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by

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Abstract

In this paper we study the local and global injectivity of spatial deformations of shearable nonlinearly elastic rods. We adopt an analytical condition introduced by Ciarlet & Nečas in nonlinear elasticity to ensure global injectivity in that case. In particular we verify the existence of an energy minimizing equilibrium state without self-penetration which may be also restricted by a rigid obstacle. Furthermore we consider the special situation where the ends of the rod are glued together. In that case we can still impose topological restrictions as, e.g., that the shape of the rod belongs to a given knot type. Again we show the existence of a globally injective energy minimizer which now in addition respects the topological constraints. Note that the investigation of super-coiled DNA molecules is an important application of the presented results.

1 Introduction

Our experience tells us that matter cannot interpenetrate and, thus, the deformation of an elastic body in nature is always restricted by the presence of other deformable or rigid bodies and by the presence of itself. Often the deformation of the considered body is not too large and the case that other bodies are touched can be excluded. However the possibility of self-contact is always present. Nevertheless this phenomenon, which is also related to the global injectivity of deformations, is usually ignored in treatments in elasticity due to analytical difficulties. Another class of very natural but scarcely regarded constraints are topological restrictions as, e.g., that a closed elastic tube forming some knot cannot change the knot type during deformation as long as it does not intersect itself. In the present paper we want to show how constraints of the mentioned kind can be treated for a general class of shearable nonlinearly elastic rods which can freely deform in space. In particular we show the existence of energy minimizing equilibrium states which might have self-contact but which do not intersect itself and where the deformation can still be restricted by a rigid obstacle. Furthermore we study rods which are glued together at its ends and we verify the existence of globally injective energy minimizers respecting topological constraints such as some prescribed knot type and some prescribed link type.

A general analytical condition ensuring global injectivity for deformations in three-dimensional nonlinear elasticity was introduced in Ciarlet & Nečas [3] and the existence of an energy minimizer subjected to that side condition was verified. A slight generalization of these results can be found in Schuricht [10]. We are now interested in the question how global injectivity of deformations and self-contact can be modeled for elastic rods and whether we can verify the existence of corresponding equilibrium states. For planar deformations of elastic curves Ball [2] has proposed some approach to describe self-contact and he verified the existence of solutions without self-intersection in some sense. But these results cannot be extended to deformations in space. We readily realize that the treatment of global injectivity for elastic rods which are merely idealized as some deformable curve in space does not make sense. For a reasonable model of self-contact we in fact have to consider the elastic rod as a slender but "thick" three-dimensional body for which we prevent self-penetration. Certainly there are different ways of how to "fatten" the elastic curve. But if we have fixed a rule for doing that, then we understand contact as a real self-touching of the body in a three-dimensionally and geometrically exact way (which sometimes is also called "hard" contact). This approach can be found in v.d. Mosel [13] and in Gonzalez et al. [7] for some special situations and it is carried out in the present paper with different analytical tools in a much more general way. An alternative approach would be, e.g., to consider the elastic curve as an electrically charged wire generating repulsive forces according to a suitable potential (cf. v.d. Mosel [12] and citations therein). This way some kind of "soft" self-contact can be modeled which, however, is not intended to study in this paper.

The Cosserat theory describing the deformation of nonlinearly elastic rods in space which can suffer flexure, shear, extension, and torsion is a suitable theory as basis for the task to model self-contact. Though mathematically one-dimensional, this theory allows a mechanically natural and geometrically exact three-dimensional interpretation of deformed configurations, i.e., it precisely tells us how to fatten the rod based on the underlying three-dimensional deformation. Since that theory also allows general nonlinear constitutive relations and inhomogeneities in the geometry and the mechanical properties of the rod, it is sufficiently general to cover a large classs of problems. In this paper we show how the condition of Ciarlet & Nečas [3] can be adopted to that theory in order to exclude self-penetration. Due to the special structure which deformations in the rod theory have, this condition implies even stronger injectivity results for the three-dimensionally deformed states of the rod than for general three-dimensional deformations in elasticity as shown in [3]. Here a local injectivity result for rods, which is again stronger than the general three-dimensional analogue, is an essential ingredient for the proof. Since the set of all configurations which satisfy the constraints ensuring local and global injectivity is weakly closed in a suitable Banach space, we can verify the existence of energy minimizing equilibrium configuration without self-penetration and without intersecting a given rigid obstacle. Note that the special case of planar deformations is already studied in Schuricht [9].

Let us now imagine that we take both ends of the "thick" rod and glue them together. We realize that we have a lot of freedom in doing so and, this way, we can influence the realized equilibrium state essentially. Before we stick the ends together we can, e.g., form a prescribed knot and if the ends already touch, then we can still rotate the terminal cross-sections around the rod axis before we fix them. Thus, as long as we prevent self-penetration, we can in fact obtain infinitely many different solutions though the boundary conditions are always the same. That we can identify these solutions we have to impose topological constraints as the knot type of some deformed mid curve of the rod and the link type between that mid curve and some curve on the boundary of the rod. Mathematically a knot type is characterized by an isotopy class. The link type, which fixes the freedom in rotation of the terminal cross-section as described above, can be specified by a homotopy class. We show that these topological side conditions define weakly closed sets in a suitable Banach space which finally enables us to verify the existence of a globally injective equilibrium state respecting these topological constraints.

