positive definite imaginary part, the so called Riemann matrix [27].

frames with given time and space orientation that are compatible to the decomposition $\mathbf{V}^a(\mathcal{S}) = T\mathcal{S} \oplus N\mathcal{S}$ is $B(\mathcal{S}, SO(2) \times SO(1,1)) \approx B(\mathcal{S}, SO(2)) + B(\mathcal{S}, SO(1,1))$.

The spacetime Levi-Civita covariant derivative ∇_a defines a covariant derivative on $\mathbf{V}^a(\mathcal{S})$ by $\delta_e X^a := \Pi^a_b \Pi^f_e \nabla_f \big(\Pi^b_c X^c\big) + O^a_b \Pi^f_e \nabla_f \big(O^b_c X^c\big)$. This derivative annihilates both the spacetime metric and the projections Π^a_b and O^a_b; and hence annihilates the intrinsic metric q_{ab}. Furthermore, it is symmetric: $(\delta_c\delta_d - \delta_d\delta_c)\phi = 0$ for any smooth function ϕ on $\mathcal{S}$. Geometrically, δ_e is the covariant derivative on $\mathbf{V}^a(\mathcal{S})$ determined by a connection on the sum of the principal $SO(2)$– and $SO(1,1)$ bundles above. The commutativity of the structure group $SO(2) \times SO(1,1)$ implies that δ_e does not 'mix' the tangential and normal sections. In particular, the connection coefficient corresponding to the vertical part of the connection can be represented by the 1-form $A_a := \Pi^b_a(\nabla_b t_c)v^c = v^c \delta_a t_c$. The curvature of δ_a, defined by $-f^a{}_{bcd}X^b := (\delta_c\delta_d - \delta_d\delta_c)X^a$, is

$$f^a{}_{bcd} = -{}^{\perp}\varepsilon^a{}_b\Big(\delta_c A_d - \delta_d A_c\Big) + \frac{1}{2}\,{}^{\mathcal{S}}R\Big(\Pi^a_c q_{bd} - \Pi^a_d q_{bc}\Big), \tag{2.1}$$

where ${}^{\mathcal{S}}R$ is the curvature scalar of the intrinsic curvature of $(\mathcal{S}, q_{ab})$. This is the sum of the curvatures corresponding to the $SO(1,1)$-connection on $N\mathcal{S}$ and the $SO(2)$-connection on $T\mathcal{S}$.

Suppose that (M, g_{ab}) admits a spinor structure, and let $\mathbf{S}^A(\mathcal{S})$ be the pull back to $\mathcal{S}$ of the bundle of unprimed spinors over M. Let $t_{AA'}$, $v_{AA'}$ be the spinor form of the normals to $\mathcal{S}$, and let us define $\gamma^A{}_B := 2t^{AA'}v_{BA'}$. This defines the bundle automorphism $\gamma : \mathbf{S}^A(\mathcal{S}) \to \mathbf{S}^A(\mathcal{S}) : \lambda^A \mapsto \gamma^A{}_B\lambda^B$, and $\gamma^A{}_A = 0$, $\gamma^A{}_B\gamma^B{}_C = \delta^A_C$ hold. Thus $\gamma^A{}_B$ has two eigenvalues, ± 1, and $\pi^{\pm A}{}_B := \frac{1}{2}(\delta^A_B \pm \gamma^A{}_B)$ are the projections of $\mathbf{S}^A(\mathcal{S})$ to the bundle $\mathbf{S}^{A\pm}(\mathcal{S})$ of the ± 1 (right handed/left handed) eigenspinors, respectively. If o^A is a left handed and ι^A is a right handed spinor normalized by $o_A\iota^A = 1$, then they form a GHP spinor dyad [3] on $\mathcal{S}$. The projection Π^a_b can be expressed by $\gamma^A{}_B$, too: $\Pi^a_b = \frac{1}{2}(\delta^A_B\delta^{A'}_{B'} - \gamma^A{}_B\bar{\gamma}^{A'}{}_{B'})$. Thus the area 2-form on $\mathcal{S}$ and on the orthogonal 2-planes, respectively, are $\varepsilon_{cd} = \frac{\mathrm{i}}{2}(\gamma_{CD}\varepsilon_{C'D'} - \bar{\gamma}_{C'D'}\varepsilon_{CD})$ and ${}^{\perp}\varepsilon_{cd} = -\frac{1}{2}(\gamma_{CD}\varepsilon_{C'D'} + \bar{\gamma}_{C'D'}\varepsilon_{CD})$. $\gamma_{AB} = \gamma_{(AB)} = 2o_{(A}\iota_{B)}$ can also be considered as a complex fibre metric on $\mathbf{S}^A(\mathcal{S})$. $\pi^{\pm a}{}_b := \pi^{\pm A}{}_B\bar{\pi}^{\mp A'}{}_{B'}$ are the projections of the complexified tangent bundle $T\mathcal{S} \otimes \mathbb{C}$ (in fact, the complexified Lorentzian vector bundle $\mathbf{V}^a(\mathcal{S}) \otimes \mathbb{C}$) to the sub-bundles $T^{(1,0)}\mathcal{S}$ and $T^{(0,1)}\mathcal{S}$ of the complex tangent vectors spanned by $m^a := o^A\bar{\iota}^{A'}$ and $\bar{m}^a := \iota^A\bar{o}^{A'}$, respectively, i.e. γ_{AB} determines the complex structure [26] of $\mathcal{S}$, too. The analogous projections $o^{\pm a}{}_b := \pi^{\pm A}{}_B\bar{\pi}^{\pm A'}{}_{B'}$ define respectively the two null normals $n^a := \iota^A\bar{\iota}^{A'}$ and $l^a := o^A\bar{o}^{A'}$ up to scale. The bundle of the GHP spinor dyads is a principal bundle $\tilde{B}(\mathcal{S}, GL(1,\mathbb{C}))$, which is a double covering bundle of $B(\mathcal{S}, SO(2) \times SO(1,1))$. In particular, if $\mathcal{S}$ is a topological 2-sphere, then $\tilde{B}(\mathcal{S}, GL(1,\mathbf{C})) \approx S^3 \times (0,\infty)$, and since $B(\mathcal{S}, SO(2) \times SO(1,1))$ is trivial for $g = 1$, its covering $\tilde{B}(\mathcal{S}, GL(1,\mathbf{C}))$ is also trivial. The derivative δ_e extends naturally to $\mathbf{S}^A(\mathcal{S})$, by the requirement that it should annihilate both ε_{AB} and γ_{AB}. For its curvature we obtain

