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Sufficiency theorems for local minimizers of the multiple integrals of the calculus of variations

by

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Abstract

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$. Consider the functional

$$I(u) := \int_\Omega f(x, u, Du)\, dx,$$

over the class of Sobolev functions $W^{1,q}(\Omega; \mathbf{R}^N)$ $(1 \le q \le \infty)$ for which the integral on the right is well defined. In this paper we establish sufficient conditions on a given function u_0 and f to ensure that u_0 provides an L^r local minimizer for I where $1 \le r \le \infty$. The case $r = \infty$ is somewhat known and there is a considerable literature on the subject treating the case $\min(n, N) = 1$, mostly based on the field theory of the calculus of variations. The main contribution here is to present a set of sufficient conditions for the case $1 \le r < \infty$. Our proof is based on an indirect approach and is largely motivated by an argument of Hestenes [17] relying on the concept of "directional convergence".

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain (open connected set) and let $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$. In this paper we consider functionals of the form

$$I(u) = \int_\Omega f(x, u, Du)\, dx, \tag{1.1}$$

over the class of admissible functions

$$\mathcal{F}^q := \{u \in W^{1,q}(\Omega; \mathbf{R}^N) : \text{the integral (1.1) is well defined}\}, \tag{1.2}$$

where $1 \le q \le \infty$. By well defined we mean that the integrand is a measurable function on Ω and that at least one of the functions $f^+ = \max\{f(\cdot, u(\cdot), Du(\cdot)), 0\}$ or $f^- = \min\{f(\cdot, u(\cdot), Du(\cdot)), 0\}$ has a finite integral. It is therefore clear that $I : \mathcal{F}^q \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$. The spaces $W^{1,q}(\Omega; \mathbf{R}^N)$ appearing in (1.2) are the usual Sobolev spaces of vector-valued functions defined over Ω and the terminology we use in this paper are in accordance with [1], [13] and [36].

Throughout this paper we assume that Ω has a Lipschitz boundary $\partial\Omega$ with $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$ where $\partial\Omega_1$ and $\partial\Omega_2$ are disjoint relatively open subsets of $\partial\Omega$ and $\mathcal{H}^{n-1}(N) = 0$. Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure. We denote the unit outward normal to the boundary at a point x by $\nu(x)$. Let us also mention that unless otherwise stated we shall use the summation convention on the indices.

A major question in calculus of variations is to formulate an appropriate set of sufficient conditions on f and a given $u_0 \in \mathcal{F}^q$ to ensure that u_0 provides a local minimizer for I. Of course the notion of local minimizer depends very much on the choice of the topology. To make this clear let us fix $u_0 \in \mathcal{F}^q$ and $\partial\Omega_1$ as described above and set

$$\mathcal{A}^q_{u_0}(\partial\Omega_1) := \{u \in \mathcal{F}^q : (u - u_0)|_{\partial\Omega_1} = 0\},$$

where the boundary values are to be interpreted in the sense of traces. We now proceed by giving the following

Definition 1.1. *Let $1 \le r \le \infty$. The point $u_0 \in \mathcal{F}^q$ is an L^r (respectively $W^{1,r}$) local minimizer of I if and only if there exists $\varepsilon > 0$ such that*

$$I(u_0) \le I(u)$$

for all $u \in \mathcal{A}^q_{u_0}(\partial\Omega_1)$ satisfying

$$||u - u_0||_{L^r(\Omega;\mathbf{R}^N)} < \varepsilon \quad (\textit{respectively} \quad ||u - u_0||_{W^{1,r}(\Omega;\mathbf{R}^N)} < \varepsilon).$$

It can be easily checked that if f is of class C^2 and the minimizer u_0 is of class C^1 then

(i) $\delta I(u_0, \varphi) := \frac{d}{dt} I(u_0 + t\varphi)|_{t=0} = 0,$

$(ii)^-$ $\delta^2 I(u_0, \varphi) := \frac{d^2}{dt^2} I(u_0 + t\varphi)|_{t=0} \ge 0,$

first for all variations $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$ and then by a density argument for all $\varphi \in W^{1,2}(\Omega; \mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$. Condition (i) known as the Euler- Lagrange equation (or the equation of first variation), is equivalent to u_0 being a weak solution to the system

$$\begin{cases} \frac{\partial}{\partial x_j}(f_{P_{ij}}(x, u, Du)) = f_{u_i}(x, u, Du) & \text{in } \Omega \\ f_{P_{ij}}(x, u, Du)\nu_j(x) = 0 & \text{on } \partial\Omega_2, \end{cases}$$

for i=1, ..., N. We often call a solution to the above system a *stationary point* of I.
Condition $(ii)^-$ states that the quadratic form

$$\int_\Omega \left(f_{P_{ij}P_{kl}}(x, u_0, Du_0)\varphi_{i,j}\varphi_{k,l} + 2 f_{P_{ij}u_k}(x, u_0, Du_0)\varphi_{i,j}\varphi_k + f_{u_k u_l}(x, u_0, Du_0)\varphi_k\varphi_l \right) dx,$$

is nonnegative. It is well-known that if this condition is slightly strengthened, that is
(ii) There exists $\gamma > 0$ such that $\delta^2 I(u_0, \varphi) \ge \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$ for all $\varphi \in W^{1,2}(\Omega; \mathbf{R}^N)$ satisfying $\varphi|_{\partial\Omega_1} = 0$,
then (i) and (ii) would imply u_0 to be a *weak* local minimizer in $\mathcal{A}^1_{u_0}(\partial\Omega_1)$. Recall that the terms weak and *strong* local minimizers are standard in the calculus of variations and refer to local minimizers in $W^{1,\infty}$ and L^∞ topologies respectively.

A sufficiency theorem for strong local minimizers of I in the case $n = N = 1$ was first properly formulated and proved by Weierstrass. His proof is based on the novel idea of constructing a *field of extremals* or what is known today as the field theory of the calculus of variations (cf. Bliss [7], Bolza [8], Hestenes [16]). Since then there have been numerous attempts on the one hand to extend the proof to higher dimensions i.e. $n > 1$ and on the other hand to find alternative ways to avoid the construction of such fields. In [20] Levi gave a proof for the planar case (when $n = N = 1$) that avoids the use of field of extremals. Levi's method is refered to as an expansion method since it is based on first expanding the total variation by application of the Taylor's formula and then showing it to be positive by the use of certain integral inequalities and properties of the *Weierstrass excess function.* Motivated by some earlier work by other people, in [23] Morrey has outlined how to extend Weierstrass's ideas to the higher dimensional case $n > 1$ and $N = 1$. In particular he has proved the existence of a field of extremals under the hypotheses in the theorem and has presented the appropriate divergence free structure for the path independent integral.

Using a completely different technique in [16] Hestenes gave an indirect proof (proof by contradiction) for the case $n = N = 1$ and later extended this to the case $n > 1$ and $N = 1$ [17]. The main ingredients in his proof are the concept of directional convergence and certain integral inequalities developed by McShane and later by Reid [24]. It is part of our aim to present an updated version of Hestenes argument as firstly it largely motivates the proof of the main results in this paper, and secondly because it seems to be quite unknown to the researchers in the field.

Let us fix $N = 1$ and recall the sufficient conditions to be satisfied by the stationary point $u_0 \in C^1(\overline{\Omega})$ to be a strong local minimizer for I in $\mathcal{A}^1_{u_0}(\partial\Omega)$ when f is of class C^2.

- *The pointwise positivity of the second variation*: For all nonzero $\varphi \in W_0^{1,2}(\Omega)$, $\delta^2 I(u_0, \varphi) > 0$.
- *The strengthened condition of Legendre*: There exists $\gamma > 0$ such that

$$f_{p_i p_j}(x, u_0(x), \nabla u_0(x))\lambda_i \lambda_j \geq \gamma |\lambda|^2$$

for all $x \in \overline{\Omega}$ and all $\lambda \in \mathbf{R}^n$.

- *The strengthened condition of Weierstrass*: There exists $\varepsilon > 0$ such that

$$E_f(x, u, p, q) := f(x, u, q) - f(x, u, p) - f_{p_i}(x, u, p)(q_i - p_i) \geq 0$$

for all $x \in \overline{\Omega}$, $|u - u_0(x)| < \varepsilon$, $|p - \nabla u_0(x)| < \varepsilon$ and all $q \in \mathbf{R}^n$.

The first two conditions and their relation to condition (ii) introduced earlier are studied in Sections 2, 4 and 5 of this paper. The function E_f in the third condition is known as the Weierstrass excess function. It is obvious that $f(x, u, \cdot)$ is convex at a point p if and only if $E_f(x, u, p, q) \geq 0$ for all $q \in \mathbf{R}^n$. Thus the last condition is a convexity requirement on f with respect to the gradient argument. It is important to point out that this convexity assumption has a central role in all the arguments mentioned earlier for the case $N = 1$. For the general case $N \geq 1$, it was shown by Meyers [21] (see also Ball [2]) that if u_0 is a C^1 strong local minimizer of I, the function $f(x, u_0(x), \cdot)$ is *quasiconvex* at $Du_0(x)$ for each $x \in \Omega$. (The smoothness of u_0 can be relaxed cf. e.g. [19] and [31]). Recall that a continuous function $f : \mathbf{R}^{N\times n} \to \mathbf{R}$ is quasiconvex at $A \in \mathbf{R}^{N\times n}$ (cf. Morrey [22], Ball [2]) if and only if

$$f(A) \leq \int_Q f(A + D\varphi)\, dx$$

for all $\varphi \in W_0^{1,\infty}(Q; \mathbf{R}^N)$, where $Q \subset \mathbf{R}^n$ is the unit n-cube. It is known that quasiconvexity is weaker than convexity and coincides with the latter when $N = 1$. Thus in the above list the last condition is a somewhat reasonable strengthened version of the necessary condition just described. However when $N > 1$ it is far more stronger than being necessary. A major question in this regard is to formulate a set of sufficient conditions for strong local minimizers in the case $N > 1$ that is based on quasiconvexity. This seems to be an open problem.

Let us point out that in almost all the sufficiency proofs mentioned earlier the function f is assumed to be smooth (at least of class C^2) and the stationary point u_0 is assumed to be of class C^1. Hence a further open problem would be to present sufficient conditions for local minimizers of I under less smoothness assumptions.

We remark that recently Ball and James (unpublishes work) have given a direct proof for the sufficiency theorem in the case $n = 1$, under slightly weaker hypotheses. There it is shown that the conclusion of the theorem follows when the first condition is replaced by

- u_0 is a weak local minimizer of I.

As a further remark in [28] and [29] Sivaloganathan has employed the idea of a divergence free structure to treat the case $N > 1$ under weaker convexity assumptions but in a special case. (cf. also the work of Ball and Murat [5]).

Another interesting issue is to formulate sufficient conditions for u_0 to be an L^r local minimizer of I when $r < \infty$. In this general format this appears to be a difficult question. The example

$$I(u) = \int_\Omega |\nabla u|^2 (1 - u^2)\, dx,$$

with $u_0 = 0$ shows that the above sufficient conditions in general would not give rise to such local minimizers. Indeed here $u_0 = 0$ is a strong local minimizer of I in $\mathcal{A}_0^1(\partial\Omega_1)$ but not an L^r local minimizer for any $r < \infty$. To show this let us take a nonzero $\varphi \in C_0^\infty(\mathbf{R}^n)$ with supp $\varphi \subset B$. Here $C_0^\infty(\mathbf{R}^n)$ stands for the class of smooth functions with compact support in $\mathbf{R}^n$. Let us also assume that $B \subset \Omega$ and consider the sequence $\varphi_\varepsilon(x) := \varepsilon^{-\alpha}\varphi(x/\varepsilon)$ for $\varepsilon \to 0^+$ and some $\alpha > 0$ to be specified later. Clearly $\varphi_\varepsilon \to 0$ in $L^r(\Omega)$ if $\alpha < n/r$. Moreover

$$\begin{aligned} I(\varphi_\varepsilon) &= \varepsilon^{-2(\alpha+1)} \int_\Omega |\nabla\varphi\left(\varepsilon^{-1}x\right)|^2 \left(1 - \varepsilon^{-2\alpha}|\varphi\left(\varepsilon^{-1}x\right)|^2\right) dx \\ &= \varepsilon^{n-2(\alpha+1)} \left(\int_\Omega |\nabla\varphi(x)|^2\, dx - \varepsilon^{-2\alpha}\int_\Omega |\nabla\varphi(x)|^2|\varphi(x)|^2\, dx\right). \end{aligned}$$

Hence if we choose φ such that the second integral on the right is nonzero, it follows that for any $1 \le r < \infty$ we can find $0 < \alpha < n/r$ such that $\varphi_\varepsilon \to 0$ in $L^r(\Omega)$ while $I(\varphi_\varepsilon) < I(0)$ for ε sufficiently small.