In Gonzalez et al. [7] the same problem as discussed in the previous paragraph is studied for rods with homogeneous circular cross-sections by means of completely different analytical methods. There a bound for the global radius of curvature, which is a nonlocal geometric quantity, is used to ensure global injectivity of deformations. Though this approach provides merely an approximation for self-contact in the shearable case, in the case of unshearable inextensible rods it allows the derivation of the Euler-Lagrange equation as necessary condition for energy minimizing states and the verification of further regularity properties (cf. Schuricht & v.d. Mosel [11]). In contrast, the approach in the present paper permits much more general existence results, but it seems to be unsuitable for the derivation of the Euler-Lagrange equation. Note that a rigorous derivation (i.e., without hypothetical regularity assumptions) of the Euler-Lagrange equation for a problem in nonlinear elasticity taking into account self-contact was never done before.

The problem of elastic rings subjeted to topological constraints has an important application in modeling super-coiled DNA molecules, i.e., molecules which wrap around itself due to torsional stresses. It has been observed that enzyms can control the global shape of the molecule through topological changes, i.e., enzyms can cut the strands of the double helix of DNA and then glue them together in a different way which increases or relaxes torsional stresses and which influences the shape of the molecule globally.

In Section 2 of this paper we present the Cosserat theory for nonlinearly elastic rods as necessary for our purposes. Conditions which ensure local and global injectivity for deformed configurations of rods are investigated in Section 3. In particular we discusse how far they define weakly closed sets in a suitable Banach space. The formulation of analytical conditions describing rigid obstacles and topological constraints can be found in Section 4 and it is shown that such constraints provide weakly closed sets in some Banach space. In Section 5 we first verify the existence of rods without self-intersection and perhaps restricted by a rigid obstacle in a general way. Then we derive the existence of an energy minimizing state for closed rods respecting also topological constraints as discussed above.

Notation. By int A, cl A, and vol A we denote the interior, the closure, and the Lebesgue measure of a set A. The set of all real 3×3-matrices is given by $\mathbb{R}^{3\times 3}$ and $SO(3)$ is the subset of all proper rotation matrices. Weak and strong convergence in a Banach space is expressed by $\rightharpoonup$ and$\rightarrow$, respectively. $\mathcal{C}(\Omega)$, $\mathcal{L}(\Omega)$, and $\mathcal{W}^{1,p}(\Omega)$ stand for the space of continuous functions, the Lebesgue space of $p-$integrable functions, and the Sobolev space of $p-$integrable functions with generalized $p-$integrable derivative, respectively.

2 Rod theory

In this section we formulate the special Cosserat theory which describes the behavior of nonlinearly elastic rods that can undergo large deformations in space by suffering flexure, torsion, extension, and shear. For a more comprehensive presentation see Antman [1, Chap. VIII].

We suppose that the deformed position field of a slender elastic body can be given in the form

$$\boldsymbol{p}(s,\zeta_1,\zeta_2) = \boldsymbol{r}(s) + \zeta_1 \boldsymbol{d}_1(s) + \zeta_2 \boldsymbol{d}_2(s) \quad \text{for } (s,\zeta_1,\zeta_2) \in \Omega\,, \tag{2.1}$$

where

$$\Omega \equiv \{(s,\zeta_1,\zeta_2)|\ s \in [0,L],\ (\zeta_1,\zeta_2) \in \mathcal{A}(s)\}\,.$$

Here $\boldsymbol{r} : [0,L] \to \mathbb{R}^3$ describes the deformed configuration of some material curve in the body, the so-called *base curve* (e.g., the curve of centroids or a suitable boundary curve). $\boldsymbol{d}_1(s)$, $\boldsymbol{d}_2(s)$ are orthogonal unit vectors describing the orientation of the cross-section at s. We interpret s as length parameter and ζ_1, ζ_2 as thickness parameters of the rod. $\mathcal{A}(s) \subset \mathbb{R}^2$ are uniformly bounded parameter sets for the cross-sections. We assume that $\mathcal{A}(s)$ is the closure of an open set with $0 \in \mathcal{A}(s)$ for all $s \in [0,L]$. To exclude certain mechanically irrelevant cases we furthermore demand that Ω is the closure of an open set in $\mathbb{R}^3$. For the stress free reference configuration, which has not to be straight or uniform in thickness, we assume that s is the arc-length of the base curve and that the cross-sections are orthogonal to the base curve. With

$$\boldsymbol{d}_3 \equiv \boldsymbol{d}_1 \times \boldsymbol{d}_2$$

the vectors $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$, which we call *directors*, form a right-handed orthonormal basis for each s which we can also identify with an orthogonal matrix $\boldsymbol{D} = (\boldsymbol{d}_1|\boldsymbol{d}_2|\boldsymbol{d}_3) \in SO(3)$ (the right hand side denotes the matrix with columns $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$).