$$f^A{}_{Bcd} = -\frac{1}{2}\gamma^A{}_B\Big(\big(\delta_c A_d - \delta_d A_c\big) - \frac{1}{4}\,{}^{\mathcal{S}}R\big(\varepsilon_{C'D'}\gamma_{CD} - \varepsilon_{CD}\bar{\gamma}_{C'D'}\big)\Big). \tag{2.2}$$

We can define the curvature scalar of δ_e by $f := f_{abcd}\frac{1}{2}(\varepsilon^{ab} - \mathrm{i}\,{}^{\perp}\varepsilon^{ab})\varepsilon^{cd} = \mathrm{i}\gamma^A{}_B f^B{}_{Acd}\varepsilon^{cd} = {}^{\mathcal{S}}R - 2\mathrm{i}\delta_c(\varepsilon^{cd}A_d)$, which is four times the so-called *complex* Gauss curvature of [17]. Then by the Gauss–Bonnet theorem $\oint_{\mathcal{S}} f\,\mathrm{d}\mathcal{S} = \oint_{\mathcal{S}} {}^{\mathcal{S}}R\,\mathrm{d}\mathcal{S} = 8\pi(1-g)$, and hence δ_e can be flat only if $\mathcal{S}$ is a torus.

3 The line bundles $E(p,q)$

Let $\rho_{(p,q)} : GL(1,\mathbb{C}) \times \mathbb{C} \to \mathbb{C} : (\lambda, z) \mapsto \lambda^{-p}\bar{\lambda}^{-q}\,z$, which is a left action of $GL(1,\mathbb{C})$ on $\mathbb{C}$ precisely when $p - q \in \mathbb{Z}$ (although $p + q$ may be arbitrary complex number, we assume that it is real), and let $E(p,q)$ denote the complex line bundle associated to $\tilde{B}(\mathcal{S}, GL(1,\mathbb{C}))$ with the group representation $\rho_{(p,q)}$. $E(p,q)$ is called the bundle of (p,q)-type scalars over $\mathcal{S}$. They have the following elementary properties [4, 28, 29] :

1. The complex conjugate bundle is $\overline{E(p,q)} = E(q,p)$.

2. The tensor product bundle is $E(p,q) \otimes E(r,s) = E(p+r, q+s)$.

3. $E(p,p)$ is a trivial vector bundle for any $p \in \mathbb{R}$, and all the $E(p,q)$'s are trivial over the torus $\mathcal{S} \approx S^1 \times S^1$. Note that the trivial vector bundles admit nowhere vanishing sections.

4. For any fixed, nowhere vanishing section h of $E(1,1)$,

$$\langle \phi, \psi \rangle_{(p,q)} := \oint_{\mathcal{S}} |h|^{-(p+q)} \phi \bar{\psi} \, \mathrm{d}\mathcal{S} \tag{3.1}$$

defines a positive definite Hermitian scalar product on the (infinite dimensional) vector space $E^\infty_{(p,q)}$ of the smooth sections of $E(p,q)$. Such a section might be, for example, the everywhere positive function $h := t_{AA'} o^A \bar{o}^{A'}$.

5. $E^\infty_{(0,0)} = C^\infty(\mathcal{S}, \mathbb{C})$, and for the spinor bundle $\mathbf{S}^A(\mathcal{S}) \to E(1,0) \oplus E(-1,0) : \omega^A \mapsto (\omega^A o_A, \omega^A \iota_A)$ is a vector bundle isomorphism.

6. For the complexified tangent and normal bundles $T\mathcal{S} \otimes \mathbb{C} \to E(1,-1) \oplus E(-1,1) : X^a \mapsto (X^a m_a, X^a \bar{m}_a)$ and $N\mathcal{S} \otimes \mathbb{C} \to E(1,1) \oplus E(-1,-1) : V^a \mapsto (V^a l_a, V^a n_a)$ are vector bundle isomorphisms.