Theorem 3.2 provides to some extent an answer to the question raised above. It is a generalization of an earlier work by the author in [30] for functionals in the form (1.1) with $f(x,u,Du) = |Du|^2 + F(x,u)$. We remark that in this latter case Brezis and Nirenberg [9] have recently established conditions for a weak local minimizer of I to be a $W^{1,2}$ local minimizer. Our results here as well as in [30] are in the somewhat same direction as their's but obviously in the more general context of L^r (and $W^{1,r}$) local minimizers. Indeed we here show that the growth of the second derivatives of f namely f_{uu} and f_{up} determine the exponent $1 \le r \le \infty$ and we present an exact expression for this dependence. We shall also see that unlike the case in e.g. [9] our proof does not rely on $N = 1$ and hence the result is valid for $N \ge 1$.

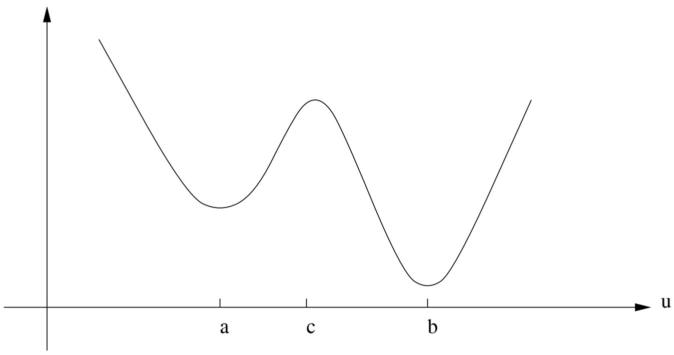

Figure 1: The double-well potential F.

As a simple example let us consider the case $f(x,u,Du) = |Du|^2 + F(u)$ where $F \in C^2(\mathbf{R}^N;\mathbf{R})$ is a usual double-well potential with two local minima occuring at $u = a$ and $u = b$ (cf. Fig. 1). As F is bounded from below here $\mathcal{F}^2$ coincides with the Sobolev space $W^{1,2}(\Omega;\mathbf{R}^N)$. It is obvious that $u_2 = b$ is the global minimum of I over $\mathcal{F}^2$. We would however like to know about the stationary point $u_1 = a$. According to Theorem 3.2 u_1 is an L^1 local minimizer of I in $\mathcal{A}^2_{u_1}(\emptyset)$ (which is clearly not a global minimizer). This is surprisingly independent of how deep the second well is, i.e. how large the quantity $F(a) - F(b)$ might get. To check this we only need to verify condition (ii) of the theorem as condition (i) is satisfied by any stationary point of F. But

$$\delta^2 I(u_1,\varphi) = \int_\Omega \left(2|D\varphi|^2 + F_{u_iu_j}(a)\varphi_i\varphi_j\right) dx \ge \gamma||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $\varphi \in W^{1,2}(\Omega;\mathbf{R}^N)$ and for some $\gamma > 0$ provided the Hessian of F at a is strictly positive definite.

Let us end this introduction by describing briefly the plan of this paper. In Section 2 we present some of the important necessary conditions satisfied by various kinds of local minimizers. These are mainly of second order and hence both the function f and the stationary point u_0 are assumed to have the required degrees of smoothness. In addition we mention the appropriate strengthened version of these conditions and also derive some of their basic consequences. In Section 3 we state the sufficiency Theorems 3.1 and 3.2 but postpone their proofs to the end of Section 4. We begin Section 4 by proving some auxiliary results on weak convergence and lower semicontinuity of variational integrals. We also study the question of positivity for quadratic forms and its relation to some of the necessary conditions stated in Section 2. Finally in Section 5, as a simple example we show how the local stability result of Sivaloganathan [29] can be achieved without any need for the construction of local fields and Hamilton-Jacobi theory. For other applications of our results we refer the reader to [6].

2 Preliminaries

We start this section by discussing the necessary conditions satisfied by different kinds of local minimizers of the functional (1.1) and mention the appropriate strengthened version of these conditions suitable for the sufficiency theorems appearing in the subsequent sections. We shall assume $N = 1$. Recall that $\Omega \subset \mathbf{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$. Moreover $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$, where $\partial\Omega_1$ and $\partial\Omega_2$ are disjoint, relatively open and $\mathcal{H}^{n-1}(N) = 0$.

Proposition 2.1. (The necessary condition of Legendre) *Let $f \in C^2(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n; \mathbf{R})$ and let $u_0 \in C^1(\overline{\Omega})$ be a weak local minimizer of I. Then for every $x \in \overline{\Omega}$ and for all $\lambda \in \mathbf{R}^n$*

$$(\mathbf{L}) \qquad f_{p_i p_j}(x, u_0(x), \nabla u_0(x))\lambda_i \lambda_j \geq 0.$$

It is worth noting that condition $(\mathbf{L})$ is actually a consequence of $\delta^2 I(u_0, \varphi) \geq 0$ for $\varphi \in W_0^{1,2}(\Omega)$, which itself is a necessary condition to be satisfied by any weak local minimizer of class C^1. The proof of Proposition 2.1 now follows by applying Proposition 2.2 to the functional $J(\varphi) = \delta^2 I(u_0, \varphi)$ with $\varphi_0 = 0$ being a global minimizer.

Definition 2.1. *The function $u_0 \in C^1(\overline{\Omega})$ is said to satisfy the strengthened condition of Legendre if and only if there exist $\gamma > 0$ such that*

$$(\mathbf{L}^+) \qquad f_{p_i p_j}(x, u_0(x), \nabla u_0(x))\lambda_i \lambda_j \geq \gamma|\lambda|^2,$$

for all $x \in \overline{\Omega}$ and all $\lambda \in \mathbf{R}^n$.

The Weierstrass excess function E_f corresponding to f was defined earlier in Section 1. The following condition restricts the values of $\nabla u_0(x)$ for any strong local minimizer to the set where $f(x, u_0(x), \cdot)$ is convex.

Proposition 2.2. (The necessary condition of Weierstrass) *If $u_0 \in C^1(\overline{\Omega})$ is a strong local minimizer of I then*

$$(\mathbf{W}) \qquad E_f(x, u_0(x), \nabla u_0(x), q) \geq 0$$

for every $x \in \overline{\Omega}$ and for all $q \in \mathbf{R}^n$.

The proof of this proposition is well-known and can be found in e.g. [16].

Definition 2.2. *The function $u_0 \in C^1(\overline{\Omega})$ is said to satisfy the strengthened condition of Weierstrass if and only if there exists $\varepsilon > 0$ such that*

$$(\mathbf{W}^+) \qquad E_f(x, u, p, q) \geq 0$$

for all $x \in \overline{\Omega}$, for all $u \in \mathbf{R}^n$ with $|u - u_0(x)| < \varepsilon$, for all $p \in \mathbf{R}^n$ with $|p - \nabla u_0(x)| < \varepsilon$ and for all $q \in \mathbf{R}^n$.

In the study of sufficiency theorems for strong local minimizers of I essential use is made of the L-function defined by

$$L(t) := (1+t^2)^{\frac{1}{2}} - 1 \qquad \text{for } t \in \mathbf{R}.$$

This function was first introduced by E.J. McShane and later used by various authors in particular Reid [24] and Hestenes [16], [17] (see also Bliss [7]). It can be easily checked that L is convex and satisfies

$$(i) \quad L(t) \le |t| \quad \text{ and } \quad (ii) \quad L(t) \le \frac{t^2}{2}. \tag{2.1}$$

Further properties of this function are stated in the following

Proposition 2.3. (McShane, Reid [24]) *Let L be as above and let $\alpha > 0$. Then*

$$\begin{aligned} (i) \quad & \alpha \min(1,\alpha)\, L(t) \le L(\alpha t) \le \alpha \max(1,\alpha)\, L(t), \\ (ii) \quad & t \min(\alpha, t) \le \left((1+\alpha^2)^{\frac{1}{2}} + 1\right) L(t). \end{aligned}$$

The fact that the L-function is quadratic near the origin and grows linearly at infinity makes it a favourable candidate for acting as a lower bound on the growth of the Weierstrass excess function. This is stated more clearly in the following

Proposition 2.4. (McShane, Reid [24], Hestenes [16, 17]) *Let $f \in C^2(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n; \mathbf{R})$ and $u_0 \in C^1(\overline{\Omega})$ satisfy $(\mathbf{L}^+)$ and $(\mathbf{W}^+)$. Then there exist α, $\varepsilon > 0$ such that*

$$E_f(x, u, p, q) \ge \alpha L(|q - p|) \tag{2.2}$$

for all $x \in \overline{\Omega}$, $|u - u_0(x)| < \varepsilon$, $|p - \nabla u_0(x)| < \varepsilon$, and all $q \in \mathbf{R}^n$.

The following property of the L-function was suggested and proved by McShane for the case $n = 1$ and later reformulated and used by Reid (cf. [24]). It was extended to $n > 1$ and $N = 1$ by Hestenes (cf. [17]). Here we consider the case where both n, $N \ge 1$.

Proposition 2.5. *Given $\delta > 0$ there exist C_1, $C_2 > 0$ such that*

$$\begin{aligned} (i) \quad & \int_\Omega (L(|Du|) - C_1 |u|^2)\, dx \ge 0, \\ (ii) \quad & \int_\Omega (L(|Du|) - C_2 |u|\, |Du|)\, dx \ge 0, \end{aligned}$$

for all $u \in W_0^{1,1}(\Omega; \mathbf{R}^N)$ satisfying $||u||_{L^\infty(\Omega;\mathbf{R}^N)} \le \delta$.

Proof. Let $a, b \in \mathbf{R}^n$ be such that $(a,b) := (a_1, b_1) \times ... \times (a_n, b_n) \supset \Omega$. Given $1 \le i \le n$, let (x', t) denote the n-tuple $(x_1, ..., x_i = t, ..., x_n)$ and set $u = 0$ in $(a,b)\backslash\overline{\Omega}$. It follows now (cf. for example [13] pp. 164) that the function $u(x', \cdot) \in W_0^{1,1}((a_i, b_i); \mathbf{R^N})$ for $\mathcal{L}^{n-1}$-almost every x'. Now for fix i and x' we define

$$v(t) := \int_{a_i}^t |u_{,i}(x', z)|\, dz.$$

Clearly $|u(x', t)| \le v(t)$ for all $t \in (a_i, b_i)$ and $v'(t) = |u_{,i}(x', t)|$ for a.e. $t \in (a_i, b_i)$. It follows from Jensen's inequality for convex functions and Proposition 2.3 (i) that

$$\begin{aligned} ⨍_{a_i}^{b_i} L(|u_{,i}(x', t)|)\, dt \quad & \ge \quad L\left((b_i - a_i)^{-1} v(b_i)\right) \\ & \ge \quad (b_i - a_i)^{-1} \min(1, (b_i - a_i)^{-1}) L(v(b_i)). \end{aligned}$$

Moreover

$$\begin{aligned}\int_{a_i}^{b_i} |u(x',t)|\,|u_{,i}(x',t)|\,dt &\le \int_{a_i}^{b_i} \min(\delta, v(t))\,v'(t)\,dt \\ &\le \min(\int_{a_i}^{b_i} \delta v'(t)\,dt, \int_{a_i}^{b_i} v(t)v'(t)\,dt) \\ &\le \frac{1}{2}v(b_i)\min(2\delta, v(b_i)) \\ &\le \frac{1}{2}\left((1+4\delta^2)^{\frac{1}{2}}+1\right)L(v(b_i))\end{aligned}$$

where in the last inequality we have used Proposition 2.3 (ii). We can therefore deduce that

$$\int_{a_i}^{b_i} L\left(|u_{,i}(x',t)|\right)\,dt \ge \int_{a_i}^{b_i} C|u(x',t)|\,|u_{,i}(x',t)|\,dt, \tag{2.3}$$

for some $C > 0$ independent of i. As $|u(x',t)|^2 \le 2\int_{a_i}^{b_i} |u(x',t)|\,|u_{,i}(x',t)|\,dt$, it follows from (2.3) that

$$\int_{a_i}^{b_i} \left(L\left(|u_{,i}(x',t)|\right) - C_1|u(x',t)|^2\right)\,dt \ge 0.$$

A further integration of (2.3) and the above inequality gives

$$\int_\Omega \left(L(|u_{,i}|) - C|u|\,|u_{,i}|\right)\,dx \ge 0$$

and

$$\int_\Omega \left(L(|u_{,i}|) - C_1|u|^2\,dx\right) \ge 0.$$

The result follows by recalling that $L(|Du|) \ge L(|u_{,i}|)$ for a.e. $x \in \Omega$ and setting $C_2 = C/n$. □

Corresponding to the L-function introduced earlier we can assign the functional

$$R : W_0^{1,1}(\Omega) \to [0,\infty)$$

where

$$R(u) = \int_\Omega L(|\nabla u|)\,dx.$$

Note that this functional is non-homogeneous and non-subadditive. Moreover when u_0 is sufficiently smooth, $R(u_0)$ represents the difference between the "area" of the hyper-surfaces corresponding to $u = u_0$ and $u = 0$. In Section 4 we shall discuss further properties of this functional.