A deformed configuration of the rod is obviously determined by functions $\boldsymbol{r} : [0,L] \to \mathbb{R}^3$ and $\boldsymbol{D} : [0,L] \to SO(3)$ where it is reasonable to choose $\boldsymbol{r} \in \mathcal{W}^{1,q}([0,L],\mathbb{R}^3)$ and $\boldsymbol{D} \in \mathcal{W}^{1,p}([0,L],\mathbb{R}^{3\times 3})$ with $p, q \geq 1$. It can be shown that each such configuration uniquely corresponds to an element

$$\boldsymbol{w} \equiv (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0) \quad \text{where} \quad \boldsymbol{u} \equiv (u_1,u_2,u_3),\ \ \boldsymbol{v} \equiv (v_1,v_2,v_3)$$

in the space

$$\boldsymbol{X}^{p,q} \equiv \mathcal{L}^p([0,L],\mathbb{R}^3) \times \mathcal{L}^q([0,L],\mathbb{R}^3) \times \mathbb{R}^3 \times SO(3) \tag{2.2}$$

such that

$$\begin{aligned}
\boldsymbol{d}_k'(s) &= \Big(\sum_{j=1}^{3} u_j \boldsymbol{d}_j\Big) \times \boldsymbol{d}_k \quad \text{for a.e. } s \in [0,L]\,, \quad k = 1,2,3 && (2.3)\\
\boldsymbol{r}'(s) &= \sum_{j=1}^{3} v_j \boldsymbol{d}_j \quad \text{for a.e. } s \in [0,L]\,, && (2.4)\\
\boldsymbol{r}(0) &= \boldsymbol{r}_0\,, \quad \boldsymbol{D}(0) = \boldsymbol{D}_0
\end{aligned}$$

(cf. Gonzalez et al. [7]). We call $\boldsymbol{u}$, $\boldsymbol{v}$ the *strains* of the problem. By the notation $\boldsymbol{p}[\boldsymbol{w}]$, $\boldsymbol{r}[\boldsymbol{w}]$, etc. we indicate that the values $\boldsymbol{p}$, $\boldsymbol{r}$, etc. are calculated for $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0)$. The special case of a theory for unshearable and inextensible rods can be obtained by simply setting

$$\boldsymbol{v} = (0,0,1)\,.$$

To ensure that configurations preserve orientation and are locally invertible it is asssumed that

$$\det \frac{\partial \boldsymbol{p}(s,\zeta_1,\zeta_2)}{\partial(s,\zeta_1,\zeta_2)} > 0 \quad \text{a.e. on } \Omega \tag{2.5}$$

which is equivalent to

$$v_3(s) > \zeta_1 u_2(s) - \zeta_2 u_1(s) \quad \text{for a.e. } (s,\zeta_1,\zeta_2) \in \Omega \tag{2.6}$$

by (2.1). Due to the structural assumptions about Ω and $\mathcal{A}(s)$ this is also equivalent to the one-dimensional inequality

$$v_3(s) > V(u_1(s), u_2(s), s) \quad \text{for a.e. } s \in [0,L] \tag{2.7}$$

where

$$V(u_1,u_2,s) \equiv \max_{(\zeta_1,\zeta_2)\in\mathcal{A}(s)} \zeta_1 u_2 - \zeta_2 u_1\,.$$

Obviously $V(0,0,s) = 0$ and $V(u_1,u_2,s) > 0$ for $|u_1| + |u_2| \neq 0$ by $0 \in \mathcal{A}(s)$. As an upper envelope of a family of linear functions, $V(\cdot,\cdot,s)$ is convex and obviously continuous (cf. also Antman [1, Chap. VIII.6]). In Section 3 we discuss how far (2.7) really ensures local injectivity of a deformation $\boldsymbol{p}$.

The rod is called *hyperelastic* if the material response can be described by a stored energy density $W : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,L] \to (-\infty, +\infty]$ depending on $(\boldsymbol{u}, \boldsymbol{v}, s)$ where $W(\cdot,\cdot,s)$ is convex and such that the total elastic energy is given by

$$E_{\mathrm{s}}(\boldsymbol{w}) = E_{\mathrm{s}}(\boldsymbol{u},\boldsymbol{v}) = \int_0^L W(\boldsymbol{u}(s), \boldsymbol{v}(s), s)\, ds\,.$$

The derivatives $\boldsymbol{n} \equiv W_{\boldsymbol{v}}$ and $\boldsymbol{m} \equiv W_{\boldsymbol{u}}$ provide the force and the moment, respectively, exerted from the material on one side of a cross-section to the material on the other side of the cross-section (cf. Antman [1, Chap. VIII.7]). For our analysis we assume that:

– $W(\cdot,\cdot,s)$ is continuous for all $s \in [0, L]$,
– $W(\boldsymbol{u}, \boldsymbol{v}, \cdot)$ is measurable for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$,
– $W(\boldsymbol{u}, \boldsymbol{v}, s) \geq \gamma(s)$ for some $\gamma \in \mathcal{L}^1([0, L])$.

