# Max-Planck-Institut
## für Mathematik
## in den Naturwissenschaften
## Leipzig


# A Marstrand type theorem for measures with cube density in general dimension

(revised version: March 2003)

by


*Andrew Lorent*



Preprint no.: 90 2000

# A MARSTRAND TYPE THEOREM FOR MEASURES WITH CUBE DENSITY IN GENERAL DIMENSION

ANDREW LORENT

Abstract. With a view to generalising rectifiability and density results to more general spaces we prove the following: Let $H^s$ denote Hausdorff $s$ measure in $l_\infty^n$. Let $s \in (0,2]$. Let $S \subset l_\infty^n$ be a subset of positive locally finite Hausdorff $s$-measure with the property

$$\lim_{r \to 0} \frac{H^s\left(B_r(x) \cap S\right)}{\alpha(s)2^{-s}r^s} = 1 \quad \text{for } H^s \text{ a.e. } x \in S$$

then $s$ is an integer and $S$ has a weak tangent at almost every point.

## 1. Introduction

One of the most attractive results in Geometric Measure Theory is the following.

**Theorem 1** (Marstrand). *Let $\mu$ be a Radon measure on $\mathbb{R}^n$ with the property*

$$0 < \lim_{r \to 0} \frac{\mu\left(B_r(x)\right)}{r^s} < \infty \tag{1}$$

*for $\mu$ a.e. $x \in \mathrm{Spt}\mu$. Then*

- *$s$ is an integer,*
- *for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ there exists an $s$-plane $V$ going through $x$ with the following property: For any $\epsilon > 0$,*

$$\liminf_{r \to 0} \frac{\mu\left(B_r(x) \backslash N_{\epsilon r}(V)\right)}{r^s} = 0 \tag{2}$$

*where $N_{\epsilon r}(V) := \{z \in \mathbb{R}^n : \mathrm{dist}(z, V) < \epsilon r\}$.*

We say a measure $\mu$ having property (2) at point $x \in \mathrm{Spt}\mu$ has a *weak $s$-tangent* at $x$.

A central conjecture in classical Geometric Measure Theory was the conjecture that Radon measures in $\mathbb{R}^n$ having positive finite $s$-density (in the sense of (1)) almost everywhere are $s$-rectifiable. Marstrand's Theorem was one of the key results in the history of this conjecture. By greatly developing the methods Marstrand introduced in his proof of Theorem 1, Preiss [12] proved the conjecture in 1986.

The proof of Theorem 1 (and of all subsequent developments) relies essentially on symmetry properties of the Euclidean unit ball. Specifically the existence of an inner product is used fundamentally for even the most basic estimates obtainable by Preiss/Marstrand methods. Earlier Besicovitch type methods for sets in $\mathbb{R}^n$ with Hausdorff $m$-measure density 1 also depend heavily on specific geometrical properties of the Euclidean unit ball, in this case the "rotundity" of the unit ball. These methods are essentially combinatorial and seem to have no application to general Radon measures, in fact they cannot even be applied to Spherical Hausdorff measure. However they do have a certain robustness in the sense that they admit some generalisation outside Euclidean space. In [13], Preiss and Tišer proved rectifiability for sets of finite Hausdorff 1-measure with lower density $> \frac{2+\sqrt{46}}{12}$ (thus giving a slight improvement of the $\frac{3}{4}$ estimate of Besicovitch for 1-sets in Euclidean space) in general metric spaces. By extending the proof of rectifiability of sets with Hausdorff $n$-measure density 1 in general codimension [11], [8], Chlebík [3] proved rectifiability of sets of finite Hausdorff $n$-measure in $l_2$ with density 1.

In contrast, until very recently there has been no progress in generalising Preiss/Marstrand theorems outside Euclidean space.

As is standard we let $l_\infty^n$ denote $\mathbb{R}^n$ with the sup norm. Let $C_r(x) := \{z : \|z - x\|_\infty < r\}$.

The motivation for the study of measures with density properties in $l_\infty^n$ is two fold.

Firstly as any metric space can be isometrically embedded into $l_\infty$ and given the unusual nature of $l_\infty^n$ even as a finite dimensional normed vector space, as a model for making the first steps in generalising rectifiability and density theorems, $l_\infty^n$ has long been considered the natural starting place. In [6] the author proved rectifiability for measures in $l_\infty^3$ satisfying a strong (uniform) 2-density condition.

The second motivation comes from the well known fact that any $2m$ sided centrally symmetric polytope in $\mathbb{R}^n$ can by obtained as a slice by an $n$-plane through a cube in $\mathbb{R}^m$, (see [1] lecture 2). Given the results of [6] the most promising conjecture generalising rectifiability and density results is the conjecture that Marstrand's theorem holds for polytope density, (i.e. the conjecture that Theorem 1 is true for measures with hypothesis (1) with respect to a centrally symmetric polytope). While proving Marstrand's theorem for measures with $s$-density in $l_\infty^n$ for some range of $s>0$ is equivalent to proving Marstrand's theorem for polytope density for the same range of $s>0$ (by the fact that a polytope can be realised as a slice through a cube). Unfortunately, we need to use a stronger form of density (as is given by (4)) and for density condition (4) there is no such equivalence between polytope density and density in $l_\infty^n$.

If $\tilde{\mu} = H^s_{\lfloor A}$ where $A$ is a set of positive finite Hausdorff $s$-measure with density 1, then $\tilde{\mu}$ satisfies condition (4), in fact condition (4) is (roughly speaking) a reformation for Radon measures of the most important additional property the density hypothesis of measure $\tilde{\mu}$ has over density hypothesis (1). Theorems for Radon measures of the form $\tilde{\mu}$ have often proceeded the same theorems proved for general Radon measures with density condition (1), see [2], [8], [10].

Before stating our results we need some background. Following Federer, given a metric space $(M,d)$ we define Hausdorff $s$-measure in $(M,d)$ by Carathéodory's construction taking our initial set function $\zeta$ to be defined by $\zeta(S) = \alpha(s)2^{-s}\left((\mathrm{diam}\,(S))\right)^s$ for any set $S$, where diam is of course taken with respect to the metric $d$ and $\alpha(s) = \frac{\Gamma\left(\frac{1}{2}\right)^s}{\Gamma\left(\frac{s}{2}+1\right)}$. $\Gamma$ denotes the standard Euler function (see [4] 2.10 for the details).

**Theorem 2.** *Let $s\in(0,2]$. Let $H^s$ denote Hausdorff measure in $l_\infty^n$.*

*Let $S\subset l_\infty^n$ be a subset of positive locally finite Hausdorff $s$-measure with the property*

$$\lim_{r\to 0}\frac{H^s\left(C_r(x)\cap S\right)}{\alpha(s)2^{-s}r^s} = 1 \quad \text{for } H^s \text{ a.e. } x\in S. \tag{3}$$

*Then*

- *$s$ is an integer.*
- *For any $\epsilon>0$, for $H^s$ a.e. $x\in S$ there exists an $s$-plane $V$ going through $x$ such that*

$$\liminf_{r\to 0}\frac{H^s\left(C_r(x)\cap S\backslash N_{\epsilon r}(V)\right)}{r^s} = 0.$$

By ([4] 2.10.18(3)) Theorem 2 follows from the following more techinical theorem.

**Theorem 3.** *Let $s\in(0,2]$. Let $\mu$ be a Radon measure on $l_\infty^n$ with the following property;*

$$0 < \lim_{r\to 0}\frac{\mu\left(C_r(x)\right)}{r^s} = \limsup_{r\to 0,\ z\in C_r(x)}\frac{\mu\left(C_r(z)\right)}{r^s} < \infty \tag{4}$$

*for $\mu$ a.e. $x\in\mathrm{Spt}\mu$, then the following is true:*

- *$s$ is an integer.*
- *Given any $\epsilon>0$, for $\mu$ a.e. $x\in\mathrm{Spt}\mu$ there exists an $s$-plane $V$ going through $x$ such that*

$$\liminf_{r\to 0}\frac{\mu\left(C_r(x)\backslash N_{\epsilon r}(V)\right)}{r^s} = 0.$$

We state our theorem for measures having $s$-density conditions for $s\in(0,2]$, this is a technical restriction that *potentially* could be removed, see the remark preceding Lemma 8. As previously mentioned a more fundamental restriction is that we use much stronger density conditions than in Theorem 1. However our methods also yield a relatively simple proof of the following theorem:

**Theorem 4.** *Let $\mu$ be a Radon measure on $l^3_\infty$ with the property that*

$$0 < \lim_{r \to 0} \frac{\mu\left(C_r\left(x\right)\right)}{r^2} < \infty$$

*then for any $\epsilon > 0$, for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ there exists an 2-plane $V$ going through $x$ such that*

$$\liminf_{r \to 0} \frac{\mu\left(C_r\left(x\right) \backslash N_{\epsilon r}\left(V\right)\right)}{r^2} = 0.$$

However Theorem 4 is a weak consequence of Theorem 5 [6].

## 2. Background

2.1. **Elementary notation.** As mentioned in the introduction we will be much concerned with properties of the unit ball in $l^n_\infty$.

Let $e_1, e_2, \dots e_n$ be orthonormal vectors forming the canonical basis of $l^n_\infty$. Define $\|\|$ to be the sup norm, so $\|x\| = \max\{|e_1 \cdot x|, |e_2 \cdot x|, \dots |e_n \cdot x|\}$. We will let $e_{j+n} = -e_j$ for $j \in \{1, 2, \dots n\}$.

Let $B_r\left(x\right)$ denote the open ball of radius $r > 0$ centered on $x$ with respect to the Euclidean norm. For any $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$, Let $A \triangle B := (A \backslash B) \cup (B \backslash A)$. We define a cubic annulus by $A\left(x, a, b\right) := C_b\left(x\right) \backslash \overline{C_a\left(x\right)}$.

Given set $A \subset \mathbb{R}^n$, $\epsilon > 0$ let $N_\epsilon\left(A\right) := \{z \in \mathbb{R}^n : \mathrm{dist}\left(z, A\right) < \epsilon\}$ where $\mathrm{dist}\left(z, A\right) := \inf\{\|z - x\| : x \in A\}$. Given a set of vectors $v_k \in \mathbb{R}^n$, say $k = 1, 2, \dots m$ we will denote by $\langle v_1, v_2, \dots v_m \rangle$ the linear span of these vectors. If $\tau$ is a linear subspace of $\mathbb{R}^n$ we let $P_\tau : \mathbb{R}^n \to \tau$ denote the orthogonal projection onto $\tau$.

In order to get information from Preiss/Marstrand type methods in $l^n_\infty$ it will be necessary to consider "sectors" of the cube where a "sector" is given by the convex hull of the center point and a particular face of the cube, see figure 1. We define this formally as follows.

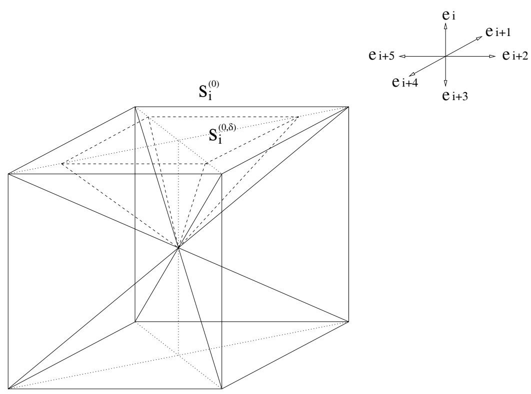


FIGURE 1

Let $T^j_r\left(0\right) := \left\{x \in \overline{C_r\left(0\right)} : x \cdot e_j = r\right\}$. We define

$$S^{(0)}_j := \bigcup_{r \geq 0} T^j_r\left(0\right),$$

so $S^{(0)}_{n+j} := -S^{(0)}_j$ for $j = 1, \dots n$. And $S^{(x)}_j := S^{(0)}_j + x$ for any $x \in \mathbb{R}^n$. Let

$$\hat{S}^{(0,\delta)}_j := \bigcup_{r \geq 0} T^j_r \cap C_{(1-\delta)r}\left(re_j\right).$$

And $\hat{S}^{(x,\delta)}_i := \hat{S}^{(0,\delta)}_i + x$ for any $x \in \mathbb{R}^n$.

Let $G\left(m, n\right)$ be the Grassmannian manifold of all m-dimensional subspaces of $\mathbb{R}^n$. Let

$$d\left(A, B\right) := \inf\{\|x_1 - x_2\| : x_1 \in A, x_2 \in B\}.$$

Define also

$$d_{\mathrm{euc}}\left(A, B\right) := \inf\{|x_1 - x_2| : x_1 \in A, x_2 \in B\}.$$

A $\beta$-Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$ is a map $f$ with the property

$$|f(x) - f(y)| \leq \beta \|x - y\|,$$

(note the use of the sup norm).

Given a Radon measure $\mu$ on $l^n_\infty$, a point $x \in \mathrm{Spt}\mu$ will be said to have *square cone density* at $x$ if and only if for some $\delta > 0$ the following holds;

$$\liminf_{r \to 0} \frac{\mu\left(\left(\hat{S}_i^{(x,\delta)} \cap \hat{S}_{i+n}^{(x,\delta)}\right) \cap C_r(x)\right)}{r^s} > 0 \quad \forall\, i \in \{1, 2, \ldots n\} \tag{5}$$

We say a Radon measure $\mu$ on $l^n_\infty$ is an *$s$-uniform measure* if and only if

$$\mu(C_r(x)) = r^s \ \ \forall\, x \in \mathrm{Spt}\mu, r > 0. \tag{6}$$

We say and $s$-uniform measure on $l^n_\infty$ satisfies the *controlled complement* condition if and only if

$$\mu(C_r(x)) \leq r^s \ \ \forall\, x \in \mathbb{R}^n, r > 0, j \in \{1, 2, \ldots n\}. \tag{7}$$

Finally we say a Radon measure $\mu$ on $l^n_\infty$ is a *symmetric measure* if and only if

$$\mu\left(S_j^{(x)} \cap C_r(x)\right) = \mu\left(S_{j+n}^{(x)} \cap C_r(x)\right) \ \ \forall\, x \in \mathrm{Spt}\mu, r > 0. \tag{8}$$

A set $S \subset \mathbb{R}^n$ of locally finite $H^m$ measure is *$m$-rectifiable* if and only if there exist countably many $C^1$ submanifolds $G_k$ of dimension $m$ such that

$$H^m\left(S \backslash \left(\bigcup_{k \in \mathbb{N}} G_k\right)\right) = 0.$$

A set $S \subset \mathbb{R}^n$ of locally finite $H^m$ measure is *purely $m$-unrectifiable* if and only if for every $C^1$ submanifold $G$ of dimension $m$ we have

$$H^m(S \cap G) = 0.$$

A measure $\mu$ is said to be an A.D. (Alfors David) $s$-measure if and only if there exist numbers $\delta_0 > 0$, $\beta > \alpha > 0$ such that

$$\alpha r^s < \mu(B_r(x)) < \beta r^s \ \text{ for all } x \in \mathrm{Spt}\mu,\ r \in (0, \delta_0).$$

A set $S$ is said to be an A.D. $s$-set if and only if $\mu := H^s_{\lfloor S}$ is an A.D. $s$-measure.

2.2. **Tangent Measures.** We will make extensive use of some elementary results about tangent measures. In order to simplify we will define tangent measures only for measures with positive finite $s$-density, the lemmas we will state hold true in much more generality, see [11].

Given $a \in \mathbb{R}^n$ and $r > 0$ define $T_{a,r}(x) = \frac{(x-a)}{r}$. Note that $T_{a,r_n\sharp}\mu(A) = \mu(r_n A + a)$, $A \subset \mathbb{R}^n$. Suppose $\mu$ is a Radon measure on $\mathbb{R}^n$ with positive finite $s$-density a.e. we say $\nu$ is a *tangent measure* of $\mu$ at a point $a \in \mathbb{R}^n$ if there exists a sequence $(r_n)$ of positive numbers, such that $r_n \to 0$ and $\frac{T_{a,r_n\sharp}\mu}{r_n^s} \rightharpoonup \nu$ as $n \to \infty$.

We will denote by $\mathrm{Tan}(\mu, x)$ the set of tangent measures to $\mu$ at $x$ and we will denote by $\widetilde{\mathrm{Tan}}(\mu, x)$, the set of supports of tangent measures to $\mu$ at $x$.

Note that by definition of weak convergence, if measure $\mu$ at point $x$ is such that

$$\liminf_{r \to 0} \frac{\mu(B_r(x) \backslash N_{\epsilon r}(V))}{r^s} = 0$$

for every $\epsilon > 0$, then for some subsequence $r_n \to 0$ we have that $\nu := \lim_{n \to \infty} \frac{T_{x,r_n\sharp}\mu}{r_n^s}$ is such that $\mathrm{Spt}\nu \subset V$ for some $V \in G(s, n)$. If $\nu$ is in addition $s$-uniform it is an elementary exercise of differentiation of measures to see that $V = \lambda H^s_{\lfloor V}$ for some $\lambda > 0$. See Step 1 and Step 2 of the proof of Lemma 18 [6].

**Lemma 1.** *Suppose $\mu$ measures $l_\infty^n$ with the property that for $\mu$ a.e. $x \in \mathrm{Spt}\mu$*

$$0 < \lim_{r\to 0} \frac{\mu(C_r(x))}{r^s} = \limsup_{r\to 0,\ z\in C_r(x)} \frac{\mu(C_r(z))}{r^s} < \infty.$$

*Then for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ every $\nu \in \mathrm{Tan}(\mu, x)$ is a measure with the following two properties:*

- *For some $\alpha > 0$ we have $\nu = \alpha\tilde{\nu}$ where $\tilde{\nu}$ is an $s$-uniform measure with the controlled complement condition, i.e. $\tilde{\nu}$ satisfies (6) and (7).*
- *$0 \in \mathrm{Spt}\nu$.*

This is a version of Corollary 14.7 [11]. The proof of Corollary 14.7 is carried out for Euclidean balls but the same proof applies here.

So note, using tangent measure notation, a measure $\mu$ with positive finite density having a weak $s$-tangent $V$ at $x$ is equivalent to $\widetilde{\mathrm{Tan}}(\mu, x) \cap G(s, n) \neq \emptyset$.