The following proposition plays an important role in the proof of Theorem 3.2. It was first stated and proved for the case $s = 0$ in [30].

Proposition 2.6. *Let* $1 \le q$, $0 \le s < q$, $p > q - s$ *and define*

$$r := r(n,p,q,s) = \max(1, n(\frac{p}{q-s} - 1)).$$

Then for any $\lambda > 0$ *there exists* $\varepsilon > 0$ *such that*

$$J(u) := \int_\Omega (|Du|^q + |u|^q - \lambda|Du|^s|u|^p)\,dx \ge \frac{1}{2}||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)},$$

for all $u \in W^{1,q}(\Omega;\mathbf{R}^N)$ *satisfying* $||u||_{L^r(\Omega;\mathbf{R}^N)} < \varepsilon$.

Proof. First note that

$$J(u) \geq ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} - \lambda \int_\Omega ((|Du|^s + |u|^s)|u|^p)dx.$$

Moreover,

$$\int_\Omega (|Du|^s + |u|^s)|u|^p dx \leq C_1 ||u||^s_{W^{1,q}(\Omega;\mathbf{R}^N)} (\int_\Omega |u|^{p\frac{q}{q-s}} dx)^{\frac{q-s}{q}}.$$

We shall now consider three distinct cases.
(i) $1 \leq q < n$. Then

$$\begin{aligned} \int_\Omega |u|^{p\frac{q}{q-s}} dx &= \int_\Omega |u|^q |u|^{q(\frac{p}{q-s}-1)} dx \\ &\leq (\int_\Omega |u|^{q^*} dx)^{\frac{q}{q^*}} (\int_\Omega |u|^{n(\frac{p}{q-s}-1)} dx)^{\frac{q}{n}}, \end{aligned}$$

and

$$(\int_\Omega |u|^{q^*} dx)^{\frac{q}{q^*}} \leq C ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)}.$$

Therefore

$$\begin{aligned} J(u) &\geq ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} \left(1 - \lambda C_2 \left(\int_\Omega |u|^{n(\frac{p}{q-s}-1)} dx\right)^{\frac{q-s}{n}}\right) \\ &\geq \frac{1}{2} ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} \end{aligned}$$

provided $||u||_{L^r(\Omega;\mathbf{R}^N)}$ is sufficiently small.
(ii) $2 \leq n \leq q$. Setting $t = \frac{nq}{n+q}$, it can be checked that $t^* = q$ and $1 \leq \frac{n}{2} \leq t < n$ for the given range of q. Thus

$$\begin{aligned} \int_\Omega |u|^{p\frac{q}{q-s}} dx &= \int_\Omega \left(|u|^{\frac{p}{q-s}}\right)^q dx \\ &\leq C \left(\int_\Omega \left(|u|^{\frac{p}{q-s}t} + |u|^{\left(\frac{p}{q-s}-1\right)t} |Du|^t\right) dx\right)^{\frac{q}{t}}, \end{aligned}$$

where we have applied the embedding $W^{1,t}(\Omega) \hookrightarrow L^q(\Omega)$ to the function $|u|^{p/(q-s)}$ (note that $p > q-s$). Using Hölder's inequality we can now write

$$\begin{aligned} \int_\Omega |u|^{\frac{p}{q-s}t} dx &= \int_\Omega |u|^{\frac{nq}{n+q}\left(\frac{p}{q-s}-1\right)} |u|^{\frac{nq}{n+q}} dx \\ &\leq \left(\int_\Omega |u|^{n\left(\frac{p}{q-s}-1\right)} dx\right)^{\frac{q}{n+q}} \left(\int_\Omega |u|^q dx\right)^{\frac{n}{n+q}}. \end{aligned}$$

Similarly

$$\int_\Omega |u|^{\left(\frac{p}{q-s}-1\right)t} |Du|^t dx \leq \left(\int_\Omega |u|^{n\left(\frac{p}{q-s}-1\right)} dx\right)^{\frac{q}{n+q}} \left(\int_\Omega |Du|^q dx\right)^{\frac{n}{n+q}}.$$

Therefore

$$\int_\Omega |u|^{p\frac{q}{q-s}} dx \leq C\, ||u||^q_{W^{1,q}(\Omega;\mathbf{R}^N)} \left(\int_\Omega |u|^{n\left(\frac{p}{q-s}-1\right)} dx\right)^{\frac{q}{n}},$$

and so the result follows similar to that in Case (i).
(iii) $n = 1$. Without loss of generality let $\Omega = (0,1)$. We can now write

$$\begin{aligned}\int_0^1 |u|^{p\frac{q}{q-s}}\,dx &= \int_0^1 |u|^{\left(\frac{p}{q-s}-1\right)}\,|u|^{1+\frac{p}{q-s}(q-1)}\,dx \\ &\leq \||u|^{1+\frac{p}{q-s}(q-1)}\|_{L^\infty(0,1)} \int_0^1 |u|^{\left(\frac{p}{q-s}-1\right)}\,dx.\end{aligned}$$

Applying the embedding $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$ to the function $|u|^{1+\frac{p}{q-s}(q-1)}$ and using a Hölder inequality we have

$$\int_0^1 |u|^{p\frac{q}{q-s}}\,dx \leq C\left(\int_0^1 |u|^{\left(\frac{p}{q-s}-1\right)}\,dx\right)\left(\int_0^1 |u|^{q\frac{p}{q-s}}\,dx\right)^{\frac{q-1}{q}}\left(\int_0^1 (|u|^q + |u_x|^q)\,dx\right)^{\frac{1}{q}},$$

and so the result follows immediately. □

It is possible to show that the exponent r defined in Proposition 2.6 is sharp. We shall refer the interested reader to [30] for a proof of this. We end this section with the following
Example. Let $p_1, p_2 > 0$ and $\lambda_1, \lambda_2 > 0$ be given. Then setting $r = \max(1, np_1/2, np_2)$, there exist $\varepsilon > 0$ such that

$$\int_\Omega (|Du|^2 + |u|^2 - \lambda_1|u|^{p_1+2} - \lambda_2|Du||u|^{p_2+1})\,dx \geq \frac{1}{2}\|u\|^2_{W^{1,2}(\Omega;\mathbf{R}^N)},$$

for all $u \in W^{1,2}(\Omega;\mathbf{R}^N)$ satisfying $\|u\|_{L^r(\Omega;\mathbf{R}^N)} < \varepsilon$.

3 Statement of the sufficiency theorems

In this section we state the sufficiency theorems for local minimizers of I. The proofs are given in Section 4. Recall that corresponding to the functional (1.1) we assign the class of admissible functions $\mathcal{F}^q$ as in (1.2) and for a fix $u_0 \in \mathcal{F}^q$ and $\partial\Omega_1 \subset \partial\Omega$ we set

$$\mathcal{A}^q_{u_0}(\partial\Omega_1) = \{u \in \mathcal{F}^q : (u-u_0)|_{\partial\Omega_1} = 0\}.$$

We can now state the following

Theorem 3.1. (The fundamental sufficiency theorem) *Let $\Omega \subset \mathbf{R}^n$ be as described earlier and consider the functional (1.1) with $f \in C^2(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n;\mathbf{R})$. Let $u_0 \in C^1(\overline{\Omega})$ satisfy $(\mathbf{W}^+)$ and*

$$(i)\quad \delta I(u_0,\varphi) = 0 \quad \text{and} \quad (ii)\quad \delta^2 I(u_0,\varphi) \geq \gamma\|\varphi\|^2_{W^{1,2}(\Omega)}$$

for some $\gamma > 0$ and all $\varphi \in W^{1,2}_0(\Omega)$. Then there exist σ, $\rho > 0$ such that

$$I(u) - I(u_0) \geq \sigma R(u - u_0)$$

for all $u \in \mathcal{A}^1_{u_0}(\partial\Omega)$ satisfying $\|u - u_0\|_{L^\infty(\Omega)} < \rho$.

Remark 3.1. It follows from (ii) that u_0 satisfies $(\mathbf{L}^+)$. In Section 4, Corollary 4.1, we shall prove more, namely that the pointwise positivity of the second variation at u_0 together with $(\mathbf{L}^+)$ are equivalent to (ii).

Theorem 3.2. (A sufficiency theorem for L^r local minimizers $1 \leq r \leq \infty$) *Let I, Ω and f be as in Theorem 3.1 and $\partial\Omega_1 \subset \partial\Omega$ as above. Let $u_0 \in C^1(\overline{\Omega})$ satisfy*

$$(i)\quad \delta I(u_0,\varphi) = 0 \quad \text{and} \quad (ii)\quad \delta^2 I(u_0,\varphi) \geq \gamma\|\varphi\|^2_{W^{1,2}(\Omega)}$$

for some $\gamma > 0$ *and all* $\varphi \in W^{1,2}(\Omega)$ *with* $\varphi|_{\partial\Omega_1} = 0$.
(1) (*The case* $r = \infty$.) *Assume that there exist* α, $\varepsilon > 0$ *such that*

$$E_f(x, u, \nabla u_0(x), q) \geq \alpha |q - \nabla u_0(x)|^2 \tag{3.1}$$

for all $x \in \overline{\Omega}$, $|u - u_0(x)| < \varepsilon$ *and* $q \in \mathbf{R}^n$. *Then there exist* σ, $\rho > 0$ *such that*

$$I(u) - I(u_0) \geq \sigma ||u - u_0||^2_{W^{1,2}(\Omega)}. \tag{3.2}$$

for all $u \in \mathcal{A}^2_{u_0}(\partial\Omega_1)$ *provided* $||u - u_0||_{L^\infty(\Omega)} < \rho$.
(2) (*The case* $1 \leq r < \infty$.) *Assume that there exists* $\alpha > 0$ *such that*

$$E_f(x, u, \nabla u_0(x), q) \geq \alpha |q - \nabla u_0(x)|^2 \tag{3.3}$$

for all $x \in \overline{\Omega}$, $u \in \mathbf{R}$, *and* $q \in \mathbf{R}^n$. *Furthermore let for some* p_1, $p_2 > 0$

$$|f_{uu}(x, u, \nabla u_0(x))| \leq C(1 + |u|^{p_1}) \quad \text{and} \quad |f_{up}(x, u, \nabla u_0(x))| \leq C(1 + |u|^{p_2}), \tag{3.4}$$

for some $C > 0$ *and all* $x \in \overline{\Omega}$. *Then there exist* σ, $\rho > 0$ *such that* (3.2) *holds for all* $u \in \mathcal{A}^2_{u_0}(\partial\Omega_1)$ *provided* $||u - u_0||_{L^r(\Omega)} < \rho$ *where* $r = r(n, p_1, p_2) = \max(1, np_1/2, np_2)$.

Note that in the above theorem (case (1)), the lower bound on $I(u) - I(u_0)$ is sharper than that of Theorem 3.1, as $||\nabla(u - u_0)||^2_{L^2(\Omega;\mathbf{R}^n)} \geq 2R(u - u_0)$. We have achieved this by imposing a quadratic growth on f with respect to the gradient at infinity (cf. (3.1)).

