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Strong versus weak local minimizers for the perturbed Dirichlet functional

by

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Strong versus weak local minimizers for the perturbed Dirichlet functional

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Abstract

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $F : \Omega \times \mathbf{R}^N \to \mathbf{R}$. In this paper we consider functionals of the form

$$I(u) := \int_\Omega \left(\frac{1}{2}|Du|^2 + F(x,u) \right) dx,$$

where the admissible function u belongs to the Sobolev space of vector-valued functions $W^{1,2}(\Omega; \mathbf{R}^N)$ and is such that the integral on the right is well defined. We state and prove a sufficiency theorem for L^r local minimizers of I where $1 \le r \le \infty$. The exponent r is shown to depend on the dimension n and the growth condition of F and an exact expression is presented for this dependence. We discuss some examples and applications of this theorem.

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain (open connected set) and let $F : \Omega \times \mathbf{R}^N \to \mathbf{R}$. We consider functionals of the form

$$I(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 + F(x,u) \right) dx, \tag{1.1}$$

where the admissible u belongs to the class of vector-valued functions

$$\mathcal{F} := \left\{ u \in W^{1,2}(\Omega; \mathbf{R}^N) : \text{ the integral (1.1) is well defined} \right\}.$$

By well defined we mean that $F(x, u(x))$ is a measurable function on Ω and at least one of the functions $F^+ := \max\{F(\cdot, u(\cdot)), 0\}$ or $F^- := \min\{F(\cdot, u(\cdot)), 0\}$ has a finite integral. It is therefore to be understood that $I : \mathcal{F} \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$. To specify the growth of F we assume that there are constants $C > 0$ and $p \ge 1$ such that

$$F(x,u) \ge -C(1 + |u|^p)$$

for all $x \in \Omega$ and all $u \in \mathbf{R}^N$.

Throughout this paper we assume that Ω has a Lipschitz boundary $\partial\Omega$ with $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$ where $\partial\Omega_1$ and $\partial\Omega_2$ are disjoint relatively open subsets of $\partial\Omega$ and $\mathcal{H}^{n-1}(N) = 0$. Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure. We denote the unit outward normal to the boundary at a point x by $\nu(x)$.

Functionals of the form (1.1) and their corresponding Euler-Lagrange equation appear in many contexts. Because of their relatively simple structure they have attracted much attention and there is a considerable literature on issues such as existence and multiplicity of their *critical points*. In this paper we shall be mainly concerned with the nature of such critical points. More specifically we aim to classify such points as various local minimizers and hence understand the "local geometry" of I. For this let us assume that $u_0 \in \mathcal{F}$ is given and $\partial\Omega_1$ is as described and set

$$\mathcal{A}_{u_0}(\partial\Omega_1) := \{u \in \mathcal{F} : (u - u_0)|_{\partial\Omega_1} = 0\},$$

where the boundary values are to be interpreted in the sense of traces. We can now state the following

Definition 1.1. *Let $1 \le r \le \infty$. The function $u_0 \in \mathcal{F}$ is said to be an L^r (respectively $W^{1,r}$) local minimizer of I if and only if there exists $\varepsilon > 0$ such that*

$$I(u_0) \le I(u)$$

for all $u \in \mathcal{A}_{u_0}(\partial\Omega_1)$ satisfying $||u - u_0||_{L^r(\Omega;\mathbf{R}^N)} < \varepsilon$ (respectively $||u - u_0||_{W^{1,r}(\Omega;\mathbf{R}^N)} < \varepsilon$).

We shall also borrow a standard terminology from calculus of variations. By a *weak* local minimizer we mean a $W^{1,\infty}$ local minimizer whereas a *strong* local minimizer refers to an L^∞ local minimizer. It can be easily checked that if F is of class C^2 and $u_0 \in \mathcal{F}$ is a weak local minimizer of class $L^\infty(\Omega;\mathbf{R}^N)$ in $\mathcal{A}_{u_0}(\partial\Omega_1)$ then

$$(i) \quad \delta I(u_0,\varphi) := \frac{d}{dt} I(u_0 + t\varphi)|_{t=0} = 0 \quad \text{and} \quad (ii)^- \quad \delta^2 I(u_0,\varphi) := \frac{d^2}{dt^2} I(u_0 + t\varphi)|_{t=0} \ge 0,$$

first for all variations $\varphi \in C^\infty(\overline{\Omega};\mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$, and then by a density argument for all $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$. Condition (i) is known as the Euler-Lagrange equation and is equivalent to u_0 being a weak solution of the semi-linear elliptic system

$$\begin{cases} \Delta u = F_u(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_2. \end{cases}$$

We often call a solution to this system a critical point of I. Condition $(ii)^-$ simply states that the first eigenvalue of the linear operator $-\Delta + F_{uu}(x,u_0)$ subject to zero Dirichlet boundary conditions on $\partial\Omega_1$ is nonnegative. By slightly strengthening condition $(ii)^-$, that is
(ii) There exists $\gamma > 0$ such that $\delta^2 I(u_0,\varphi) \ge \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$ for all $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$,
we can achieve the following (cf. Theorem 2.1)

$$(i) \text{ and } (ii) \Longrightarrow u_0 \text{ is an } L^r \text{ local minimizer of } I \text{ in } \mathcal{A}_{u_0}(\partial\Omega_1)\ ,$$

where $r := \max(1, \frac{n}{2}(p-2))$.

It is well known that conditions (i) and (ii) imply u_0 to be a weak local minimizer in $\mathcal{A}_{u_0}(\partial\Omega_1)$. Appealing to the special structure of I, in Proposition 3.1 we improve this to u_0 being a strong local minimizer and then by the use of a truncation operator and an inequality proved in Lemma 3.1 we establish the result for the correct L^r.

In general one can not get the above conclusion without imposing any restrictions on the growth of F from below. As an example consider the case where $F(x,u) = -\lambda e^{|u|^2}/2$ with $\lambda > 0$ and assume that $\partial\Omega_1 \ne \emptyset$. It can then be verified that for any choice of $\lambda \le \lambda_1$ where $\lambda_1 = \lambda_1(\partial\Omega_1) > 0$ denotes the first eigenvalue of the Laplacian with zero Dirichlet boundary conditions on $\partial\Omega_1$ the function $u_0 = 0$ is a strong local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega_1)$ (cf. Proposition 3.1). However u_0 *is not* an L^r local minimizer of I for any $r < \infty$. The proof of this claim is similar to that of Lemma 3.2; for any such r we can construct a sequence $\varphi_\varepsilon \to 0$ in $L^r(\Omega;\mathbf{R}^N)$ such that $I(\varphi_\varepsilon) < I(0)$. It is clear that here F does not satisfy any growth of the type mentioned in the theorem.

The study of local minimizers can be ultimately related to the study of dynamical stability for equilibrium points of special class of dynamical systems. The connection is based on the application of Lyapunov type arguments. Indeed following a result of Ball and Marsden [2] an equilibrium point of a dynamical system endowed with a Lyapunov function is (nonlinearly) stable if it furnishes a local minimizer for the given Lyapunov function. Let us note that the actual statement of this result involves the notion of a *potential well* instead of a local minimizer however as we shall see later (cf. Remark 2.2) the local minimizers obtained by the application of Theorem 2.1 do lie in potential wells. We refer the reader to [1] and [2] for a detailed discussion on this.

In [6] Brezis and Nirenberg study functionals similar to (1.1) for the case $N = 1$. To give a brief description of their result let us assume that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a given Carathéodory integrand. In addition let there be constants $C > 0$ and $q \geq 1$ with $q \leq (n+2)/(n-2)$ for $n \geq 3$ and $q < \infty$ for $n \leq 2$ such that

$$|f(x,u)| \leq C(1 + |u|^q)$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$ and let

$$F(x,u) := \int_0^u f(x,s)\,ds.$$

Then any weak local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega)$ is a $W^{1,2}$ local minimizer.

The proof is by contradiction. If u_0 is *not* a $W^{1,2}$ local minimizer there would exist a sequence $u^{(k)} \to u_0$ in $W^{1,2}(\Omega)$ such that $I(u^{(k)}) < I(u_0)$. Using elliptic regularity theory they show that the convergence of this sequence (or a more regular sequence retaining the property $I(u^{(k)}) < I(u_0)$) can be "improved" to $u^{(k)} \to u_0$ in $W^{1,\infty}(\Omega)$ giving the desired contradiction. This idea has also been used in an earlier work by De Figueiredo [8].