**Lemma 2.** *Suppose $\mu$ measures $l_\infty^n$, then for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ every $\nu \in \mathrm{Tan}(\mu, x)$ has the following two properties:*

$$(1) \qquad T_{z,1\sharp}\nu \in \mathrm{Tan}(\mu, x) \quad \text{for all } z \in \mathrm{Spt}\nu.$$

$$(2) \qquad \mathrm{Tan}(\nu, z) \subset \mathrm{Tan}(\mu, x) \quad \text{for all } z \in \mathrm{Spt}\nu.$$

This is a slightly weaker form of Theorem 14.16 [11], again the proof in [11] is for Euclidean space, but it applies in $l_\infty^n$ without change.

## 3. Reduction of Problem

Let $\mu$ be a measure on $\mathbb{R}^n$ with the property

$$0 < \lim_{r\to 0} \frac{\mu(C_r(x))}{r^s} < \infty$$

for $\mu$ a.e. $x \in \mathrm{Spt}\mu$.

Now suppose our Theorem is false, so in the case where $s$ is an integer, for some subset $B \subset \mathrm{Spt}\mu$ of positive $\mu$ measure we have that $\mathrm{Spt}\mu$ has no weak $s$-tangents at any point $x \in B$. Now let $y_0$ be one of the $\mu$ almost all points in $B$ such that Lemmas 1 and 2 hold true. Take any $\nu_0 \in \mathrm{Tan}(\mu, y_0)$, by Lemma 2 for any $z \in \mathrm{Spt}\nu_0$ we have

$$\mathrm{Tan}(\nu_0, z) \subset \mathrm{Tan}(\mu, y_0).$$

Suppose $\mathrm{Spt}\nu_0$ has a weak $s$-tangent at point $z \in \mathrm{Spt}\nu_0$, i.e. for some $V \in G(s, n)$ we have $V \in \widetilde{\mathrm{Tan}}(\nu_0, z)$, then $V \in \widetilde{\mathrm{Tan}}(\mu, y_0)$ and so $\mathrm{Spt}\mu$ has a weak $s$-tangent at $y_0$, contradiction. Thus $\mathrm{Spt}\nu_0$ has no weak $s$-tangents at any point of its support and so the measure is purely $s$-unrectifiable.

Now by Lemma 1, $\nu_0$ is (after a multiplication by a positive constant) an $s$-uniform measure with the controlled complement condition. So all we need to do is to prove our theorem is to show that any Radon satisfying (6) and (7) can only exist when $s$ is an integer and that this measure has to have a weak $s$-tangent somewhere in its support, this will follow essentially from repeated applications of Proposition 1.

## 4. Sketch of the Proof

4.1. **Dimension reduction.** One of the starting points for the results of this paper is the observation that an $s$-uniform measure $\nu$ on $l_\infty^n$ whose support is contained in the graph of a 1-Lipschitz function (defined from subspace $e_i^\perp$ where $e_i$ is an element of the orthonormal basis of $l_\infty^n$) can be pushed forwards (via the projection onto $e_i^\perp$) to form an $s$-uniform measure on $l_\infty^{n-1}$. As such if we are able to prove that $\mathrm{Spt}\nu$ is contained in the graph of a 1-Lipschitz function, we can obtain an $s$-uniform measure $\nu_1$ with the same properties in a space with one less dimension.

As is implied in the last paragraph, we will be arguing on a (finite) sequence of measures, each one defined from the last and each having sucessively better properties. Pushing forward measures onto subspaces will be one of the methods we will use to obtain new measures, the other will be via the taking of tangent measures.

By virtue of Lemma 2 the measure we eventually show is supported on an $s$-plane can be found as the $n$-th tangent measure of a sequence tangent measures of tangent measures (i.e. $\nu_n \in \mathrm{Tan}(\nu_{n-1}, x_{n-1})$, $\nu_{n-1} \in \mathrm{Tan}(\nu_{n-2}, x_{n-2})$,... $\nu_1 \in \mathrm{Tan}(\mu, x)$). Informally speaking; Lemma 2 alows us to "blow up" the measure we are dealing with at any time to obtain a new one with stronger properties. This turns out to be extremely powerfull. However the measure we eventually obtain that is supported on an $s$-plane will be found through a combination of taking tangent measures and pushing forward meaures onto subspaces. So in our proof of existance of weak tangents for measure $\mu$ we can not simply apply Lemma 2. We need a lemma to the effect that if a "pushed forward" measure $\nu$ has a weak $s$-tangent, then the measure it has been pushed forward *from* also has an $s$-tangent. This the contents of Lemma 10.

So the main difficulty is to show that we can find measure $\nu$ (obtained from a sequence of taking tangent measures and pushing measures onto subspaces) supported on a 1-Lipschitz graph. Note that this is equivilent to the following statment.

$$\left(S_i^{(x)} \cup S_{i+n}^{(x)}\right) \cap \mathrm{Spt}\nu = \emptyset \quad \forall x \in \mathrm{Spt}\nu. \tag{9}$$

There are four main ingredients from which we will achieve a proof of the existence of measure $\nu$ satisfying (9). Firstly we will show that tangent measures to our original measure $\mu$ has a property we call *measure symmetry* .

4.2. **Measure symmetry.** Formally *measure symmetry* is defined as follows: For $\nu$ a.e. $x \in \mathrm{Spt}\nu$, for any $i \in \{1, 2, \dots n\}$, $r > 0$ we have

$$\nu\left(S_i^{(x)} \cap C_r(x)\right) = \nu\left(S_{i+n}^{(x)} \cap C_r(x)\right).$$

Recall a measure having this property is known as a *symmetric measure*, see (8). Originally this property was proved for unrectifiable 2-uniform measures in $l_\infty^3$ in [6] by using certain integral estimates originating from [9]. On our context, measure symmetry "comes for free" as a consequence of our stronger density condition (4), a proof comes from simply differentiating the function $x \longrightarrow \int_{C_r(x)} r - \|z - x\| d\nu z$ and noting that this function achieves its maximum on points of $\mathrm{Spt}\nu$. In Euclidean rectifiability and densities, condition (4) also implies strong symmetry properties of the support of the measure. In particular it can be (reasonably easily) shown that that the tangent measures to a measure in Euclidean space satisfying density condition (4) are supported on subspaces, and consequently these measures are rectifiable.

4.3. **Monotonicity.** Measure symmetry in $l_\infty^n$ turns out to be extremely powerful. As a consequence of measure symmetry we can prove a property we call *monotonicity* . Monotonicity is our second ingredient. First some background. From standard arguments via the theory of differentiation of measures, for any $x \in \mathrm{Spt}\nu$ for $L^1$ a.e. $r > 0$ we can induce a "slicing measure" $\nu_r$ on $\partial C_r(x)$ such that for any $A \subset l_\infty^n$ we have

$$\nu(A) = \int_{r>0} \nu_r\left(\partial C_r(x) \cap A\right) dL^1 r.$$

For any $i \in \{1, 2, \dots n\}$ we define $f_i^{(x)}(r) = \nu_r\left(\partial C_r(x) \cap S_i^{(x)}\right)$. By *monotonicity* we mean that that $\nu$ has the property that $f_i^{(x)}$ is a monotonic non-decreasing function for any $i \in \{1, 2, \dots n\}$ and any $x \in \mathrm{Spt}\nu$.

The proof of this property is essentially an elementary trick. Measure symmetry in the cube $C_r(x)$ implies measure symmetry on the boundary $\partial C_r(x)$ and by the fact that (which we get for free by arguing by contradiction and assuming that none of the tangent measures of $\nu$ are supported on, or near 1-Lipschitz graphs) $\mathrm{Spt}\nu$ must approach $x$ from inside many different sectors, we can use measure symmetry on boundaries to "push" the measure of $\partial C_r(x) \cap S_i^{(x)}$ slightly upward inside $S_i^{(x)} \backslash C_r(x)$ (see figure 3 for a suggestion of the argument) and hence in this way we get a "weak form" of monotonicity. By pursuing this argument in a more careful way it is possible to show $f_i^{(x)}$ is monotonic. This result first appeared (with slightly weaker hypotheses, for measures in $l_\infty^3$) in [6], Lemma 9. The proof we give here is similar but simpler.

4.4. **Touching point arguments.** The third ingredient in the proof of (9) comes from a refinement of a technique originating from Besicovitch usually called "touching point arguments". This refinement uses the fact that the support of the measure must have projection zero onto almost all subspaces of one dimension lower than the ambient space (since, arguing by contradiction, $\mathrm{Spt}\nu$ is unrectifiable or of dimension less than the ambient space minus one) to show that for $\nu$ almost all $x \in \mathrm{Spt}\nu$, for some $i \in \{1, 2, \ldots 2n\}$ we have

$$\liminf_{r \to 0} \frac{\nu\left(S_i^{(x)} \cap C_r(x)\right)}{r^s} = 0.$$

This result first appears in [6] Lemma 14 in $l_\infty^3$ and see [7] Lemma 1 for the result with $(n-1)$ sets in $\mathbb{R}^n$. Unfortunately in this paper we need the result for $s$-sets in $\mathbb{R}^n$ so we need to repeat (with minor adaptions) many arguments from [6], [7].

In combination these three ingredients yield a tangent measure $\nu$ with measure symmetry, monotonicity and the property that

$$\left(S_i^{(0)} \cup S_{i+n}^{(0)}\right) \cap \mathrm{Spt}\nu = \emptyset. \tag{10}$$

Now note that if, (9) was not true then we can find a point $x \in \mathrm{Spt}\nu$ such that (by monotonicity)

$$f_i^{(x)}(r) > 0 \text{ for all } r > 0. \tag{11}$$

4.5. **Bounds on $f_i^{(x)}$.** Our fourth ingredient is to show that for an $s$-uniform measure $\nu$ (with measure symmetry, monotonicity) that satisfies (10) for which $\mathrm{Spt}\nu$ is not contained in a Lipschitz graph we can find $x \in \mathrm{Spt}\nu$ such that $f_i^{(x)}$ is strictly positive but is *bounded* . This can be shown in any number of ways (see for example Lemma 19 [6]), the essential point being that $\mathrm{Spt}\nu \cap S_i^{(x)} \subset S_i^{(x)} \backslash S_i^{(0)}$ and the measure of $\left(S_i^{(x)} \backslash S_i^{(0)}\right) \cap C_r(0)$ can be shown to be of order $r\|x\|$. The approach we take in this paper is "combinatorial" and relies heavily on measure symmetry and the possibility of taking repeated tangent measures. Once $f_i^{(x)}$ is bounded, by using measure symmetry it is not hard to see that for any $y \in S_i^{(x)} \cap \mathrm{Spt}\nu$, $f_i^{(y)}$ must also be bounded and $\sup_{r>0} f_i^{(x)} = \sup_{r>0} f_i^{(y)}$. This has the consequence that for sufficiently large $R > 0$, $\mathrm{Spt}\nu \cap A(x, R, \infty) \cap S_i^{(x)} = \emptyset$. From this point on it is not hard to gain a contradiction, see figure 7 for a suggestion of the argument.

So (11) is not true and hence for every $x \in \mathrm{Spt}\nu$ we have

$$\left(S_i^{(x)} \cup S_{i+n}^{(x)}\right) \cap \mathrm{Spt}\nu = \emptyset$$

thus $\mathrm{Spt}\nu$ is contained in the graph of a 1-Lipschitz function. This is how the proof works.

## 5. Proof of Theorem

**Proof** [of Theorem 3]

So as already argued, if our Theorem is false we have the existence of an $s$-uniform measure in $l_\infty^n$ satisfying the controlled complement condition, which in the case where $s$ is an integer has no weak $s$-tangent at any point of its support.

First we begin with some preliminary lemmas.

## 6. Preliminary Lemmas

**Lemma 3.** *Let integer $n \geq 1$. If $\nu$ is an $s$-uniform measure on $l_\infty^n$ with the property that $\nu\left(\partial S_i^{(x)}\right) = 0$ for any $x \in \mathrm{Spt}\nu$, $i \in \{1, 2, \ldots 2n\}$ and the property that $\nu(C_r(z)) \leq r^s$ for all $z \in l_\infty^n$. Then $\nu$ is a symmetric measure, i.e. for any $x \in \mathrm{Spt}\nu$ the following is true;*

$$\nu\left(S_j^{(x)} \cap C_r(x)\right) = \nu\left(S_{j+n}^{(x)} \cap C_r(x)\right)$$

*for every $j \in \{1, 2, \ldots n\}$, $0 < r < \infty$.*

**Proof** Let $r>0$, $y\in \mathrm{Spt}\nu$.
*Step 1:* We will show function $h_r:\mathbb{R}^n\to\mathbb{R}$ given by

$$h_r(y)=\int_{C_r(y)} r-\|y-z\|d\nu z$$

is a smooth function and its derivatives are

$$\frac{\partial h_r}{\partial x_i}(y)=\nu\left(S_i^{(y)}\cap C_r(y)\right)-\nu\left(S_{i+n}^{(y)}\cap C_r(y)\right) \tag{12}$$

for each $i\in\{1,2,\dots n\}$.

Take some $y\in\mathrm{Spt}\nu$ and some $i\in\{1,2,\dots 2n\}$. To start with let

$$q^{(y)}(z):=\begin{cases} r-\|z-y\| & z\in C_r(y)\\ 0 & z\notin C_r(y)\end{cases}$$

Since

$$h_r(y+he_i)-h_r(y)=\int_{\mathbb{R}^n} q^{(y+he_i)}(z)-q^{(y)}(z)\,d\nu z. \tag{13}$$

We will start by evaluating $q^{(y+he_i)}(z)-q^{(y)}(z)$ in various regions of $C_r(y+he_i)\cap C_r(y)$.

To start, if $z\in S_i^{(y+he_i)}\cap C_r(y)$ we have

$$\begin{aligned} q^{(y+he_i)}(z)-q^{(y)}(z) &= -\|z-(y+he_i)\|+\|z-y\| \\ &= h \end{aligned} \tag{14}$$

and similarly if $z\in S_{i+n}^{(y)}\cap C_r(y+he_i)$ we have

$$\begin{aligned} q^{(y+he_i)}(z)-q^{(y)}(z) &= -\|z-(y+he_i)\|+\|z-y\| \\ &= -h. \end{aligned} \tag{15}$$

For any $j\in\{1,2,\dots 2n\}\setminus\{i,i+n\}$ we have that for $S_j^{(y)}\cap S_j^{(y+he_i)}$ we have

$$q^{(y+he_i)}(z)-q^{(y)}(z)=0. \tag{16}$$

We know that for all $z\in C_r(y)\cap C_r(y+he_i)$

$$\left|q^{(y+he_i)}(z)-q^{(y)}(z)\right|\le h.$$

And for all $z\in C_r(y)\triangle C_r(y+he_i)$

$$\left|q^{(y+he_i)}(z)\right|+\left|q^{(y)}(z)\right|\le 2h.$$

Since $\nu\left(\bigcup_{i\in\{1,2\dots 2n\}}\partial S_i^{(y)}\right)=0$ so we know for each $j\in\{1,2,\dots 2n\}$

$$h^{-1}\left(\int_{\left(C_r(y)\cap\left(S_j^{(y)}\triangle S_j^{(y+he_i)}\right)\right)}\left|q^{(y+he_i)}(z)-q^{(y)}(z)\right|d\nu z\right)\to 0 \text{ as } h\to 0. \tag{17}$$

Similarly the terms on the boundary have no influence,

$$h^{-1}\left(\int_{C_r(y)\triangle C_r(y+he_i)}\left|q^{(y+he_i)}(z)\right|+\left|q^{(y)}(z)\right|d\nu z\right)\to 0 \text{ as } h\to 0. \tag{18}$$

So putting together (14), (15), (16), (17), (18) we get

$$\lim_{h\to 0} h^{-1}\left(\int_{\mathbb{R}^n} q^{(y+he_i)}(z)-q^{(y)}(z)\,d\nu z\right)=\nu\left(S_i^{(y)}\cap C_r(y)\right)-\nu\left(S_{i+n}^{(y)}\cap C_r(y)\right). \tag{19}$$

And putting (13) together with (19) we establish (12) and so we have completed Step 1.

*Step 2:* We show $h$ has zero derivative on the points of $\mathrm{Spt}\nu$.

Now we note by the Fubini type theorem given by 1.15 [11], for any $y \in \mathbb{R}^n$ we have

$$\begin{aligned} h(y) &= \int_{C_r(y)} r - \|y-z\| d\nu z \\ &= \int_0^r \nu\left(z \in C_r(y) : r - \|y-z\| \geq t\right) dL^1 t \\ &= \int_0^r \nu\left(C_{r-t}(y)\right) dL^1 t \\ &\leq \int_0^r (r-t)^s \, dL^1 t \\ &= \frac{r^{s+1}}{s+1}. \end{aligned}$$

With equality when $y \in \mathrm{Spt}\nu$. Hence $\sup\{h_r(z) : z \in \mathbb{R}^n\} = \frac{r^{s+1}}{s+1}$ and $h_r$ achieves this supremum on the points of $\mathrm{Spt}\nu$. So for all points $y \in \mathrm{Spt}\nu$ we have $\frac{\partial h_r}{\partial x_i}(y) = 0$ for each $i \in \{1,2,\dots n\}$ and by (12), this completes the proof. □

Remark

Lemma 3 states that any $s$-uniform measures satisfying the controlled complement condition and for which $\nu\left(\partial S_i^{(x)}\right) = 0$ for all $x \in \mathrm{Spt}\nu$ and for all $i \in \{1,2,\dots 2n\}$, is a symmetric measure (recall definitions (6), (7), (8)).

Remark

As in [11], p.139 given our $s$-uniform measure $\nu$ on $l_\infty^n$ we can induce a measure (sometimes called slicing measure) $\nu_r^{(z)}$ on $\partial C_r(z)$ for $L^1$ a.e. $r > 0$ for any $z \in \mathrm{Spt}\nu$, such that

$$\int_a^b \int_{\partial C_r(z)} \phi(x) \ \ d\nu_r^{(z)} x \, dLr = \int_{A(z,a,b)} \phi(x) \ d\nu x$$

for all $\phi \in C_0(\mathbb{R}^n)$, $0 < a < b < \infty$. When the situation is unambiguous we will drop the superscript and so the measure will be denoted by $\nu_r$. Note that if $x \in S_i^{(y)}$ with $y, x \in \mathrm{Spt}\nu$ then we have

$$\nu_r^{(y)}(A) = \nu_{r-\|x-y\|}^{(x)}(A)$$

for any $A \subset \partial C_r(y) \cap S_i^{(x)}$. Both measures $\nu_r^{(y)}$ and $\nu_{r-\|x-y\|}^{(x)}$ are derived via differentiation of measures and hence are defined from $\nu$ "locally" so this is to be expected.