We also remark that in the special case $f(x, u, \nabla u) = |\nabla u|^2 + F(x, u)$ condition (3.3) trivially holds with $\alpha = 1$ and that $r = r(n, p_1, p_2) = \max(1, \frac{n}{2}p_1)$. In this way we recover the results in [30] (cf. also [9]).

4 Proofs

This section is devoted to the proof of Theorems 3.1 and 3.2. We first discuss some auxiliary results and postpone the proofs of these theorems to Subsection 4.5.

4.1 Lower semicontinuity of quadratic forms

Let $\{a^{(k)}_{ij}\}$ be a given sequence of measurable functions on Ω and such that $a^{(k)}_{ij} \to a_{ij}$ (the mode of convergence to be specified later). In this subsection we study the question of lower semicontinuity in the following setting:

$$\int_\Omega a_{ij}(x)\varphi_{,i}\varphi_{,j}\, dx \leq \liminf_{k\to\infty} \int_\Omega a^{(k)}_{ij}(x)\varphi^{(k)}_{,i}\varphi^{(k)}_{,j}\, dx$$

when $\varphi^{(k)} \rightharpoonup \varphi$ in $W^{1,2}(\Omega)$. As replacing $a^{(k)}_{ij}(x)$ with $(a^{(k)}_{ij}(x) + a^{(k)}_{ji}(x))/2$ does not change the integrands we can assume without loss of generality that the sequence $\{a^{(k)}_{ij}\}$ is symmetric, that is $a^{(k)}_{ij}(x) = a^{(k)}_{ji}(x)$ for a.e. $x \in \Omega$ and for all $1 \leq i, j \leq n$. We shall start by recalling two well-known results, namely:

Lemma 4.1. (A refinement of Fatou's Lemma) *Let* $\{u^{(k)}\}$ *be a sequence of measurable functions such that* $\{(u^{(k)})^-\}$ *is uniformly integrable and* $u^{(k)} \to u$ *a.e.. Then*

$$\int_\Omega u\, dx \leq \liminf_{k\to\infty} \int_\Omega u^{(k)}\, dx.$$

The proof of this lemma is an easy exercise in measure theory and we refer the reader to [18] and [25]. We also need the following

Lemma 4.2. *Let $a \in L^\infty(\Omega; \mathbf{R}^{n\times n})$ satisfy the following version of Legendre's condition*

$$a_{ij}(x)\lambda_i\lambda_j \geq 0 \quad \text{for a.e.} \quad x \in \Omega, \tag{4.1}$$

for all $\lambda \in \mathbf{R}^n$. Then the variational integral $J(\varphi) = \int_\Omega a_{ij}(x)\varphi_{,i}\varphi_{,j}\,dx$ is sequentially weakly lower semicontinuous on $W^{1,2}(\Omega)$.

An almost immediate consequence of this is the following

Proposition 4.1. *Let $a^{(k)} \to a$ in $L^\infty(\Omega; \mathbf{R}^{n\times n})$ with (a_{ij}) satisfying (4.1). Then*

$$\int_\Omega a_{ij}(x)\varphi_{,i}\varphi_{,j}\,dx \leq \liminf_{k\to\infty} \int_\Omega a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx$$

when $\varphi^{(k)} \rightharpoonup \varphi$ in $W^{1,2}(\Omega)$.

Proof. We claim that

$$\liminf_{k\to\infty} \int_\Omega a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx = \liminf_{k\to\infty} \int_\Omega a_{ij}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx.$$

Indeed this can be seen from

$$\begin{aligned}|\int_\Omega a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx - \int_\Omega a_{ij}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx| &\leq \int_\Omega |a^{(k)} - a||\nabla\varphi^{(k)}|^2 dx \\ &\leq ||a^{(k)} - a||_{L^\infty(\Omega;\mathbf{R}^{n\times n})}||\varphi^{(k)}||^2_{W^{1,2}(\Omega)}.\end{aligned}$$

An application of Lemma 4.2 now completes the proof. □

We can now make the following general statement concerning the lower semicontinuity question raised at the beginning of this section.

Proposition 4.2. *Let $\{a_{ij}^{(k)}\}$ be a sequence of measurable functions such that $a_{ij}^{(k)} \to a_{ij}$ a.e. and let $\varphi^{(k)} \rightharpoonup \varphi$ in $W^{1,2}(\Omega)$. Furthermore assume (a_{ij}) satisfies (4.1) and that the sequence $\{\left(a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\right)^-\}$ is uniformly integrable. Then*

$$\int_\Omega a_{ij}(x)\varphi_{,i}\varphi_{,j}\,dx \leq \liminf_{k\to\infty} \int_\Omega a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx.$$

Remark 4.1. The hypotheses of this proposition are weaker than those of Lemma 4.1 in the sense that the weak convergence of $\{\nabla\varphi^{(k)}\}$ in $L^2(\Omega; \mathbf{R}^n)$ does not imply any kind of pointwise convergence. This proposition exhibits how convexity can "handle" weak convergence in the context of lower semicontinuity.

Proof. An application of Egoroff's Theorem to the sequence $\{a^{(k)}\}$ shows that $a^{(k)} \to a$ almost uniformly in Ω. This means that for a sequence $\{\Omega^{(l)}\}$ of measurable subsets of Ω shrinking to zero i.e. $\Omega^{(l+1)} \subset \Omega^{(l)}$ and $\mathcal{L}^n(\Omega^{(l)}) \to 0$,

$$a^{(k)} \to a \qquad L^\infty(\Omega\backslash\Omega^{(l)}; \mathbf{R}^{n\times n}) \quad \text{for all } l.$$

By the uniform integrability condition, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E \left(a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\right)^- dx \leq \varepsilon \quad \text{whenever } \mathcal{L}^n(E) < \delta.$$

Therefore for l sufficiently large

$$\begin{aligned}\int_\Omega a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx &\geq \int_{\Omega\setminus\Omega^{(l)}} a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx - \int_{\Omega^{(l)}}\left(a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\right)^{-}dx\\ &\geq \int_{\Omega\setminus\Omega^{(l)}} a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx - \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and applying the previous proposition to the first term we obtain

$$\liminf_{k\to\infty}\int_\Omega a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx \geq \int_{\Omega\setminus\Omega^{(l)}} a_{ij}\varphi_{,i}\varphi_{,j}\,dx - \varepsilon.$$

The result now follows by letting $l \to \infty$ and an application of Lebesgue's Theorem on monotone convergence. □

By slightly modifying the above proof we can also show that

Proposition 4.3. *Let $\{a_{ij}^{(k)}\}$ be a sequence of measurable functions such that $a_{ij}^{(k)} \to a_{ij}$ almost uniformly and let $\nabla\varphi^{(k)} \rightharpoonup \nabla\varphi$ in $L^2(\Omega_A;\mathbf{R}^n)$ for each measurable $\Omega_A \subset \Omega$ on which $a_{ij}^{(k)} \to a_{ij}$ uniformly. Furthermore assume (a_{ij}) satisfies* (4.1) *and that $\{\left(a_{ij}^{(k)}\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\right)^{-}\}$ is uniformly integrable. Then*

$$\int_\Omega a_{ij}(x)\varphi_{,i}\varphi_{,j}\,dx \leq \liminf_{k\to\infty}\int_\Omega a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}dx.$$

We now turn our attention to quadratic functionals of the form

$$J(\varphi) = \int_\Omega (a_{ij}(x)\varphi_{,i}\varphi_{,j} + b_i(x)\varphi_{,i}\varphi + c(x)\varphi^2)\,dx$$

where the coefficients a, b and c are all bounded measurable functions and where (a_{ij}) satisfies the following version of the strengthened Legendre condition, namely there exists $\gamma > 0$ such that

$$a_{ij}(x)\lambda_i\lambda_j \geq \gamma|\lambda|^2$$

for a.e. $x \in \Omega$ and all $\lambda \in \mathbf{R}^n$.

Proposition 4.4. *Let J be as above and assume $J(\varphi) > 0$ for all nonzero $\varphi \in W^{1,2}(\Omega)$ satisfying $\varphi|_{\partial\Omega_1} = 0$. Then there exists $\lambda_1 > 0$ such that*

$$J(\varphi) \geq \lambda_1||\varphi||^2_{W^{1,2}(\Omega)}$$

for all $\varphi \in W^{1,2}(\Omega)$ with $\varphi|_{\partial\Omega_1} = 0$.

Recalling that the second variation of I at any point $u_0 \in C^1(\overline{\Omega})$ is a quadratic functional of the above type, more specifically

$$\delta^2 I(u_0,\varphi) = \int_\Omega (f_{p_ip_j}(x,u_0,\nabla u_0)\varphi_{,i}\varphi_{,j} + 2f_{p_iu}(x,u_0,\nabla u_0)\varphi_{,i}\varphi + f_{uu}(x,u_0,\nabla u_0)\varphi^2)\,dx,$$

we can state the following

Corollary 4.1. *Let I, Ω, u_0 be as in Theorem* 3.1*. Then condition (ii) is equivalent to* $(\mathbf{L}^+)$ *and*

$$\delta^2 I(u_0,\varphi) > 0$$

for all nonzero $\varphi \in W^{1,2}(\Omega)$ satisfying $\varphi|_{\partial\Omega_1} = 0$.

Remark 4.2. This result is still true when $N > 1$ if $(\mathbf{L}^+)$ is replaced by the *strong ellipticity condition* (cf. Section 5) and provided $\partial\Omega_1 = \partial\Omega$. For the case $\partial\Omega_2 \neq \emptyset$ an additional condition known as *the complementing condition* should hold for every $x \in \partial\Omega_2$ (cf. [26]).

Proof of Proposition 4.4.
Step 1. By a standard minimization of J over the unit sphere in $L^2(\Omega)$ it follows that there exists $\alpha > 0$ such that

$$J(\varphi) \geq \alpha ||\varphi||^2_{L^2(\Omega)}, \tag{4.2}$$

for all $\varphi \in W^{1,2}(\Omega)$ satisfying $\varphi|_{\partial\Omega_1} = 0$, so that

$$J_1(\varphi) := J(\varphi) - \frac{\alpha}{2}||\varphi||^2_{L^2(\Omega)} \geq \frac{\alpha}{2}||\varphi||^2_{L^2(\Omega)}.$$

Step 2. We now claim that

$$J_1(\varphi) \geq \beta ||\nabla\varphi||^2_{L^2(\Omega;\mathbf{R}^n)},$$

for some $\beta > 0$. Indeed if this were not the case there would be a sequence of nonzero functions $\{\varphi^{(k)}\}$ such that

$$\frac{1}{k}||\nabla\varphi^{(k)}||^2_{L^2(\Omega;\mathbf{R}^n)} > J_1(\varphi^{(k)}) \geq \frac{\alpha}{2}||\varphi^{(k)}||^2_{L^2(\Omega)} > 0.$$

Note that from this it follows that $||\nabla\varphi^{(k)}||_{L^2(\Omega;\mathbf{R}^n)} \neq 0$ and so letting $\psi^{(k)} = \varphi^{(k)}/||\nabla\varphi^{(k)}||_{L^2(\Omega;\mathbf{R}^n)}$ and appealing to the quadratic nature of J_1, we get

$$\frac{1}{k} > \int_\Omega (a_{ij}\psi^{(k)}_{,i}\psi^{(k)}_{,j} + b_i\psi^{(k)}_{,i}\psi^{(k)} + (c - \frac{\alpha}{2})(\psi^{(k)})^2)dx \geq \frac{\alpha}{2}||\psi^{(k)}||^2_{L^2(\Omega)}. \tag{4.3}$$

The boundedness of the sequence $\{\psi^{(k)}\}$ in $W^{1,2}(\Omega)$ implies that by passing to a subsequence,

$$\psi^{(k)} \rightharpoonup \psi \quad \text{in } W^{1,2}(\Omega), \qquad \psi^{(k)} \to \psi \quad \text{in } L^2(\Omega),$$

and so $\psi = 0$. By passing to the limit in (4.3) as $k \to \infty$

$$0 \geq \liminf_{k\to\infty} \int_\Omega a_{ij}\psi^{(k)}_{,i}\psi^{(k)}_{,j}dx + \int_\Omega (b_i\psi_{,i}\psi + (c - \frac{\alpha}{2})\psi^2)\,dx.$$