7. For the complex structure $\Lambda^{(1,0)} T^* \mathcal{S} \simeq T^{(0,1)} \mathcal{S} \to E(1,-1) : \tilde{z}^a \mapsto \tilde{z}^a m_a$ and $\Lambda^{(0,1)} T^* \mathcal{S} \simeq T^{(1,0)} \mathcal{S} \to E(-1,1) : z^a \mapsto z^a \bar{m}_a$ are vector bundle isomorphisms. $\Lambda^{(1,0)} T^* \mathcal{S}$, spanned by the differential of the *holomorphic* coordinates, is called the canonical bundle of the Riemann surface $\mathcal{S}$ [20].

The covariant derivative δ_a on $\mathbf{S}^A(\mathcal{S})$ defines a covariant derivative $\eth_a$ of $\phi \in E^\infty_{(p,q)}$ for any $p, q \in \mathbb{Z}$ by

$$\eth_e \phi := \delta_e (\phi \iota^{A_1} \ldots \iota^{A_m} o_{B_1} \ldots o_{B_n} \bar{\iota}^{C'_1} \ldots \bar{\iota}^{C'_r} \bar{o}_{D'_1} \ldots \bar{o}_{D'_s}) o_{A_1} \ldots o_{A_m} \iota^{B_1} \ldots \iota^{B_n} \bar{o}_{C'_1} \ldots \bar{o}_{C'_r} \bar{\iota}^{D'_1} \ldots \bar{\iota}^{D'_s}, \tag{3.2}$$

where $m, n, r, s = 0, 1, 2, \ldots$ such that $p = m - n$ and $q = r - s$. If $p \in (0,1)$, then let $\eth_a$ be the covariant derivative on $E^\infty_{(p,p)}$ for which $\eth_a(\phi^{1/p}) = \frac{1}{p} \phi^{\frac{1-p}{p}} \eth_a \phi$ for any $\phi \in E^\infty_{(p,p)}$. Then by the Leibniz rule and property 2. above $\eth_a$ can be extended to $E(p,q)$ for any $p, q \in \mathbb{R}$ satisfying $p - q \in \mathbb{Z}$. The curvature of $\eth_a$ on $E(p,q)$, defined by $-\mathcal{F}_{ab} v^a w^b \phi := v^a \eth_a (w^b \eth_b \phi) - w^a \eth_a (v^b \eth_b \phi) - [v,w]^a \eth_a \phi$, is $\mathcal{F}_{ab} = \frac{\mathrm{i}}{4}(-pf + q\bar{f}) \varepsilon_{ab}$, and hence the integral of the first Chern class of $E(p,q)$ is $c_1(p,q) := -\oint_{\mathcal{S}} \frac{\mathrm{i}}{4\pi} \mathcal{F}_{ab} = -(p-q)(1-g)$. In particular, the vanishing of $c_1(p,q)$ characterizes the trivial line bundles, and for the canonical bundle it is $c_1(1,-1) = 2(g-1)$, in accordance with [20, pp.110].

The covariant directional derivatives, $\eth\phi := m^e \eth_e \phi$ and $\eth'\phi := \bar{m}^e \eth_e \phi$, the edth and edth-prime operators, are elliptic differential operators $\eth : E^\infty_{(p,q)} \to E^\infty_{(p+1,q-1)}$ and $\eth' : E^\infty_{(p,q)} \to E^\infty_{(p-1,q+1)}$, and hence, because of the compactness of $\mathcal{S}$, their kernel is finite dimensional [24, 25]. The formal adjoint of the edth and edth-prime operators with respect to the Hermitian scalar product $\langle . | . \rangle_{(p,q)}$ above are

$$\begin{aligned} (\eth_{(p,q)})^\dagger &= -|h|^{(p+q)} \eth'_{(-q+1,-p-1)} |h|^{-(p+q)}, \\ (\eth'_{(p,q)})^\dagger &= -|h|^{(p+q)} \eth_{(-q-1,-p+1)} |h|^{-(p+q)}. \end{aligned} \tag{3.3}$$

They are also elliptic, and hence the analytic index of the edth and edth-prime operators, defined by $\mathrm{ind}(\eth_{(p,q)}) := \dim\ker \eth_{(p,q)} - \dim\ker (\eth_{(p,q)})^\dagger$ and $\mathrm{ind}(\eth'_{(p,q)}) := \dim\ker \eth'_{(p,q)} - \dim\ker (\eth'_{(p,q)})^\dagger$, are finite [24, 25]. In terms of these notions the Riemann–Roch theorem of the theory of Riemann surfaces [20, pp.111] takes the form

$$\begin{aligned} \mathrm{ind}(\eth_{(p,q)}) &= (1 + p - q)(1 - g), \\ \mathrm{ind}(\eth'_{(p,q)}) &= (1 - p + q)(1 - g), \end{aligned} \tag{3.4}$$

which are just Baston's formulae [29].

For any real 1-form ω_a one has $\omega := \omega_a m^a \in E^\infty_{(1,-1)}$ and $\bar\omega := \omega_a \bar m^a \in E^\infty_{(-1,1)}$; and $2\delta_{[a}\omega_{b]} m^a \bar m^b = \eth\bar\omega - \eth'\omega$, $2\delta_{(a}\omega_{b)} m^a \bar m^b = \eth\bar\omega + \eth'\omega$ and $(\delta_a\omega_b) m^a m^b = \eth\omega$ hold. Therefore, ω_a is a real harmonic 1-form iff $\eth'\omega = 0$, and ω_a is (the dual of) a real conformal Killing vector iff $\eth\omega = 0$.