**Lemma 4.** *Let integer $n \geq 1$. Let $\nu$ be a symmetric $s$-uniform measure on $l_\infty^n$ with the property that $\nu\left(\partial S_i^{(x)}\right) = 0$ for any $x \in \mathrm{Spt}\nu$ and any $i \in \{1,2,\dots 2n\}$ then for any $z \in \mathrm{Spt}\nu$ the following is true*

$$\nu_r\left(S_j^{(z)} \cap \partial C_r(z)\right) = \nu_r\left(S_{j+n}^{(z)} \cap \partial C_r(z)\right)$$

*for any $j \in \{1,2\dots n\}$, for $L^1$ a.e. $r > 0$.*

**Proof** Where $\nu_r$ is defined we have for any $i \in \{1,2,\dots n\}$

$$\begin{aligned} \nu_r\left(S_j^{(z)} \cap \partial C_r(z)\right) &= \lim_{h\to 0} \frac{\nu\left(A(z, r-h, r+h) \cap S_j^{(z)}\right)}{2h} \\ &= \lim_{h\to 0} \frac{\nu\left(A(z, r-h, r+h) \cap S_{j+n}^{(z)}\right)}{2h} \\ &= \nu_r\left(S_{j+n}^{(z)} \cap \partial C_r(z)\right). \quad \square \end{aligned}$$

**Lemma 5.** *Given $n \geq 2$ and $s \in (0,n)$ if $\mu$ is a symmetric $s$-uniform measure on $l_\infty^n$ and has the following two properties;*

- $\mu\left(\partial S_i^{(x)}\right) = 0$ *for every $i \in \{1,2,\dots 2n\}$, $x \in \mathrm{Spt}\mu$*

- *$\mu$ has square cone density (i.e. satisfies (5)) at $\mu$ almost every point*

*then firstly $s \geq 1$ and secondly for $\mu$ a.e. $y \in \mathrm{Spt}\mu$ and any $i \in \{1, 2, \ldots n\}$ the function*

$$f_i^{(y)}(s) = \mu_s\left(\partial C_s(y) \cap S_i^{(y)}\right)$$

*is monotonic non decreasing function that is locally Lipschitz away from 0.*

**Proof**

We break the proof into 3 steps:

Firstly let $G$ denote the set of points of $\mathrm{Spt}\mu$ with square cone density.

*Step 1:* Given $y \in G$, $i \in \{1, 2, \ldots 2n\}$ we will show that for any $s > 0$

$$\partial C_s(y) \cap S_i^{(y)} \cap \mathrm{Spt}\mu \neq \emptyset.$$

Suppose not, then we can find interval $(a, b) \subset \mathbb{R}$ which contains $s$ and has the following properties,

- $\mu\left(A(y, a, b) \cap S_i^{(y)}\right) = 0.$
- For every $\epsilon > 0$ we can find $s \in (a - \epsilon, a)$ such that

$$\mu_s^{(y)}\left(\partial C_s(y) \cap S_i^{(y)}\right) > 0.$$

Note by measure symmetry, we have that the interval $(a, b)$ has the same properties for $S_{i+n}^{(y)}$.

Let $\epsilon > 0$, $\delta > 0$. Now since $y \in G$ we must be able to find $z \in \left(\hat{S}_{i+n}^{(y,\delta)} \cup \hat{S}_i^{(y,\delta)}\right) \cap C_{\frac{b-a}{100}}(y) \cap G$, by the above remark we lose no generality in assuming $z \in S_{i+n}^{(y)}$ because if we had $z \in S_i^{(y)}$ we could use the properties of $(a, b)$ with respect to $S_{i+n}^{(y)}$ and argue in exactly the same way to get, by measure symmetry, exactly the same conclusion. So pick $s \in \left(a - \frac{\|y-z\|}{200}, a\right)$ such that

$$\mu_s^{(y)}\left(\partial C_s(y) \cap S_i^{(y)}\right) > 0.$$

Figure 2 gives an impression of how the argument works.

So since $\partial C_s(y) \cap S_i^{(y)} \subset \partial C_{s+\|z-y\|}(z) \cap S_i^{(z)}$ we have that by measure symmetry

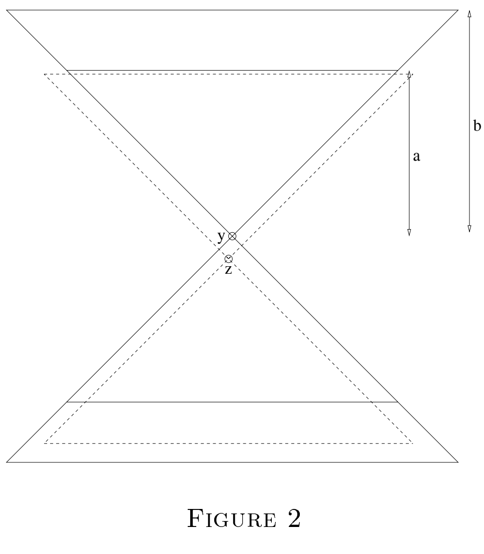


FIGURE 2

$$\mu_{s+\|z-y\|}^{(z)}\left(\partial C_{s+\|z-y\|}(z) \cap S_{i+n}^{(z)}\right) > 0.$$

However $\partial C_{s+\|z-y\|}(z) \cap S_{i+n}^{(z)} \subset A(y, a, b) \cap S_{i+n}^{(y)}$ and by our construction

$$d\left(\partial C_{s+\|z-y\|}(z) \cap S_{i+n}^{(z)}, \partial\left(A(y, a, b) \cap S_{i+n}^{(y)}\right)\right) > 0.$$

So we can find a point $z_0 \in \operatorname{int}\left(A(y,a,b) \cap S_{i+n}^{(y)}\right) \cap \operatorname{Spt}\mu$ and so for some small $\delta > 0$ we have $C_\delta(z_0) \subset A(y,a,b) \cap S_{i+n}^{(y)}$ thus $\mu\left(A(y,a,b) \cap S_{i+n}^{(y)}\right) > 0$, and by measure symmetry this is a contradiction.

*Step 2:* $f_i^{(y)}$ is monotonic non-decreasing.
This follows from the same kind of reflection trick as Step 1. Figure 3 illustrates how the argument works. Given $r_1 < r_2$ we can pick $z \in S_{i+n}^{(y)} \cap \partial C_{\frac{r_2 - r_1}{2}}(y) \cap \operatorname{Spt}\mu$. Since

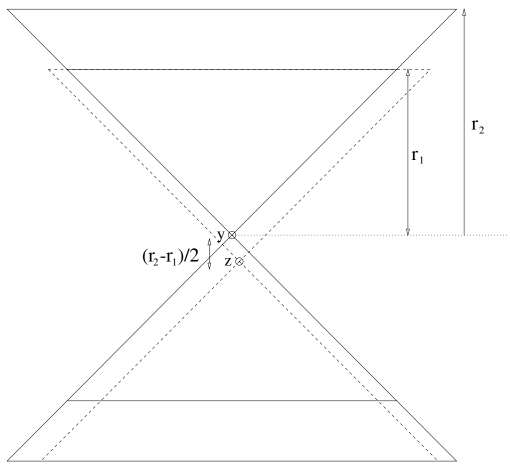


FIGURE 3

$$r_1 + \|y - z\| = r_1 + \frac{r_2 - r_1}{2} = \frac{r_1 + r_2}{2}$$

we have

$$\partial C_{r_1}(y) \cap S_i^{(y)} \subset \partial C_{\frac{r_1+r_2}{2}}(z) \cap S_i^{(z)} \tag{20}$$

And as

$$\partial C_{\frac{r_1+r_2}{2}}(z) \cap S_{i+n}^{(z)} \subset \partial C_{r_2}(y) \cap S_{i+n}^{(y)}. \tag{21}$$

So we have monotonicity because by (20), (21) and measure symmetry we have

$$\begin{aligned}
\mu_{r_1}^{(y)}\left(\partial C_{r_1}(y) \cap S_i^{(y)}\right) &\leq \mu_{\frac{r_1+r_2}{2}}^{(z)}\left(\partial C_{\frac{r_1+r_2}{2}}(z) \cap S_i^{(z)}\right) \\
&= \mu_{\frac{r_1+r_2}{2}}^{(z)}\left(\partial C_{\frac{r_1+r_2}{2}}(z) \cap S_{i+n}^{(z)}\right) \\
&\leq \mu_{r_2}^{(y)}\left(\partial C_{r_2}(y) \cap S_{i+n}^{(y)}\right) \\
&= \mu_{r_2}^{(y)}\left(\partial C_{r_2}(y) \cap S_i^{(y)}\right).
\end{aligned}$$

Now since $y \in G$ it is clear that $\mu\left(S_i^{(y)} \cap C_\alpha(y)\right) > 0$ for any $\alpha > 0$ and so we know that $f_i^{(y)}(r) = \mu_r\left(\partial C_r(y) \cap S_i^{(y)}\right) > 0$ for all $r > 0$.

*Step 3:* $s \geq 1$.
Suppose not, so $0 < s < 1$, then since $\mu_r(\partial C_r(y)) = s r^{s-1}$ the measure on the boundary tends to infinity as $r \to 0$. So for some small $0 < r_0 < \infty$ and some $i \in \{1, 2, \ldots 2n\}$ we must have

$$f_i^{(y)}(r_0) = \mu_{r_0}\left(\partial C_{r_0}(y) \cap S_i^{(y)}\right) > 2s$$

but as $f_i^{(y)}(1) \leq s$ so this contradicts monotonicity.

*Step 4:* Lipschitzness.
We know that for a.e. $r > 0$

$$\sum_{k=1}^{n} f_k^{(y)}(r) = \frac{sr^{s-1}}{2}.$$

Let $t(r) := \frac{s}{2} r^{s-1}$, so $t$ is locally Lipschitz away from 0. So given $r_1 < r_2$

$$\begin{aligned} f_i^{(y)}(r_2) - f_i^{(y)}(r_1) &= \frac{s}{2}\left(r_2^{s-1} - r_1^{s-1}\right) - \left(\sum_{k\in\{1,2,\ldots n\}\setminus\{i\}} f_k^{(y)}(r_2)\right) + \sum_{k\in\{1,2,\ldots n\}\setminus\{i\}} f_k^{(y)}(r_1) \\ &\leq \frac{s}{2}\left(r_2^{s-1} - r_1^{s-1}\right) \\ &= t(r_2) - t(r_1) \qquad (22) \end{aligned}$$

by monotonicity. $\square$

## 7. Main Proposition

**Proposition 1.** *Given integer $n \geq 2$ and real number $s \in (0,2]$, if $\mu$ is a symmetric $s$-uniform measure on $l_\infty^n$ where either*

- *$s$ is an integer and* $\mathrm{Spt}\mu$ *is supported on a purely unrectifiable $s$-set*
- *$s$ is not an integer*

*then for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ we can find a tangent measure $\nu \in \mathrm{Tan}(\mu, x)$ such that $\mathrm{Spt}\nu \cap C_1(0)$ is contained in the graph of a 1-Lipschitz function from $e_k^\perp$ to $\langle e_k \rangle$ for some $k \in \{1,2,\ldots n\}$.*

**Proof** Suppose not, so we have some subset $B \subset \mathrm{Spt}\mu$ of positive $\mu$ measure for which Proposition 1 is false.
Let $\Gamma_k$ denote the set of graphs of 1-Lipschitz functions from $e_k^\perp$ to $\langle e_k\rangle$ for $k \in \{1,2,\ldots n\}$. Let $\rho_1(A,B) := \sup\{d(z,B) : z \in A\}$, so this is half the Hausdorff metric.

*Step 1:*
We will show that for $\mu$ a.e. $x \in B$ and for every $k \in \{1,2,\ldots n\}$

$$\inf\{\rho_1(\mathrm{Spt}\nu \cap C_1(0), G \cap C_1(0)) : \nu \in \mathrm{Tan}(\mu,x), G \in \Gamma_k\} > 0. \qquad (23)$$

Suppose not, then we must be able to find $x \in B$ such that Lemma 1 and Lemma 2 hold true and for some $k \in \{1,2,\ldots n\}$

$$\inf\{\rho_1(\mathrm{Spt}\nu \cap C_1(0), G \cap C_1(0)) : \nu \in \mathrm{Tan}(\mu,x), G \in \Gamma_k\} = 0.$$

So for every $m \in \mathbb{N}$ we can find $\nu_m \in \mathrm{Tan}(\mu,x)$ and $G_m \in \Gamma_k$ such that

$$\rho_1(\mathrm{Spt}\nu_m \cap C_1(0), G_m \cap C_1(0)) < 2^{-m},$$

thus $\mathrm{Spt}\nu_m \cap C_1(0) \subset N_{2^{-m}}(G_m \cap C_1(0))$. Now by Ascoli-Arzela Theorem we can find a subsequence such that $G_{k_m} \cap C_1(0)$ converges in Hausdorff metric to $G \cap C_1(0)$ where $G$ is a 1-Lipschitz graph. Thus for any $\epsilon$ we can find an $N_\epsilon \in \mathbb{N}$ such that for all $m \geq N_\epsilon$

$$\begin{aligned} \mathrm{Spt}\nu_{k_m} \cap C_1(0) &\subset N_{2^{-k_m}}(G_{k_m} \cap C_1(0)) \\ &\subset N_\epsilon(G \cap C_1(0)). \end{aligned}$$

So by definition of tangent measure for every $p \in \mathbb{N}$ we can find $r_p > 0$ such that

$$\frac{T_{x,r_p\sharp}\,\mu\left(C_1(0)\setminus N_{2^{-p}}(G)\right)}{r_p^s} < 2^{-p}.$$

Let $\nu = \lim_{p\to\infty} \frac{T_{x,r_p\sharp}\,\mu}{r_p^s}$, $\nu$ is a tangent measure of $\mu$ at $x$ and $\mathrm{Spt}\nu \cap C_1(0)$ is contained in the graph of a 1-Lipschitz function from $e_k^\perp$ to $\langle e_k\rangle$, contradiction. Thus we have shown (23) and completed the Step 1.

So pick some $x_0 \in B$ for which inequality (23) and Lemma 1, Lemma 2 hold true. Let

$$\delta_1 = \inf \left\{ \rho_1 \left( \mathrm{Spt}\nu \cap C_1(0), G \cap C_1(0) \right) : \nu \in \mathrm{Tan}(\mu, x_0), G \in \bigcup_{k=1}^{n} \Gamma_k \right\}.$$

We will prove some general properties about tangent measures $\nu \in \mathrm{Tan}(\mu, x_0)$.

Firstly note that by Lemma 2, for any $\nu \in \mathrm{Tan}(\mu, x_0)$ if $\nu_1 \in \mathrm{Tan}(\nu, z)$ for some $z \in \mathrm{Spt}\nu$ then

$$\inf \left\{ \rho_1 \left( \mathrm{Spt}\nu_1 \cap C_1(0), G \cap C_1(0) \right) : G \in \Gamma_k \right\} \geq \delta_1 \text{ for all } k \in \{1, 2, \dots n\} \tag{24}$$

Let $\epsilon_1 > 0$ be some small number depending on $\delta_1$ whose size will be determined later.

*Step 2:*

Given $\nu \in \mathrm{Tan}(\mu, x_0)$ we will show that for $\nu$ a.e. $x \in \mathrm{Spt}\nu$

$$\limsup_{r \to 0} \frac{\nu\left( C_r(x) \cap \left( \hat{S}_i^{(x,\epsilon_1)} \cup \hat{S}_{i+n}^{(x,\epsilon_1)} \right) \right)}{r^s} \geq \epsilon_1^s \tag{25}$$

for any $i \in \{1, 2, \dots n\}$.

Suppose not and we can find subset $B_2 \subset \mathrm{Spt}\nu$ such that for some $i \in \{1, 2, \dots n\}$

$$\limsup_{r \to 0} \frac{\nu\left( C_r(x) \cap \left( \hat{S}_i^{(x,\epsilon_1)} \cup \hat{S}_{i+n}^{(x,\epsilon_1)} \right) \right)}{r^s} < \epsilon_1^s \tag{26}$$

for each $x \in B_2$. So we can extract some subset $B_3 \subset B_2$ of positive $\nu$ measure such that for some small number $\sigma_1 > 0$

$$\frac{\nu\left( C_r(x) \cap \left( \hat{S}_i^{(x,\epsilon_1)} \cup \hat{S}_{i+n}^{(x,\epsilon_1)} \right) \right)}{r^s} < \epsilon_1^s \quad \forall\ r \in (0, \sigma_1),\ \ x \in B_3. \tag{27}$$

We will show that this implies

$$C_{\frac{\sigma_1}{2}}(x) \cap \left( \hat{S}_i^{(x,4\epsilon_1)} \cup \hat{S}_{i+n}^{(x,4\epsilon_1)} \right) \cap \mathrm{Spt}\nu = \emptyset \quad \forall\ x \in B_3. \tag{28}$$

Suppose not and we can pick a point $x \in B_3$ such that

$$C_{\frac{\sigma_1}{2}}(x) \cap \left( \hat{S}_i^{(x,4\epsilon_1)} \cup \hat{S}_{i+n}^{(x,4\epsilon_1)} \right) \cap \mathrm{Spt}\nu \neq \emptyset$$

then let $z \in C_{\frac{\sigma_1}{2}}(x) \cap \left( \hat{S}_i^{(x,4\epsilon_1)} \cup \hat{S}_{i+n}^{(x,4\epsilon_1)} \right) \cap \mathrm{Spt}\nu$ and let $r = \|z - x\|$. As we have

$$C_{2r\epsilon_1}(z) \subset C_{2r}(x) \cap \left( \hat{S}_i^{(x,\epsilon_1)} \cup \hat{S}_{i+n}^{(x,\epsilon_1)} \right)$$

so we have $\nu\left( C_{2r}(x) \cap \left( \hat{S}_i^{(x,\epsilon_1)} \cup \hat{S}_{i+n}^{(x,\epsilon_1)} \right) \right) > (2r)^s \epsilon_1^s$, and this contradicts (27). Thus we have established (28). Let $x_3 \in B_3$ be a density point of $B_3$. Since from Lemma 14.5 [11] we know that $\mathrm{Tan}(\nu, x_3) = \mathrm{Tan}\left(\nu_{\lfloor B_3}, x_3\right)$. So any $\nu_1 \in \mathrm{Tan}(\nu, x_3)$ has the property

$$\left( \hat{S}_i^{(x,5\epsilon_1)} \cup \hat{S}_{i+n}^{(x,5\epsilon_1)} \right) \cap \mathrm{Spt}\nu_1 = \emptyset \text{ for all } x \in \mathrm{Spt}\nu_1. \tag{29}$$

Let $K = P_{e_i^\perp}(\mathrm{Spt}\nu_1)$, we define a map $g : K \to \langle e_i \rangle$ by

$$g(x) = P_{e_i^\perp}^{-1}(x).$$

From equation (29) it is clear that $g$ is a well defined $\frac{1}{(1-5\epsilon_1)}$-Lipschitz map from $K$. Further more every point of $\mathrm{Spt}\nu_1$ lives in the graph of $g$, see figure 4. Now let $h : e_i^\perp \to \langle e_i \rangle$ be defined by

$$h(x) := \inf \left\{ g(y) \cdot e_i + \frac{\|x - y\|}{(1 - 5\epsilon_1)} : y \in K \right\} e_i$$

for any $x \in e_i^\perp$. This is a $\frac{1}{(1-5\epsilon_1)}$-Lipschitz map from $e_i^\perp$ extending $g$, see Theorem 2.10.44. [4].