Our hypotheses in Theorem 2.1 are stronger than that of [6] both in terms of the smoothness of F and the starting assumption of u_0 being a weak local minimizer (conditions (i) and (ii)). Nevertheless the result here is stated for vector-valued functions for which the argument in [6] does not seem to extend. Moreover we specify the local minimizers in L^r and present an exact expression for r in terms of p and n. Finally we do not impose any upper bound on the exponent p.

As a simple application of Theorem 2.1 let us consider the case where $N = 1$ and $F = F(u)$ is a usual double-well potential with two local minima occuring at $u = a$ and $u = b$ (Fig. 1). As F is bounded from below here $\mathcal{F}$ coincides with the Sobolev space $W^{1,2}(\Omega)$.

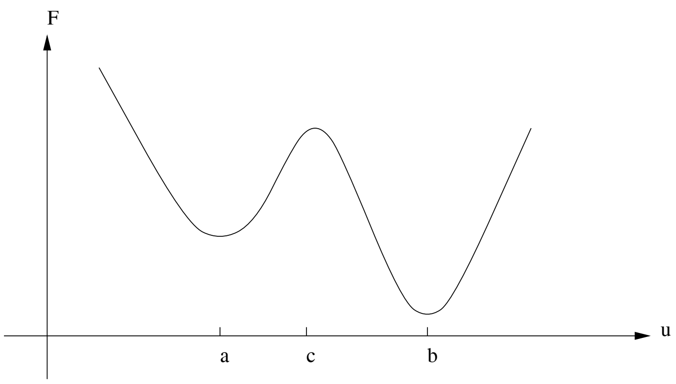

Figure 1: The double-well potential F.

It is obvious that $u_2 = b$ is the global minimum of I over $\mathcal{F}$. We would however like to know about the critical point $u_1 = a$. According to Theorem 2.1 u_1 is an L^1 local minimizer of I in $\mathcal{A}_{u_1}(\emptyset)$ (which is clearly not a global minimizer). This is surprisingly independent of how deep the second well is, i.e. how large the quantity $F(a) - F(b)$ might get. To check this we only need to verify condition (ii) of the theorem as condition (i) is automatically satisfied by any critical point of F (cf. Section 4 for a refinement of this argument). But

$$\delta^2 I(u_1, \varphi) = \int_\Omega \left(|\nabla\varphi|^2 + F''(a)\varphi^2 \right) dx \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega)}$$

for all $\varphi \in W^{1,2}(\Omega)$ where $\gamma = \min(1, F''(a)) > 0$.

It is also worthwhile noting that the third critical point of F, namely the local maximum could still correspond to a local minimizer of I depending on the size of Ω and $F''(c)$, though we definitely need to restrict the competing functions to coincide with $u_3 = c$ on some sufficiently large portion of the boundary. We discuss this fact more in Section 4.

We end this introduction by noting that in [16] we establish sufficient conditions for L^r local minimizers (with $1 \le r \le \infty$) for a larger class of functionals using a somewhat different method. The proof of Theorem 2.1 as presented here can be viewed as an alternative way of achieving the results in [16] (cf. also [15]).

2 Statement of the main result

In this section we state the main result of this paper. The proof will be presented in Section 3. Let us recall from the previous section that $\Omega \subset \mathbf{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$. Moreover $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$ where $\partial\Omega_1$ and $\partial\Omega_2$ are disjoint relatively open subsets of $\partial\Omega$ and $\mathcal{H}^{n-1}(N) = 0$. Corresponding to the functional (1.1) and a given function $u_0 \in \mathcal{F}$ we associate the class

$$\mathcal{A}_{u_0}(\partial\Omega_1) := \{u \in \mathcal{F} : (u - u_0)|_{\partial\Omega_1} = 0\}.$$

We can now state the following

Theorem 2.1. *Let $F \in C^2(\overline{\Omega} \times \mathbf{R}^N; \mathbf{R})$ and assume that there are constants $C > 0$ and $p \ge 1$ such that*

$$F(x, u) \ge -C(1 + |u|^p) \tag{2.1}$$

for all $x \in \Omega$ and all $u \in \mathbf{R}^N$. Furthermore let $u_0 \in \mathcal{F}$ be of class $L^\infty(\Omega; \mathbf{R}^N)$ and satisfy

$$(i) \quad \delta I(u_0, \varphi) = 0 \qquad \textit{and} \qquad (ii) \quad \delta^2 I(u_0, \varphi) \ge \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^n)}$$

for all $\varphi \in W^{1,2}(\Omega; \mathbf{R}^N)$ with $\varphi|_{\partial\Omega_1} = 0$ and some $\gamma > 0$. Finally let $r = r(n, p, 2) = \max(1, \frac{n}{2}(p - 2))$. Then there exist $\sigma, \rho > 0$ such that

$$I(u) - I(u_0) \ge \sigma ||u - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $u \in \mathcal{A}_{u_0}(\partial\Omega_1)$ satisfying $||u - u_0||_{L^r(\Omega;\mathbf{R}^N)} < \rho$.

Remark 2.1. It is clear that for the choices of $1 \le p \le 2 + \frac{2}{n}$, the corresponding $r(n, p, 2) = 1$, and so the conclusion of the theorem would not be affected if we replace the growth condition in this case by $F(x, u) \ge -C_0(1 + |u|^{2+\frac{2}{n}})$ where $C_0 > 0$ is such that

$$-C(1 + |u|^p) \ge -C_0(1 + |u|^{2+\frac{2}{n}}).$$

Therefore there is no loss of generality in assuming $p \ge 2 + \frac{2}{n}$.

Remark 2.2. Following Ball and Marsden [2] we say that $u_0 \in \mathcal{F}$ lies in an L^r *potential well* of I if and only if for every $\varepsilon > 0$ sufficiently small there exists $\delta > 0$ such that

$$I(u) - I(u_0) \ge \delta$$

for all $u \in \mathcal{A}_{u_0}(\partial\Omega_1)$ such that $||u - u_0||_{L^r(\Omega;\mathbf{R}^N)} = \varepsilon$.

It follows immediately from Theorem 2.1 that for $n \ge 3$ when $p \le 2^* := \frac{2n}{n-2}$ (and for $n \le 2$ when $p < \infty$), u_0 lies in an L^r potential well of I for $r = r(n, p, 2)$. Indeed if this were not true (consider the case $n \geq 3$) for some sequence $\{u^{(k)}\}$ and $\varepsilon_0 > 0$ satisfying $||u^{(k)} - u_0||_{L^r(\Omega;\mathbf{R}^N)} = \varepsilon_0$, we would have

$$\sigma||u^{(k)} - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} \leq I(u^{(k)}) - I(u_0) < \frac{1}{k},$$

and so $u^{(k)} \to u_0$ in $W^{1,2}(\Omega;\mathbf{R}^N)$. Hence since $p \leq 2^*$ implies $r \leq 2^*$, $u^{(k)} \to u_0$ in $L^r(\Omega;\mathbf{R}^N)$, a contradiction. The case $n \leq 2$ is similar. □

Remark 2.3. The key point in restricting the graph of F to lie above the graph of $-C(1+|u|^p)$ is to avoid certain "spike-shaped" functions having energies lower than that of u_0. Indeed in Lemma 3.2 we use this idea to construct a counterexample whenever the choice of the topology L^r is incompatible with the growth of F, namely $r < \max(1, n(\frac{p}{2} - 1))$.

Remark 2.4. In the case when $F = F(u)$ it is shown in [7] that if $\Omega \subset \mathbf{R}^n$ is convex then every sufficiently smooth weak local minimizer of I in $\mathcal{A}_{u_0}(\emptyset)$ is necessarily a constant function. Of course this claim is not true when F depends on x or when the domain Ω is non convex. A particular counterexample for the latter case is constructed in [13].