This together with the fact that

$$\int_\Omega a_{ij}\psi^{(k)}_{,i}\psi^{(k)}_{,j}dx \geq \gamma$$

contradicts $\psi = 0$. The proof is thus complete. □

A Direct Proof for Proposition 4.4. By step 1 of the previous proof we can assume (4.2). Clearly for any $\varepsilon_1 > 0$ we have

$$\int_\Omega \left((1+\varepsilon_1)a_{ij}\varphi_{,i}\varphi_{,j} + b_i\varphi_{,i}\varphi + c\varphi^2\right)\,dx \geq \int_\Omega \left(\varepsilon_1 a_{ij}\varphi_{,i}\varphi_{,j} + \alpha\varphi^2\right)\,dx,$$

or alternatively,

$$\int_\Omega \left(a_{ij}\varphi_{,i}\varphi_{,j} + \frac{1}{1+\varepsilon_1}\left(b_i\varphi_{,i}\varphi + c\varphi^2\right)\right)\,dx \geq \frac{\varepsilon_1\gamma}{1+\varepsilon_1}||\nabla\varphi||^2_{L^2(\Omega;\mathbf{R}^n)} + \frac{\alpha}{1+\varepsilon_1}||\varphi||^2_{L^2(\Omega)}.$$

Now we wish to prove

$$\int_\Omega \left(a_{ij}\varphi_{,i}\varphi_{,j} + \frac{1+\sigma}{1+\varepsilon_1}\left(b_i\varphi_{,i}\varphi + c\varphi^2\right)\right)\,dx \geq \beta||\varphi||^2_{L^2(\Omega)},$$

for some $\beta > 0$ and $\sigma > \varepsilon_1$. Then this would imply the desired uniform positivity in the $W^{1,2}$ sense. However the left side of the above inequality can be written as

$$\int_\Omega \left(a_{ij}\varphi_{,i}\varphi_{,j} + \frac{1}{1+\varepsilon_1}\left(b_i\varphi_{,i}\varphi + c\varphi^2\right) + \frac{\sigma}{1+\varepsilon_1}\left(b_i\varphi_{,i}\varphi + c\varphi^2\right)\right) dx$$

$$\begin{aligned} \geq\ & \frac{\varepsilon_1\gamma}{1+\varepsilon_1}||\nabla\varphi||^2_{L^2(\Omega;\mathbf{R}^n)} + \frac{\alpha}{1+\varepsilon_1}||\varphi||^2_{L^2(\Omega)} \\ & -\frac{M\sigma}{1+\varepsilon_1}\left(||\nabla\varphi||_{L^2(\Omega;\mathbf{R}^n)}||\varphi||_{L^2(\Omega)} + ||\varphi||^2_{L^2(\Omega)}\right) \\ \geq\ & \frac{1}{1+\varepsilon_1}\left((\varepsilon_1\gamma - \sigma M\frac{s}{2})||\nabla\varphi||^2_{L^2(\Omega;\mathbf{R}^n)} + \left(\alpha - \sigma\frac{M}{2s} - M\sigma\right)||\varphi||^2_{L^2(\Omega)}\right), \end{aligned}$$

with $\max(||b||_{L^\infty(\Omega;\mathbf{R}^n)}, ||c||_{L^\infty(\Omega)}) \leq M$ and $s > 0$ arbitrary. Therefore we require

$$\varepsilon_1\gamma - \sigma M\frac{s}{2} > 0, \qquad \alpha - \sigma M(\frac{1}{2s} + 1) > 0,$$

plus the fact that $\varepsilon_1 < \sigma$. These can be written as

$$1 < \frac{\sigma}{\varepsilon_1} < \frac{\gamma}{M}\frac{2}{s}, \qquad \sigma < \frac{\alpha}{M}\frac{2s}{1+2s}. \tag{4.4}$$

So we proceed as follows; choose $s > 0$ such that $1 < 2\gamma/Ms$. With this s (now fixed) select $\sigma > 0$ to satisfy the second inequality in (4.4), then find ε_1 to fit the first inequality. □

4.2 Some auxiliary results on weak convergence

We shall start this section by recalling the following well-known lemma. A proof can be found in e.g. [10].

Lemma 4.3. *Let p_1, $p_2 \in [1,\infty]$ such that $1/p_1 + 1/p_2 \leq 1$. Assume $\{\varphi^{(k)}\}$,$\{\psi^{(k)}\}$ are given sequences such that $\varphi^{(k)} \to \varphi$ in $L^{p_1}(\Omega)$ and $\psi^{(k)} \rightharpoonup \psi$ in $L^{p_2}(\Omega)$. Then $\varphi^{(k)}\psi^{(k)} \rightharpoonup \varphi\psi$ in $L^r(\Omega)$, whenever $1/r \geq 1/p_1 + 1/p_2$ with the usual interpretations for ∞.*

Note that in the particular case where p_1 and p_2 are conjugate exponents the product sequence converges weakly in $L^1(\Omega)$ to the product of the limits.

Proposition 4.5. *Assume $n \geq 3$ and let $\{\varphi^{(k)}\}$ be a bounded sequence in $W^{1,2}(\Omega)$. Then by passing to a subsequence if necessary*

$$\begin{aligned} &(i) && \varphi^{(k)} \rightharpoonup \varphi && \text{in } W^{1,2}(\Omega), \\ &(ii) && (\varphi^{(k)})^2 \rightharpoonup \varphi^2 && \text{in } W^{1,1^*}(\Omega). \end{aligned}$$

Furthermore if $\{b^{(k)}\}$ and $\{c^{(k)}\}$ are sequences such that $b^{(k)} \to b$ in $L^n(\Omega;\mathbf{R}^n)$ and $c^{(k)} \to c$ in $L^{\frac{n}{2}}(\Omega)$, then

$$\begin{aligned} &(iii) && c^{(k)}(\varphi^{(k)})^2 \rightharpoonup c\varphi^2 && \text{in } L^1(\Omega), \\ &(iv) && b_i^{(k)}\varphi^{(k)}\varphi_{,i}^{(k)} \rightharpoonup b_i\varphi\varphi_{,i} && \text{in } L^1(\Omega). \end{aligned} \tag{4.5}$$

Proof of Proposition 4.5. As a result of the reflexivity of $W^{1,2}(\Omega)$ and the compactness of the imbedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $p < 2^*$, it follows that by passing to a subsequence if necessary (we do not re-label this) we have

$$\varphi^{(k)} \rightharpoonup \varphi \quad \text{in } W^{1,2}(\Omega), \qquad \varphi^{(k)} \to \varphi \quad a.e. \quad \text{in } \Omega.$$

Now let $\psi^{(k)} = (\varphi^{(k)})^2$. Then $\nabla\psi^{(k)} = 2\varphi^{(k)}\nabla\varphi^{(k)}$ *a.e.* and therefore by a simple application of Hölder's inequality we have that since $1^*/2^* + 1^*/2 = 1$

$$\int_\Omega |\nabla\psi^{(k)}|^{1^*} dx \le C_0 \left(\int_\Omega |\varphi^{(k)}|^{2^*} dx\right)^{\frac{1^*}{2^*}} \left(\int_\Omega |\nabla\varphi^{(k)}|^2 dx\right)^{\frac{1^*}{2}} < C_1.$$

Also

$$\int_\Omega |\psi^{(k)}|^{1^*} dx = \int_\Omega |\varphi^{(k)}|^{1^*.2} dx < C_2,$$

as $1^*.2 < 2^*$. Therefore $\{\psi^{(k)}\}$ is bounded in $W^{1,1^*}(\Omega)$ and so by passing to a further subsequence $\psi^{(k)} \rightharpoonup \chi$ for some χ in $W^{1,1^*}(\Omega)$. The pointwise convergence $\psi^{(k)} \to \varphi^2$ now implies $\chi = \varphi^2$ a.e..

To show the next part we first recall that $(\varphi^{(k)})^2 \rightharpoonup \varphi^2$ in $L^{1^{**}}(\Omega) = L^{\frac{n}{n-2}}(\Omega)$ as a result of the continuity of the imbedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. An application of Lemma 4.3 with $p_1 = n/2$, $p_2 = n/(n-2)$ and $r = 1$ now implies the weak convergence $c^{(k)}(\varphi^{(k)})^2 \rightharpoonup c\varphi^2$. The other case is similar. □

Proposition 4.6. (Hestenes [17]) *Let $\{\varphi^{(k)}\}$ be a sequence in $W_0^{1,1}(\Omega)$ such that*

$$(i) \quad \sup_k \int_\Omega |\varphi^{(k)}|^2 \, dx < \infty,$$

$$(ii) \quad \sup_k \int_\Omega |\varphi^{(k)}|\,|\nabla\varphi^{(k)}|\, dx < \infty, \tag{4.6}$$

and $\varphi^{(k)} \to \varphi$ a.e. in Ω for some $\varphi \in W_0^{1,2}(\Omega)$. Then for any $b \in C(\overline{\Omega}; \mathbf{R}^n)$

$$\lim_{k\to\infty} \int_\Omega b_i(x)\varphi^{(k)}\varphi^{(k)}_{,i}\, dx = \int_\Omega b_i(x)\varphi\varphi_{,i}\, dx.$$

It is important to note that in this proposition the restriction on the functions $\varphi^{(k)}$ to vanish on the boundary is essential.

Proof. We shall give this in two steps.

Step 1. We prove the result for the case when $b \in C^1(\overline{\Omega}; \mathbf{R}^n)$. It follows from (4.6) that the sequence $\{(\varphi^{(k)})^2\}$ is bounded in $W_0^{1,1}(\Omega)$. (Note that (4.6) implies that for each k the weak derivative $\nabla(\varphi^{(k)})^2 = 2\varphi^{(k)}\nabla\varphi^{(k)}$ cf. e.g. [14] pp. 151.) Thus it follows from the compactness of the imbedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ that by passing to a subsequence if necessary $(\varphi^{(k)})^2 \to \varphi^2$ in $L^1(\Omega)$. Now

$$b_i\varphi^{(k)}\varphi^{(k)}_{,i} = \frac{1}{2}\frac{\partial}{\partial x_i}(b_i(\varphi^{(k)})^2) - \frac{1}{2}b_{i,i}(\varphi^{(k)})^2,$$

and hence an application of the divergence theorem shows that

$$\int_\Omega b_i\varphi^{(k)}\varphi^{(k)}_{,i} dx = -\frac{1}{2}\int_\Omega b_{i,i}(\varphi^{(k)})^2 dx.$$

This implies the conclusion.

Step 2. We now consider the general case when $b \in C(\overline{\Omega}; \mathbf{R}^n)$. By approximation it follows that for any given $\varepsilon > 0$ there exists $b^* \in C^1(\overline{\Omega}; \mathbf{R}^n)$ such that $||b - b^*||_{L^\infty(\Omega;\mathbf{R}^n)} < \varepsilon$. Therefore

$$\begin{aligned} |\int_\Omega b_i(\varphi^{(k)}\varphi^{(k)}_{,i} - \varphi\varphi_{,i})\, dx| &\le \int_\Omega |b_i - b_i^*|\,|\varphi^{(k)}\varphi^{(k)}_{,i} - \varphi\varphi_{,i}|\, dx + |\int_\Omega b_i^*(\varphi^{(k)}\varphi^{(k)}_{,i} - \varphi\varphi_{,i})\, dx| \\ &\le \varepsilon C + \int_\Omega b_i^*(\varphi^{(k)}\varphi^{(k)}_{,i} - \varphi\varphi_{,i})\, dx \\ &\le \varepsilon(C+1), \end{aligned}$$

provided k is sufficiently large (we have used the result from step 1 for the second integral). The proof is complete since ε is arbitrary. □

Corollary 4.2. *Let* $\{\varphi^{(k)}\}$ *be as above and let* $\{b^{(k)}\}$ *be a sequence such that* $b^{(k)} \to b$ *in* $L^\infty(\Omega;\mathbf{R}^n)$ *with* $b \in C(\overline{\Omega};\mathbf{R}^n)$. *Then*

$$\lim_{k\to\infty}\int_\Omega b_i^{(k)}(x)\varphi^{(k)}\varphi_{,i}^{(k)}\,dx = \int_\Omega b_i(x)\varphi\varphi_{,i}\,dx.$$

Furthermore if $c^{(k)} \to c$ *in* $L^n(\Omega)$ *then*

$$c^{(k)}(\varphi^{(k)})^2 \rightharpoonup c\varphi^2 \quad \text{in } L^1(\Omega).$$

(*Compare with* (4.5).)