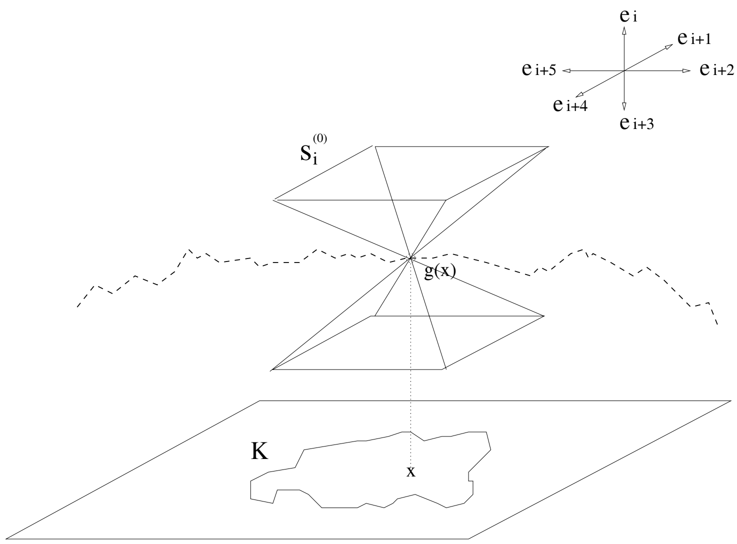


FIGURE 4

Now assuming $\epsilon_1$ is small enough we can certainly find a 1-Lipschitz map $T : e_i^{\perp} \to \langle e_i \rangle$ such that if we let $G = \left\{ (x_1, \ldots, T(x), \ldots x_n) : x \in C_1(0) \cap e_i^{\perp} \right\}$ then we have

$$\rho_1 \left( \left\{ (x_1, \ldots, h(x), \ldots x_n) : x \in C_1(0) \cap e_i^{\perp} \right\}, G \right) < \frac{\delta_1}{2}.$$

And so $\mathrm{Spt}\nu \cap C_1(0) \subset N_{\frac{\delta_1}{2}}(G)$, this contradicts (24) and so we have established (25).

Recall notation; any point satisfying inequality (25) is known as a point having *square-cone density* (see (5)). Also note that any $\nu \in \mathrm{Tan}(\mu, x_0)$ has to be such that

$$\nu\left(\partial S_i^{(x)}\right) = 0 \tag{30}$$

for $i \in \{1, 2, \ldots 2n\}$, $x \in \mathrm{Spt}\nu$, since if this was not true then we could take a tangent measure $\nu_1$ in a density point of the set $\partial S_i^{(x)} \cap \mathrm{Spt}\nu$. So $\mathrm{Spt}\nu_1$ would be contained in an $(n-1)$-plane that is the image of a 1-Lipschitz map from some $e_k^{\perp}$ for some $k \in \{1, 2, \ldots 2n\}$, which contradicts (24). Now we will break off the proof of Proposition 1 to establish some more properties of measures $\nu \in \mathrm{Tan}(\mu, x_0)$. The coming Lemmas will rely heavily on equations (25) and (30). The first thing we should note however is that by (25) and (30), measure $\nu$ satisfies the hypotheses of Lemma 3, so we know from this point on (in the proof of Proposition 1) that $s \in [1, 2]$.

### 7.1. **Properties of measures in** $\mathrm{Tan}(\mu, x_0)$.

**Lemma 6.** *If* $\tilde{\nu} \in \mathrm{Tan}(\mu, x_0)$ *with the property* $\tilde{\nu}\left(S_i^{(0)}\right) = 0$ *then we can find* $\nu \in \mathrm{Tan}(\tilde{\nu}, 0)$ *and* $x \in \mathrm{Spt}\nu$ *such that:*

- *The function* $f_i^{(x)}(r) = \nu_r\left(\partial C_r(x) \cap S_i^{(x)}\right)$ *is monotonic non-decreasing and bounded with the property that* $f_i^{(x)}(r) > 0$ *for all* $r > 0$.
- *There exist some large number* $R > 0$ *and some small number* $\vartheta > 0$ *such that*

$$N_{\vartheta}\left(\left(\partial S_i^{(x)} \cup \partial S_{i+n}^{(x)}\right)\right) \cap A(x, R, \infty) \cap \mathrm{Spt}\nu = \emptyset.$$

To avoid having to keep track of constants we make the assumption that any $\tilde{\nu} \in \mathrm{Tan}(\mu, x_0)$ is an $s$-uniform measure with the controlled component condition (i.e. not just a scalar multiple of such a measure)

*Step 1:* First we find a tangent measure with convenient properties.

Let

$$B_1 = \left\{ j \in \{1, 2, \dots n\} : \tilde{\nu} \left( S_j^{(0)} \cap C_\alpha(0) \right) = 0 \text{ for some } \alpha > 0 \right\}. \tag{31}$$

If we take any $\tilde{\nu}_1 \in \operatorname{Tan}(\tilde{\nu}, 0)$ then we will have $\tilde{\nu}_1 \left( S_j^{(0)} \right) = 0$ for every $j \in B_1$. So let

$$B_2 = \left\{ j \in \{1, 2, \dots n\} \backslash B_1 : \tilde{\nu}_1 \left( S_j^{(0)} \cap C_\alpha(0) \right) = 0 \text{ for some } \alpha > 0 \right\}. \tag{32}$$

Any $\tilde{\nu}_2 \in \operatorname{Tan}(\tilde{\nu}_1, 0)$ will have the property $\tilde{\nu}_2 \left( S_j^{(0)} \right) = 0$ for every $j \in B_1 \cup B_2$. Recall, by Lemma 2 we know $\nu_2 \in \operatorname{Tan}(\tilde{\nu}, 0)$. We can keep doing this until we have a $\nu \in \operatorname{Tan}(\tilde{\nu}, 0)$ and a set $B \subset \{1, 2, \dots n\}$ such that

- $\nu \left( S_j^{(0)} \right) = 0$ for every $j \in B$,
- $\nu \left( S_k^{(0)} \cap C_\alpha(0) \right) > 0$ for all $\alpha > 0$ whenever $k \in \{1, 2, \dots n\} \backslash B$.

If we have that $B = \{1, 2, \dots n\}$ then by measure symmetry we have $\nu \left( S_j^{(0)} \cup S_{j+n}^{(0)} \right) = 0$ for all $j \in \{1, 2, \dots n\}$ and so $\nu(\mathbb{R}^n) = 0$, contradiction. So we know $B$ is a proper subset of $\{1, 2, \dots n\}$.

Let $\{i_1, i_2, \dots i_p\}$ be an ordering of the set $\{1, 2, \dots n\} \backslash B$, $i_1$ being the smallest number in this set, $i_2$ the second smallest and so on.

Pick $x_1 \in S_{i_1}^{(0)} \cap \operatorname{Spt}\nu$ such that $d_1 := \frac{d\left(x_1, \partial S_{i_1}^{(0)}\right)}{2} > 0$. We know we can find such an $x_1$ because $\nu \left( \partial S_{i_1}^{(0)} \right) = 0$. So we have $C_{d_1}(0) \subset S_{i_1+n}^{(x_1)}$. Next pick $x_2 \in C_{d_1}(0) \cap S_{i_2}^{(0)} \cap \operatorname{Spt}\nu$ for which $d_2 := \frac{d\left(x_2, \partial S_{i_2}^{(0)}\right)}{2} > 0$. The situation is as shown in Figure 5. We can continue picking

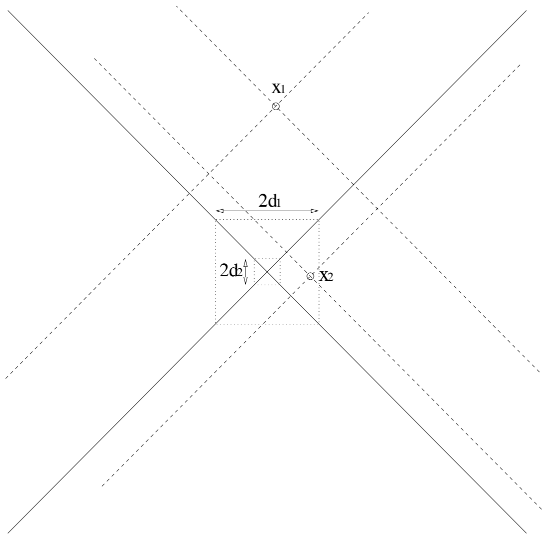


FIGURE 5

points

$$x_k \in C_{d_{k-1}}(0) \cap \operatorname{int} \left( S_{i_k}^{(0)} \right) \cap \operatorname{Spt}\nu \tag{33}$$

until we finally have a point $x_p \in \operatorname{int} \left( S_{i_p}^{(0)} \right) \cap \operatorname{Spt}\nu$ with the property that $x_p \in \operatorname{int} \left( S_{i_k+n}^{(x_k)} \right)$ for $k \in \{1, 2, \dots p-1\}$.

So for some small $\varepsilon > 0$ we have

- $$C_{2\varepsilon}(x_p) \subset \operatorname{int} \left( S_{i_p}^{(0)} \right), \tag{34}$$
- $$C_{2\varepsilon}(x_p) \subset \operatorname{int} \left( S_{i_k+n}^{(x_k)} \right) \quad \text{for } k \in \{1, 2, \dots p-1\}. \tag{35}$$

We must be able to find $x\in C_{\varepsilon}(x_p)$ such that $\nu\left(S_i^{(x)}\cap C_{\varepsilon}(x)\right)>0$ by square cone density, (i.e. inequality (25)).

*Step 2:* We will show the function $f_i^{(x)}$ is bounded.

Note that since $x\in\mathrm{int}\left(S_{i_p}^{(0)}\right)$, for all large enough $r>\|x\|$ we have that

$$\partial C_{r-\|x\|}(0)\cap S_{i_p+n}^{(0)}\subset\partial C_r(x)\cap S_{i_p+n}^{(x)}. \tag{36}$$

Secondly note that since from (35) we know that $x\in\mathrm{int}\left(S_{i_k+n}^{(x_k)}\right)$ for $k\in\{1,2,\dots p-1\}$ thus $x_k\in\mathrm{int}\left(S_{i_k}^{(x)}\right)$ for all $k\in\{1,2,\dots p-1\}$ and we have

$$\partial C_{r-\|x-x_k\|}(x_k)\cap S_{i_k}^{(x_k)}\subset\partial C_r(x)\cap S_{i_k}^{(x)}. \tag{37}$$

As (from (33) and (34)) $x_k\in S_{i_k}^{(0)}$ for all $k\in\{1,2,\dots p\}$ and so we have that for all large enough $s>\|x\|$

$$\partial C_{s-\|x_k\|}(0)\cap S_{i_k+n}^{(0)}\subset\partial C_s(x_k)\cap S_{i_k+n}^{(x_k)}. \tag{38}$$

Let $k\in\{1,2,\dots p-1\}$. Working backwards we see (38) implies $f_{i_k+n}^{(x_k)}(s)\geq f_{i_k+n}^{(0)}(s-\|x_k\|)$ for all large enough $s>0$. And (37) implies $f_{i_k}^{(x)}(r)\geq f_{i_k}^{(x_k)}(r-\|x-x_k\|)$ for large enough $r>0$. Recall that (by measure symmetry) $f_{i_k}^{(x_k)}=f_{i_k+n}^{(x_k)}$, putting these in equalities together gives us that for each $k\in\{1,2,\dots p-1\}$

$$f_{i_k}^{(x)}(r)\geq f_{i_k}^{(0)}(r-\|x_k\|-\|x-x_k\|) \tag{39}$$

for all large enough $r>0$. Finally by (36), for all large enough $r>0$

$$f_{i_p+n}^{(x)}(r)\geq f_{i_p+n}^{(0)}(r-\|x\|). \tag{40}$$

Let $\beta>0$ be some number bigger than $\|x\|$ and $\max\{\|x_k\|+\|x-x_k\|:k\in\{1,2,\dots p-1\}\}$. By monotonicity and measure symmetry and equations (39), (40) we have that for all $k\in\{1,2,\dots p\}$ and for all large enough $r>0$

$$f_{i_k}^{(x)}(r)\geq f_{i_k}^{(0)}(r-\beta). \tag{41}$$

Now as

$$\frac{sr^{s-1}}{2}=\frac{v_r^{(x)}(\partial C_r(x))}{2}=\sum_{k=1}^{n}f_k^{(x)}(r)$$

and clearly $i\notin\{i_1,i_2,\dots i_p\}$ (recall $\tilde{\nu}\left(S_i^{(0)}\right)=0$ and the way we arrived at $\{i_1,i_2,\dots i_p\}$ from (31) and (32)) so by using (41) we have

$$\begin{aligned}f_i^{(x)}(r)&=\frac{sr^{s-1}}{2}-\sum_{k\in\{1,2,\dots n\}\setminus\{i\}}f_k^{(x)}(r)\\&\leq\frac{sr^{s-1}}{2}-\sum_{k=1}^{p}f_{i_k}^{(x)}(r)\\&\overset{(41)}{\leq}\frac{sr^{s-1}}{2}-\sum_{k=1}^{p}f_{i_k}^{(0)}(r-\beta).\end{aligned} \tag{42}$$

By the way we arrived at the set $\{i_k:k=1,2,\dots p\}$ we have

$$\sum_{k=1}^{p}f_{i_k}^{(0)}(r-\beta)=\frac{v_{(r-\beta)}^{(0)}\left(\partial C_{(r-\beta)}(0)\right)}{2}=\frac{s(r-\beta)^{s-1}}{2}.$$

Now for $r>1+\beta$ since (recall $\nu\in\mathrm{Tan}(\mu,x_0)$ satisfies the hypotheses of Lemma 3) $s\in[1,2]$ we have that $\frac{s}{2}\left(r^{s-1}-(r-\beta)^{s-1}\right)\leq\frac{s}{2}\beta$.

So applying this to (42) we have

$$f_i^{(x)}(r) \le \frac{s}{2}\beta \tag{43}$$

for all large enough $r > 0$.

*Step 3:* We will show that for some small $\theta > 0$ for sufficiently large $R > 0$ we have

$$N_\theta\left(\partial S_i^{(x)}\right) \cap A(x, R, \infty) \cap \mathrm{Spt}\nu = \emptyset.$$

Since $\nu\left(S_i^{(x)} \cap C_\varepsilon(x)\right) > 0$ we can pick $x_1 \in \mathrm{int}\left(S_i^{(x)}\right) \cap C_\varepsilon(x) \cap \mathrm{Spt}\nu$ and by measure symmetry we can also pick a point $x_2 \in \mathrm{int}\left(S_{i+n}^{(x)}\right) \cap C_\varepsilon(x) \cap \mathrm{Spt}\nu$. We can assume $x_1$ and $x_2$ are points for which the functions $f_i^{(x_1)}$ and $f_i^{(x_2)}$ are monotonic non-decreasing and locally Lipschitz away from 0. Let $\vartheta_1 := d\left(x_1, \partial S_i^{(x)}\right)$ and $\vartheta_2 := d\left(x_2, \partial S_{i+n}^{(x)}\right)$. Since $f_i^{(x)} = f_{i+n}^{(x)}$ is bounded and $f_{i+n}^{(x_2)}(r) \le f_{i+n}^{(x)}(r + \|x_2 - x\|)$ we know $f_{i+n}^{(x_2)} = f_i^{(x_2)}$ is bounded. Just as easily we can observe $f_i^{(x_1)}$ is bounded. So let $\alpha_1 := \sup\left\{f_i^{(x_1)}(r) : r > 0\right\}$, $\alpha_2 := \sup\left\{f_i^{(x_2)}(r) : r > 0\right\}$ and $\alpha := \sup\left\{f_i^{(x)}(r) : r > 0\right\}$. Firstly we want to show $\alpha_2 \le \alpha$. Suppose not, let $\delta = \alpha_2 - \alpha$.

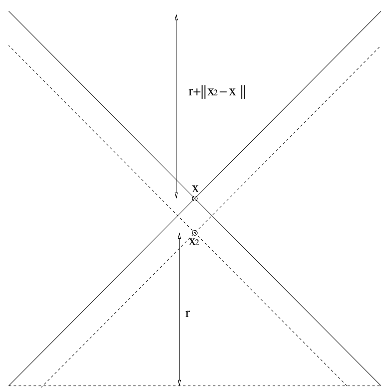


FIGURE 6

We can find an $r > 0$ such that $f_i^{(x_2)}(r) > \alpha_2 - \frac{\delta}{2}$. See figure 6. So and since $\partial C_r(x_2) \cap S_{i+n}^{(x_2)} \subset \partial C_{r+\|x-x_2\|}(x) \cap S_{i+n}^{(x)}$ by measure symmetry

$$\begin{aligned} f_{i+n}^{(x)}(r + \|x_2 - x\|) &\ge f_{i+n}^{(x_2)}(r) \\ &= f_i^{(x_2)}(r) \\ &> \alpha_2 - \frac{\delta}{2}. \end{aligned}$$

Hence by measure symmetry $f_i^{(x)}(r + \|x_2 - x\|) > \alpha_2 - \frac{\delta}{2} > \alpha$, contradiction. So we have $\alpha_2 \le \alpha$.