In the case where condition (ii) of the theorem fails we still have the following (cf. [14] and [16])

Proposition 2.1. (Local stability of critical points) *Let Ω, I and F be as in Theorem* 2.1 *and let $u_0 \in \mathcal{F}$ be of class $L^\infty(\Omega;\mathbf{R}^N)$ and satisfy condition* (*i*). *Then for every $x_0 \in \Omega$ there exist $\delta(x_0)$, $\rho(x_0) > 0$ such that for any variation $\varphi \in W_0^{1,2}(\Omega;\mathbf{R}^N)$ vanishing outside $B_\delta(x_0)$ and satisfying $u_0 + \varphi \in \mathcal{F}$ we have*

$$I(u_0) \leq I(u_0 + \varphi)$$

provided $||\varphi||_{L^r(B_\delta(x_0);\mathbf{R}^N)} < \rho$.

By imposing an upper bound on the exponent p we can obtain a result similar to that of Brezis and Nirenberg [6], namely:

Proposition 2.2. *Let Ω, I and u_0 be as in Theorem* 2.1 *where F now satisfies the growth condition* (2.1) *with $1 \leq p \leq 2^*$ when $n \geq 3$ and $1 \leq p < \infty$ when $n \leq 2$. Then u_0 is a $W^{1,2}$ local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega_1)$.*

The proof of this proposition is an immediate consequence of Theorem 2.1 and the continuity of the imbedding $W^{1,2}(\Omega;\mathbf{R}^N) \hookrightarrow L^r(\Omega;\mathbf{R}^N)$ for $r \leq 2^*$ when $n \geq 3$ and $r < \infty$ when $n \leq 2$. More precisely there exists a constant $C > 0$ (cf. [10] pp. 138, or [18] pp. 56) such that $||u||_{L^r(\Omega;\mathbf{R}^N)} \leq C||u||_{W^{1,2}(\Omega;\mathbf{R}^N)}$. We recall that $r \leq 2^*$ whenever $p \leq 2^*$.

Closely connected to this is the following

Proposition 2.3. *Let $F : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory integrand and assume that there are constants $C > 0$ and $p \geq 1$ such that*

$$F(x,u) \geq -C(1 + |u|^p)$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}^N$. Let $u_0 \in \mathcal{F}$ be of class $L^\infty(\Omega;\mathbf{R}^N)$ and assume $I(u_0) < +\infty$. Then if u_0 is a $W^{1,2}$ local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega_1)$ it is an L^r local minimizer where r is as in Theorem 2.1.

Note that here we do not impose any *a priori* regularity on F. Also the exponent p is not bounded from above.

Proposition 2.4. *Assume $n \geq 3$ and let $F \in C^2(\overline{\Omega} \times \mathbf{R}^N;\mathbf{R})$ with its second derivative with respect to u satisfying the following Hölder type condition*

$$|F_{uu}(x,u) - F_{uu}(x,v)| \leq C\left(1 + |u|^{p-2-\alpha} + |v|^{p-2-\alpha}\right)|u - v|^\alpha \tag{2.2}$$

for all $x \in \Omega$*, all* $u, v \in \mathbf{R}^N$ *and for some* $C > 0$*,* $2 < p \leq 2^*$ *and* $0 < \alpha \leq \min(1, p-2)$*. Let* $u_0 \in W^{1,2}(\Omega; \mathbf{R}^N)$ *be such that*

$$(i) \quad \delta I(u_0, \varphi) = 0 \quad \text{and} \quad (ii) \quad \delta^2 I(u_0, \varphi) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $\varphi \in W^{1,2}(\Omega, \mathbf{R}^N)$ *satisfying* $\varphi|_{\partial\Omega_1} = 0$ *where* $\gamma > 0$*. Then* u_0 *is an* L^r *local minimizer of* I *in* $\mathcal{A}_{u_0}(\partial\Omega_1)$ *where* r *is as in Theorem 2.1.*

3 Proofs

In this section we prove the main results in this paper. In our analysis an important role is played by the functional

$$J(u) := \int_\Omega (|Du|^q + |u|^q - \lambda |u|^p) \, dx,$$

defined over $W^{1,q}(\Omega; \mathbf{R}^N)$ where $1 \leq p, q$ and $\lambda > 0$. We shall start by studying the local geometry of J in a neighbourhood of the point $u_0 = 0$. Having a proper understanding of this we then proceed to the original functional given by (1.1).

Lemma 3.1. *Let* $1 \leq q < p$ *and define*

$$r := r(n, p, q) = \max(1, \frac{n}{q}(p-q)). \tag{3.1}$$

Then for given $\lambda > 0$ *there exists* $\varepsilon > 0$ *such that*

$$J(u) \geq \frac{1}{2} ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)},$$

for all $u \in W^{1,q}(\Omega; \mathbf{R}^N)$*, satisfying* $||u||_{L^r(\Omega;\mathbf{R}^N)} < \varepsilon$*.*

Proof. We shall consider three distinct cases.
(i) $1 \leq q < n$ and $p > q$. It follows from the Sobolev embedding Theorem that $W^{1,q}(\Omega; \mathbf{R}^N)$ can be continuously imbedded in $L^{q^*}(\Omega; \mathbf{R}^N)$. This means that there is a constant $C = C(n, q)$, such that

$$||u||_{L^{q^*}(\Omega;\mathbf{R}^N)} \leq C ||u||_{W^{1,q}(\Omega;\mathbf{R}^N)}$$

for all $u \in W^{1,q}(\Omega; \mathbf{R}^N)$. Furthermore, an application of Hölder's inequality implies that

$$\begin{aligned} \int_\Omega |u|^p dx = \int_\Omega |u|^q |u|^{p-q} dx &\leq \left(\int_\Omega |u|^{q(\frac{n}{n-q})} dx \right)^{\frac{n-q}{n}} \left(\int_\Omega |u|^{\frac{n}{q}(p-q)} dx \right)^{\frac{q}{n}} \\ &\leq C_1 ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} ||u||^{p-q}_{L^{\frac{n}{q}(p-q)}(\Omega;\mathbf{R}^N)}. \end{aligned}$$

Therefore

$$\begin{aligned} J(u) &= ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} - \lambda ||u||^p_{L^p(\Omega;\mathbf{R}^N)} \\ &\geq ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} \left(1 - \lambda C_1 ||u||^{p-q}_{L^{\frac{n}{q}(p-q)}(\Omega;\mathbf{R}^N)} \right), \end{aligned} \tag{3.2}$$

and the result follows.
(ii) $2 \leq n \leq q$ and $p > q$. Setting $s = \frac{nq}{n+q}$ it can be checked that $s^* = q$ and $1 \leq \frac{n}{2} \leq s < n$ for the given range of q. Thus

$$\begin{aligned} \int_\Omega |u|^p dx &= \int_\Omega \left(|u|^{\frac{p}{q}} \right)^q dx \\ &\leq C \left(\int_\Omega \left(|u|^{\frac{p}{q}s} + |u|^{\frac{p-q}{q}s} |Du|^s \right) dx \right)^{\frac{q}{s}}, \end{aligned}$$

where we have applied the embedding $W^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$ to the function $|u|^{p/q}$ (note that $p > q$). Using Hölder's inequality we can now write

$$\begin{aligned}\int_\Omega |u|^{\frac{p}{q}s} dx &= \int_\Omega |u|^{n\frac{p-q}{n+q}} |u|^{\frac{nq}{n+q}} dx \\ &\leq \left(\int_\Omega |u|^{\frac{n}{q}(p-q)} dx\right)^{\frac{q}{n+q}} \left(\int_\Omega |u|^q dx\right)^{\frac{n}{n+q}}.\end{aligned}$$

Similarly

$$\int_\Omega |u|^{\frac{p-q}{q}s} |Du|^s\, dx \leq \left(\int_\Omega |u|^{\frac{n}{q}(p-q)} dx\right)^{\frac{q}{n+q}} \left(\int_\Omega |Du|^q\, dx\right)^{\frac{n}{n+q}}.$$