This way we in particular ensure that $E_{\mathrm{s}}(\cdot)$ is weakly lower semicontinuous on the Banach space

$$\boldsymbol{Y}^{p,q} \equiv \mathcal{L}^p([0,L],\mathbb{R}^3) \times \mathcal{L}^q([0,L],\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$$

(cf. Dacorogna [4]). Note that $\boldsymbol{X}^{p,q}$ is a closed subset of $\boldsymbol{Y}^{p,q}$ but not a linear subspace.

Analogously as for planar rods in Schuricht [8], external forces exerted to the rod can be described by a vector valued regular Borel measure $\mathfrak{f}$ on Ω and the corresponding potential energy is given by

$$E_{\mathrm{p}}(\boldsymbol{w}) = -\int_{\Omega} \boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2) \cdot d\mathfrak{f}\,.$$

In Section 5 we are looking for minimizers of the total energy

$$E(\boldsymbol{w}) \equiv E_{\mathrm{s}}(\boldsymbol{w}) + E_{\mathrm{p}}(\boldsymbol{w})$$

subjected to further side conditions.

To simplify notation we set

$$\boldsymbol{X} \equiv \boldsymbol{X}^{p,q}, \quad \boldsymbol{Y} \equiv \boldsymbol{Y}^{p,q}$$

and we assume that $1 < p, q < \infty$ for the rest of this paper. Note that parameters p, q for $\boldsymbol{X}$ and $\boldsymbol{Y}$ have to be always the same.

3 Local and global injectivity

3.1 Formulation of the results

According to our experience, the place occupied by one body cannot be occupied by another body at the same time. The same must be true for each part of a deformable body and one says that the body cannot penetrate itself. We invoke this property of matter into our theory by the mathematical demand that the mapping $\boldsymbol{p}$ given in (2.1) has to be (globally) injective on int Ω. Since this condition is hard to treat analytically, it is mostly neglected and merely a local condition like (2.5) is usually taken into account in problems of continuum mechanics. If deformations are continuously differentiable, then local injectivity can be ensured this way. In 3-dimensional nonlinear elasticity, however, deformations belonging to $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^3)$ are usually considered and a condition like (2.5) ensures local injectivity only in some generalized sense (cf. Fonseca & Gangbo [6]). But, due to the speccial structure (2.1) which we impose to deformations of our rods, we will get local injectivity for deformed rods satisfying (2.7). Since condition (2.7) does not define a weakly closed set in $\boldsymbol{Y}$, it cannot be used to verify minimizers of the energy. Therefore we still consider some more sophisticated situation ensureing local injectivity. Then we consider a slightly generalized situation which is important for existence results. Finally we adopt a condition introduced by Ciarlet & Nečas [3] for problems in 3-dimensional elasticity to our setting to ensure global injectivity of deformations $\boldsymbol{p}$ on $\operatorname{int}\Omega$.

Let us start with a proposition which states local injectivity on the basis of (2.7). The proof essentially uses the special structure of $\boldsymbol{p}$ according to (2.1) and can be found at the end of this section.

Proposition 3.1 *Assume that* $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0) \in \boldsymbol{X}$ *satisfies the orientation preserving condition* (2.7)*. Then the mapping* $(s, \zeta_1, \zeta_2) \to \boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2)$ *is locally injective on* $\operatorname{int}\Omega$*. Furthermore, this mapping is open on* $\operatorname{int}\Omega$*, i.e., the images of open sets are open.*

Since we are interested in existence results, which we want to derive by variational methods, we need side conditions which define weakly closed sets in a suitable Banch space. This, however, is not the case for condition (2.7). By this reason we have to consider the relaxed condition

$$v_3(s) \geq V(u_1(s), u_2(s), s) \quad \text{a.e. on } [0, L]\,. \tag{3.2}$$

Here we also allow deformations $\boldsymbol{p}$ where portions of the rod are compressed to zero volume. By the natural growth condition that

$$W(\boldsymbol{u}, \boldsymbol{v}, s) \to \infty \quad \text{as} \quad v_3 - V(u_1, u_2, s) \to 0\,, \tag{3.3}$$

i.e., the elastic energy approaches infinity under complete compression, we can enforce that equality in (3.2) can only occur on a set of measure zero for configurations with finite energy. Though (3.2) itself does not ensure local injectivity of $\boldsymbol{p}$, we can supplement Proposition 3.1 with the following more sophisticated result.