Since $x \in S_i^{(x_2)}$ so $\partial C_r(x) \cap S_i^{(x)} \subset \partial C_{r+\|x-x_2\|}(x_2) \cap S_i^{(x_2)}$ we can argue in exactly the same way to show $\alpha_2 \ge \alpha$, thus $\alpha_2 = \alpha$. With an identical argument we can see that $\alpha_1 = \alpha$. So we have $\alpha = \alpha_1 = \alpha_2$.

Let $\vartheta = \frac{\min\{\vartheta_1, \vartheta_2\}}{8}$. We can find $R > 0$ such that $f_i^{(x_1)}\left(\frac{R}{2}\right) > \alpha - \frac{\vartheta^{s-1}}{4}$. Now suppose we had some point $z \in A(x, R, \infty) \cap N_\vartheta\left(\partial S_i^{(x)}\right) \cap \mathrm{Spt}\nu$ then we could fit the cube

$$C_\vartheta(z) \subset \left(S_i^{(x_2)} \backslash S_i^{(x_1)}\right) \cap A\left(x, \frac{R}{2}, \infty\right),$$

so

$$\int_{A(x_2,\|z-x_2\|-\vartheta,\|z-x_2\|+\vartheta)} \nu_p\left(\partial C_p(x_2)\cap\left(S_i^{(x_2)}\backslash S_i^{(x_1)}\right)\right)\geq\vartheta^s.$$

So we must be able to find a point $p\in[\|z-x_2\|-\vartheta,\|z-x_2\|+\vartheta]$ such that

$$\nu_p\left(\partial C_p(x_2)\cap\left(S_i^{(x_2)}\backslash S_i^{(x_1)}\right)\right)\geq\frac{\vartheta^{s-1}}{2}.$$

Now $\nu_p\left(\partial C_p(x_2)\cap\left(S_i^{(x_2)}\backslash S_i^{(x_1)}\right)\right)=f_i^{(x_2)}(p)-f_i^{(x_1)}(p-\|x_1-x_2\|)$. So

$$\begin{aligned} f_i^{(x_2)}(p) &= f_i^{(x_1)}(p-\|x_1-x_2\|)+\nu_p\left(\partial C_p(x_2)\cap\left(S_i^{(x_2)}\backslash S_i^{(x_1)}\right)\right) \\ &\geq \alpha+\frac{\vartheta^{s-1}}{2}-\frac{\vartheta^{s-1}}{4} \\ &= \alpha+\frac{\vartheta^{s-1}}{4}, \end{aligned}$$

contradiction.

So we have $N_\vartheta\left(\partial S_i^{(x)}\right)\cap A(x,R,\infty)\cap\mathrm{Spt}\nu=\emptyset$. For identical reasons we have $N_\vartheta\left(\partial S_{i+n}^{(x)}\right)\cap A(x,R,\infty)\cap\mathrm{Spt}\nu=\emptyset$. $\square$

**Lemma 7.** *There can be no $\tilde{\nu}\in\mathrm{Tan}(\mu,x_0)$ with the property that $\tilde{\nu}\left(S_i^{(0)}\right)=0$ for some $i\in\{1,2,\ldots 2n\}$.*

**Proof**

Suppose not, and so we have $\tilde{\nu}\in\mathrm{Tan}(\mu,x_0)$ with $\tilde{\nu}\left(S_i^{(0)}\right)=0$. We know by Lemma 6 that we can find $\nu\in\mathrm{Tan}(\tilde{\nu},0)$ and $x\in\mathrm{Spt}\nu$ such that

- The function $f_i^{(x)}(r):=\nu_r\left(\partial C_r(x)\cap S_i^{(x)}\right)$ is monotonic non-decreasing and bounded with the property that $f_i^{(x)}(r)>0$ for all $r>0$.
- There exists some large number $R>\|x\|$ and some small number $\vartheta>0$ such that

$$N_\vartheta\left(\left(\partial S_i^{(x)}\cup\partial S_{i+n}^{(x)}\right)\right)\cap A(x,R,\infty)\cap\mathrm{Spt}\nu=\emptyset. \tag{44}$$

Since $f_i^{(x)}(s)>0$ for all $s>0$ by Lemma 5 we can pick some point $y\in A(x,3R,\infty)\cap\mathrm{int}\left(S_i^{(x)}\right)\cap\mathrm{Spt}\nu$ for which the hypothesis of Lemma 5 are satisfied (recall this follows from (25) and (30)). And so as $y$ has the property that $f_j^{(y)}$ is a positive non-decreasing locally Lipschitz function for every $j\in\{1,2,\ldots n\}$. So in particular $\nu\left(S_j^{(y)}\cap C_r(y)\right)>0$ for every $j\in\{1,2,\ldots n\}$ and every $r>0$.

Now pick the $j\in\{1,2,\ldots 2n\}\setminus\{i,i+n\}$ for which $y\cdot e_j>x\cdot e_j$. For some small $\sigma_0\in\left(0,\frac{\vartheta}{4}\right)$ we can pick some $z_1\in\mathrm{int}\left(S_{i+n}^{(y)}\right)\cap C_{\sigma_0}(y)\cap\mathrm{Spt}\nu$ such that the function $f_j^{(z_1)}$ is Lipschitz monotonic non-decreasing and $f_j^{(z_1)}(s)>0$ for all $s>0$, assuming we choose $\sigma_0$ small enough we also have $z_1\in S_i^{(x)}$.

We will pick a chain of points crawling forward from $z_1$ in direction $e_j$ in the obvious way. Firstly we pick $z_2\in A\left(z_1,\frac{\vartheta}{8},\frac{\vartheta}{4}\right)\cap S_j^{(z_1)}\cap\mathrm{Spt}\nu$ (using the fact that $f_j^{(z_1)}(s)>0$ for all $s>0$) and we can assume the point $z_2$ we picked is such that $f_j^{(z_2)}$ is positive monotonic non-decreasing. So we can use this to pick a point $z_3\in A\left(z_2,\frac{\vartheta}{8},\frac{\vartheta}{4}\right)\cap S_j^{(z_2)}\cap\mathrm{Spt}\nu$ with the property that $f_j^{(z_3)}$ is positive monotonic non-decreasing. In this way we build up a sequence of points $(z_n)$ where $z_{n+1}\in A\left(z_n,\frac{\vartheta}{8},\frac{\vartheta}{4}\right)\cap S_j^{(z_n)}$ for all $n\in\mathbb{N}$, see figure 7. We will show that

$$z_n\in S_i^{(x)}\ \ \forall\, n\in\mathbb{N}. \tag{45}$$

Firstly recall that we know $z_1\in S_i^{(x)}$. We also know $z_k\in S_j^{(z_1)}$, consider the following expression

$$(z_k-x)\cdot e_j \quad=\quad (z_k-z_1)\cdot e_j+(z_1-y)\cdot e_j+(y-x)\cdot e_j.$$

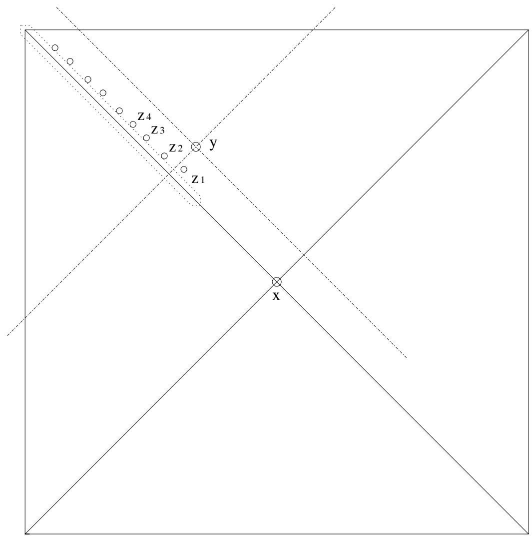


FIGURE 7

Case 1: $(z_k - z_1) \cdot e_j > \frac{3R}{2}$, since $\|z_1 - y\|$ is small and $(y-x)\cdot e_j > 0$ we have that $z_k \in A\left(x, \frac{5R}{4}, \infty\right)$.

Case 2: $(z_k - z_1)\cdot e_j \le \frac{3R}{2}$ since $z_k \in S_j^{(z_1)}$ we have that $\|z_k - z_1\| \le \frac{3R}{2}$ and thus by smallness of $\|z_1 - y\|$ we have $\|z_k - y\| \le \|z_k - z_1\| + \|z_1 - y\| < \frac{7R}{4}$ and so since $\|x-y\| > 3R$ we have $\|z_k - x\| \ge \|x-y\| - \|z_k - y\| > \frac{5R}{4}$. Thus

$$z_k \in A\left(x, \frac{5R}{4}, \infty\right) \tag{46}$$

for all $k \in \mathbb{N}$.

Recall; we want to show (45). Now let $m \in \mathbb{N}$ be the first number such that $z_{m+1} \notin S_i^{(x)}$. So $z_m \in S_i^{(x)}$ and as $\|z_m - z_{m+1}\| \le \frac{\vartheta}{4}$ so $d\left(z_m, \partial S_i^{(x)}\right) \le \frac{\vartheta}{4}$. And thus by (46) $N_\vartheta\left(\left(\partial S_i^{(x)} \cup \partial S_{i+n}^{(x)}\right)\right) \cap A(x, R, \infty) \cap \mathrm{Spt}\nu \ne \emptyset$ which contradicts (44). So we have shown (45).

Now recall also that for each $k \in \mathbb{N}$ we know $z_k \in S_j^{(z_{k-1})} \cap A\left(z_{k-1}, \frac{\vartheta}{8}, \frac{\vartheta}{4}\right)$ and so $z_k \in A\left(z_1, k\frac{\vartheta}{8}, \infty\right)$. So we know

$$(z_k - x)\cdot e_i \overset{(45)}{=} \|z_k - x\| \ge \|z_k\| - \|x\| \ge \|z_k - z_1\| - \|z_1\| - \|x\| \to \infty \text{ as } k \to \infty. \tag{47}$$

Now recall $z_1 \in \mathrm{int}\left(S_{i+n}^{(y)}\right) \cap \mathrm{Spt}\nu$ so let $\sigma_1 := d\left(z_1, S_i^{(y)}\right)$, since $z_k \in S_j^{(z_1)}$ we have

$$d\left(z_k, S_i^{(y)}\right) \ge \sigma_1 \tag{48}$$

for all $k \in \mathbb{N}$. Let $\sigma = \min\left\{\frac{\vartheta}{8}, \frac{\sigma_1}{2}\right\}$. So by (44), (45), (46) and (48) we know the set $\{z_k : k \ge 1\}$ are trapped in $S_i^{(x)} \backslash S_i^{(y)}$ and stay distance $\sigma$ away from the boundaries $\partial S_i^{(y)}$ and $\partial S_i^{(x)}$.

We also know $f_i^{(x)}$ is bounded and monotonic (recall, we obtained tangent measure $\nu$ by using Lemma 6) and so let $L_1 := \lim_{r\to\infty} f_i^{(x)}(r)$. Since $y \in S_i^{(x)}$ the function $f_i^{(y)}$ is also bounded. And as we have argued before at the end of Lemma 6 we have that $\lim_{r\to\infty} f_i^{(y)}(r) = L_1$.

Now we can take some $r_0 > 0$ such that

$$f_i^{(x)}(r) > L_1 - \frac{\sigma^{s-1}}{4} \quad \text{and} \quad f_i^{(y)}(r) > L_1 - \frac{\sigma^{s-1}}{4}$$

for all $r \in (r_0, \infty)$. And by (47) we can pick some $q \in \mathbb{N}$ such that $(z_q - x) \cdot e_i \geq 2r_0$. Now as already noted by (44), (45), (46) and (48) we know

$$C_\sigma(z_q) \subset \left(S_i^{(x)} \backslash S_i^{(y)}\right) \cap A\left(x, (z_q - x) \cdot e_i - \sigma, (z_q - x) \cdot e_i + \sigma\right),$$

so

$$\int_{(z_q-x)\cdot e_i-\sigma}^{(z_q-x)\cdot e_i+\sigma} \nu_s\left(\partial C_s(x) \cap \left(S_i^{(x)} \backslash S_i^{(y)}\right)\right) dLs \geq \sigma^s.$$

Thus there must exist some $s \in ((z_q - x) \cdot e_i - \sigma, (z_q - x) \cdot e_i + \sigma)$ such that

$$\nu_s\left(\partial C_s(x) \cap \left(S_i^{(x)} \backslash S_i^{(y)}\right)\right) \geq \frac{\sigma^{s-1}}{2}.$$

However as

$$f_i^{(x)}(s) - f_i^{(y)}(s - (y - x) \cdot e_i) = \nu_s\left(\partial C_s(x) \cap \left(S_i^{(x)} \backslash S_i^{(y)}\right)\right)$$

we have that

$$\begin{aligned} f_i^{(x)}(s) &\geq \frac{\sigma^{s-1}}{2} + f_i^{(y)}(s - (y - x) \cdot e_i) \\ &\geq L_1 + \frac{\sigma^{s-1}}{4}, \end{aligned}$$

contradiction. □

Remark. The proof of Proposition 1 now follows if we can show the existence of some tangent measure $\tilde{\nu} \in \mathrm{Tan}(\mu, x_0)$ with the property that $\tilde{\nu}\left(S_i^{(0)}\right) = 0$. This is a consequence of the results of the next section.

## 8. Strengthened Touching Point Arguments

The results and the methods of this section are basically a reworking of the methods already used in [6] (for entirely the same purpose, in $l^3_\infty$) and [7]. As such there is essentially nothing in this section that can be said to be original. However for completeness we include most of the necessary details.

The following lemma is a co-dimension $(n-2)$ version of a result that was first proved for 2-sets in $\mathbb{R}^3$ in [6] Lemma 14 and for $(n-1)$ sets in $\mathbb{R}^n$ in [7] Lemma 1.

Since the support of the measures we deal with are either $(n-1)$-unrectifiable or of dimension $s < n-1$ we know from the the projection theorems of Federer and Marstrand that the projection of our set onto almost any choice of $(n-1)$ subspace will have zero $(n-1)$ Lebesgue measure. Roughly speaking, what the coming lemma says is that the complement of the projection of this set can not have big holes inside it. To put it another way, having zero projection implies that we can "fill" the space with many parallel cylinders running up through the complement of our set. Lemma 8 says that (so long as we are on a sufficiently small scale) the diameters of the cylinders can not be too big.

The point of the lemma is the following: In [2] and all subsequent papers in the field one of the basic methods has been so called "touching point arguments" which establish the existence of a tangent measure (to a measure with positive finite density in Euclidean space) on one side of a half space. There is no way to use this fact to show that $\mathrm{Spt}\mu$ is emptied out from one of the sectors $S_j^{(x)}$ (where $x \in \mathrm{Spt}\mu$) because it is easy to see there exists an $(n-1)$ plane cutting through all the sectors of a cube centered on $x$. Only when $s = 2$ is this not the case. And thus if we project the measure onto a 2-dimensional subspace we can hope to empty a sector.

In Euclidean space the standard method is to use the fact that for any point $x$ in the support of the measure and for any $r > 0$, in the ball $B_r(x)$ there must be a sub-ball of diameter $\lambda r$ which does not intersect any point of the support. The sub-ball is expanded until it touches the first point of the subset of the support of the measure which (arguing by contradiction) has the property that it is (quantitatively) always onto both sides of any half-space. Assuming $x$ is a density point of this subset and the scale is small enough we have a contradiction.

In our case, when we expand the empty cylinders it may turn out that the first point of the subset they touch is at the very top of the cube $C_r(x)$ (hence any small cube centered on this point will be half outside $C_r(x)$) and as such we can not use the fact that $x$ is a density point. The situation is salvaged however if the empty cylinders can not be too big because then there will be many small sub-cubes at the boundary of the cube $C_r(x)$. See figure 8, and as we have very sharp estimates for the measure on the boundary of the cube, we get a contradiction.

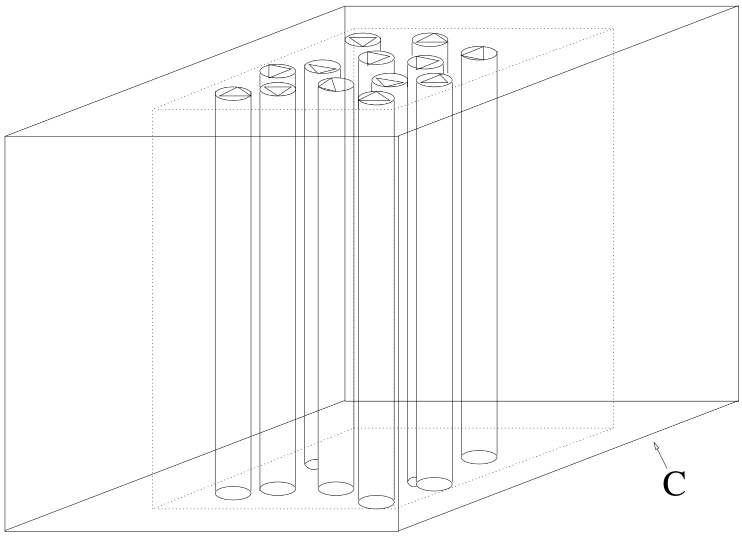


FIGURE 8

**Lemma 8.** *Given integer $n \geq 3$, let $s \in (0,2]$. Given an A.D. s-set $S_0 \subset \mathbb{R}^n$ and a closed subset $S_1 \subset S_0$ with the property that there exists 2-plane $V$ and number $r_0 > 0$ such that for some small $\lambda_1 > 0$ and $\rho > 0$ we have*

$$\frac{H^s\left(B_r(x) \cap X^+(x,\phi,\rho) \cap S_0\right)}{r^s} > \lambda_1 \tag{49}$$

*for all $\phi \in V \cap S^{n-1}$ and all $x \in S_1$, $r \in (0,r_0)$. Then we can find constants $\kappa_\rho^{\lambda_1} > 0$ and $\vartheta_\rho^{\lambda_1} > 0$ such that the following statement is true:*

*Let $K(x,V,\alpha) := P_V^{-1}(B_\alpha(x) \cap V)$ denote the cylinder. If $d \in (0,r_0)$ is such that*

$$\frac{H^s\left(B_{4d}(x) \backslash S_1\right)}{d^s} \leq \epsilon \tag{50}$$

*then for all $z \in (V+x) \cap K\left(x,V,\kappa_\rho^{\lambda_1} d\right)$ we have that*

$$K\left(z,V,\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1} d\right) \cap S_1 \cap B_{2d}(x) \neq \emptyset.$$

**Proof**

To simplify notation, let $\mu := H^s_{\lfloor S_0}$. We will assume that $V = \langle e_1, e_2\rangle$ (if not then we just define everything in terms of a different co-ordinate system $\{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n\}$ where $V = \langle \varepsilon_1, \varepsilon_2\rangle$).