Therefore

$$\int_\Omega |u|^p\, dx \leq C\, ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} ||u||^{p-q}_{L^{\frac{n}{q}(p-q)}(\Omega;\mathbf{R}^N)},$$

and so the result follows similar to that of (3.2).
(iii) $n = 1$ and $p > q$. Without loss of generality let $\Omega = (0,1)$. If $q = 1$ the proof is trivial so assume $q > 1$. We can now write

$$\begin{aligned}\int_0^1 |u|^p\, dx &= \int_0^1 |u|^{\frac{1}{q}(p-q)} |u|^{p-\frac{1}{q}(p-q)}\, dx \\ &\leq |||u|^{p-\frac{1}{q}(p-q)}||_{L^\infty(0,1)} \int_0^1 |u|^{\frac{1}{q}(p-q)}\, dx.\end{aligned}$$

Applying the embedding $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$ to the function $|u|^{p-\frac{1}{q}(p-q)}$ (again $p - \frac{1}{q}(p-q) = p(1-\frac{1}{q}) + 1 > 1$) and using a Hölder inequality we have

$$\int_0^1 |u|^p\, dx \leq C\left(\int_0^1 |u|^{\frac{1}{q}(p-q)}\, dx\right)\left(\int_0^1 |u|^p\, dx\right)^{\frac{q-1}{q}} \left(\int_0^1 (|u|^q + |u_x|^q)\, dx\right)^{\frac{1}{q}},$$

and so the result follows immediately. □

The main question that arises now is if the exponent r defined by (3.1) is sharp. Regarding this we can state the following

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *the exponent* r *given by* (3.1) *is sharp.*

Proof. Let $r < \frac{n}{q}(p-q)$. We construct a sequence $\varphi_\varepsilon \to 0$ in $L^r(\Omega;\mathbf{R}^N)$ satisfying $J(\varphi_\varepsilon) < 0$ when ε is sufficiently small. For this, choose $\varphi \in C_0^\infty(\mathbf{R}^n;\mathbf{R}^N)$ with supp$\varphi \subset B$ where B denotes the unit ball in $\mathbf{R}^n$, and take $x_0 \in \Omega$. Define

$$\varphi_\varepsilon(x) = \varepsilon^{-\alpha}\varphi\left(\frac{x - x_0}{\varepsilon}\right)$$

for some $\alpha > 0$ to be specified later. It can be seen that $\varphi_\varepsilon \to 0$ in $L^r(\Omega;\mathbf{R}^N)$ if $r < \frac{n}{\alpha}$. Also

$$\begin{aligned}J(\varphi_\varepsilon) &= \int_\Omega (|D\varphi_\varepsilon|^q + |\varphi_\varepsilon|^q - \lambda|\varphi_\varepsilon|^p)\, dx \\ &= \varepsilon^{n-q(\alpha+1)} \int_B (|D\varphi|^q + \varepsilon^q|\varphi|^q - \lambda\varepsilon^{q-\alpha(p-q)}|\varphi|^p)\, dx.\end{aligned}$$

Hence by selecting

$$\frac{q}{p-q} < \alpha < \frac{n}{r},$$

which is possible according to our assumption on r, $J(\varphi_\varepsilon) < 0$ for ε sufficiently small. The proof is thus complete. □

It is clear that the case $1 \le p < q$ is of no interest. Indeed taking the sequence $\varphi_\varepsilon = \varepsilon\varphi$ where $\varphi \in C_0^\infty(\Omega; \mathbf{R}^N)$ it follows that $u_0 = 0$ is not even a weak local minimizer of J. Now to see how the conclusion of Lemma 3.1 is affected when $p = q$, consider the functional

$$J(u) = \int_\Omega (|Du|^2 - \lambda |u|^2)\, dx$$

over $\{u \in W^{1,2}(\Omega; \mathbf{R}^N) : u|_{\partial\Omega_1} = 0\}$ where we also assume that $\partial\Omega_1 \neq \emptyset$. It is clear that $J(u) \ge 0$ if and only if $\lambda \le \lambda_1$ where as before $\lambda_1 = \lambda_1(\partial\Omega_1) > 0$ denotes the first eigenvalue of the Laplacian over Ω with zero Dirichlet boundary condition on $\partial\Omega_1$. As loosely speaking, λ_1 increases as $\mathcal{L}^n(\Omega)$ decreases, given $\lambda > 0$ (no matter how large) we can always assure $J \ge 0$ by requiring $\mathcal{L}^n(\Omega)$ to be sufficiently small. We can however prove a more general statement, namely

Lemma 3.3. *Let $1 \le q$ and $\lambda > 0$ be given. Then there exists $\delta > 0$ such that*

$$\int_\Omega (|Du|^q - \lambda |u|^q)\, dx \ge 0$$

for any $u \in W^{1,q}(\Omega; \mathbf{R}^N)$ satisfying $\mathcal{L}^n(\{x : u(x) \neq 0\}) < \delta$.

Proof. If not, there would exist $\lambda > 0$ and a sequence of nonzero functions $\{u^{(k)}\}$ such that

$$\int_\Omega |Du^{(k)}|^q dx < \lambda \int_\Omega |u^{(k)}|^q dx,$$

and $\mathcal{L}^n(\{x : u^{(k)}(x) \neq 0\}) < \frac{1}{k}$. Setting $v^{(k)} = u^{(k)}/||u^{(k)}||_{L^q(\Omega;\mathbf{R}^N)}$ it follows that for all k $||v^{(k)}||_{L^q(\Omega;\mathbf{R}^N)} = 1$,

$$\int_\Omega |Dv^{(k)}|^q dx < \lambda,$$

and $\mathcal{L}^n(\{x : v^{(k)}(x) \neq 0\}) < 1/k$. Using the compactness of the imbedding $W^{1,q}(\Omega; \mathbf{R}^N) \hookrightarrow L^q(\Omega; \mathbf{R}^N)$ we deduce that there exists $v \in L^q(\Omega; \mathbf{R}^N)$ such that by passing to a subsequence if necessary $v^{(k)} \to v$ in $L^q(\Omega; \mathbf{R}^N)$ and so $||v||_{L^q(\Omega;\mathbf{R}^N)} = 1$. This is a contradiction as $v^{(k)} \to 0$ in measure. □

Remark 3.1. Note that in the above lemma the choice of the boundary values of u is irrelevant.

We are now prepared to pass on to the general case, namely the functional I given by (1.1). As pointed out earlier the positivity of the second variation at a sufficiently smooth critical point would imply it to be a weak local minimizer. For the functional I however, this immediately implies the critical point to be a strong local minimizer. The exact statement of this claim is presented in the following

Proposition 3.1. *Let Ω, I and u_0 be as in Theorem 2.1 and $F \in C^2(\overline{\Omega} \times \mathbf{R}^N; \mathbf{R})$. Assume (i) and (ii) hold. Then there exist δ, $\sigma > 0$ such that*

$$I(u) - I(u_0) \ge \sigma ||u - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $u \in \mathcal{A}_{u_0}(\partial\Omega_1)$ satisfying $||u - u_0||_{L^\infty(\Omega;\mathbf{R}^N)} \le \delta$.

Proof. As $u_0 \in L^\infty(\Omega; \mathbf{R}^N)$ it can be easily checked that

$$\delta^2 I(u_0, \varphi) = \int_\Omega \left(|D\varphi|^2 + F_{u_i u_j}(x, u_0)\varphi_i\varphi_j\right)\, dx.$$

Now setting $u = u_0 + \varphi$ we can use the Taylor expansion of F to write

$$I(u_0 + \varphi) - I(u_0)$$

$$\begin{aligned}
&= \int_\Omega \left(\frac{1}{2}|D(u_0+\varphi)|^2 + F(x, u_0+\varphi)\right) dx - \int_\Omega \left(\frac{1}{2}|Du_0|^2 + F(x, u_0)\right) dx \\
&= \frac{1}{2}\int_\Omega \left(|D\varphi|^2 + F_{u_i u_i}(x, u_0 + \theta(x)\,\varphi)\varphi_i\varphi_j\right) dx \\
&\geq \frac{1}{2}\int_\Omega \left(|D\varphi|^2 + F_{u_i u_j}(x, u_0)\varphi_i\varphi_j\right) dx - \frac{1}{2}\int_\Omega |F_{uu}(x, u_0 + \theta\,\varphi) - F_{uu}(x, u_0)||\varphi|^2 dx \\
&\geq \frac{\gamma}{4}||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}
\end{aligned}$$

where $\theta(x)$ takes values between 0 and 1 and the last inequality holds provided $||\varphi||_{L^\infty(\Omega;\mathbf{R}^N)}$ is sufficiently small. The proof is thus complete. □