A cross section ϕ of $E(p,q)$ is called holomorphic/anti-holomorphic if $\eth'\phi = 0$ or $\eth\phi = 0$, respectively. The point $P \in \mathcal{S}$ is said to be the zero of the holomorphic section ϕ with order m if ϕ and its first $(m-1)$ $\eth$-derivatives vanish at P, but its mth $\eth$-derivative is not zero there. It is not difficult to show that P is a zero of the holomorphic cross section ϕ with order m if and only if there is a holomorphic section ψ and a holomorphic function f on some open neighbourhood $W \subset \mathcal{S}$ of P such that $\phi = f\psi$ and ψ is nonzero on W and f has a single zero with order m at P. The meromorphic/anti-meromorphic sections are defined analogously with the only difference that we allow them to have isolated poles. The order of the pole $P \in \mathcal{S}$ of the meromorphic section ϕ is defined to be n if for some holomorphic function f, defined on an open neighbourhood W of P and having P as a zero with order n, $f\phi$ is holomorphic on W, while it is not holomorphic for functions f' whose zero at P is only of order $n' < n$. Note that by the compactness of $\mathcal{S}$ the meromorphic sections have finitely many zeros and poles. The degree of a meromorphic section ϕ of $E(p,q)$ with zeros of order $m_1, \dots, m_k$ and poles of order $n_1, \dots, n_l$ is defined by $\deg(\phi) := \sum_{i=1}^k m_i - \sum_{j=1}^l n_j$. By a theorem of the theory of Riemann surfaces [20, pp.103] this degree depends only on the line bundle: $\deg(\phi) = c_1(p,q)$. In the analogous statement on the anti-meromorphic sections ψ of $E(p,q)$ one has the first Chern class of the complex conjugate bundle: $\deg(\psi) = c_1(q,p)$.

4 The calculation of $\dim\ker\eth_{(p,q)}$ and $\dim\ker\eth'_{(p,q)}$

We calculate the dimension of the kernels using basically the technique of the appendix of [22]. Thus first we recall the basic formulae. As a consequence of the similarity transformations (3.3) between $(\eth_{(p,q)})^\dagger$ and $\eth'_{(-q+1,-p-1)}$ and between $(\eth'_{(p,q)})^\dagger$ and $\eth_{(-q-1,-p+1)}$, the dimension of the kernel spaces of $(\eth_{(p,q)})^\dagger$ and $\eth'_{(-q+1,-p-1)}$, and of $(\eth'_{(p,q)})^\dagger$ and $\eth_{(-q-1,-p+1)}$ are the same. By substituting these into Baston's formulae (3.4) above, we obtain

$$\begin{aligned} \dim\ker\eth_{(p,q)} &= (1+p-q)(1-g) + \dim\ker\eth'_{(-q+1,-p-1)}, \\ \dim\ker\eth'_{(p,q)} &= (1-p+q)(1-g) + \dim\ker\eth_{(-q-1,-p+1)}. \end{aligned} \tag{4.1}$$

By the Leibniz rule $\eth_{(p+p',q+q')}(\phi\psi) = \psi\eth_{(p,q)}\phi + \phi\eth_{(p',q')}\psi$ for any $\phi \in E^\infty_{(p,q)}$ and $\psi \in E^\infty_{(p',q')}$, one has the inequality

$$\dim\ker\eth_{(p+p',q+q')} \geq \max\{\dim\ker\eth_{(p,q)}, \dim\ker\eth_{(p',q')}\} \tag{4.2}$$

whenever $(\dim\ker\eth_{(p,q)}) \cdot (\dim\ker\eth_{(p',q')}) \neq 0$. There is a similar inequality for the edth-prime operator. Finally, the last ingredient that we need is

$$\dim\ker\eth_{(0,0)} = \dim\ker\eth'_{(0,0)} = 1, \tag{4.3}$$

which is just the generalized Liouville theorem [20, pp.6]: Every holomorphic/anti-holomorphic function on a compact Riemann surface is constant. Then by (4.2) and (4.3)

$$\begin{aligned} (\dim\ker\eth_{(p,q)})(\dim\ker\eth_{(-p,-q)}) = 0 \quad &\text{or} \quad \dim\ker\eth_{(p,q)} = \dim\ker\eth_{(-p,-q)} = 1, \\ (\dim\ker\eth'_{(p,q)})(\dim\ker\eth'_{(-p,-q)}) = 0 \quad &\text{or} \quad \dim\ker\eth'_{(p,q)} = \dim\ker\eth'_{(-p,-q)} = 1, \end{aligned} \tag{4.4}$$

for any $p, q \in \mathbb{R}$, $p - q \in \mathbb{Z}$.