Proof. The first part follows by noting that

$$\lim_{k\to\infty}\int_\Omega b_i^{(k)}\varphi^{(k)}\varphi_{,i}^{(k)}dx = \lim_{k\to\infty}\int_\Omega b_i\varphi^{(k)}\varphi_{,i}^{(k)}dx.$$

Indeed

$$\begin{aligned} &\left|\int_\Omega b_i^{(k)}\varphi^{(k)}\varphi_{,i}^{(k)}dx - \int_\Omega b_i\varphi^{(k)}\varphi_{,i}^{(k)}\,dx\right| \\ &\leq \int_\Omega |b^{(k)}-b|\,|\varphi^{(k)}||\nabla\varphi^{(k)}|\,dx \\ &\leq C||b^{(k)}-b||_{L^\infty(\Omega;\mathbf{R}^n)}. \end{aligned}$$

For the second part assume $n > 1$. Then it follows that the sequence $\{(\varphi^{(k)})^2\}$ is bounded in $L^{1^*}(\Omega)$ and thus $(\varphi^{(k)})^2 \rightharpoonup \varphi^2$ in $L^{1^*}(\Omega)$. (Note the pointwise convergence given in the proposition.) The result is now a consequence of Lemma 4.3. The case $n = 1$ is similar. □

4.3 Some convergence properties related to R

The positive functional R was defined earlier in Section 2. The fact that it is non homogeneous and non-subadditive makes it far from being a norm over $W_0^{1,1}(\Omega)$; however it has some features similar to that of the norm $||\cdot||_{W^{1,1}(\Omega)}$. We start by showing that they have the same convergent sequences. (We note that the results in this subsection are due to Hestenes [17] and are presented in a shorter and updated form for the convenience of the reader).

For a sequence $\{v^{(k)}\}$ in $W_0^{1,1}(\Omega)$, the convergence $R(v^{(k)}) \to 0$ implies $\nabla v^{(k)} \to 0$ in measure. Recall that a sequence of measurable functions $\{g^{(k)}\}$ converge to zero in measure if and only if for any $\varepsilon > 0$

$$\mathcal{L}^n(\{x \in \Omega; |g^{(k)}(x)| > \varepsilon\}) \to 0 \quad \text{as } k \to \infty.$$

Infact one can say more, namely

Proposition 4.7. $||v^{(k)}||_{W^{1,1}(\Omega)} \to 0$ *if and only if* $R(v^{(k)}) \to 0$.

Proof.
(a) The implication ($\Longrightarrow$) is trivial.
(b) To show ($\Longleftarrow$) we consider

$$\begin{aligned} ||\nabla u||^2_{L^1(\Omega;\mathbf{R}^n)} &= \left(\int_\Omega |\nabla u|\,dx\right)^2 \\ &\leq \left(\int_\Omega \frac{|\nabla u|}{\left(1+\frac{1}{2}L(\nabla u)\right)^{\frac{1}{2}}}\left(1+\frac{1}{2}L(\nabla u)\right)^{\frac{1}{2}}dx\right)^2 \\ &\leq \left(\int_\Omega L(\nabla u)dx\right)\left(\int_\Omega (2+L(\nabla u))dx\right). \end{aligned}$$

Therefore an application of Poincaré's inequality shows that

$$||u||^2_{W^{1,1}(\Omega)} \leq CR(u)\,(1+R(u))\,,$$

for some $C > 0$. The proof is thus complete. □

Assume now that the sequence of nonzero functions $\{v^{(k)}\}$ in $W_0^{1,1}(\Omega)$ is such that $R(v^{(k)}) \to 0$ and define a corresponding sequence of variations

$$\varphi^{(k)} = \frac{v^{(k)}}{R^{\frac{1}{2}}(v^{(k)})}.$$

In what follows we shall explore some properties of this sequence.

Proposition 4.8. *The sequence $\{\varphi^{(k)}\}$ is weakly relatively compact in $W_0^{1,1}(\Omega)$.*

Proof. The result follows by showing that $\{\nabla\varphi^{(k)}\}$ is weakly relatively compact in $L^1(\Omega;\mathbf{R}^n)$. For this let E be a measurable subset of Ω, then

$$\begin{aligned}\left(\int_E |\nabla\varphi^{(k)}|\,dx\right)^2 &\leq \int_E \frac{|\nabla\varphi^{(k)}|^2}{1+\frac{1}{2}L(\nabla v^{(k)})}\,dx \int_E \left(1+\frac{1}{2}L(\nabla v^{(k)}\right)\,dx \\ &\leq 2\left(\mathcal{L}^n(E)+\frac{1}{2}R(v^{(k)})\right).\end{aligned}$$

Since $R(v^{(k)}) \to 0$, it follows that the sequence $\{\nabla\varphi^{(k)}\}$ is uniformly integrable and thus according to the Dunford-Pettis criterion, sequentially weakly relatively compact in $L^1(\Omega;\mathbf{R}^n)$. □

As a consequence of the above proposition we can now assume that there exists $\varphi \in W_0^{1,1}(\Omega)$ such that by passing to a subsequence (we do not re-label this) $\varphi^{(k)} \rightharpoonup \varphi$ in $W^{1,1}(\Omega)$.

An application of Egoroff's Theorem to the sequence $\{\nabla v^{(k)}\}$ implies the existence of a sequence $\{\Omega^{(l)}\}$ of measurable subsets of Ω, shrinking to zero, such that $\nabla v^{(k)} \to 0$ in $L^\infty(\Omega\backslash\Omega^{(l)};\mathbf{R}^n)$ for each l. Using this we can now improve the weak convergence of the sequence of variations $\{\varphi^{(k)}\}$ as stated in the following

Proposition 4.9. *The sequence $\{\nabla\varphi^{(k)}\}$ lies in $L^2(\Omega\backslash\Omega^{(l)};\mathbf{R}^n)$ for sufficiently large k (depending on l) and the variation φ belongs to $W_0^{1,2}(\Omega)$. Furthermore $\nabla\varphi^{(k)} \rightharpoonup \nabla\varphi$ in $L^2(\Omega\backslash\Omega^{(l)};\mathbf{R}^n)$ for each l.*

Proof. Consider the sequence $\{z^{(k)}\}$, where

$$z^{(k)} = \frac{\nabla\varphi^{(k)}}{(1+\frac{1}{2}L(\nabla v^{(k)}))^{\frac{1}{2}}}.$$

Then

$$||z^{(k)}||^2_{L^2(\Omega;\mathbf{R}^n)} = \int_\Omega \frac{|\nabla\varphi^{(k)}|^2}{1+\frac{1}{2}L(\nabla(v^{(k)})}\,dx = 2,$$

and so by passing to a subsequence $z^{(k)} \rightharpoonup z$ in $L^2(\Omega;\mathbf{R}^n)$. Now let $g \in L^\infty(\Omega;\mathbf{R}^n)$. Then

$$\begin{gathered}\left|\int_{\Omega\backslash\Omega^{(l)}} (g\cdot z^{(k)} - g\cdot\nabla\varphi^{(k)})\,dx\right| \\ \leq ||g||_{L^\infty(\Omega;\mathbf{R}^n)}||\nabla\varphi^{(k)}||_{L^1(\Omega;\mathbf{R}^n)}\left|\left|\frac{1}{(1+\frac{1}{2}L(\nabla v^{(k)}))^{\frac{1}{2}}}-1\right|\right|_{L^\infty(\Omega\backslash\Omega^{(l)})}.\end{gathered}$$

The convergence of the last term to zero in the above inequality implies that $z = \nabla\varphi$ for a.e. $x \in \Omega\backslash\Omega^{(l)}$ and so for a.e. $x \in \Omega$; therefore $\varphi \in W_0^{1,2}(\Omega)$. In addition

$$\int_{\Omega\backslash\Omega^{(l)}} |\nabla\varphi^{(k)} - z^{(k)}|^2\,dx = \frac{1}{2}\int_{\Omega\backslash\Omega^{(l)}} |z^{(k)}|^2\left(L(\nabla v^{(k)})\right)^2\,dx \to 0$$

and therefore

$$\nabla\varphi^{(k)} \rightharpoonup \nabla\varphi \qquad in \qquad L^2(\Omega\backslash\Omega^{(l)};\mathbf{R}^n).$$

This completes the proof. □

4.4 A lower semicontinuity property for I

Given $a \in C(\overline{\Omega}\times\mathbf{R};\mathbf{R})$, $b \in C^1(\overline{\Omega}\times\mathbf{R};\mathbf{R}^n)$ consider the functional

$$J(u) = \int_\Omega (a(x,u) + b_i(x,u)u_{,i})\, dx.$$

The following is an extension of a lower semicontinuity property of J by Lindeberg and Hestenes cf. [17] to the case where $\partial\Omega_2 \neq \emptyset$.

Lemma 4.4. *Let $u_0 \in C^1(\overline{\Omega})$ satisfy $b_i(x,u_0(x))\ \nu_i(x) = 0$ on $\partial\Omega_2$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|J(u) - J(u_0)| \leq \varepsilon\,(1 + R(u-u_0))\,,$$

for all $u \in W^{1,1}(\Omega)$ satisfying

$$||u-u_0||_{L^\infty(\Omega)} < \delta, \qquad (u-u_0)|_{\partial\Omega_1} = 0.$$

Corollary 4.3. *Let I, f, u_0 and Ω be as in Theorem 3.1. Then there exists $\beta > 0$ such that to any $\varepsilon > 0$ there corresponds a $\delta > 0$ satisfying*

$$I(u) - I(u_0) \geq \beta R(u-u_0)dx - \varepsilon,$$

whenever

$$||u-u_0||_{L^\infty(\Omega)} < \delta, \qquad (u-u_0)|_{\partial\Omega_1} = 0.$$

Before presenting the proof of this corollary and Lemma 4.4, let us make a little remark about some algebraic manipulations here and in the proof of Theorems 3.1 and 3.2. As we are dealing with integrals taking their values in the set of extended real numbers, that is $\overline{\mathbf{R}} = \mathbf{R}\cup\{-\infty,+\infty\}$ the "identity" $\int(g+h)\,dx = \int g\,dx + \int h\,dx$ is not in general valid. (Consider for example the case where $g = -h \geq 0$ with g having an infinite integral). However if at least one of g or h has a finite integral the above equality holds. Thus special attention need to be paid on such manipulations specially in Subsection 4.5.