We shall now focus on proving Theorem 2.1. The main idea here is to "truncate" a given function u in such a way that the resulting $\overline{u}$ lies in a suitable L^∞ neighbourhood of u_0. The issue is then to use the growth condition on F and the contribution of the gradient term to control the remaining part and this is possible when $||u - u_0||_{L^r(\Omega;\mathbf{R}^N)}$ is sufficiently small. To put this in a more precise form we shall proceed by giving the following

Definition 3.1. *Let $\tau, \sigma \in \mathbf{R}^N$ and let $\mathbf{Q}^N_{(\tau,\sigma)} = [\tau_1, \sigma_1] \times ... \times [\tau_N, \sigma_N]$. Furthermore assume that $0 \in$ int $\mathbf{Q}^N_{(\tau,\sigma)}$. We define the Truncation Map $T_{(\tau,\sigma)} : \mathbf{R}^N \to \mathbf{Q}^N_{(\tau,\sigma)}$ associated to the pair (τ, σ) as*

$$T_{(\tau,\sigma)}(a) := \max(\tau_i, \min(a_i, \sigma_i))$$

for $1 \leq i \leq N$ with $a = (a_1, ..., a_N)$. We define the corresponding Truncation Operator

$$\mathbf{T}_{(\tau,\sigma)} : L^1(\Omega, \mathbf{R}^N) \to L^\infty(\Omega, \mathbf{R}^N)$$

by

$$\mathbf{T}_{(\tau,\sigma)}(u)(x) := T_{(\tau,\sigma)}(u(x))$$

for a.e. $x \in \Omega$.

Note that the operator $\mathbf{T}$ here is well defined in the sense that if $u_1 = u_2$ a.e. then $\mathbf{T}_{(\tau,\sigma)}(u_1) = \mathbf{T}_{(\tau,\sigma)}(u_2)$ a.e.. Also it is clear that

$$||\mathbf{T}_{(\tau,\sigma)}(u)||_{L^\infty(\Omega;\mathbf{R}^N)} \leq \max\{|\tau_i|, |\sigma_i| \quad 1 \leq i \leq N\}.$$

Proof of Theorem 2.1. For a given $u \in \mathcal{A}_{u_0}(\partial\Omega_1)$ we write $u = u_0 + \varphi$ and denote $\overline{\varphi} = \mathbf{T}_{(-\delta e, \delta e)}(\varphi)$ where $e = (1, ..., 1) \in \mathbf{R}^N$. It follows from the previous proposition that there exist $\sigma, \delta > 0$ (we may suppose that $\sigma < \frac{1}{2}$) such that

$$I(u_0 + \overline{\varphi}) - I(u_0) \geq \sigma||\overline{\varphi}||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for any φ such that $u_0 + \varphi \in \mathcal{A}_{u_0}(\partial\Omega_1)$. It can also be shown (cf. [10] pp. 129) that

$$\overline{\varphi}_{i,j} = \begin{cases} \varphi_{i,j} & \text{for a.e. } x \in \{|\varphi_i| < \delta\} \\ 0 & \text{for a.e. } x \in \{\varphi_i \geq \delta\} \cup \{\varphi_i \leq \delta\}. \end{cases}$$

Hence

$$I(u_0 + \varphi) - I(u_0 + \overline{\varphi})$$

$$\begin{aligned}
&= \int_\Omega \left(\frac{1}{2}|D(u_0+\varphi)|^2 + F(x,u_0+\varphi)\right)\,dx - \int_\Omega \left(\frac{1}{2}|D(u_0+\overline{\varphi})|^2 + F(x,u_0+\overline{\varphi})\right)\,dx \\
&= \int_\Omega \left(\frac{1}{2}|D(\varphi-\overline{\varphi})|^2 + F(x,u_0+\varphi) - F(x,u_0+\overline{\varphi})\right)\,dx, \\
&\quad + \int_\Omega \left(D\overline{\varphi}\cdot D(\varphi-\overline{\varphi}) + Du_0\cdot D(\varphi-\overline{\varphi})\right)\,dx, \\
&= \int_\Omega \left(\frac{1}{2}|D(\varphi-\overline{\varphi})|^2 + F(x,u_0+\varphi) - F(x,u_0+\overline{\varphi}) - F_u(x,u_0)\cdot(\varphi-\overline{\varphi})\right)\,dx,
\end{aligned}$$

where we have used condition (i) and the trivial identity $D\overline{\varphi}\cdot D(\varphi-\overline{\varphi}) = 0$ for a.e. $x\in\Omega$. Therefore

$$I(u_0+\varphi) - I(u_0)$$

$$\begin{aligned}
&= I(u_0+\varphi) - I(u_0+\overline{\varphi}) + I(u_0+\overline{\varphi}) - I(u_0) \\
&\geq \int_\Omega \left(\beta(|D\varphi|^2 + |\overline{\varphi}|^2) + F(x,u_0+\varphi) - F(x,u_0+\overline{\varphi}) - F_u(x,u_0)\cdot(\varphi-\overline{\varphi})\right)\,dx \\
&\geq \beta\|\varphi\|^2_{W^{1,2}(\Omega;\mathbf{R}^N)} - C\int_\Omega |\varphi|^p\,dx,
\end{aligned}$$

where $\beta = \min(\frac{1}{2},\sigma) > 0$ and $p > 2$ (cf. Remark 2.1). The result now follows from Lemma 3.1. □

The idea of using a truncation operator in the proof of Theorem 2.1 is to some extent motivated by the *Weierstrass field theory* of the calculus of variations. There to show that a given critical point furnishes a strong local minimizer for the functional under study one has to imbedd the given function into a *field of extremals* (or more generally a *Mayer field*). Then one tries to establish the minimality properties by certain convexity arguments. We refer the interested reader to the books of G. Bliss [4], O. Bolza [5] or the more recent books of M. Giaquinta and S. Hildebrandt [11] for a detailed discussion on this.

Proof of Proposition 2.1. For $\varphi \in W^{1,2}_0(\Omega;\mathbf{R}^N)$

$$\begin{aligned}
\delta^2 I(u_0,\varphi) &= \int_\Omega \left(|D\varphi|^2 + F_{u_iu_j}(x,u_0)\varphi_i\varphi_j\right)\,dx \\
&\geq \int_{B_\delta} \left(|D\varphi|^2 - C|\varphi|^2\right)\,dx
\end{aligned}$$

where in the last inequality we have assumed that $\varphi\in W^{1,2}_0(\Omega;\mathbf{R}^N)$ vanishes outside $B_\delta(x_0)\subset\Omega$ for some $\delta=\delta(x_0)$ to be specified below and the constant C is such that $|F_{uu}(x,u_0(x))|\leq C$. It follows now from Lemma 3.3 that $\delta^2 I(u_0,\varphi) > 0$ for $\varphi\neq 0$ provided δ is sufficiently small. Moreover selecting $0<\alpha\leq\lambda_1(\partial B_\delta(x_0)) - C$, we have

$$\int_{B_\delta} \left(|D\varphi|^2 - C|\varphi|^2\right)\,dx \geq \alpha\int_{B_\delta}|\varphi|^2\,dx,$$

for all $\varphi\in W^{1,2}_0(B_\delta(x_0);\mathbf{R}^N)$. Thus for $\varepsilon>0$ sufficiently small

$$\begin{aligned}
\delta^2 I(u_0,\varphi) &\geq \varepsilon\int_{B_\delta}|D\varphi|^2\,dx + (1-\varepsilon)\int_{B_\delta}\left(|D\varphi|^2 - \frac{C}{1-\varepsilon}|\varphi|^2\right)\,dx \\
&\geq \beta\|\varphi\|^2_{W^{1,2}(B_\delta;\mathbf{R}^N)},
\end{aligned}$$

where $0<\beta\leq\min(\varepsilon,\frac{\alpha}{2}(1-\varepsilon))$. The result is now a consequence of Theorem 2.1 with $\Omega = B_\delta(x_0)$. □