Proposition 3.4 *Let* W *satisfy* (3.3)*, let* $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0) \in \boldsymbol{X}$ *satisfy* (3.2)*, and let*

$$E_{\mathrm{s}}(\boldsymbol{u}, \boldsymbol{v}) = \int_0^L W(\boldsymbol{u}(s), \boldsymbol{v}(s), s)\, ds < \infty\,.$$

Then the mapping $(s, \zeta_1, \zeta_2) \to \boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2)$ *is locally injective and open on* $\operatorname{int}\Omega$.

This way we justify the use of the relaxed condition (3.2) in cases where (3.3) is met. The fact that the mapping $\boldsymbol{p}[\boldsymbol{w}](\cdot)$ is open will be an important ingredient in the proof of Theorem 3.7 below. The proof of the proposition is postponed to the end of this section.

Now we have seen how local injectivity can be ensured. But it can still happen that parts of the rod which are far away from each other in the reference configuration penetrate each other after large deformation. To prevent this we adapt an inequality condition introduced by Ciarlet & Nečas [3] to our rod theory

$$\int_\Omega \det \frac{\partial \boldsymbol{p}(s, \zeta_1, \zeta_2)}{\partial (s, \zeta_1, \zeta_2)}\, d(s, \zeta_1, \zeta_2) \leq \operatorname{vol} \boldsymbol{p}[\boldsymbol{w}](\Omega) \tag{3.5}$$

where 'vol' denotes the Lebesgue measure. Roughly speaking we demand that the volume occupied by the deformed rod is not smaller than the sum over the volumes occupied by disjoint parts of the rod. By (2.1) inequality (3.5) is obviously equivalent to

$$\int_\Omega v_3(s) - \zeta_1 u_2(s) + \zeta_2 u_1(s)\, d(s, \zeta_1, \zeta_2) \leq \operatorname{vol} \boldsymbol{p}[\boldsymbol{w}](\Omega)\,. \tag{3.6}$$

Theorem 3.7 *Let W satisfy* (3.3), *let* $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0) \in \boldsymbol{X}$ *satisfy* (3.2), (3.6), *and let* $E_{\mathrm{s}}(\boldsymbol{u}, \boldsymbol{v}) < \infty$. *Then the mapping* $(s, \zeta_1, \zeta_2) \to \boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2)$ *is (globally) injective on* $\operatorname{int} \Omega$.

Observe that we merely get injectivity on $\operatorname{int} \Omega$ and not on Ω. Hence it is not excluded that deformed points of the rod corresponding to parameters on the boundary $\partial\Omega$ may coincide. This is a very natural fact and just means that the deformed elastic body can have self-contact. Note further that in the general 3-dimensional case, studied by Ciarlet & Nečas, the strains must be p-integrable with some $p > 3$ to get a comparable result. However, due to the special structure of deformations $\boldsymbol{p}$ given in (2.1), in our case $p, q > 1$ is sufficient. The proof of the theorem, which is given at the end of this section, uses also Proposition 3.4.

As we already mentioned, for our variational problem we need side conditions which provide weakly closed sets in $\boldsymbol{Y}$ in order to apply the direct methods of calculus of variations. The next lemma states that this is the case for conditions (3.2) and (3.6).

Lemma 3.8 *The sets*

$$\begin{aligned} \boldsymbol{X}_1 &\equiv \{(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0) \in \boldsymbol{X} \,|\, (\boldsymbol{u}, \boldsymbol{v}) \text{ satisfies(3.2)}\} \quad \text{and} \\ \boldsymbol{X}_2 &\equiv \{\boldsymbol{w} \in \boldsymbol{X} \,|\, \boldsymbol{w} \text{ satisfies(3.6)}\} \end{aligned} \tag{3.9}$$

are weakly closed in $\boldsymbol{Y}$.

3.2 Proofs

Proof of Proposition 3.1. Since $\boldsymbol{w} \in \boldsymbol{X}$ is fixed, we suppress the dependence of $\boldsymbol{p}$, $\boldsymbol{d}_i$ etc. on $\boldsymbol{w}$. Let us first show that $\boldsymbol{p}(\cdot,\cdot,\cdot)$ is locally injective on $\operatorname{int} \Omega$. For this reason we fix any $(s_0, \zeta_{10}, \zeta_{20}) \in \operatorname{int} \Omega$. We obviously can find $\varepsilon > 0$ and $0 < \delta < 1$ such that

$$\operatorname{cl} U_{\varepsilon, 3\delta} \subset \operatorname{int} \Omega \tag{3.10}$$

where

$$U_{\varepsilon,\delta} \equiv \{(s, \zeta_1, \zeta_2) \in \mathbb{R}^3 \mid |s - s_0| < \varepsilon,\ |\zeta_1 - \zeta_{10}| < \delta,\ |\zeta_2 - \zeta_{20}| < \delta\}.$$

Since the parameter sets $\mathcal{A}(s)$ are supposed to be uniformly bounded, there is $\tilde{r} > 0$ such that

$$|\zeta_1|,\ |\zeta_2| < \tilde{r} \quad \text{for all } (s, \zeta_1, \zeta_2) \in \Omega .$$