*Step 1:*

Let $a \in \partial K(x,V,r)$ and $W_a$ denote the $(n-1)$ tangent plane of the boundary of the cylinder at point $a$, let $n_a \in W_a^{\perp} \cap S^{n-1}$ be the unit normal pointing "inwards" towards the center of the cylinder. We will show there exists a constant $\xi_\rho > 0$ such that

$$X^+(a,n_a,\rho) \cap B_{\xi_\rho r}(a) \subset K(x,V,r).$$

As $P_V(a) \in \partial K(x,V,r) \cap V$ it can be easily seen that there exists some constant $\xi_\rho > 0$ such that

$$P_V\left(X^+(a,n_a,\rho) \cap B_{\xi_\rho r}(a)\right) \subset B_r(x) \cap V.$$

Thus $X^{+}(a,n_a,\rho)\cap B_{\xi_\rho r}(a)\subset P_V^{-1}(B_r(x)\cap V)=K(x,V,r)$.

*Step 2:*
Now suppose $a\in\partial K(x,V,r)$ and $n_a\in S^{n-1}\cap V$ such that

$$X^{+}(a,n_a,\rho)\cap B_{\xi_\rho r}(a)\subset K(x,V,r)$$

and we have that

$$\mu\left(B_{\xi_\rho r}(a)\cap X^{+}(a,n_a,\rho)\cap S_1\right)\geq\frac{\lambda_1}{2}(\xi_\rho r)^s, \tag{51}$$

we will show there exists some number $\zeta_\rho^{\lambda_1}>0$ with the property that

$$X^{+}(a,n_a,\rho)\cap K\left(x,V,\left(1-\zeta_\rho^{\lambda_1}\right)r\right)\cap S_1\neq\emptyset. \tag{52}$$

By the fact the $S_0$ is A.D. we have some constant $\beta_{\lambda_1}>0$ such that

$$\mu\left(X^{+}(a,n_a,\rho)\cap A(a,\beta_{\lambda_1}r,\xi_\rho r)\cap S_1\right)\geq\frac{\lambda_1}{4}(\xi_\rho r)^s. \tag{53}$$

Now it should be clear that

$$P_V\left(X^{+}(a,n_a,\rho)\cap A(a,\beta_{\lambda_1}r,\xi_\rho r)\right)\subset P_V(K(x,V,r))$$

and in fact there exits some constant $\zeta_\rho^{\lambda_1}>0$ (letting $d_H$ denote Hausdorff metric) such that

$$d_H\left(P_V\left(X^{+}(a,n_a,\rho)\cap A(a,\beta_{\lambda_1}r,\xi_\rho r)\right),\partial P_V(K(x,V,r))\right)>\zeta_\rho^{\lambda_1}r.$$

Hence from (53) we have

$$X^{+}(a,n_a,\rho)\cap B_{\xi_\rho r}(a)\cap K\left(x,V,\left(1-\zeta_\rho^{\lambda_1}\right)r\right)\cap S_1\neq\emptyset.$$

Now we define the slab

$$H(z,V,\alpha):=\{y\in\mathbb{R}^n:|(y-z)\cdot e_i|<\alpha,\ \text{ for each } i=3,4,\dots n\}.$$

Let

$$\phi_k=\sum_{j=0}^{k}\xi_\rho\kappa_\rho^{\lambda_1}d\left(1-\zeta_\rho^{\lambda_1}\right)^j. \tag{54}$$

Define

$$\Gamma_k:=H(z,V,\phi_k)\cap K\left(z,V,\left(1-\zeta_\rho^{\lambda_1}\right)^k\kappa_\rho^{\lambda_1}d\right). \tag{55}$$

And define the cone as follows.

$$\Psi^u\left(z,V,\kappa_\rho^{\lambda_1},d\right):=\left\{y\in C_d(z):|P_V(y-z)|\leq\kappa_\rho^{\lambda_1}\left|P_{\langle e_k\rangle}(y-(z+de_k))\right|\text{ for each }k\in\{3,\dots n\}\right\}.$$

And define

$$\Psi^d\left(z,V,\kappa_\rho^{\lambda_1},d\right):=\left\{y\in C_d(z):|P_V(y-z)|\leq\kappa_\rho^{\lambda_1}\left|P_{\langle e_k\rangle}(y-(z-de_k))\right|\text{ for each }k\in\{3,\dots n\}\right\}.$$

Finally define

$$\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)=\Psi^u\left(z,V,\kappa_\rho^{\lambda_1},d\right)\cap\Psi^d\left(z,V,\kappa_\rho^{\lambda_1},d\right).$$

*Step 3:*
We will show

$$\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)\subset\bigcup_{j=0}^{\infty}\Gamma_j. \tag{56}$$

To start with note that

$$\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)\subset C_d(z)\subset\{y\in\mathbb{R}^n:|(y-z)\cdot e_i|<d\text{ for }i=3,4,\dots n\}.$$

So for any $y\in\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)\cap H(z,V,\phi_0)$ we have for any $k\in\{3,\dots n\}$

$$|P_V(y-z)|\leq\kappa_\rho^{\lambda_1}\min\left\{\left|P_{\langle e_k\rangle}(y-(z+de_k))\right|,\left|P_{\langle e_k\rangle}(y-(z-de_k))\right|\right\}\leq\kappa_\rho^{\lambda_1}d$$

thus (recall definition (55))

$$\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)\cap H(z,V,\phi_0)\subset\Gamma_0.$$

We will argue by induction, see figure 9. Suppose we have that

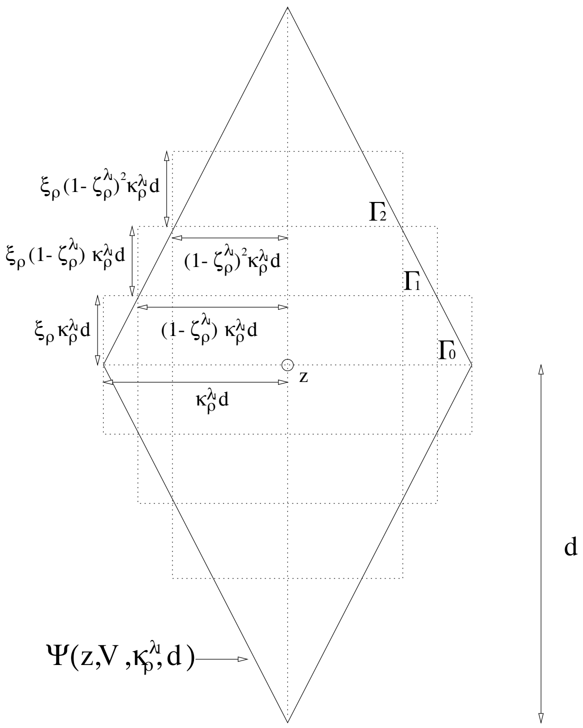


FIGURE 9

$$\Psi\left(z, V, \kappa_\rho^{\lambda_1}, d\right) \cap H\left(z, V, \phi_k\right) \subset \bigcup_{j=0}^{k} \Gamma_j. \tag{57}$$

Note that for $\alpha_0 \in (0, d)$ we have (recall, $V = \langle e_1, e_2 \rangle$)

$$\begin{aligned} & \left(z+\alpha_0\left(e_3+e_4+\ldots e_n\right)+V\right) \cap \partial \Psi\left(z, V, \kappa_\rho^{\lambda_1}, d\right) \\ & = \left\{ \begin{array}{lll} y \in \mathbb{R}^n & : & y \cdot e_{k_1} = \left(z+\alpha_0\left(e_3+\ldots e_n\right)\right) \cdot e_{k_1} \text{ for all } k_1 \in \{3, \ldots n\} \text{ and } \\ & & \left|P_V(y-z)\right| = \kappa_\rho^{\lambda_1}\left|P_{\langle e_{k_2}\rangle}\left(y-\left(z+d e_{k_2}\right)\right)\right| \text{ for some } k_2 \in \{3, \ldots n\} \end{array} \right\} \\ & = \left\{y \in \mathbb{R}^n : y \cdot e_k = z \cdot e_k+\alpha_0 \text{ for } k \in \{3,4, \ldots n\},\ \left|P_V(y-z)\right| = \kappa_\rho^{\lambda_1}\left|\alpha_0-d\right|\right\} \end{aligned} \tag{58}$$

which is the boundary of a circle of radius $\kappa_\rho^{\lambda_1}\left(d-\alpha_0\right)$ in the 2-plane $z+\alpha_0\left(e_3+e_4+\ldots e_n\right)+V$. In the same way, for $\alpha_1 \in (-d, 0)$ we have

$$\begin{aligned} & \left(z+\alpha_1\left(e_3+e_4+\ldots e_n\right)+V\right) \cap \partial \Psi\left(z, V, \kappa_\rho^{\lambda_1}, d\right) \\ & = \left\{y \in \mathbb{R}^n : y \cdot e_k = z \cdot e_k+\alpha_1 \text{ for } k \in \{3,4, \ldots n\},\ \left|P_V(y-z)\right| = \kappa_\rho^{\lambda_1}\left|\alpha_1+d\right|\right\} \end{aligned} \tag{59}$$

which is the boundary of a circle radius $\kappa_\rho^{\lambda_1}\left(d-\left|\alpha_1\right|\right)$ in the 2-plane $z+\alpha_1\left(e_3+e_4+\ldots e_n\right)+V$.

Now let $\vartheta_{k+1}$ be the radius of the two congruent spheres given by $\partial \Psi\left(z, V, \kappa_\rho^{\lambda_1}, d\right) \cap \partial H\left(z, V, \phi_k\right)$. By equations (58), (59) we have that

$$\begin{aligned} \vartheta_{k+1} & = \kappa_\rho^{\lambda_1} d-\kappa_\rho^{\lambda_1} \phi_k \\ & = \kappa_\rho^{\lambda_1} d-\kappa_\rho^{\lambda_1}\left(\sum_{j=0}^{k} \xi_\rho \kappa_\rho^{\lambda_1} d\left(1-\zeta_\rho^{\lambda_1}\right)^j\right) \\ & = \kappa_\rho^{\lambda_1} d-\xi_\rho\left(\kappa_\rho^{\lambda_1}\right)^2 d\left(\sum_{j=0}^{k}\left(1-\zeta_\rho^{\lambda_1}\right)^j\right). \end{aligned} \tag{60}$$

And note

$$\sum_{j=0}^{k}\left(1-\zeta_{\rho}^{\lambda_1}\right)^{j} = \frac{1}{\zeta_{\rho}^{\lambda_1}}\left(1-\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k+1}\right).$$

Recall $\kappa_{\rho}^{\lambda_1}=\frac{\zeta_{\rho}^{\lambda_1}}{\xi_{\rho}}$ so putting the above expression into (60) we get

$$\vartheta_{k+1} = \kappa_{\rho}^{\lambda_1} d\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k+1}$$

which is exactly the width of the cylinder $\Gamma_{k+1}$. Now as $\vartheta_{k+1}$ is the biggest radius of the spheres given by $\psi^{\perp}$ slices of

$$\partial\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap\left(H\left(z,V,\phi_{k+1}\right)\backslash H\left(z,V,\phi_{k}\right)\right).$$

So we know that $\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap\left(H\left(z,V,\phi_{k+1}\right)\backslash H\left(z,V,\phi_{k}\right)\right)$ is "thin" enough to fit into $\Gamma_{k+1}$. As it is also, by definition, short enough to fit into $\Gamma_{k+1}$. So by inductive assumption (57) we have

$$\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap H\left(z,V,\phi_{k+1}\right)\subset\bigcup_{j=0}^{k+1}\Gamma_j.$$

This establishes Step 3.

*Step 4:*

Recall $z\in V+x$ such that $\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\subset B_{4d}(x)$.

Suppose $\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap S_1\neq\emptyset$. Let $k_1=\max\left\{k:\Gamma_k\cap\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap S_1\neq\emptyset\right\}$. We will show $k_1$ is sufficiently big so that $\lambda_1\left(1-\zeta_{\rho}^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_{\rho}\kappa_{\rho}^{\lambda_1}d\right)^{s}<2\epsilon d^{s}$.

We argue by contradiction and assume otherwise, so

$$\lambda_1\left(1-\zeta_{\rho}^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_{\rho}\kappa_{\rho}^{\lambda_1}d\right)^{s}\geq 2\epsilon d^{s}. \tag{61}$$

We know that $\Gamma_{k_1+1}\cap\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap S_1=\emptyset$. Let $y_1\in\Gamma_{k_1}\cap\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\cap S_1$, let $h=|P_V(y_1-z)|$ so by definition of $\Gamma_k$ (see (55)) we know that

$$h\in\left(\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k_1+1}\kappa_{\rho}^{\lambda_1}d,\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k_1}\kappa_{\rho}^{\lambda_1}d\right). \tag{62}$$

Now $y_1\in\Psi\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)\subset B_{2d}(x)$ and as $h\xi_{\rho}\leq\xi_{\rho}\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k_1}\kappa_{\rho}^{\lambda_1}d=\zeta_{\rho}^{\lambda_1}\left(1-\zeta_{\rho}^{\lambda_1}\right)d<d$

$$X^{+}\left(y_1,n_{y_1},\rho\right)\cap B_{h\xi_{\rho}}(y_1)\subset B_d(y_1)\subset B_{4d}(x). \tag{63}$$

And by the fact that $y_1\in S_1$ (recall (49)) we know that

$$\mu\left(B_{h\xi_{\rho}}(y_1)\cap X^{+}\left(y_1,n_{y_1},\rho\right)\right)\geq\lambda_1\left(\xi_{\rho}h\right)^{s}.$$

By using (61), (62) we know $\lambda_1\left(\xi_{\rho}h\right)^{s}\geq 2\epsilon d^{s}$ and so from (63) and the density estimate (50) we know that we have

$$\begin{aligned}\mu\left(B_{h\xi_{\rho}}(y_1)\cap X^{+}\left(y_1,n_{y_1},\rho\right)\cap S_1\right) &\geq \lambda_1\left(\xi_{\rho}h\right)^{s}-\epsilon d^{s}\\ &\geq \frac{\lambda_1}{2}\left(\xi_{\rho}h\right)^{s}\end{aligned}$$

and so by equations (51), (52) we can pick $y_2\in X^{+}\left(y_1,n_{y_1},\rho\right)\cap B_{h\xi_{\rho}}(y_1)\cap S_1$ such that

$$y_2\in K\left(z,V,\left(1-\zeta_{\rho}^{\lambda_1}\right)h\right)\subset K\left(z,V,\left(1-\zeta_{\rho}^{\lambda_1}\right)^{k_1+1}\kappa_{\rho}^{\lambda_1}d\right). \tag{64}$$

Now $y_1\in\Psi^{u}\left(z,V,\kappa_{\rho}^{\lambda_1},d\right)$ which means for each $j\in\{3,4,\dots n\}$ we have

$$h=|P_V(y_1-z)|\leq\kappa_{\rho}^{\lambda_1}\left|P_{\langle e_j\rangle}\left(y_1-(z+de_j)\right)\right|.$$

Thus from (64)

$$|P_V(y_2-z)|\leq\left(1-\zeta_{\rho}^{\lambda_1}\right)h\leq\left(1-\zeta_{\rho}^{\lambda_1}\right)\kappa_{\rho}^{\lambda_1}\left|P_{\langle e_j\rangle}\left(y_1-(z+de_j)\right)\right|. \tag{65}$$

And since $y_2 \in B_{h\xi_\rho}(y_1)$

$$\begin{aligned} \left|P_{\langle e_j\rangle}(y_2-(z+de_j))\right| &\geq \left|P_{\langle e_j\rangle}(y_1-(z+de_j))\right| - h\xi_\rho \\ &\geq \left|P_{\langle e_j\rangle}(y_1-(z+de_j))\right| - \xi_\rho\kappa_\rho^{\lambda_1}\left|P_{\langle e_j\rangle}(y_1-(z+de_j))\right| \\ &= \left|P_{\langle e_j\rangle}(y_1-(z+de_j))\right|\left(1-\zeta_\rho^{\lambda_1}\right). \end{aligned} \tag{66}$$

Putting (65) and (66) together we have for each $j \in \{3,4,\ldots n\}$

$$|P_V(y_2-z)| \leq \kappa_\rho^{\lambda_1}\left|P_{\langle e_j\rangle}(y_2-(z+de_j))\right|.$$

So

$$y_2 \in \Psi^u\left(z,V,\kappa_\rho^{\lambda_1},d\right).$$

In the same way $y_1 \in \Psi^d\left(z,V,\kappa_\rho^{\lambda_1},d\right)$ implies

$$y_2 \in \Psi^d\left(z,V,\kappa_\rho^{\lambda_1},d\right).$$

Hence we have that

$$y_2 \in \Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right).$$

Now from (56) we know that $(y_2-z)$ is "short" enough (i.e. $|(y_2-z)\cdot e_k|$ is small enough for each $k \in \{3,4,\ldots n\}$) and from (64) we know $(y_2-z)$ "thin" enough (i.e. $|P_V(y_2-z)|$ is small enough) for (recall the definition of $\Gamma_k$, see (55))

$$y_2 \in \Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right) \cap \left(\bigcup_{j=k_1+1}^{\infty}\Gamma_j\right) \cap S_1$$

thus contradicting the maximality of $k_1$.