Proof of Proposition 2.3. Assume first that $n \geq 3$ and consider the following three distinct cases.
Case (1) $1 \leq p \leq 2 + \frac{2}{n}$. As for any p within this range $r = 1$, without loss of generality we can take $p = 2 + \frac{2}{n}$. Assume now that the conclusion of the proposition were false. Then for some sequence $\{u^{(k)}\}$ in $\mathcal{A}_{u_0}(\partial\Omega_1)$ we have $u^{(k)} \to u_0$ in $L^1(\Omega; \mathbf{R}^N)$ and

$$I(u^{(k)}) < I(u_0). \tag{3.3}$$

From the growth condition on F and the assumption $u_0 \in L^\infty(\Omega; \mathbf{R}^N)$ it follows that there exist $C, \beta > 0$ such that

$$F(x,u) \geq -C\left(1 + |u - u_0(x)|^{2+\frac{2}{n}}\right) + \beta|u - u_0(x)|^2,$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}^N$. Using (3.3) and the fact that $I(u_0) < +\infty$ we can write

$$\gamma||u^{(k)} - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} - C\int_\Omega |u^{(k)} - u_0|^{2+\frac{2}{n}}\,dx + \int_\Omega Du_0 \cdot D(u^{(k)} - u_0)\,dx \leq C_1$$

for some constant C_1 where $\gamma = \min(\frac{1}{2}, \beta) > 0$. Applying Lemma 3.1 and recalling the convergence $u^{(k)} \to u_0$ in $L^1(\Omega; \mathbf{R}^N)$ implies that

$$\frac{\gamma}{2}||u^{(k)} - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} - ||Du_0||_{L^2(\Omega;\mathbf{R}^{N\times n})}||u^{(k)} - u_0||_{W^{1,2}(\Omega;\mathbf{R}^N)} \leq C_1$$

for sufficiently large k. Thus the sequence $\{u^{(k)}\}$ is bounded in $W^{1,2}(\Omega; \mathbf{R}^N)$. As $p < 2^*$, it follows from the compactness of the imbedding $W^{1,2}(\Omega; \mathbf{R}^N) \hookrightarrow L^p(\Omega; \mathbf{R}^N)$ that by passing to a subsequence $u^{(k)} \rightharpoonup u_0$ in $W^{1,2}(\Omega; \mathbf{R}^N)$, $u^{(k)} \to u_0$ in $L^p(\Omega; \mathbf{R}^N)$ and $u^{(k)} \to u_0$ a.e. in Ω.

If $u^{(k)} \to u_0$ in $W^{1,2}(\Omega; \mathbf{R}^N)$ the contradiction is immediate. If not, there would exist $\varepsilon > 0$ such that by passing to a further subsequence $||u^{(k)} - u_0||_{W^{1,2}(\Omega;\mathbf{R}^N)} > 2\varepsilon$. Taking into account the convergence $u^{(k)} \to u_0$ strongly in $L^2(\Omega; \mathbf{R}^N)$ and weakly in $W^{1,2}(\Omega; \mathbf{R}^N)$ we can rewrite this as

$$\int_\Omega |Du_0|^2 dx + \varepsilon \leq \int_\Omega |Du^{(k)}|^2\,dx \tag{3.4}$$

for k large enough. Applying Fatou's lemma to $F(x,u) + C|u|^{2+\frac{2}{n}}$ which is clearly bounded from below we obtain

$$\int_\Omega F(x, u_0)\,dx \leq \liminf_{k\to\infty} \int_\Omega F(x, u^{(k)})\,dx.$$

This together with (3.4) implies that

$$I(u_0) < \liminf_{k\to\infty} I(u^{(k)})$$

which is a contradiction.
Case (2) $2 + \frac{2}{n} < p < \frac{2n}{n-2}$. The argument in this case is similar to that of case (1). Indeed let $\{u^{(k)}\}$ be a sequence satisfying $u^{(k)} \to u_0$ in $L^r(\Omega; \mathbf{R}^N)$ and $I(u^{(k)}) < I(u_0)$. Proceeding in a similar way as in case (1), it follows that $\{u^{(k)}\}$ is bounded in $W^{1,2}(\Omega; \mathbf{R}^N)$ and thus by passing to a subsequence $u^{(k)} \rightharpoonup u_0$ in $W^{1,2}(\Omega; \mathbf{R}^N)$. Now if this convergence is not strong, by passing to a further subsequence $I(u_0) < \liminf_{k\to\infty} I(u^{(k)})$ and this clearly is a contradiction.
Case (3) $p \geq \frac{2n}{n-2}$. The main difference between this case and the other two cases is that the boundedness of the sequence $\{u^{(k)}\}$ in $W^{1,2}(\Omega; \mathbf{R}^N)$ alone, does not provide any information about the sequence lying in $L^p(\Omega; \mathbf{R}^N)$. However here $p \leq \frac{n}{2}(p-2) = r(n,p,2)$ and hence $u^{(k)} \to u_0$ in $L^r(\Omega; \mathbf{R}^N)$ implies $u^{(k)} \to u_0$ in $L^p(\Omega; \mathbf{R}^N)$ which is all we need. The proof proceeds now as in cases (1) and (2).
The case $n \leq 2$ is similar and so we shall not reproduce the proof. □

Using a similar argument as in the proof of this proposition we can state the following:

Proposition 3.2. *Assume Ω, I and F are as given in Proposition 2.3 and that F satisfies the growth condition $F(x,u) \geq -C(1+|u|^p)$ for a.e. $x \in \Omega$, all $u \in \mathbf{R}^N$ and some $p < 2^*$. Let $\{u^{(k)}\}$ be a sequence such that $u^{(k)} \rightharpoonup u_0$ in $W^{1,2}(\Omega;\mathbf{R}^N)$ and $I(u^{(k)}) \to I(u_0)$ where $I(u_0) < +\infty$. Then $u^{(k)} \to u_0$ in $W^{1,2}(\Omega;\mathbf{R}^N)$.*

In other words the convergence of the functional improves the mode of convergence (cf. [9] and [17]). Note that the choice of the exponent p in this proposition is optimal in the sense that in general the result would not hold if $n \geq 3$ with $p \geq 2^*$. To show this let $p = 2^*$, $N = 1$, $\Omega = B$ and consider the sequence

$$u^{(k)}(x) = k^{\frac{n-2}{2}}\varphi(kx),$$

where the nonzero function $\varphi \in C_0^\infty(\mathbf{R}^n)$ with supp $\varphi \subset B$. Then

$$\int_B |\nabla u^{(k)}|^2 \, dx = \int_B |\nabla \varphi|^2 \, dx$$

which is a positive constant and therefore $\{u^{(k)}\}$ is bounded in $W^{1,2}(B)$ and $u^{(k)} \rightharpoonup 0$ (but not strongly) in $W^{1,2}(B)$. Setting $F(x,u) = -C|u|^{2^*}$ it follows that

$$I(u^{(k)}) = \int_B \left(\frac{1}{2}|\nabla\varphi|^2 - Ck^{2^*(\frac{n-2}{2})-n}|\varphi|^{2^*} \right) \, dx.$$

But then for the choice of $C = \left(\frac{1}{2}\int_B |\nabla\varphi|^2 \, dx\right) / \left(\int_B |\varphi|^{2^*} \, dx\right) > 0$ we have $I(u^{(k)}) \to I(0)$. However it is not the case that $u^{(k)} \to 0$ in $W^{1,2}(B)$. □

Proof of Proposition 2.4. Setting $v = 0$ in (2.2) it follows that $|F_{uu}(x,u)| \leq C(1+|u|^{2^*-2})$ for all $x \in \overline{\Omega}$ and all $u \in \mathbf{R}^N$. Integrating this twice it follows that the functional I is well defined (and finite) over $W^{1,2}(\Omega;\mathbf{R}^N)$. As in Remark 2.1, there is no loss of generality if we assume $p \geq 2 + \frac{2}{n}$. Indeed if $2 < p < 2 + \frac{2}{n}$, we can replace the constant C by a suitable $C_0 > 0$ such that $C(1+|u|^{p-2-\alpha}+|v|^{p-2-\alpha}) \leq C_0(1+|u|^{\frac{2}{n}-\alpha}+|v|^{\frac{2}{n}-\alpha})$.