We assume that $\varepsilon > 0$ is so small that

$$\int_{s_0-\varepsilon}^{s_0+\varepsilon} |u_3(s)|\, ds\,, \int_{s_0-\varepsilon}^{s_0+\varepsilon} |v_1(s)|\, ds\,, \int_{s_0-\varepsilon}^{s_0+\varepsilon} |v_2(s)|\, ds < \min\left\{\frac{\delta}{16}, \frac{\delta}{16\,\tilde{r}}\right\} \tag{3.11}$$

and

$$\boldsymbol{d}_3(\sigma_1) \cdot \boldsymbol{d}_3(\sigma_2) > \frac{1}{2} \quad \text{for all } \sigma_1, \sigma_2 \in [s_0 - \varepsilon, s_0 + \varepsilon] \tag{3.12}$$

(recall that $\boldsymbol{d}_3(\cdot)$ is continuous). By assumption $(\boldsymbol{u}, \boldsymbol{v})$ satisfies the orientation preserving condition (2.7) and thus also (2.6). Hence, by (3.10),

$$v_3(s) - \zeta_1 u_2(s) + \zeta_2 u_1(s) > 2\delta(\,|u_1(s)| + |u_2(s)|\,) \quad \text{on } U_{\varepsilon,\delta}\,. \tag{3.13}$$

We will now show that $\boldsymbol{p}$ is injective on $U_{\varepsilon,\delta}$. For contrary suppose that there are different points $(s_1,\zeta_{11},\zeta_{21}),(s_2,\zeta_{12},\zeta_{22}) \in U_{\varepsilon,\delta}$, $s_1 < s_2$, with

$$\boldsymbol{p}(s_1,\zeta_{11},\zeta_{21}) = \boldsymbol{p}(s_2,\zeta_{12},\zeta_{22}) \tag{3.14}$$

Observe that the case $s_1 = s_2$ easily implies $\zeta_{11} = \zeta_{12}$, $\zeta_{21} = \zeta_{22}$ and thus can be excluded. Using

$$\tilde{\boldsymbol{p}}(s) \equiv \boldsymbol{p}(s,\zeta_{12},\zeta_{22})\,, \quad \Delta s \equiv s_2 - s_1\,, \quad \tilde{s}(t) \equiv s_1 + t\Delta s$$

we consider

$$\begin{aligned}
\Delta\boldsymbol{p} &\equiv \tilde{\boldsymbol{p}}(s_2) - \tilde{\boldsymbol{p}}(s_1) \\
&= \Delta s \int_0^1 \tilde{\boldsymbol{p}}'(s_1 + t\Delta s)\, dt \\
&= \Delta s \int_0^1 \boldsymbol{r}'(\tilde{s}(t)) + \zeta_{12}\boldsymbol{d}_1'(\tilde{s}(t)) + \zeta_{22}\boldsymbol{d}_2'(\tilde{s}(t))\, dt \\
&= \Delta s \int_0^1 \Big(\sum_{j=1}^3 v_j\boldsymbol{d}_j + \zeta_{12}\sum_{j=1}^3 u_j\boldsymbol{d}_j \times \boldsymbol{d}_1 + \zeta_{22}\sum_{j=1}^3 u_j\boldsymbol{d}_j \times \boldsymbol{d}_2\Big)\, dt \quad \text{(by (2.3), (2.4))} \\
&= \Delta s \int_0^1 (v_1 - \zeta_{22}u_3)\boldsymbol{d}_1 + (v_2 + \zeta_{12}u_3)\boldsymbol{d}_2 + (v_3 - \zeta_{12}u_2 + \zeta_{22}u_1)\boldsymbol{d}_3\; dt
\end{aligned} \tag{3.15}$$

where $\tilde{s}(t)$ is the argument of all functions in the last two lines. We claim to show that

$$\Delta\boldsymbol{p}\cdot\tilde{\boldsymbol{d}} > 0 \quad \text{where} \quad \tilde{\boldsymbol{d}} \equiv \boldsymbol{d}_3(s_1)\,. \tag{3.16}$$

By $\boldsymbol{p}(s_1,\zeta_{11},\zeta_{21})\cdot\tilde{\boldsymbol{d}} = \tilde{\boldsymbol{p}}(s_1)\cdot\tilde{\boldsymbol{d}}$ this contradicts (3.14) and would verify the global injectivity of $\boldsymbol{p}$ on $U_{\varepsilon,\delta}$.