So we have established Step 4 and thus we know (recall (61)) that

$$\lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho\kappa_\rho^{\lambda_1}d\right)^s < 2\epsilon d^s. \tag{67}$$

Let $\theta = \left(1-\zeta_\rho^{\lambda_1}\right)^{k_1}\kappa_\rho^{\lambda_1}d$ recall, this is the width of cylinder $\Gamma_{k_1}$. So

$$2\epsilon d^s > \lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho\kappa_\rho^{\lambda_1}d\right)^s = \lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^s\xi_\rho^s\theta^s.$$

Thus

$$\theta < \frac{(2\epsilon)^{\frac{1}{s}}d}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}.$$

So we have shown that for any $z \in V+x$ such that $\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right) \subset B_{4d}(x)$ if we know that

$$\Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right) \cap S_1 \neq \emptyset$$

then

$$K\left(z,V,\frac{(2\epsilon)^{\frac{1}{s}}d}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}\right) \cap B_{2d}(x) \cap S_1 \neq \emptyset.$$

Since we know that for all $z \in (V+x) \cap K\left(x,V,\kappa_\rho^{\lambda_1}d\right)$ we have

$$x \in \Psi\left(z,V,\kappa_\rho^{\lambda_1},d\right)$$

and letting $\vartheta_\rho^{\lambda_1} = \frac{2^{\frac{1}{s}}}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}$ completes the proof. □

**Lemma 9.** *Let $n \geq 3$ be integer and let $s \in (0,2]$. Given $s$-uniform measure $\mu$ on $l_\infty^n$ (*Spt$\mu$ *is a purely unrectifiable in the case $s=2$). For any $\rho > 0$ for $\mu$ a.e. $x \in$* Spt$\mu$ *we have*

$$\liminf_{r\to 0}\frac{\mu\left(\hat{S}_j^{(x,\rho)} \cap C_r(x)\right)}{r^s} = 0,$$

*for some $j \in \{1,2,\ldots 2n\}$.*

**Proof** As the proof is basically identical to the proof of Lemma 15 [6] we only sketch the arguments.

Suppose for some $\rho > 0$ and some subset $B \subset \mathrm{Spt}\mu$ we have that

$$\mu\left(\hat{S}_j^{(x,\rho)} \cap C_r(x)\right) \geq \lambda r^s \tag{68}$$

for all $j \in \{1, 2, n+1, n+2\}$, $x \in B$ and $r \in (0, r_0)$ where $r_0 > 0$ is some small number. Assume $B$ is compact. Let $M > 0$ be some large number such that $B \subset C_M(0)$. Let $S := C_M(0) \cap \mathrm{Spt}\mu$. Note that it is immediate from Theorem 6.9 [11] that $\gamma := H^s_{\lfloor S}$ is an A.D. $s$-measure. Infact by Theorem 6.9. there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1\mu(A) \leq \gamma(A) \leq c_2\mu(A) \tag{69}$$

for any $A \subset \mathbb{R}^n$. Let $\epsilon > 0$ be very small and $x_0$ be a density point of $B$. Let $r > 0$ be such that $H^s(C_{8r}(x_0) \backslash B) < \epsilon r^s$.

Let $V = \langle e_1, e_2 \rangle$. We know that the orthogonal projection of $B$ onto a subspace arbitrarily close to $V$ has zero Lebesgue measure. As before, to simplify notation we assume the projection onto $V$ has zero Lebesgue measure. So we have many empty cylinders (parallel to $V^\perp$) running up through our cube $C$.

By (68), (69) we can apply Lemma 8 and so we know that the cylinders inside $K\left(x_0, V, \kappa_\rho^\lambda r\right) \cap C_r(x_0)$ have width less than $\beta_\lambda \epsilon^{\frac{1}{s}} r$ and thus there are approximately $P := \left[\frac{\left(\kappa_\rho^\lambda\right)^2}{\left(\epsilon^{\frac{1}{s}}\beta_\lambda\right)^2}\right]$ of them.

See figure 8. Now we expand these cylinders until they touch the first point of our set $B$, call this point $z$. Since for some $j \in \{1, 2, n+1, n+2\}$, $\hat{S}_j^{(z,\rho)}$ (with a reasonable amount of $\mathrm{Spt}\mu$ in it) must stick inside our cylinder, so most of these points must be very close to the boundary of $C$.

So in fact we must have (roughly) $P$ disjoint cubes

$$\left\{C_{\beta_\lambda \epsilon^{\frac{1}{s}} r}(z_1), \ldots C_{\beta_\lambda \epsilon^{\frac{1}{s}} r}(z_P)\right\} \subset C_{r\left(1+2\epsilon^{\frac{1}{s}}\beta_\lambda\right)}(x_0) \backslash C_{r\left(1-2\epsilon^{\frac{1}{s}}\beta_\lambda\right)}(x_0)$$

and thus

$$\begin{aligned} \frac{\left(\kappa_\rho^\lambda\right)^2}{2\left(\epsilon^{\frac{1}{s}}\beta_\lambda\right)^2}\epsilon\left(\beta_\lambda r\right)^s &\leq P\epsilon\left(\beta_\lambda r\right)^s \\ &\leq \left(\left(1+2\epsilon^{\frac{1}{s}}\beta_\lambda\right)^s - \left(1-2\epsilon^{\frac{1}{s}}\beta_\lambda\right)^s\right) r^s \end{aligned}$$

as $s \in (0, 2]$ this gives a contradiction for small enough $\epsilon > 0$.

Thus for $\mu$ a.e. $x \in \mathrm{Spt}\mu$

$$\liminf_{r \to 0} \frac{\mu\left(\hat{S}_j^{(x,\rho)} \cap C_r(x)\right)}{r^s} = 0$$

for some $j \in \{1, 2, n+1, n+2\}$.

## 9. Proposition 1 continued.

Recall, we are arguing by contradiction and have established in Lemmas 6, 7 some strong properties of tangent measures $\nu \in \mathrm{Tan}(\mu, x_0)$. All that remains for us to get a contradiction is to show that there exists $\lambda \in \mathrm{Tan}(\mu, x_0)$ such that $\lambda\left(S_i^{(0)}\right) = 0$. So to begin with take (arbitrary) $\nu \in \mathrm{Tan}(\mu, x_0)$.

As we know from Lemma 9 so for $\nu$ a.e. $x$ we can find a sequence $r_n \to 0$ such that

$$\frac{\nu\left(\hat{S}_j^{\left(x,2^{-n}\right)} \cap C_{r_n}(x)\right)}{r_n^s} \to 0 \tag{70}$$

as $n \to \infty$. This must imply that $\frac{\nu\left(S_j^{(x)} \cap C_{r_n}(x)\right)}{r_n^s} \to 0$ because otherwise the tangent measure $\tilde{\nu} := \lim_{n\to\infty} \frac{T_{x,r_n} \sharp \nu}{r_n^s}$ will be such that $\tilde{\nu}\left(\partial S_j^{(0)}\right) > 0$, which we have already shown is impossible for measures in $\text{Tan}(\mu, x_0)$ (i.e. because it contradicts (24)). Taking tangent measure $\lambda := \lim_{n\to\infty} \frac{T_{x,r_n} \sharp \nu}{r_n^s}$ we have that $\lambda\left(S_j^{(0)}\right) = 0$. By Lemma 2 $\lambda \in \text{Tan}(\mu, x_0)$ and thus the existence of $\lambda$ contradicts Lemma 7. This completes the proof of Proposition 1. □

## 10. Proof of Theorem 3 continued.

### 10.1. Intermediate Lemma.

**Lemma 10.** *Given an $s$-uniform measure $\mu$ on $l_\infty^n$, $s \in \{1, 2, \dots n-1\}$ and*

$$\text{Spt}\mu \subset \left\{f(t)\, e_i + t : t \in e_i^\perp\right\}$$

*where $f : e_i^\perp \to \mathbb{R}$ is 1-Lipschitz. Let $\nu := P_{e_i^\perp} \sharp \mu$. Suppose for some point $a \in \text{Spt}\nu$ we have that*

$$\widetilde{\text{Tan}}(\nu, a) \cap G(s, n-1) \neq \emptyset$$

*then for some $q \in \text{Spt}\mu$*

$$\widetilde{\text{Tan}}(\mu, q) \cap G(s, n) \neq \emptyset.$$

**Proof**

First some notation. It may help to distinguish between cubes in $l_\infty^n$ and cubes in $e_i^\perp$. So we will let $\tilde{C}_r(x)$ denote the cube in subspace $e_k^\perp$ of radius $r > 0$ centered at $x \in e_k^\perp$.

Let $b := f(a)\, e_i + a$. Recall notation; $T_{x,r}(z) := \frac{z-x}{r}$. Let $r_n \to 0$ be a sequence such that

$$\lim_{n\to\infty} \frac{T_{a,r_n} \sharp \nu}{r_n^s} = \lambda H^s_{\lfloor V} \tag{71}$$

for some $V \in G(s, n-1)$, (recall explanation given at the start of section 2.2).

*Step 1:*

We will prove we can find some subsequence $(r_{k_n})$ of $(r_n)$ such that measure

$$\tilde{\mu} := \lim_{n\to\infty} \frac{T_{b,r_{k_n}} \sharp \mu}{r_{k_n}^s}. \tag{72}$$

is such that $\text{Spt}\tilde{\mu} \cap P_{e_i^\perp}^{-1}\left(\tilde{C}_1(0)\right)$ is contained in the graph of a 1-Lipschitz function.

To begin with we will show that

$$J_m := T_{b,r_m}\left(\left\{f(t)\, e_i + t : t \in e_i^\perp\right\}\right)$$

is a 1-Lipschitz graph. Take $x, y \in \left\{f(t)\, e_i + t : t \in e_i^\perp\right\}$. Let $x_i := x \cdot e_i$, $y_i := y \cdot e_i$. Note that since $|x_i - y_i| \leq \|P_{e_i^\perp}(x-y)\|$ we have

$$\begin{aligned}
\left|\left(T_{b,r_m}(x) - T_{b,r_m}(y)\right) \cdot e_i\right| &= \left|\frac{x_i}{r_m} - \frac{y_i}{r_m}\right| \\
&\leq \frac{1}{r_m}\|P_{e_i^\perp}(x-y)\| \\
&= \left\| \begin{pmatrix} \frac{x_1}{r_m} \\ \dots \\ \frac{x_{i-1}}{r_m} \\ \frac{x_{i+1}}{r_m} \\ \dots \\ \frac{x_n}{r_m} \end{pmatrix} - \begin{pmatrix} \frac{y_1}{r_m} \\ \dots \\ \frac{y_{i-1}}{r_m} \\ \frac{y_{i+1}}{r_m} \\ \dots \\ \frac{y_n}{r_m} \end{pmatrix} \right\| \\
&= \|P_{e_i^\perp}\left(T_{b,r_m}(x)\right) - P_{e_i^\perp}\left(T_{b,r_m}(y)\right)\|.
\end{aligned} \tag{73}$$

Its easy to see that $P_{e_i^\perp}(J_m) = e_i^\perp$. Define $G_m : e_i^\perp \to \mathbb{R}$ by

$$G_m(x) := \left(P_{e_i^\perp}^{-1}(x) \cap J_m\right) \cdot e_i$$

and by (73) we know that $G_m$ is 1-Lipschitz. So by Ascoli-Azerla we can find a subsequence $(l_m)$ of $(k_m)$ such that $G_{l_m}$ converges uniformly on $P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right)$ to $G$, for some 1-Lipschitz function $G$.

And since $(l_m)$ is a subsequence of $(k_m)$

$$\frac{T_{b,r_{l_m}\,\sharp}\mu}{r^s_{l_m}} \rightharpoonup \tilde{\mu}. \tag{74}$$

Note that $x \in \mathrm{Spt} T_{b,r_n\,\sharp}\mu$ is equivalent to the fact that

$$T_{b,r_n\,\sharp}\mu\left(B_\epsilon(x)\right) = \mu\left(r_n B_\epsilon(x) + b\right) > 0 \;\; \forall\, \epsilon > 0,$$

which is equivalent to

$$x r_n + b = T^{-1}_{b,r_n}(x) \in \mathrm{Spt}\mu.$$

Thus

$$\mathrm{Spt} T_{b,r_n\,\sharp}\mu = \{T_{b,r_n}(y) : y \in \mathrm{Spt}\mu\} \subset J_n = \left\{G_n(x)\, e_i + x : x \in e_i^{\perp}\right\}.$$

Since $G_n$ converge uniformly to $G$ inside $P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right)$, for any $\epsilon > 0$ we can find $M$ sufficiently large so that

$$J_m \cap P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right) \subset N_\epsilon\left(\left\{G(x)\, e_i + x : x \in \tilde{C}_1(0)\right\}\right)$$

for all $m > M$.

By definition of weak convergence, we have (recall (74))

$$\mathrm{Spt}\tilde{\mu} \cap P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right) \subset N_\epsilon\left(\left\{G(x)\, e_i + x : \tilde{C}_1(0)\right\}\right)$$

and as this is true for every $\epsilon > 0$ we have shown Step 1.

*Step 2:*

We will show that $\mathrm{Spt}\tilde{\mu} \cap P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right) \subset P^{-1}_{e_i^{\perp}}(V)$.

Firstly by definition of weak tangent, for any $\epsilon > 0$ we have

$$\lim_{n\to\infty} \frac{\nu\left(\tilde{C}_{r_n}(a) \backslash N_{\epsilon r_n}(a+V)\right)}{r_n^s} = 0.$$

Note that from the definition of measure $\nu$ and the fact that $P_{e_i^{\perp}}(a) = P_{e_i^{\perp}}(b)$

$$\begin{aligned} \nu\left(\tilde{C}_{r_n}(a) \backslash N_{\epsilon r_n}(a+V)\right) &= T_{a,r_n\,\sharp}\nu\left(\tilde{C}_1(0) \backslash N_\epsilon(V)\right) \\ &= T_{a,r_n\,\sharp}\mu\left(P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0) \backslash N_\epsilon(V)\right)\right) \\ &= T_{b,r_n\,\sharp}\mu\left(P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0) \backslash N_\epsilon(V)\right)\right). \end{aligned} \tag{75}$$

So

$$\lim_{n\to\infty} \frac{T_{b,r_n\,\sharp}\mu\left(P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0) \backslash N_\epsilon(V)\right)\right)}{r_n^s} = \lim_{n\to\infty} \frac{\nu\left(\tilde{C}_{r_n}(a) \backslash N_{\epsilon r_n}(a+V)\right)}{r_n^s} = 0.$$

Thus (recall definition (72))

$$\tilde{\mu}\left(P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0) \backslash N_\epsilon(V)\right)\right) = 0$$

for all $\epsilon > 0$. Hence we finally have

$$\mathrm{Spt}\tilde{\mu} \cap P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0)\right) \subset P^{-1}_{e_i^{\perp}}\left(\tilde{C}_1(0) \cap V\right)$$

and we have shown Step 2.

*Step 3:*

We will show that for some point $c \in \mathrm{Spt}\tilde{\mu}$ we have $\widetilde{\mathrm{Tan}}(\tilde{\mu}, c) \cap G(s,n) \neq \emptyset$.

Firstly note that by putting Step 1 and Step 2 together we have

$$\begin{aligned}\mathrm{Spt}\tilde{\mu} \cap P_{e_i^\perp}^{-1}\left(\tilde{C}_1(0)\right) &\subset \left\{G(x)e_i + x : x \in e_i^\perp\right\} \cap P_{e_i^\perp}^{-1}\left(\tilde{C}_1(0) \cap V\right) \\ &= \left\{G(x)e_i + x : x \in \tilde{C}_1(0) \cap V\right\}. \end{aligned} \tag{76}$$

By (71)

$$\tilde{\nu} := \lim_{n\to\infty} \frac{T_{a,r_{l_n}} \sharp \nu}{r_{l_n}^s} = \lambda H^s_{\lfloor V}. \tag{77}$$

Let $x \in V$, as before, from the definition of measure $\tilde{\nu}$

$$\lambda 2^s \epsilon^s = \tilde{\nu}\left(\tilde{C}_\epsilon(x)\right) = \lim_{n\to\infty} \frac{T_{a,r_{l_n}} \sharp \nu\left(\tilde{C}_\epsilon(x)\right)}{r_{l_n}^s} = \lim_{n\to\infty} \frac{T_{a,r_{l_n}} \sharp \mu\left(P_{e_i^\perp}^{-1}\left(\tilde{C}_\epsilon(x)\right)\right)}{r_{l_n}^s}. \tag{78}$$

So since $P_{e_i^\perp}(a) = P_{e_i^\perp}(b)$ (recall definition (74))

$$\begin{aligned}\tilde{\mu}\left(P_{e_i^\perp}^{-1}(C_\epsilon(x))\right) &= \lim_{n\to\infty} \frac{T_{b,r_{l_n}} \sharp \mu\left(P_{e_i^\perp}^{-1}\left(\tilde{C}_\epsilon(x)\right)\right)}{r_{l_n}^s} \\ &= \lim_{n\to\infty} \frac{T_{a,r_{l_n}} \sharp \mu\left(P_{e_i^\perp}^{-1}\left(\tilde{C}_\epsilon(x)\right)\right)}{r_{l_n}^s} \\ &\overset{(78)}{=} \lambda 2^s \epsilon^s.\end{aligned}$$

Thus $\mathrm{Spt}\tilde{\mu} \cap P_{e_i^\perp}^{-1}(x) \neq \emptyset$. Now $G : V \to \mathbb{R}$ is a 1-Lipschitz function from an $s$-dimensional subspace to $\mathbb{R}$. So by Rademacher's Theorem it is differentiable for $L^s$ a.e. $x \in V$.