We now claim that the second variation of I at a point $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ is given by

$$\delta^2 I(u,\varphi) = \int_\Omega \left(|D\varphi|^2 + F_{u_iu_j}(x,u)\varphi_i\varphi_j\right) \, dx.$$

To justify this we calculate explicitly the expression $\frac{d^2}{dt^2}I(u+t\varphi)|_{t=0}$ for an arbitrary $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$. As the dependence on the gradient in I is quadratic (and therefore the corresponding part in this expression has a simple form) we shall concentrate on the second term only. We first compute the first variation

$$\begin{aligned} \frac{d}{dt}\int_\Omega F(x,u+t\varphi)\,dx|_{t=0} &= \lim_{t\to 0}\int_\Omega \frac{F(x,u+t\varphi)-F(x,u)}{t}\,dx \\ &= \lim_{t\to 0}\int_\Omega F_{u_j}(x,u+t\alpha(x)\varphi)\varphi_j\,dx, \\ &= \int_\Omega F_{u_j}(x,u)\varphi_j\,dx \end{aligned}$$

where $0 \leq \alpha(x) \leq 1$, and in the last step we have used Lebesgue's theorem on dominated convergence as $|F_u(x,u)| \leq C(1+|u|^{2^*-1})$ and $u \in L^{2^*}(\Omega;\mathbf{R}^N)$. In a similar way

$$\frac{d^2}{dt^2}\int_\Omega F(x,u+t\varphi)\,dx|_{t=0} = \int_\Omega F_{u_iu_j}(x,u)\varphi_i\varphi_j\,dx.$$

We now claim that $I : W^{1,2}(\Omega;\mathbf{R}^N) \to \mathbf{R}$ is of class C^2. To show this we need only verify

$$\sup\{|\delta^2 I(u;\varphi) - \delta^2 I(v;\varphi)| : ||\varphi||_{W^{1,2}(\Omega;\mathbf{R}^N)} \leq 1\} \to 0$$

as $v \to u$ in $W^{1,2}(\Omega;\mathbf{R}^N)$. But this is a consequence of the Hölder condition (2.2).

We shall now proceed by showing the positivity of the second variation in an L^r neighbourhood of u_0. Indeed

$$\begin{aligned}\delta^2 I(u,\varphi) &\geq \delta^2 I(u_0,\varphi) - |\delta^2 I(u,\varphi) - \delta^2 I(u_0,\varphi)| \\ &\geq \gamma||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} - \int_\Omega |F_{uu}(x,u) - F_{uu}(x,u_0)|\,|\varphi|^2\,dx \\ &\geq \gamma||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} - C\int_\Omega \left(1+|u|^{p-2-\alpha}+|u_0|^{p-2-\alpha}\right)|u-u_0|^\alpha|\varphi|^2 dx.\end{aligned}$$

Applying a generalized Hölder's inequality with the exponents

$$p_1 = \frac{n}{2}\frac{(p-2)}{p-2-\alpha}, \qquad p_2 = \frac{n}{2\alpha}(p-2), \qquad p_3 = \frac{2^*}{2},$$

$(\sum_{i=1}^3 p_i^{-1} = 1)$, we have

$$\begin{aligned}\delta^2 I(u,\varphi) &\geq \gamma||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)} \\ &\quad -C\left(\left(\int_\Omega (1+|u|^{p-2-\alpha}+|u_0|^{p-2-\alpha})^{p_1}\,dx\right)^{\frac{1}{p_1}} ||u-u_0||^\alpha_{L^r}\,||\varphi||^2_{L^{2^*}}\right) \\ &\geq \gamma||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}\left(1 - C_1||u-u_0||^\alpha_{L^r(\Omega;\mathbf{R}^N)}\right) \\ &\geq \frac{\gamma}{2}||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}\end{aligned}$$

provided $||u-u_0||_{L^r(\Omega;\mathbf{R}^N)}$ is sufficiently small. The result now follows by writing the Taylor expansion of I. □

4 Concluding remarks and some examples

In this section we consider some examples related to the functional (1.1) when $F = F(u)$. We start by considering a functional with no global minimum as it is unbounded from below.

Example 1. Let μ_1, $\mu_2 > 0$ be given and $0 < p_1 < 2 < p_2$. Consider the case where

$$F(u) = \mu_1|u|^{p_1} - \mu_2|u|^{p_2}.$$

Then $u_0 = 0$ is an L^r local minimizer of I in $\mathcal{A}_{u_0}(\emptyset)$ where $r = r(n,p_2,2)$.

To show this we first note that there exist ν_1, $\nu_2 > 0$ such that

$$\mu_1|u|^{p_1} - \frac{1}{2}|u|^2 - \mu_2|u|^{p_2} \geq \nu_1|u|^{p_1} - \nu_2|u|^{p_2}.$$

Therefore

$$I(u) \geq \int_\Omega \left(\frac{1}{2}(|Du|^2+|u|^2) + \nu_1|u|^{p_1} - \nu_2|u|^{p_2}\right)dx.$$

The proof now proceeds as in Lemma 3.1. Notice that here F does not have the required degree of smoothness for the applicability of Theorem 2.1.

It is also worthwhile mentioning that in the case $\mu_1 = 0$ although the point $u = 0$ represents the global maximum of F, the function $u_0 = 0$ can be an L^r local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega_1)$. Here however we need to restrict to choices of $\partial\Omega_1 \neq \emptyset$. This emphasises the fact that the characterization of local minimizers depends not only on the topology L^r but the nature of $\partial\Omega_1$. To make this more transparent consider the following

Example 2. Let Ω be the unit ball in $\mathbf{R}^2$ and $N = 1$. For $0 \leq \theta \leq 1$, let $F_\theta : \mathbf{R} \to \mathbf{R}$ denote a one-parameter family of smooth functions having $u = a$ as a critical point for all θ, starting from a local minimum at $\theta = 0$ ($F_0^{''}(a) > 0$) and deforming to a local maximum at $\theta = 1$ ($F_1^{''}(a) < 0$) in such a way that $F_\theta''(a)$ is decreasing in θ. As this can always be done within a compact subset of $\mathbf{R}$ we can assume F_θ to satisfy the same growth independent of θ. Let

$$I_\theta(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F_\theta(u)\right) dx,$$

and for $0 \leq s \leq 2\pi$ consider the boundary arcs

$$\begin{cases} \Gamma_0 = S^1, \\ \Gamma_s = \{(\cos\theta, \sin\theta) : s < \theta < 2\pi\}, \text{ for } 0 < s \leq 2\pi, \end{cases}$$

where S^1 is the unit circle. Denoting by $\lambda_1(s) \geq 0$ the first eigenvalue of the Laplacian over Ω with zero Dirichlet boundary data on Γ_s, it is clear that $\lambda_1 : [0, 2\pi] \to R$ is monotone decreasing with $\lambda_1(0) > 0$ and $\lambda_1(2\pi) = 0$.

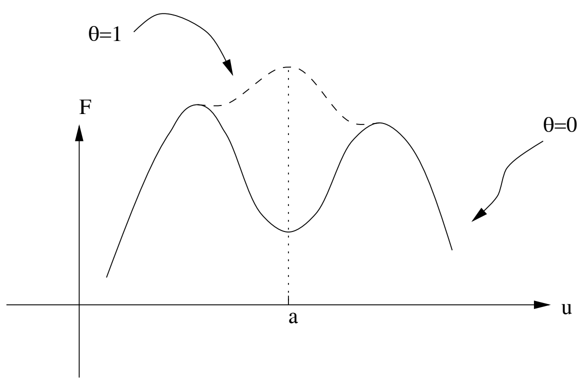

Figure 2: A one-parameter family of potentials F_θ at $\theta = 0$ and $\theta = 1$.

It thus follows from Theorem 2.1 that $u_0 = a$ is an L^r local minimizer of I_θ in $\mathcal{A}_0(\Gamma_s)$ for all $0 \leq \theta \leq 1$ (where r depends on the growth of F_θ) if only

$$\delta^2 I_\theta(u_0, \varphi) = \int_\Omega \left(|\nabla\varphi|^2 + F_\theta^{''}(a)\varphi^2\right) dx > 0$$

for all nonzero $\varphi \in W^{1,2}(\Omega)$ satisfying $\varphi|_{\Gamma_s} = 0$, that is $F_\theta^{''}(a) > -\lambda_1(s)$. Furthermore as θ increases ($F_\theta^{''}(a)$ becomes more and more negative) a larger $\lambda_1(s)$ is required to satisfy the inequality. This in turn means that the competing functions have to coincide with $u_0 = a$ on a larger portion of the boundary.