Let us introduce the notation

$$\alpha_i \equiv \int_{s_1}^{s_2} |u_i(s)|\, ds\,, \quad \alpha_{i+3} \equiv \int_{s_1}^{s_2} |v_i(s)|\, ds\,, \quad i = 1,2,3,$$

$$\beta_i \equiv \int_0^1 |u_i(\tilde{s}(t))|\, dt\,, \quad \beta_{i+3} \equiv \int_0^1 |v_i(\tilde{s}(t))|\, dt\,, \quad i = 1,2,3.$$

Clearly

$$\alpha_i = \Delta s\, \beta_i\,, \quad i = 1,\dots,6. \tag{3.17}$$

By (2.3)

$$\boldsymbol{d}_1(s) = \boldsymbol{d}_1(s_1) + \int_{s_1}^{s} \Big(-u_2(\sigma)\boldsymbol{d}_3(\sigma) + u_3(\sigma)\boldsymbol{d}_2(\sigma)\Big)\, d\sigma\,.$$

Hence

$$\|\boldsymbol{d}_1\cdot\tilde{\boldsymbol{d}}\|_{C(s_1,s_2)} \le \alpha_2 + \alpha_3\|\boldsymbol{d}_2\cdot\tilde{\boldsymbol{d}}\|_{C(s_1,s_2)}\,. \tag{3.18}$$

Analogously,

$$\|\boldsymbol{d}_2\cdot\tilde{\boldsymbol{d}}\|_{C(s_1,s_2)} \le \alpha_1 + \alpha_3\|\boldsymbol{d}_1\cdot\tilde{\boldsymbol{d}}\|_{C(s_1,s_2)}\,. \tag{3.19}$$

Pluging (3.19) into (3.18) and vice versus we obtain

$$\|\boldsymbol{d}_1 \cdot \tilde{\boldsymbol{d}}\|_{\mathcal{C}(s_1,s_2)} \le \frac{\alpha_2 + \alpha_1\alpha_3}{1-\alpha_3^2}\,, \quad \|\boldsymbol{d}_2 \cdot \tilde{\boldsymbol{d}}\|_{\mathcal{C}(s_1,s_2)} \le \frac{\alpha_1 + \alpha_2\alpha_3}{1-\alpha_3^2}\,. \tag{3.20}$$

Notice that the denominator is positive by (3.11). Using (3.15), (3.12), (3.13), (3.20) we can now estimate

$$\begin{aligned} \Delta\boldsymbol{p}\cdot\tilde{\boldsymbol{d}} &\ge \Delta s \int_0^1 \Big(\delta(|u_1|+|u_2|) - (|v_1| + \tilde{r}|u_3|)\frac{\alpha_2+\alpha_1\alpha_3}{1-\alpha_3^2} - (|v_2|+\tilde{r}|u_3|)\frac{\alpha_1+\alpha_2\alpha_3}{1-\alpha_3^2}\Big)\,dt \\ &\qquad\qquad\qquad \text{(all arguments have to be } \tilde{s}(t)) \\ &= \delta(\alpha_1+\alpha_2) - \frac{1}{1-\alpha_3^2}\Big(\alpha_1(\alpha_5 + \tilde{r}\alpha_3 + \alpha_3\alpha_4 + \tilde{r}\alpha_3^2) + \alpha_2(\alpha_4 + \tilde{r}\alpha_3 + \alpha_3\alpha_5 + \tilde{r}\alpha_3^2)\Big) \end{aligned}$$

Using (3.11) and $\delta < 1$ we obtain that

$$\frac{1}{1-\alpha_3^2} < 2$$

and, again by (3.11),

$$\Delta\boldsymbol{p}\cdot\tilde{\boldsymbol{d}} \ge \delta(\alpha_1+\alpha_2) - 2(\frac{\delta}{4}\alpha_1 + \frac{\delta}{4}\alpha_2) = \frac{\delta}{2}(\alpha_1+\alpha_2)\,.$$

If $\alpha_1 + \alpha_2 > 0$, then we have verified (3.16). Otherwise $u_1(s) = u_2(s) = 0$ a.e. on $[s_1, s_2]$ and, therefore, $\boldsymbol{d}_3(s) = \boldsymbol{d}_3(s_1) = \tilde{\boldsymbol{d}}$ on $[s_1, s_2]$ by (2.3). With (3.15) and (2.6) we then get

$$\Delta\boldsymbol{p}\cdot\tilde{\boldsymbol{d}} = \Delta s \int_0^1 v_3(\tilde{s}(t))\boldsymbol{d}_3(\tilde{s}(t))\cdot\tilde{\boldsymbol{d}}\,dt = \int_{s_1}^{s_2} v_3(s)\,ds > 0 \tag{3.21}$$

which verifies (3.16) also in the remaining case. Hence $\boldsymbol{p}$ is injective on $U_{\varepsilon,\delta}$ and by the arbitrary choice of $(s_0, \zeta_{10}, \zeta_{20}) \in \operatorname{int}\Omega$ the local injectivity of $\boldsymbol{p}$ on $\operatorname{int}\Omega$ is shown.