Now pick some point $x_0 \in V \cap \tilde{C}_{\frac{1}{2}}(0)$ for which $G$ is differentiable at $x_0$. Thus for any $\epsilon > 0$ we can find $\delta > 0$ such that for any $z \in V \cap \tilde{C}_\delta(x_0)$ we have

$$|G(z) - G(x_0) - \mathcal{D}G(x_0) \cdot (z - x_0)| < \epsilon |z - x_0|.$$

So

$$G(z) \in N_{\epsilon\delta}(G(x_0) - \mathcal{D}G(x_0) \cdot (z - x_0)) \quad \text{for all } z \in \tilde{C}_\delta(x_0) \cap V.$$

Let $W := \{(\mathcal{D}G(x_0) \cdot z)e_i + z : z \in V\}$, $W$ is an $s$ dimensional subspace. So we have

$$\left\{(G(x)e_i + x) : x \in V \cap \tilde{C}_\delta(x_0)\right\} \subset N_{\epsilon\delta}((G(x_0)e_i + x_0) + W).$$

And by (76) this implies

$$\mathrm{Spt}\tilde{\mu} \cap P_{e_i^\perp}^{-1}\left(\tilde{C}_\delta(x_0)\right) \subset N_{\epsilon\delta}((G(x_0)e_i + x_0) + W).$$

Now as $\epsilon > 0$ was arbitrary, by the same argument we can find a sequence $\delta_n \to 0$ such that

$$\mathrm{Spt}\tilde{\mu} \cap P_{e_i^\perp}^{-1}\left(\tilde{C}_{\delta_n}(x_0)\right) \subset N_{\delta_n 2^{-n}}((G(x_0)e_i + x_0) + W).$$

Let $z_0 = P_{e_i^\perp}^{-1}(x_0) \cap \left\{G(y)e_i : y \in e_i^\perp\right\}$, and let $\lambda := \lim_{n\to\infty} \frac{T_{z_0,\delta_n} \sharp \tilde{\mu}}{\delta_n^s}$. So we have

$$\mathrm{Spt}\lambda \subset W \in G(s,n).$$

As by Lemma 2, $\lambda \in \mathrm{Tan}(\tilde{\mu}, z_0) \subset \mathrm{Tan}(\mu, b)$ we have completed the proof of the lemma. □

10.2. **Proof of Theorem.** Our plan is to use Proposition 1 to repeatedly reduce the dimension of the ambient space in which the measure lives.

Recall, we are arguing by contradiction, so assuming our theorem is false we have an $s$-uniform measure $\mu$ in $l_\infty^n$ that satisfies the controlled complement condition (recall definition (6), (7)) which in the case where $s$ is an integer has no weak $s$-tangents at any point of its support.

It is worth noting that the *only* results we have at our disposal to prove the theorem are Lemmas 1, 2, 3 and Proposition 1.

So firstly by Lemma 3 we have that $\mu$ is symmetric and by Proposition 1 for $\mu$ a.e. $x \in \mathrm{Spt}\mu$ we can find a tangent measure $\tilde{\mu} \in \mathrm{Tan}(\mu, x)$ such that $\mathrm{Spt}\tilde{\mu} \cap C_1(0)$ is contained in the graph of a 1-Lipschitz function $g_1 : e_k^{\perp} \to \langle e_k \rangle$ for some $k \in \{1, 2, \ldots n\}$. In addition by Lemma 1 we know that $\tilde{\mu}$ is $s$-uniform and satisfies the controlled complement condition. Let $\tilde{\nu_1} \in \mathrm{Tan}\,(0, \tilde{\mu})$, $\tilde{\nu_1}$ is again $s$-uniform with the controlled complement condition and as we have seen before in Step 1 of Lemma 10 its entire support contained in a graph of a 1-Lipschitz function $h_1 : e_k^{\perp} \to \langle e_k \rangle$. Finally define a measure $\mu_1$ on $e_k^{\perp}$ by $\mu_1 := \tilde{\nu_1}_{\sharp P_{e_k^{\perp}}}$, i.e. for any subset $A \subset e_k^{\perp}$

$$\mu_1(A) = \tilde{\nu_1}\left(P_{e_k^{\perp}}^{-1}(A)\right).$$

See figure 10. Its clear that $\mathrm{Spt}\mu_1 = P_{e_k^{\perp}}(\mathrm{Spt}\tilde{\nu_1})$. Again to avoid confusion we will let $\tilde{C}_r(x)$

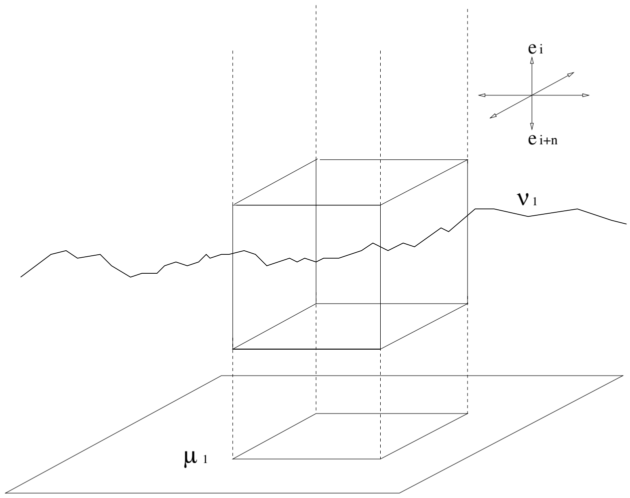


FIGURE 10

denote the cube in subspace $e_k^{\perp}$ radius $r$ centered at $x \in e_k^{\perp}$. Now for any $x \in \mathrm{Spt}\mu_1$ let $y = P_{e_k^{\perp}}^{-1}(x) \cap \mathrm{Spt}\tilde{\nu_1}$, we have that

$$\begin{aligned} \mu_1\left(\tilde{C}_r(x)\right) &= \tilde{\nu_1}\left(P_{e_k^{\perp}}^{-1}\left(\tilde{C}_r(x)\right)\right) \\ &= \tilde{\nu_1}\left(C_r(y)\right), \end{aligned}$$

by the fact that $\mathrm{Spt}\tilde{\nu_1}$ is contained in the graph of $h_1$.

Thus $\mu_1$ is an $s$-uniform measure in $e_k^{\perp}$, i.e. an $s$-uniform measure on $l_\infty^{N-1}$. We can repeatedly make this reduction until either:

$$\begin{aligned} &\text{We have an } s\text{-uniform measure } \hat{\mu} \text{ on } l_\infty^q \text{ for which} \\ &\exists\, y \in \mathrm{Spt}\hat{\mu} \text{ with } \widetilde{\mathrm{Tan}}\,(\hat{\mu}, y) \cap G(s, q) \neq \emptyset. \end{aligned} \tag{79}$$

Or

$$\text{We have an } s\text{-uniform measure } \tilde{\mu} \text{ on } l_\infty^3. \tag{80}$$

Note that if $s$ is not an integer then the hypotheses of Proposition 1 are always satisfied and hence we must have case (80). However let us argue first for $s = 2$.

*Step 1:*

So in either case, (79) or (80) we have a 2-uniform measure in some space, say $l_\infty^p$ with weak 2-tangents. This measure is arrived at by a finite sequence of blow ups (i.e. taking tangent measures) and projections down onto a subspaces.

Formally; for some $m \in \mathbb{N}$ we have a sequence measures $\mu_1 := \mu, \mu_2, \ldots \mu_m$ where for each $k \in \{2, 3, \ldots m\}$ either

$$\mu_k \in \mathrm{Tan}\,(\mu_{k-1}, x_{k-1}) \text{ for some } x_{k-1} \in \mathrm{Spt}\mu_{k-1}$$

or

$$\mu_k = P_{e^{\perp}_{p(k)}\sharp}\mu_{k-1} \text{ for some } p(k) \in \{1, 2, \dots n\}.$$

Now suppose that (in fact this is the case but we do not need it) $\mu_m = P_{e^{\perp}_{p(m)}\sharp}\mu_{m-1}$ for some $p(m) \in \{1, 2, \dots n\}$ then by Lemma 10 we know that $\mathrm{Spt}\mu_{m-1}$ has a point $b \in \mathrm{Spt}\mu_{m-1}$ such that $\widetilde{\mathrm{Tan}}(\mu_{m-1}, b) \cap G(2, p+1)$, i.e. it has a weak tangent.

If $\mu_{m-1} \in \mathrm{Tan}(\mu_{m-2}, x_{m-2})$ for some $x_{m-2} \in \mathrm{Spt}\mu_{m-2}$ then by Lemma 2 we have that $\widetilde{\mathrm{Tan}}(\mu_{m-2}, x_{m-2}) \cap G(2, p+1) \neq \emptyset$. And we can continue "pulling back" the weak tangent until we reach measure $\mu_{m-k}$ which is not a tangent measure of the previous measure, i.e. $\mu_{m-k} = P_{e^{\perp}_{p(m-k)}\sharp}\mu_{m-k-1}$ [1]. Here again we apply Lemma 10 to establish that there is a point $c \in \mathrm{Spt}\mu_{m-k-1}$ such that $\widetilde{\mathrm{Tan}}(\mu_{m-k-1}, c) \cap G(2, p+2) \neq \emptyset$. And so we can continue, until we finally have shown that there exists a point $d \in \mathrm{Spt}\mu$ such that $\mathrm{Tan}(\mu, d) \cap G(2, n) \neq \emptyset$. So we have a contradiction and for the case $s = 2$ the theorem is proved.

*Step 2:*

Now we will argue for $s \in (0, 2)$. If $s = 1$ and we had case (79) we could argue as in Step 1 and establish a contradiction. So if $s = 1$ we only need to deal with case (80) which we will do presently.

If $s \neq 1$ then only case (80) can occur (because as noted, the hypotheses of Proposition 1 are always satisfied) so whether $s = 1$ or not we only need to deal with case (80).

We again push forward the measure, this time onto a 2-subspace. So we have an $s$-uniform measure $\nu$ in $l^2_\infty$. In this case we claim:

For any $\rho > 0$, for $\nu$ a.e. $x \in \mathrm{Spt}\nu$ we have that $\liminf_{r\to 0} \frac{\nu\left(\hat{S}^{(x,\rho)}_j \cap C_r(x)\right)}{r^s} = 0$ for some $j \in \{1, 2, 3, 4\}$.

This follows from the most basic version of Besicovitch touching point argument. We will sketch the details.

Suppose the claim was false to we have a closed set $B \subset \mathrm{Spt}\nu$ of positive $\nu$ measure such that for some $\rho > 0$ and some $\delta_0 > 0$, $r_0 > 0$ we have

$$\nu\left(\hat{S}^{(x,\rho)}_j \cap C_r(x)\right) > \delta_0 r^s$$

for all $j \in \{1, 2, 3, 4\}$, $x \in B$, $r \in (0, r_0)$.

We note that for any ball $B_r(y)$ in $\mathbb{R}^2$, if we have point $z \in \partial B_r(y)$ then for some $j \in \{1, 2, 3, 4\}$ we have $\hat{S}^{(z,\rho)}_j \cap C_{\lambda_\rho r}(z) \subset B_r(y)$ where $\lambda_\rho > 0$ is some constant depending only on $\rho$. Let $\epsilon > 0$ be some small number to be decided on later, we take a density point $x \in B$ and some $r \in (0, r_0)$ such that

$$\nu(C_r(x) \backslash B) < \epsilon r^s.$$

Now let $y \in C_{\frac{r}{4}}(x) \backslash \mathrm{Spt}\nu$, take number $\alpha > 0$ such that $B_\alpha(y) \subset C_r(x) \backslash B$ and $\partial B_\alpha(y) \subset C_r(x) \cap B \neq \emptyset$. Pick $z \in \partial B_\alpha(y) \cap B$. So for some $j \in \{1, 2, 3, 4\}$ we have

$$\hat{S}^{(z,\rho)}_j \cap C_{\lambda_\rho \alpha}(z) \subset B_\alpha(y).$$

Since we have $\mathrm{Spt}\nu \cap \hat{S}^{(z,\rho)}_j \cap C_{\lambda_\rho \alpha}(z) \subset \mathrm{Spt}\nu \backslash B$ we have

$$\begin{aligned} \delta_0 (\lambda_\rho \alpha)^s &< \nu\left(\hat{S}^{(z,\rho)}_j \cap C_{\lambda_\rho \alpha}(z)\right) \\ &< \epsilon r^s. \end{aligned} \tag{81}$$

So we have $\alpha \leq \left(\frac{\epsilon}{\delta_0}\right)^{\frac{1}{s}} \frac{r}{\lambda_\rho}$ and so in this way we can show all the holes in $C_{\frac{r}{4}}(x) \cap B$ are very small compared to $r$, so we can "fill up" the 2-dimensional square $C_{\frac{r}{4}}$ with points of $\mathrm{Spt}\nu$. Formally; let $\varsigma = \left(\frac{\epsilon}{\delta_0}\right)^{\frac{1}{s}} \frac{r}{\lambda_\rho}$ for every $y \in C_{\frac{r}{4}}$ we can find a point $z_y \in C_\varsigma(y) \cap B$.

[1] in fact we do not "take a tangent measure" more than once "in between" projections down onto subspaces, however as this in no way simplifies the proof we make no special note of it.

So $\left\{C_{\varsigma}\left(z_y\right) : y \in C_{\frac{r}{4}}\right\}$ forms a cover of $C_{\frac{r}{4}}$ and we can apply the $5r$ Covering Theorem to get a disjoint set $\left\{C_{\varsigma}\left(z_{y_k}\right) : k = 1, 2, \dots p\right\}$ such that $4p\varsigma^2 \geq \frac{r^2}{100}$. However as $s \in (0,2)$ we have

$$\begin{aligned}
\nu\left(\bigcup_{k=1}^{p} C_{\varsigma}\left(z_{y_k}\right)\right) &= p\varsigma^s \\
&\geq \frac{r^2}{400}\varsigma^{s-2} \\
&\geq \frac{r^2}{400}\left(\left(\frac{\epsilon}{\delta_0}\right)^{\frac{1}{s}}\frac{r}{\lambda_\rho}\right)^{s-2} \\
&= \frac{r^s}{400}\frac{1}{\lambda_\rho^{s-2}}\left(\frac{\epsilon}{\delta_0}\right)^{\frac{s-2}{s}}.
\end{aligned}$$

So as $\epsilon \to 0$ we have $\frac{\nu(C_r(x))}{r^s} \to \infty$, which is a contradiction, so we have established our claim.

Now as before, for $\nu$ a.e. $x \in \mathrm{Spt}\nu$ we can use this to get a tangent measure $\tilde{\nu} \in \mathrm{Tan}(\nu, x)$ with the property that $\tilde{\nu}\left(S_j^{(0)}\right) = 0$. This allows us to apply Proposition 1 and find a tangent measure $\lambda \in \mathrm{Tan}(\nu, x)$ with its support inside $C_1(0)$ contained in a graph of a 1-Lipschitz function from $e_i^{\perp} \to \langle e_i \rangle$, for some $i \in \{1, 2\}$. So its immediate that $s \leq 1$. If $s = 1$ as we have seen before in Step 3 of the proof of Lemma 10; by Rademacher's Theorem and Lemma 2, $\mathrm{Spt}\nu$ has a weak 1-tangent at $x$. We can now go through the same "pulling back" procedure used in the case $s = 2$ so finally show that our measure $\mu$ has weak 1-tangents. This is contradiction, and it completes the proof of Theorem 2 for the case $s = 1$.

If $s < 1$ then we use Rademacher's Theorem to get a tangent measure $\lambda \in \mathrm{Tan}(x, \hat{\nu})$ that is supported inside a subspace $V \in G(1,2)$. We identify $V$ with the real line in the trivial way. Note that in one dimension $S_1^{(x)} \cap C_r(x) = \hat{S}_1^{(x,\delta)} \cap C_r(x) = [x, x + r)$ and $S_2^{(x)} \cap C_r(x) = \hat{S}_2^{(x,\delta)} \cap C_r(x) = (x - r, x]$. By Lemma 6, $\lambda$ is symmetric. Also note that $\lambda$ trivially has square cone density (recall definition (5)). We can then argue in the same way as Step 1 of Lemma 5 and show that $\mathrm{Spt}\lambda = \mathbb{R}$. As $s \in (0,1)$ by a simple covering and counting argument this implies $\lambda([-1,1]) = \infty$ which is a contradiction. This concludes the proof of the Theorem 2. □

## 11. Proof of Theorem 4

By the reduction of the problem given in section 3, it suffices to show that a 2-uniform measure $\mu$ has a weak tangent somewhere in its support. We argue by contradiction, so $\mu$ does not have a weak tangent anywhere in its support and so in particular is purely unrectifiable. By Lemma 5 of [6] we know $\mu$ is a symmetric measure. Hence $\mu$ satisfies the hypotheses of Proposition 1 and so we can find a tangent measure $\nu \in \mathrm{Tan}(\mu, x)$, for some $x \in \mathrm{Spt}\mu$ such that $\mathrm{Spt}\nu \cap C_1(0)$ is contained in the graph of a 1-Lipschitz function $g : e_k^{\perp} \to \langle e_k \rangle$ for some $k \in \{1, 2, 3\}$. Let $G$ denote the graph of $g$.

As we have done before we can define a 2-uniform measure $\tilde{\nu}$ on $e_k^{\perp}$ by

$$\tilde{\nu}(A) = \nu\left(P_{e_k^{\perp}}^{-1}(A)\right) \text{ for any } A \subset e_k^{\perp}.$$

Recall $\tilde{\nu}$ is 2-uniform because firstly $\mathrm{Spt}\tilde{\nu} = P_{e_k^{\perp}}(\mathrm{Spt}\nu)$ and secondly if we let $\tilde{C}_r(x)$ denote a cube of radius $r > 0$ on point $x \in \mathrm{Spt}\tilde{\nu}$ inside $e_k^{\perp}$ then as $\mathrm{Spt}\nu \cap P_{e_k^{\perp}}^{-1}\left(\tilde{C}_r(x)\right) \subset G \cap P_{e_k^{\perp}}^{-1}\left(\tilde{C}_r(x)\right)$. So $\nu\left(P_{e_k^{\perp}}^{-1}\left(\tilde{C}_r(x)\right)\right) = \nu\left(C_r\left(P_{e_k^{\perp}}^{-1}(x) \cap G\right)\right) = r^2$. Hence by a basic covering argument we see that $L^2(\mathrm{Spt}\tilde{\nu}) > 0$ (in fact $\mathrm{Spt}\tilde{\nu} = e_k^{\perp}$ but we do not need this so we do not prove it). Now by Rademacher's Theorem we must be able to find a point $x_0 \in \mathrm{Spt}\tilde{\nu}$ at which $g$ is differentiable. As a very weak consequence, this implies the set $G$ has a weak tangent $V \in G(2,3)$ at

$z_0 := P^{-1}_{e_k^{\perp}}(x_0) \cap G$. So we can find a tangent measure $\lambda \in \mathrm{Tan}(\nu, z_0)$ supported inside $V$. Hence by Lemma 2 measure $\mu$ has a weak tangent $V$ at $x$ and this contradiction completes the proof. $\square$

**Acknowledgments:**
This work was completed under the support of an EPDI fellowship, Program 2000/2002.
Thanks to Bob Jerrard for helpful comments about the presentation.

Andrew Lorent
Max Planck Institute for Mathematics,
04103 Leipzig/Germany.
lorent@mis.mpg.de

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