In the following two examples we consider situations in which condition (ii) in Theorem 2.1 fails, but still the weaker form of this condition, that is $(ii)^-$ hold (cf. Section 1). We show that the critical point u_0 can still be an L^r local minimizer but of course there is a price to pay.

Example 3. Let $G \in C(\mathbf{R}^N; \mathbf{R})$ satisfy
(H1) There exists $\alpha > 0$ such that $G(u) = 0$ for all $|u| \leq \alpha$, and
(H2) There exist constants $C > 0$ and $p \geq 1$ such that

$$G(u) \geq -C(1 + |u|^p)$$

for all $u \in \mathbf{R}^N$.

Let $\lambda_1 = \lambda_1(\partial\Omega) > 0$ and consider

$$F(u) = -\frac{1}{2}\lambda_1|u|^2 + G(u).$$

It is clear that the function $u_0 = 0$ is a strong local minimizer of I in $\mathcal{A}_{u_0}(\partial\Omega)$. We now claim that u_0 is an L^r local minimizer of I where r is as in Theorem 2.1.

To show this let us recall from spectral theory that the first and second eigenvalues of $-\Delta$ subject to zero Dirichlet boundary condition on $\partial\Omega$ satisfy $0 < \lambda_1 < \lambda_2$. Let φ_1 denote the principle normalized eigenfunction of the Laplacian. It is well known (cf. e.g. [12] pp. 214) that $\varphi_1 \in L^\infty(\Omega; \mathbf{R}^N)$. Now for any $u \in W_0^{1,2}(\Omega; \mathbf{R}^N)$ we can write $u = \beta\varphi_1 + v$ where $\beta = < u, \varphi_1 >$ (here $< \cdot, \cdot >$ stands for the $W_0^{1,2}(\Omega; \mathbf{R}^N)$ inner product) and v is the projection of u into the orthogonal complement of the eigenspace corresponding to φ_1. We can write

$$\begin{aligned} I(u) - I(u_0) &= \int_\Omega \left(\frac{1}{2}|D(\beta\varphi_1 + v)|^2 - \frac{1}{2}\lambda_1|\beta\varphi_1 + v|^2 + G(\beta\varphi_1 + v)\right) dx \\ &= \int_\Omega \left(\frac{1}{2}|Dv|^2 - \frac{1}{2}\lambda_1|v|^2 + G(\beta\varphi_1 + v)\right) dx \\ &\geq \int_\Omega \left(\varepsilon|Dv|^2 + G(\beta\varphi_1 + v)\right) dx \end{aligned}$$

where in the last inequality we have used a similar argument as in the proof of Proposition 2.1 to deduce that there exists $\varepsilon > 0$ such that

$$\int_\Omega \left(|Dv|^2 - \lambda_1|v|^2\right) dx \geq 2\varepsilon \int_\Omega |Dv|^2\, dx$$

for all $v \in W_0^{1,2}(\Omega; \mathbf{R}^N)$ satisfying $< v, \varphi_1 >= 0$. It follows from a simple application of Hölder's inequality and the relation $v = u- < u, \varphi_1 > \varphi_1$ that for any $r \geq 1$ there exists $C > 0$ such that $||u||_{L^r(\Omega;\mathbf{R}^N)} < \delta$ implies that $||v||_{L^r(\Omega;\mathbf{R}^N)} < C\delta$. Hence as φ_1 is uniformly bounded we can write

$$\begin{aligned} I(u) - I(u_0) &\geq \int_\Omega \left(\varepsilon|Dv|^2 - C|v|^p\right) dx \\ &\geq \frac{\varepsilon}{2}\int_\Omega |Dv|^2\, dx \end{aligned}$$

where we have used Lemma 3.1. This justifies the claim.

Example 4. Let $F \in C(\mathbf{R}^N; \mathbf{R})$ satisfy the assumptions (H1) and (H2) in Example 3 and assume that $n \geq 3$. We claim that the function $u_0 = 0$ is an L^r local minimizer of I in $\mathcal{A}_{u_0}(\emptyset)$ where r is as in Theorem 2.1.

It follows from (H1) and (H2) that there exists $C_0 > 0$ such that

$$\begin{aligned} \int_\Omega F(u)\, dx &\geq \int_{\{|u|>\alpha\}} F(u)\, dx \geq -C_0 \int_{\{|u|>\alpha\}} |u|^p\, dx \\ &\geq -C_0 \left(\int_{\{|u|>\alpha\}} |u|^{\frac{n}{2}(p-2)}\, dx\right)^{\frac{2}{n}} \left(\int_{\{|u|>\alpha\}} |u|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}. \end{aligned}$$

According to the Sobolev-Poincaré inequality (cf. e.g. [18]) there exists $C_1 > 0$ such that

$$\left(\int_\Omega |u - u_\Omega|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{2n}} \leq C_1 \left(\int_\Omega |Du|^2\, dx\right)^{\frac{1}{2}}$$

for all $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ where $u_\Omega = \left(\int_\Omega u\,dx\right)/\mathcal{L}^n(\Omega)$. Moreover

$$\begin{aligned}\left(\int_\Omega |u-u_\Omega|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} &\geq \left(\int_{\{|u|>\alpha\}} |u-u_\Omega|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ &\geq \frac{1}{2}\left(\int_{\{|u|>\alpha\}} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} - \mathcal{L}^n(\{|u|>\alpha\})^{\frac{n-2}{n}}|u_\Omega|^2\end{aligned}$$

where we have used the triangle inequality for the $L^{\frac{2n}{n-2}}$ norm and the inequality $(a-b)^2 \geq (1-\varepsilon)a^2 - (\frac{1}{\varepsilon}-1)b^2$ that holds for all $a, b \in \mathbf{R}$ and $\varepsilon > 0$ (by setting $\varepsilon = \frac{1}{2}$). We can therefore write

$$\begin{aligned}&I(u) - I(u_0) \\ \geq\ & C_2\left(\int_\Omega |u-u_\Omega|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} - C_0\left(\int_{\{|u|>\alpha\}} |u|^{\frac{n}{2}(p-2)}\,dx\right)^{\frac{2}{n}}\left(\int_{\{|u|>\alpha\}} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ \geq\ & \left(\int_{\{|u|>\alpha\}} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\left(\frac{1}{2}C_2 - C_0\left(\int_{\{|u|>\alpha\}} |u|^{\frac{n}{2}(p-2)}\,dx\right)^{\frac{2}{n}}\right) \\ & -C_2\mathcal{L}^n(\{|u|>\alpha\})^{\frac{n-2}{n}}|u_\Omega|^2 \\ \geq\ & \frac{1}{3}C_2\left(\int_{\{|u|>\alpha\}} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} - C_2\mathcal{L}^n(\{|u|>\alpha\})^{\frac{n-2}{n}}|u_\Omega|^2\end{aligned}$$

provided $||u||_{L^r(\Omega;\mathbf{R}^N)}$ is sufficiently small. Hence

$$\begin{aligned}I(u) - I(u_0) &\geq \frac{1}{3}C_2\mathcal{L}^n(\{|u|>\alpha\})^{\frac{n-2}{n}}\left(\alpha^2 - 3|u_\Omega|^2\right) \\ &\geq \frac{\alpha^2}{4}C_2\mathcal{L}^n(\{|u|>\alpha\})^{\frac{n-2}{n}}\end{aligned}$$

once again provided $||u||_{L^r(\Omega;\mathbf{R}^N)}$ is sufficiently small. The claim is thus justified.

A close inspection of the lower bound on $I(u) - I(u_0)$ in Examples 3 and 4 reveals that the function $u_0 = 0$, although an L^r local minimizer of I, does not lie in a potential well for this functional. This is clearly due to the fact that u_0 does not satisfy condition (ii) in Theorem 2.1 and the lower bound achieved here is different from the one obtained in the theorem.

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