First let us consider the trivial bundles, i.e. $q = p$ or $g = 1$, and fix a holonomically trivial covariant derivative ${}_0\eth_a$ on $E(p,q)$. [To see that such a connection exists, recall that there are flat connections

on the corresponding principal bundles, and their holonomy groups, being homomorphic images of the fundamental group $\pi_1(\mathcal{S})$ in the structure group, are discrete [26]. Let $\tilde{B}$ be any of these principal bundles, and $\tilde{\omega}_a$ the connection 1-form of the flat connection. Let us fix a canonical homology basis $\{a_i, b_i\}$, $i = 1, \dots, g$ of $\mathcal{S}$, let $\tilde{h}(a_i)$, $\tilde{h}(b_i)$ denote the corresponding holonomies, and define the connection 1-form $\omega_a := \tilde{\omega}_a + (A_i + \mathrm{i}B_i)\alpha^i_a + (C_i + \mathrm{i}D_i)\beta^i_a$, where A_i, B_i, C_i and D_i, $i = 1, \dots, g$, are real constants, and where$\{\alpha^i_a, \beta^i_a\}$ is a basis for the real harmonic 1-forms on $\mathcal{S}$. Then ω_a is also flat, and the corresponding holonomies along a_j and b_j, respectively, are $h(a_j) = \tilde{h}(a_j) + (A_i + \mathrm{i}B_i) \oint_{a_j} \alpha^i_a + (C_i + \mathrm{i}D_i) \oint_{a_j} \beta^i_a = \tilde{h}(a_j) + (A_i + \mathrm{i}B_i) M^i{}_j + (C_i + \mathrm{i}D_i) M^{g+i}{}_j$ and $h(b_j) = \tilde{h}(b_j) + (A_i + \mathrm{i}B_i) \oint_{b_j} \alpha^i_a + (C_i + \mathrm{i}D_i) \oint_{b_j} \beta^i_a = \tilde{h}(b_j) + (A_i + \mathrm{i}B_i) M^i{}_{g+j} + (C_i + \mathrm{i}D_i) M^{g+i}{}_{g+j}$. Since $M^I{}_J$, $I, J = 1, \dots, 2g$, is nonsingular, there exist (uniquely determined) constants A_i, B_i, C_i and D_i such that $h(a_i) = 0$ and $h(b_i) = 0$.] Then for some globally defined real 1-forms V_a, Z_a on $\mathcal{S}$ one has $\eth_a \phi = {}_0\eth_a \phi + p(V_a + \mathrm{i}Z_a)\phi + q(V_a - \mathrm{i}Z_a)\phi$ for any $\phi \in E^\infty_{(p,q)}$. Let $\phi_0 \in E^\infty_{(p,q)}$ be constant with respect to ${}_0\eth_a$, which is necessarily nowhere vanishing. Then any smooth cross section of $E(p,q)$ can be written as $\phi = f\phi_0$ for some $f \in E^\infty_{(0,0)}$, thus its edth- and edth-prime derivatives are $\eth\phi = (\eth f + pm^a(V_a + \mathrm{i}Z_a)f + qm^a(V_a - \mathrm{i}Z_a)f)\phi_0$ and $\eth'\phi = (\eth' f + p\bar{m}^a(V_a + \mathrm{i}Z_a)f + q\bar{m}^a(V_a - \mathrm{i}Z_a)f)\phi_0$, respectively. However, by (4.3) there is a one parameter family of solutions to $\eth f + pm^a(V_a + \mathrm{i}Z_a)f + qm^a(V_a - \mathrm{i}Z_a)f = 0$, and another family of solutions to $\eth' f' + p\bar{m}^a(V_a + \mathrm{i}Z_a)f' + q\bar{m}^a(V_a - \mathrm{i}Z_a)f' = 0$, therefore $\dim\ker\eth_{(p,q)} \geq 1$ and $\dim\ker\eth'_{(p,q)} \geq 1$. Then, however, (4.4) implies that

$$\dim\ker\eth_{(p,q)} = \dim\ker\eth'_{(p,q)} = 1 \quad \text{if} \quad (p-q)(1-g) = 0. \tag{4.5}$$

Therefore, in particular, on the tori the edth and edth-prime operators have one dimensional kernels, independently of the type (p,q) of the line bundle. These kernel and cokernel spaces can be visualized by drawing the sequence of the edth operators between the various line bundles (Fig. 1).

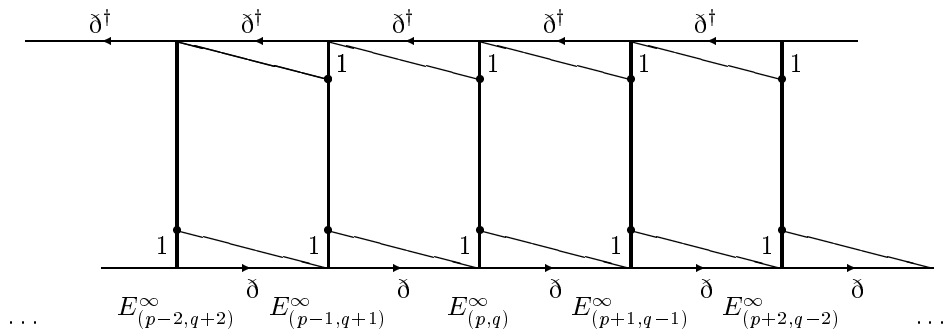

Figure 1: The sequence of line bundles and the edth and the adjoint-edth operators on the torus. The vertical lines represent the spaces of the smooth sections of the line bundles, and a piece of them, the kernels, are mapped into the zero of the next space, but not every section has a pre-image in the previous space. In particular, on the torus the kernel and cokernel spaces of the edth operators are 1 dimensional. The kernel and cokernel of the adjoint-edth operator is just the cokernel and kernel of the edth, respectively.