Proof of the corollary.

$$\begin{aligned} I(u) - I(u_0) &= \int_\Omega (f(x,u,\nabla u) - f(x,u_0,\nabla u_0))\, dx \\ &= \int_\Omega (E_f(x,u,\nabla u_0,\nabla u) + f(x,u,\nabla u_0) \\ &\quad + f_{p_i}(x,u,\nabla u_0)(u-u_0)_{,i} - f(x,u_0,\nabla u_0))\, dx \\ &\geq \alpha \int_\Omega L(|\nabla(u-u_0)|)\, dx + J(u) - J(u_0) \end{aligned}$$

where

$$J(u) = \int_\Omega (f(x,u,\nabla u_0) + f_{p_i}(x,u,\nabla u_0)(u-u_0)_{,i})\, dx,$$

and we have used Proposition 2.4. The result now follows from the lemma and recalling the natural boundary condition $f_{p_i}(x,u_0(x),\nabla u_0(x))\,\nu_i(x) = 0$ (when $\partial\Omega_2 \neq \emptyset$). □

Proof of the lemma. It can easily be checked that

$$\begin{aligned} J(u) - J(u_0) &= \int_\Omega (a(x,u) - a(x,u_0) + (b_i(x,u) - b_i(x,u_0))u_{0,i} \\ &\quad + b_i(x,u)(u-u_0)_{,i})\, dx. \end{aligned}$$

But

$$b_i(x,u)(u-u_0)_{,i} = \frac{\partial}{\partial x_i}(b_i(x,u_0)(u-u_0)) - \frac{\partial}{\partial x_i}b_i(x,u_0)(u-u_0)$$
$$+(b_i(x,u)-b_i(x,u_0))(u-u_0)_{,i},$$

and hence an application of the divergence theorem shows that

$$\int_\Omega b_i(x,u)(u-u_0)_{,i}dx \quad = \quad \int_\Omega (-\frac{\partial}{\partial x_i}b_i(x,u_0)(u-u_0)$$
$$+(b_i(x,u)-b_i(x,u_0))(u-u_0)_{,i})\,dx.$$

Therefore from the continuity assumption on a and b it follows that for $||u-u_0||_{L^\infty(\Omega)}$ sufficiently small

$$\begin{aligned}|J(u)-J(u_0)| &\leq \int_\Omega (|a(x,u)-a(x,u_0)| + |\nabla u_0|\,|b(x,u)-b(x,u_0)| \\ &\quad + |b_{i,i}(x,u_0)|\,|u-u_0| + |b(x,u)-b(x,u_0)|\,|\nabla(u-u_0)|)\,dx \\ &\leq \frac{\varepsilon}{c}\int_\Omega (1+L(|\nabla(u-u_0)|))\,dx \\ &\leq \varepsilon\left(1+\int_\Omega L(|\nabla(u-u_0)|)\,dx\right),\end{aligned}$$

where $c = \max(1, \mathcal{L}^n(\Omega))$. □

4.5 Proof of Theorems 3.1 and 3.2

We start this subsection with the proof of Theorem 3.1. Assume the conclusion were false. Then there would exist a sequence $\{u^{(k)}\} \subset \mathcal{A}^1_{u_0}(\partial\Omega)$ with $u^{(k)}$ different from u_0, $u^{(k)} \to u_0$ in $L^\infty(\Omega)$, such that

$$I(u^{(k)}) - I(u_0) < \frac{1}{k}R(u^{(k)}-u_0). \tag{4.7}$$

It follows from Corollary 4.3 that for any $\varepsilon > 0$

$$\beta R(u^{(k)}-u_0) - \varepsilon < \frac{1}{k}R(u^{(k)}-u_0),$$

for sufficiently large k. This implies that $R(u^{(k)}-u_0) \to 0$. Following Subsection 4.3 with $v^{(k)} = u^{(k)} - u_0$, we can define a corresponding sequence of variations

$$\varphi^{(k)} = \frac{u^{(k)}-u_0}{R^{\frac{1}{2}}(u^{(k)}-u_0)}.$$

We can now write

$$\begin{aligned} & I(u^{(k)}) - I(u_0) \\ = & \int_\Omega (f(x,u^{(k)},\nabla u^{(k)}) - f(x,u^{(k)},\nabla u_0) - f_{p_i}(x,u^{(k)},\nabla u_0)(u^{(k)}-u_0)_{,i} \\ & + \; f(x,u^{(k)},\nabla u_0) - f(x,u_0,\nabla u_0) \\ & + \; (f_{p_i}(x,u^{(k)},\nabla u_0) - f_{p_i}(x,u_0,\nabla u_0))(u^{(k)}-u_0)_{,i} \\ & + \; f_{p_i}(x,u_0,\nabla u_0)(u^{(k)}-u_0)_{,i})\,dx. \end{aligned} \tag{4.8}$$

However

$$E_f(x,u^{(k)},\nabla u_0,\nabla u^{(k)})$$

$$= f(x,u^{(k)},\nabla u^{(k)}) - f(x,u^{(k)},\nabla u_0) - f_{p_i}(x,u^{(k)},\nabla u_0)(u^{(k)}-u_0)_{,i}$$
$$= \int_0^1 (1-t) f_{p_i p_j}(x,u^{(k)},\nabla u_0 + t\nabla(u^{(k)}-u_0))\,dt \quad (u^{(k)}-u_0)_{,i}(u^{(k)}-u_0)_{,j}.$$

Similarly

$$f(x,u^{(k)},\nabla u_0) - f(x,u_0,\nabla u_0) = f_u(x,u_0,\nabla u_0)(u^{(k)}-u_0)$$
$$+ \int_0^1 (1-t) f_{uu}(x,u_0 + t(u^{(k)}-u_0),\nabla u_0)\,dt \quad (u^{(k)}-u_0)^2,$$

and

$$(f_{p_i}(x,u^{(k)},\nabla u_0) - f_{p_i}(x,u_0,\nabla u_0))(u^{(k)}-u_0)_{,i}$$
$$= \int_0^1 f_{p_i u}(x,u_0 + t(u^{(k)}-u_0),\nabla u_0)\,dt \quad (u^{(k)}-u_0)_{,i}(u^{(k)}-u_0).$$

Therefore combining these together and making use of condition (i) in the theorem we can write

$$I(u^{(k)}) - I(u_0)$$

$$= \int_\Omega (a_{ij}^{(k)}(x)(u^{(k)}-u_0)_{,i}(u^{(k)}-u_0)_{,j} + b_i^{(k)}(x)(u^{(k)}-u_0)_{,i}(u^{(k)}-u_0)$$
$$+ c^{(k)}(x)(u^{(k)}-u_0)^2)\,dx, \tag{4.9}$$

where

$$a_{ij}^{(k)}(x) = \int_0^1 (1-t) f_{p_i p_j}(x,u^{(k)},\nabla u_0 + t\nabla(u^{(k)}-u_0))\,dt,$$
$$b_i^{(k)}(x) = \int_0^1 f_{p_i u}(x,u_0 + t(u^{(k)}-u_0),\nabla u_0)\,dt,$$
$$c^{(k)}(x) = \int_0^1 (1-t) f_{uu}(x,u_0 + t(u^{(k)}-u_0),\nabla u_0)\,dt.$$

It can be easily checked that

$$a_{ij}^{(k)} \to \frac{1}{2} f_{p_i p_j}(\cdot,u_0(\cdot),\nabla u_0(\cdot))$$

for a.e. $x \in \Omega$ and

$$b_i^{(k)} \to f_{p_i u}(\cdot,u_0(\cdot),\nabla u_0(\cdot)), \qquad c^{(k)} \to \frac{1}{2} f_{uu}(\cdot,u_0(\cdot),\nabla u_0(\cdot))$$

in $L^\infty(\Omega)$. Dividing (4.9) by $R(u^{(k)}-u_0)$ and using (4.7) we obtain

$$\frac{1}{k} > \int_\Omega \left(a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)} + b_i^{(k)}(x)\varphi_{,i}^{(k)}\varphi^{(k)} + c^{(k)}(x)(\varphi^{(k)})^2 \right) dx. \tag{4.10}$$

Setting $u = u^{(k)} - u_0$ in Proposition 2.5 it follows that the sequence $\{\varphi^{(k)}\}$ satisfies (4.6). By passing to the limit in (4.10) and using Corollary 4.2

$$0 \geq \liminf_{k\to\infty} \int_\Omega a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx$$

$$+ \frac{1}{2}\int_\Omega \left(2 f_{p_i u}(x,u_0,\nabla u_0)\varphi_{,i}\varphi + f_{uu}(x,u_0,\nabla u_0)\varphi^2\right) dx. \tag{4.11}$$

Since

$$\liminf_{k\to\infty}\int_\Omega a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)}\,dx =$$

$$\liminf_{k\to\infty}\int_\Omega \frac{E_f(x,u^{(k)},\nabla u_0,\nabla u^{(k)})}{R(u^{(k)}-u_0)} \geq \alpha > 0,$$

(according to (2.2)) it can immediately be deduced that $\varphi \neq 0$.

Recalling the lower semicontinuity result in Proposition 4.3, Proposition 4.9 and the fact that $a_{ij}^{(k)}(x)\varphi_{,i}^{(k)}\varphi_{,j}^{(k)} \geq 0$ it follows from (4.11) that

$$0 \geq \frac{1}{2}\delta^2 I(u_0,\varphi) \tag{4.12}$$

This however contradicts (ii). The proof is thus complete. □

We shall now proceed with the proof of Theorem 3.2. The main lines of the proof are similar to that of Theorem 3.1 and for this reason we will abbreviate some of the arguments.

Proof of Theorem 3.2. We argue by contradiction. Indeed if the conclusion were false, for some sequence $\{u^{(k)}\} \subset \mathcal{A}^2_{u_0}(\partial\Omega_1)$ with $u^{(k)}$ different from u_0 we would have $u^{(k)} \to u_0$ in $L^r(\Omega)$ and

$$I(u^{(k)}) - I(u_0) < \frac{1}{k}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)}. \tag{4.13}$$

We now define the sequence of normalized variations $\varphi^{(k)} = (u^{(k)}-u_0)/||u^{(k)}-u_0||_{W^{1,2}(\Omega)}$. As $||\varphi^{(k)}||_{W^{1,2}(\Omega)} = 1$, it follows that there exists a $\varphi \in W^{1,2}(\Omega)$ such that by passing to a subsequence $\varphi^{(k)} \rightharpoonup \varphi$ in $W^{1,2}(\Omega)$, $\varphi^{(k)} \to \varphi$ in $L^2(\Omega)$ and $\varphi|_{\partial\Omega_1} = 0$. We now consider two distinct cases:
Case (1) $r = \infty$. Here we proceed in a way similar to that of (4.12) and we show that $\delta^2 I(u_0,\varphi) \leq 0$.
Case (2) $1 \leq r < \infty$. The main difference between this case and the previous one is that the convergence $u^{(k)} \to u_0$ in $L^r(\Omega)$ does not imply any uniform convergence of the sequence over Ω in order to allow us make use of the continuity of f and its derivatives. To overcome this we have imposed the growth conditions (3.4) stated in the theorem. We also need the following

Lemma 4.5. *Let I, Ω, f, p_1, p_2 and u_0 be as above and let $u^{(k)} \to u_0$ in $L^r(\Omega)$ such that*

$$I(u^{(k)}) - I(u_0) \leq \frac{1}{k}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)} + C,$$

for some $C \geq 0$. Then the sequence $\{u^{(k)}\}$ is bounded in $W^{1,2}(\Omega)$. In the case $C = 0$ we can extract a subsequence such that $u^{(k)} \to u_0$ in $W^{1,2}(\Omega)$.

Proof. It follows from the growth condition (3.4) that there exist C_1, C_2, $\beta > \alpha$ (α as in (3.3)) such that

$$f(x,u^{(k)},\nabla u_0) - f(x,u_0,\nabla u_0) \geq -C_1(1+|u^{(k)}-u_0|^{p_1+2}) + \beta|u^{(k)}-u_0|^2,$$

and similarly,

$$(f_p(x,u^{(k)},\nabla u_0) - f_p(x,u_0,\nabla u_0))\cdot\nabla(u^{(k)}-u_0) \geq -C_2(1+|u^{(k)}-u_0|^{p_2+1})|\nabla(u^{(k)}-u_0)|,$$

for a.e. $x \in \Omega$. Therefore using the expansion (4.8) and condition (3.3) we can write

$$\frac{1}{k}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)} + C$$

$$\geq \int_\Omega (\alpha|\nabla(u^{(k)}-u_0)|^2 - C_1(1+|u^{(k)}-u_0|^{p_1+2}) + \beta|u^{(k)}-u_0|^2$$

$$-C_2(1+|u^{(k)}-u_0|^{p_2+1})|\nabla(u^{(k)}-u_0)| - C_3|\nabla(u^{(k)}-u_0)|)\,dx,$$

and as $\alpha < \beta$

$$C_4 \geq \frac{\alpha}{2}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)} - \int_\Omega C_1|u^{(k)}-u_0|^{p_1+2}\,dx$$
$$-\int_\Omega C_5(1+|u^{(k)}-u_0|^{p_2+1})|\nabla(u^{(k)}-u_0)|\,dx.$$

Applying Proposition 2.6 (with $q=2$) to this inequality (cf. the example following the proposition), we get

$$C_4 \geq \frac{\alpha}{4}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)} - C_5||\nabla(u^{(k)}-u_0)||_{L^1(\Omega;\mathbf{R}^n)},$$

for sufficiently large k. Using a Hölder inequality on the last expression it follows immediately that the above can hold only if $||u^{(k)}-u_0||_{W^{1,2}(\Omega)}$ is bounded.