Although $\dim\ker\eth_{(p,q)}$ and $\dim\ker\eth'_{(p,q)}$ have been calculated for $g = 0$ in the Appendix of [22], here we give a considerably simpler calculation of them. Thus let $p \in \mathbb{R}$ and $n \in \mathbb{N}$. Then by (4.1) one has $\dim\ker\eth_{(-p+n,-p)} = 1 + n + \dim\ker\eth'_{(p+1,p-n-1)} \geq 1 + n$ and $\dim\ker\eth'_{(-p,-p+n)} = 1 + n + \dim\ker\eth_{(p-n-1,p+1)} \geq 1 + n$. Thus by (4.4)

$$\dim\ker\eth_{(p-n,p)} = \dim\ker\eth'_{(p,p-n)} = 0 \tag{4.6}$$

for any $p \in \mathbb{R}$ and $n \in \mathbb{N}$. Substituting this back to (4.1) we get

$$\dim \ker \eth_{(p+n,p)} = \dim \ker \eth'_{(p,p+n)} = 1 + n. \tag{4.7}$$

(4.5) (for $q = p$), (4.6) and (4.7) give the complete list of the dimension of the kernel spaces of $\eth$ and $\eth'$ on the spheres (see also [4]). The line bundles and the edth operators form two sequences, the one in which $p - q$ is even (i.e. if the spin weight is integer, the 'tensorial sequence'), and in which $p - q$ is odd (half-integer spin weight, the 'spinorial sequence'). These are shown by Fig. 2 and 3 in the case of vanishing boost weight: $p + q = 0$.

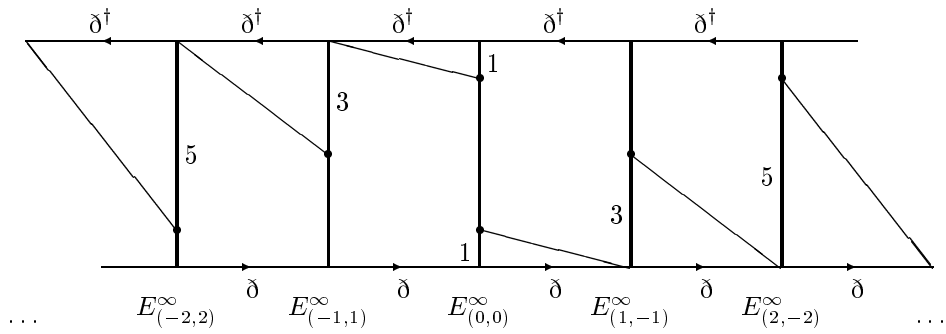

Figure 2: The tensorial series for g=0 and vanishing boost weight: $p + q = 0$. The operator $\eth$ is injective only for negative spin weights ($p < q$), and surjective only for non-negative spin weights ($p \geq q$). Thus in the tensorial series $\eth$ and $\eth'$ are never isomorphisms.

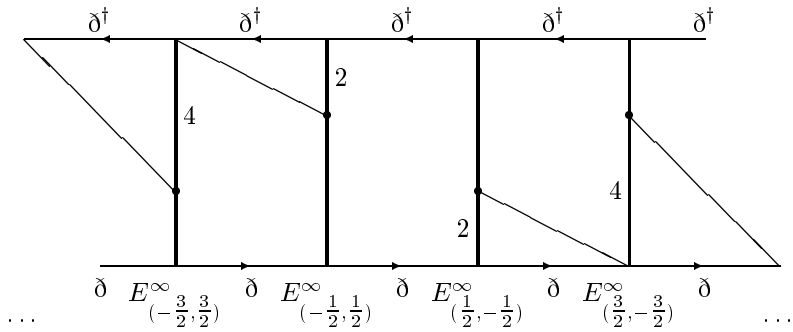

Figure 3: The spinorial series for $g = 0$ in the case of vanishing boost weight. $\eth$ is injective for negative spin weights, and surjective for spin weights greater than or equal to $-\frac{1}{2}$. Thus, $\eth$ is an isomorphism precisely between $E^\infty_{(p,p+1)}$ and $E^\infty_{(p+1,p)}$.

Finally, let $g \geq 2$. By (4.1)

$$\begin{aligned} \dim\ker \eth_{(p,p+n+1)} &= n(g-1) + \dim\ker \eth'_{(-p-n,-p-1)}, \\ \dim\ker \eth'_{(p,p-n-1)} &= n(g-1) + \dim\ker \eth_{(-p+n,-p+1)} \end{aligned} \tag{4.8}$$

for any $n \in \{0\} \cup \mathbb{N}$ and $p \in \mathbb{R}$. Then for $n \geq 2$ and any $p \in \mathbb{R}$ these, and for $n = 1$ and any $p \in \mathbb{R}$ these and (4.5) imply $\dim\ker \eth_{(p,p+n+1)} \geq 2$ and $\dim\ker \eth'_{(p,p-n-1)} \geq 2$. Therefore, by (4.4) it follows that $\dim\ker \eth_{(p,p-n-1)} = \dim\ker \eth'_{(p,p+n+1)} = 0$ for any $n \in \mathbb{N}$ and $p \in \mathbb{R}$. However, this implies that

$$\dim\ker \eth_{(p,p-n)} = \dim\ker \eth'_{(p,p+n)} = 0 \tag{4.9}$$

for any $n \in \mathbb{N}$ and $p \in \mathbb{R}$, too. To see this, suppose, on the contrary, that e.g. $\dim\ker \eth_{(p,p-1)} \geq 1$ for some $p \in \mathbb{R}$. Then by (4.2) this would imply that $\dim\ker \eth_{(2p,2p-2)} \geq 1$. Substituting (4.9) back to (4.8) we obtain

$$\dim\ker \eth_{(p,p+n+1)} = \dim\ker \eth'_{(p,p-n-1)} = \begin{cases} g & \text{if } n = 1 \\ n(g-1) & \text{if } n \geq 2 \end{cases} \tag{4.10}$$

for any $n \in \mathbb{N}$ and $p \in \mathbb{R}$. Note that for $g \geq 2$ the Riemann–Roch theorem, more precisely, (4.8) for $n = 0$, states only that two unknown dimensions are equal: $\dim\ker \eth_{(p,p+1)} = \dim\ker \eth'_{(-p,-p-1)}$. Thus to calculate them let us choose a real harmonic 1-form ω_a and define the map $\bar{\omega} : \ker\ \eth_{(0,1)} \to \ker\ \eth_{(-1,2)} : \phi \mapsto \bar{\omega}\phi$. Obviously, this is injective. By (4.10) $\dim\ker \eth_{(-1,2)} = 2(g-1)$, and $\deg(\nu) = c_1(-1,2) = 3(g-1)$ for any $\nu \in \ker\ \eth_{(-1,2)}$, whilst $\deg(\bar{\omega}) = c_1(-1,1) = 2(g-1)$. Thus the quotient $\nu/\bar{\omega}$ can be holomorphic only if the zeros of $\bar{\omega}$ are compensated by the zeros of ν, hence the map $\bar{\omega}$ is not surjective. Its cokernel is $2(g-1)$ dimensional, and hence $\dim\ker \eth_{(0,1)} = \dim\ker \eth'_{(1,0)} = g-1$. Finally, since by $\dim\ker \eth_{(p,p)} = 1$ we have $\dim\ker \eth_{(p,p+1)} \geq \max\{\dim\ker \eth_{(0,1)}, \dim\ker \eth_{(p,p)}\} = g-1$. On the other hand, by $\dim\ker \eth_{(-p,-p)} = 1$ we have $g-1 = \dim\ker \eth_{(0,1)} \geq \max\{\dim\ker \eth_{(p,p+1)}, \dim\ker \eth_{(-p,-p)}\} = \dim\ker \eth_{(p,p+1)}$. Therefore,

$$\dim\ker \eth_{(p,p+1)} = \dim\ker \eth'_{(p+1,p)} = g-1 \tag{4.11}$$

for any $p \in \mathbb{R}$. For $g \geq 2$ (4.5), (4.9)–(4.11) is the complete list of the dimension of the kernel spaces of the edth and edth-prime operators. The tensor- and spinor sequences of the edth operators are shown by Fig. 4 and 5, respectively.

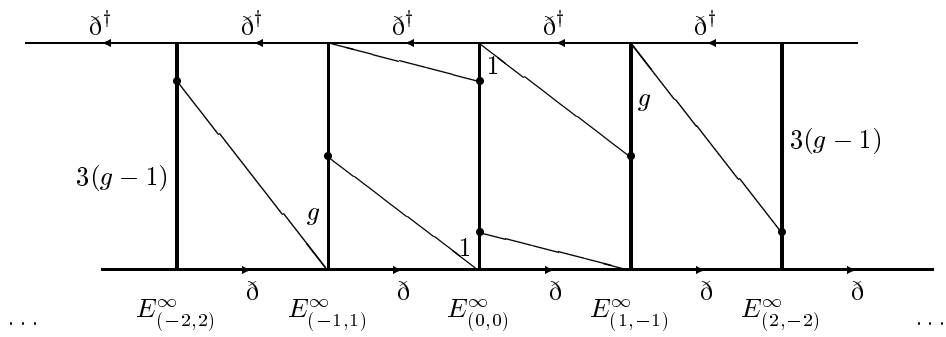

Figure 4: The tensorial series for $g > 1$ and vanishing boost weight $p + q = 0$.

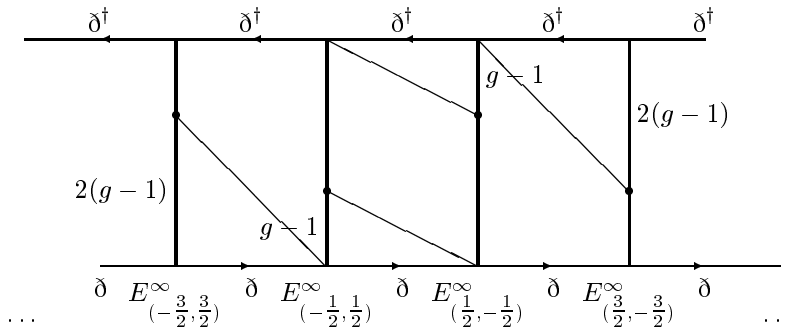

Figure 5: The spinorial series for $g > 1$ in the case of vanishing boost weight.

5 Discussion

The figures show clearly that $g = 1$ is a natural division between the spherical case ($g = 0$) and the higher genus surfaces. While on the sphere, the edth operator has non-trivial kernels for $p \geq q$, on the surfaces with higher genus the kernels are non-trivial for $p \leq q$. This is obviously related to the fact that on the sphere there exist conformal Killing vectors but no harmonic forms, while for $g \geq 2$ there are harmonic forms but no conformal Killing vectors. On the torus there is exactly one complex one of each. In fact, by (4.5), (4.7) and (4.9) the number of the independent real conformal Killing vectors is six on S^2, two on $S^1 \times S^1$ and zero on surfaces with genus $g \geq 2$. Similarly, by (4.5), (4.7) and (4.10) $\dim \ker \eth'_{(1,-1)} = g$, i.e. there are g holomorphic 1-forms on a surface with genus g which can be combined with their (anti-holomorphic) complex conjugates to yield the $2g$ real harmonic 1-forms whose existence is guaranteed by the Hodge decomposition.