If $C=0$, it follows from the previous part that by passing to a subsequence if necessary $u^{(k)} \to u_0$ in $L^2(\Omega)$ (as a result of the compactness of the imbedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$). Using (4.9) and the growth condition (3.4) we can write

$$\frac{1}{k}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)}$$

$$\geq \int_\Omega (\alpha|\nabla(u^{(k)}-u_0)|^2 - C_6(1+|u^{(k)}-u_0|^{p_1})\,|u^{(k)}-u_0|^2$$
$$-C_7(1+|u^{(k)}-u_0|^{p_2})\,|u^{(k)}-u_0|\,|\nabla(u^{(k)}-u_0)|)\,dx$$

or

$$\begin{aligned} 0 \quad &\geq \quad \frac{\alpha}{2}||u^{(k)}-u_0||^2_{W^{1,2}(\Omega)} - \int_\Omega C_8(1+|u^{(k)}-u_0|^{p_1})\,|u^{(k)}-u_0|^2\,dx \\ &\quad - \int_\Omega C_7(1+|u^{(k)}-u_0|^{p_2})\,|u^{(k)}-u_0|\,|\nabla(u^{(k)}-u_0)|\,dx \\ &\geq \quad \int_\Omega \Big(\frac{\alpha}{4}|\nabla(u^{(k)}-u_0)|^2 - C_8|u^{(k)}-u_0|^2 - C_7|u^{(k)}-u_0|\,|\nabla(u^{(k)}-u_0)|\Big)\,dx, \end{aligned}$$

where we have again used Proposition 2.6.
Setting $\varepsilon = ||u^{(k)}-u_0||_{L^2(\Omega)}$ and $t = ||\nabla(u^{(k)}-u_0)||_{L^2(\Omega;\mathbf{R}^n)}$ we have

$$0 \geq \frac{\alpha}{4}t^2 - C_7\varepsilon t - C_8\varepsilon^2,$$

or $0 \leq t \leq C_9\varepsilon$. The result follows by letting $\varepsilon \to 0$. □

We can now apply Lemma 4.5 to the sequence $\{u^{(k)}\}$ and deduce that by passing to a subsequence $u^{(k)} \to u_0$ in $W^{1,2}(\Omega)$. By passing to a further subsequence this implies that

$$a^{(k)}_{ij} \to \frac{1}{2}f_{p_ip_j}(\cdot,u_0(\cdot),\nabla u_0(\cdot))$$

for a.e. $x \in \Omega$. We shall now consider two cases.
Case (a) $n \geq 3$. It follows from the convergence $u^{(k)} \to u_0$ in $L^r(\Omega)$ that $u^{(k)} \to u_0$ in $L^{np_1/2}(\Omega)$ and $L^{np_2}(\Omega)$. Therefore the growth conditions (3.4) together with Lebesgue's theorem on dominated convergence imply that

$$b^{(k)}_i \to f_{p_iu}(\cdot,u_0(\cdot),\nabla u_0(\cdot)), \qquad c^{(k)} \to \frac{1}{2}f_{uu}(\cdot,u_0(\cdot),\nabla u_0(\cdot))$$

in $L^n(\Omega)$ and $L^{n/2}(\Omega)$ respectively. Applying Propositions 4.2 and 4.5 to (4.10) we can now deduce that $\delta^2 I(u_0,\varphi) \le 0$.
Case (b) $n \le 2$. The main problem is that in this case we can not apply Proposition 4.5 to the sequence $\{\varphi^{(k)}\}$. However this difficulty can be overcome by recalling the conclusion of Lemma 4.5. Indeed it follows from $u^{(k)} \to u_0$ in $W^{1,2}(\Omega)$, the continuity of the imbedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for $q < \infty$ ($W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ when $n = 1$) and the growth conditions (3.4) that

$$b_i^{(k)} \to f_{p_i u}(\cdot, u_0(\cdot), \nabla u_0(\cdot)), \qquad c^{(k)} \to \frac{1}{2} f_{uu}(\cdot, u_0(\cdot), \nabla u_0(\cdot))$$

in $L^q(\Omega)$ for any $q < \infty$. We can again pass to the limit in (4.10) by an application on Lemma 4.3. Thus $\delta^2 I(u_0,\varphi) \le 0$.
The only remaining task (in both cases (1) and (2)) now is to exhibit $\varphi \ne 0$. But this follows from (4.10), i.e.

$$\frac{1}{k} > \alpha ||\varphi^{(k)}||^2_{W^{1,2}(\Omega)} + \int_\Omega \left(b_i^{(k)}(x)\varphi^{(k)}_{,i}\varphi^{(k)} + (c^{(k)}(x) - \alpha)(\varphi^{(k)})^2 \right) dx,$$

or after passing to the limit and noting that $||\varphi^{(k)}||_{W^{1,2}(\Omega)} = 1$,

$$0 \ge \alpha + \frac{1}{2} \int_\Omega \left(2 f_{p_i u}(x, u_0, \nabla u_0)\varphi_{,i}\varphi + (f_{uu}(x, u_0, \nabla u_0) - 2\alpha)\varphi^2 \right) dx,$$

which is false if $\varphi = 0$. □

Remark 4.3. Similar to the case $N = 1$ we can associate to any given sufficiently smooth $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$, the Weierstrass excess function $E_f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \times \mathbf{R}^{N\times n} \to \mathbf{R}$ by setting

$$E_f(x, u, P, Q) := f(x, u, Q) - f(x, u, P) - f_P(x, u, P) \cdot (Q - P).$$

The statement of Theorem 3.2 (for simplicity here we restrict to case (1), similar comments apply to case (2) with the appropriate modifications) can now be generalized to the case $N > 1$ if condition (3.1) is replaced by its multi-dimensional analogue:
There exist $\alpha, \varepsilon > 0$ such that

$$E_f(x, u, Du_0(x), Q) \ge \alpha |Du_0(x) - Q|^2, \tag{4.14}$$

for all $x \in \overline{\Omega}$, $|u - u_0(x)| < \varepsilon$ and $Q \in \mathbf{R}^{N\times n}$.

The proof can be extended to this case without much difficulty. The important point however is that this condition is much stronger than necessary. For example it follows from (4.14) that $f(x, u_0(x), \cdot)$ is strictly convex at $Q = Du_0(x)$ for $x \in \overline{\Omega}$ which is stronger than the necessary condition of quasiconvexity (cf. Section 1).

Note that even when f fails to satisfy (4.14) it might still be possible that this condition is verified by $\tilde{f} = f + g$ where $g : \Omega \times \mathbf{R}^N \times \mathbf{R}^{N\times n} \to \mathbf{R}$ is a *null Lagrangian*, that is the integral $\int_\Omega g(x, u, Du)\, dx$ depends only on the boundary values of u. The key observation is that unlike the case when either of n or $N = 1$, in the multi-dimensional setting a null Lagrangian is not necessarily an affine function of the gradient (cf. [2] and [3]). In this case Theorem 3.2 can be translated to the multi-dimensional setting wihout the requirement of f being convex in the gradient argument (cf. e.g. Theorem 5.1). But of course the difficulty would be to find the appropriate g. We recall that in [29] Sivaloganathan shows that for a special class of non convex functions f one can establish (4.14) for a modified $\tilde{f}$ by finding a suitable null Lagrangian. He then employs this idea together with some machinery from Hamilton-Jacobi theory to establish a local stability result in nonlinear elasticity. We will later see how this follows from Theorem 3.2.

5 Local stability theorems

The positivity of the second variation of I at the stationary point u_0 is a key assumption in the sufficiency Theorems 3.1 and 3.2. In this section we show that under reasonable convexity assumptions on f this can always be "locally" true. We then apply this observation to prove local stability of stationary points. In particular we are able to obtain the result of Sivaloganathan [29] without any need for the construction of local fields and the Hamilton-Jacobi theory (cf. also Zhang [34] and [35]). To start let us recall that a C^2 function $f : \mathbf{R}^{N\times n} \to \mathbf{R}$ is strongly elliptic at $A \in \mathbf{R}^{N\times n}$ if and only if there exists $\alpha > 0$ such that

$$D^2 f(A)(\lambda\otimes\mu, \lambda\otimes\mu) \geq \alpha|\lambda|^2|\mu|^2$$

for all $\lambda \in \mathbf{R}^N$ and $\mu \in \mathbf{R}^n$.

Proposition 5.1. *Let $f \in C^2(\overline{\Omega}\times\mathbf{R}^N\times\mathbf{R}^{N\times n};\mathbf{R})$, $u_0 \in C^1(\overline{\Omega};\mathbf{R}^N)$, $x_0\in\Omega$ and $f(x_0,u_0(x_0),\cdot)$ to be strongly elliptic at $Du_0(x_0)$. Then there exist $\tau,\delta>0$ such that*

$$\delta^2 I(u_0,\varphi) \geq \tau||\varphi||^2_{W^{1,2}(B_\delta;\mathbf{R}^N)}$$

for all $\varphi \in W^{1,2}_0(B_\delta;\mathbf{R}^N)$.

Proof. Let us set $f_{ijkl}(x) := f_{P_{ij}P_{kl}}(x,u_0(x),Du_0(x))$. It follows from the strong ellipticity condition on f and an application of Plancherel's Theorem (cf. [33]) that

$$\int_\Omega f_{ijkl}(x_0)\varphi_{i,j}\varphi_{k,l}\,dx \geq \alpha\int_\Omega |D\varphi|^2\,dx$$

for some $\alpha=\alpha(x_0)>0$ and all $\varphi\in W^{1,2}_0(\Omega;\mathbf{R}^N)$. Moreover a simple continuity argument shows that

$$\begin{aligned}\int_\Omega f_{ijkl}(x)\varphi_{i,j}\varphi_{k,l}\,dx &\geq \int_\Omega f_{ijkl}(x_0)\varphi_{i,j}\varphi_{k,l}\,dx - \int_\Omega |(f_{ijkl}(x)-f_{ijkl}(x_0))\,\varphi_{i,j}\varphi_{k,l}|\,dx\\ &\geq \frac{\alpha}{2}\int_\Omega |D\varphi|^2\,dx,\end{aligned}$$

provided φ vanishes outside a sufficiently small ball around x_0. Hence

$$\begin{aligned}\delta^2 I(u_0,\varphi) &= \int_\Omega (f_{ijkl}(x)\varphi_{i,j}\varphi_{k,l} + 2f_{P_{ij}u_k}(x,u_0,Du_0)\varphi_{i,j}\varphi_k\\ &\quad + f_{u_ku_l}(x,u_0,Du_0)\varphi_k\varphi_l)\,dx\\ &\geq \int_{B_\delta(x_0)} (\frac{\alpha}{2}|D\varphi|^2 - C(\varepsilon|D\varphi|^2+\frac{1}{\varepsilon}|\varphi|^2) - C|\varphi|^2)\,dx\\ &\geq \frac{\alpha}{4}\int_{B_\delta(x_0)} (|D\varphi|^2 - C_1|\varphi|^2)\,dx\end{aligned}$$

where $\varphi\in W^{1,2}(\Omega;\mathbf{R}^N)$ is assumed to vanish outside $B_\delta(x_0)$ for some small enough $\delta>0$. By taking δ smaller if necessary we can make the right hand side in the last inequality strictly positive for nonzero φ. The result is now a consequence of Proposition 4.4. □

Combining the above proposition together with Theorem 3.2 and recalling Remark 4.3 we can now state the following (cf. [29] Theorems 2.4 and 3.2)

Theorem 5.1. *Let $f\in C^2(\overline{\Omega}\times\mathbf{R}^N\times\mathbf{R}^{N\times n};\mathbf{R})$ and $u_0\in C^1(\overline{\Omega};\mathbf{R}^N)$ be a stationary point of I. For given $x_0\in\Omega$ let there be $r,\tau>0$ and $g\in C^2(\overline{B}_r(x_0)\times\mathbf{R}^N\times\mathbf{R}^{N\times n};\mathbf{R})$ such that the integral*

$$\int_{B_r(x_0)} g(x,u_0+\varphi,Du_0+D\varphi)\,dx$$

is well defined and constant for all $\varphi \in W_0^{1,2}(B_r(x_0); \mathbf{R}^N)$ *with* $||\varphi||_{L^\infty(B_r(x_0);\mathbf{R}^N)} < \tau$. *Moreover let* $f+g$ *satisfy* (4.14) *with* Ω *replaced by* $B_r(x_0)$. *Then there exist* $\delta, \rho, \gamma > 0$ *such that for any variation* $\varphi \in W^{1,2}(\Omega; \mathbf{R}^N)$ *vanishing outside* $B_\delta(x_0)$,

$$I(u_0+\varphi) - I(u_0) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

provided $||\varphi||_{L^\infty(\Omega;\mathbf{R}^N)} < \rho$.

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