It is well known that in the case $g = 0$ the kernel of $\eth_{(\frac{1}{2},-\frac{1}{2})}$ serves as a building block for the kernels of $\eth_{(s,-s)}$ with $2s \in \mathbb{N}$ in the sense that if $\{\alpha_1, \alpha_2\}$ is a basis in $\ker \eth_{(\frac{1}{2},-\frac{1}{2})}$ then $\{\alpha_1^{2s}, \alpha_1^{2s-1}\alpha_2, \ldots, \alpha_2^{2s}\}$ is a basis in $\ker \eth_{(s,-s)}$. Similarly, on the torus $\dim \ker \eth_{(p,q)} = 1$ for any p, q, thus if $\alpha \in \ker \eth_{(\frac{1}{2},-\frac{1}{2})}$ and $\beta \in \ker \eth_{(-\frac{1}{2},\frac{1}{2})}$, then α^{2s} spans $\ker \eth_{(s,-s)}$ and β^{2s} spans $\ker \eth_{(-s,s)}$. If, however, $g = 2$ then $\dim \ker \eth_{(-\frac{1}{2},\frac{1}{2})} = 1$, but $\dim \ker \eth_{(-s,s)} \geq 2$ for $s \geq 1$, $2s \in \mathbb{N}$, and hence the elements of $\ker \eth_{(-s,s)}$ cannot be generated by the single independent element of $\ker \eth_{(-\frac{1}{2},\frac{1}{2})}$.

To understand the geometric roots of this difference between the $g \leq 1$ and $g \geq 2$ cases, recall that the elements of $\ker \eth_{(1,-1)}$ correspond to globally defined conformal Killing vectors, which generate global group actions on $\mathcal{S}$. Then $\ker \eth_{(\frac{1}{2},-\frac{1}{2})}$ is the representation space of the double covering group of this symmetry group, and hence any irreducible representation of the symmetry group is build from $\eth_{(\frac{1}{2},-\frac{1}{2})}$. On S^2 this group is $SL(2,\mathbb{C})$, on $S^1 \times S^1$ it is $U(1) \times U(1)$, but there is no such group on higher-genus 2-surfaces. The harmonic 1-forms, which correspond to the elements of $\ker \eth_{(-1,1)}$, do not generate any such group action on $\mathcal{S}$, and in lack of such a group structure the spaces $\ker \eth_{(-\frac{1}{2},\frac{1}{2})}$ and $\ker \eth_{(-s,s)}$ cannot be expected to be related as (different weight) representation spaces of a group.

Yet, it is obvious that for any holomorphic 1-form $\omega \in \ker \eth_{(-1,1)}$ and any $\alpha \in \ker \eth_{(-s,s)}$ we have $\omega\alpha \in \ker \eth_{(-s-1,s+1)}$ so that the holomorphic 1-forms do in fact map the kernels into each other. So the question arises as to whether one can obtain all the elements in $\ker \eth_{(-s,s)}$ for $s \in \mathbb{N}$ as linear combinations of the s-fold products of the g holomorphic 1-forms. However, this cannot be true in general because it is easy to see that in the case of hyper-elliptic Riemann surfaces with genus $g \geq 3$ (see [27]) these products are not sufficient to span the entire kernel because they satisfy too many linear relations. But these are the only exceptions: a rather deep result in the theory of Riemann surfaces of higher genus, the theorem

of Noether [27] states that, except for these special cases, the kernels $\ker \eth_{(-s,s)}$ in the tensorial series, i.e., for $s \in \mathbb{N}$ are generated by the s-fold products of holomorphic 1-forms. It would be interesting to have a similar result for the spinorial series.

To solve the equation $\eth_{(p,q)}\phi = \omega$ for ϕ with given ω, the cross section $\omega \in E^{\infty}_{(p+1,q-1)}$ must belong to the range of $\eth_{(p,q)}$, i.e. ω must be orthogonal (with respect to (3.1)) to $\ker(\eth_{(p,q)})^{\dagger}$. Then there is a unique solution ϕ if that is chosen to be orthogonal to $\ker \eth_{(p,q)}$. In fact, $\eth$ is a continuous linear operator with respect to the standard Sobolev norms on $E^{\infty}_{(p,q)}$ and $E^{\infty}_{(p+1,q-1)}$, and hence it is a Fredholm operator. Then this criterion of the solvability of $\eth_{(p,q)}\phi = \omega$ is just the Fredholm alternative theorem (see [25]).

In particular, to answer the question posed in the introduction on the vanishing of the Weyl tensor on the conformal boundaries with higher genus topology, we note that this leads to the equation $\eth\phi = 0$ for $\phi \in E^{\infty}_{(4,0)}$. While for $g = 0$ the appropriate kernel is trivial, this is not the case if $g \geq 1$. Then there are non-trivial solutions so that the Weyl tensor does not necessarily vanish on toroidal (and other higher genus) null-infinities.

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