The continuity and the local injectivity of $\boldsymbol{p}$ finally imply that $\boldsymbol{p}$ maps open sets onto open sets (cf. Zeidler [14, Theorem 16.C]. ◇

Proof of Proposition 3.4. We proceed exactly as in the previous proof of Proposition 3.1. However for the verification of (3.21) we argue that $v_3(s) > 0$ a.e. on $[0, L]$ due to (3.3) and $E_{\mathrm{s}}(\boldsymbol{u}, \boldsymbol{v}) < \infty$. ◇

Proof of Theorem 3.7. By the special structure of $\boldsymbol{p} \equiv \boldsymbol{p}[\boldsymbol{w}]$ according to (2.1) and by $\boldsymbol{r} \equiv \boldsymbol{r}[\boldsymbol{w}] \in \mathcal{W}^{1,q}([0,L])$, $\boldsymbol{d}_i \equiv \boldsymbol{d}_i[\boldsymbol{w}] \in \mathcal{W}^{1,p}([0,L])$, $i = 1,2,3$, there is a set $\mathcal{I}_0 \subset [0, L]$ of measure zero such that

$$\limsup_{(\tilde{s},\tilde{\zeta}_1,\tilde{\zeta}_2)\to(s,\zeta_1,\zeta_2)} \frac{\|\boldsymbol{p}(\tilde{s},\tilde{\zeta}_1,\tilde{\zeta}_2) - \boldsymbol{p}(s,\zeta_1,\zeta_2)\|}{\|(\tilde{s},\tilde{\zeta}_1,\tilde{\zeta}_2) - (s,\zeta_1,\zeta_2)\|} < \infty \quad \text{for all } (s,\zeta_1,\zeta_2) \in \Omega' \equiv \Omega \setminus \Omega_{\mathcal{I}_0}$$

where

$$\Omega_{\mathcal{I}_0} \equiv \{(s,\zeta_1,\zeta_2) \in \Omega |\; s \in \mathcal{I}_0\}\,. \tag{3.22}$$

By Federer [5, p. 241,243] we then have that

$$\int_{\Omega'} \Big(v_3(s) - \zeta_1 u_2(s) + \zeta_2 u_1(s)\Big)\, d(s,\zeta_1,\zeta_2) = \int_{\boldsymbol{p}(\Omega')} \text{card}\,\{\boldsymbol{p}^{-1}(\boldsymbol{q})\}\, d\boldsymbol{q}$$

where 'card' denotes the number of elements of a set and $\boldsymbol{p}^{-1}$ is the inverse of the mapping $\boldsymbol{p}(\cdot)$. Below we show that

$$\text{vol}\,\boldsymbol{p}(\Omega_{\mathcal{I}_0}) = 0\,. \tag{3.23}$$

Using (3.6) we thus get

$$\begin{aligned} \text{vol}\,\boldsymbol{p}(\Omega) &= \int_{\boldsymbol{p}(\Omega')} d\boldsymbol{q} \;\le\; \int_{\boldsymbol{p}(\Omega')} \text{card}\,\{\boldsymbol{p}^{-1}(\boldsymbol{q})\}\, d\boldsymbol{q} \\ &= \int_{\Omega'} \Big(v_3(s) - \zeta_1 u_2(s) + \zeta_2 u_1(s)\Big)\, d(s,\zeta_1,\zeta_2) \;\le\; \text{vol}\,\boldsymbol{p}(\Omega)\,. \end{aligned} \tag{3.24}$$

Consequently

$$\text{card}\,\{\boldsymbol{p}^{-1}(\boldsymbol{q})\} = 1 \quad \text{for almost all } \boldsymbol{q} \in \boldsymbol{p}(\Omega)\,. \tag{3.25}$$

Suppose now that there are different $(s_1,\zeta_{11},\zeta_{21})$, $(s_2,\zeta_{12},\zeta_{22}) \in \text{int}\,\Omega$ with

$$\tilde{\boldsymbol{q}} \equiv \boldsymbol{p}(s_1,\zeta_{11},\zeta_{21}) = \boldsymbol{p}(s_2,\zeta_{12},\zeta_{22})\,.$$

We can choose disjoint open balls B_1 and B_2 centered at $(s_1,\zeta_{11},\zeta_{21})$ and $(s_2,\zeta_{12},\zeta_{22})$, respectively, and both contained in int Ω. By Proposition 3.4 the sets $\boldsymbol{p}(B_1)$ and $\boldsymbol{p}(B_2)$ are open and both contain $\tilde{\boldsymbol{q}}$. Hence we have card $\{\boldsymbol{p}^{-1}(\boldsymbol{q})\} \ge 2$ on a neighborhood of $\tilde{\boldsymbol{q}}$. But this contradicts (3.25) and verifies the injectivity of $\boldsymbol{p}$ on int Ω.

We still have to prove (3.23). Denote the diameter of $\boldsymbol{p}(\Omega)$ by $\tilde{d}$ and $\tilde{\zeta}$ be an upper bound for $|\zeta_1|$, $|\zeta_2|$ as long as $(s,\zeta_1,\zeta_2) \in \Omega$. Let $(s_1,\zeta_{11},\zeta_{21})$, $(s_2,\zeta_{12},\zeta_{22}) \in \Omega$, $s_1 < s_2$, and consider

$$\varrho \equiv \Big|\Big(\boldsymbol{p}(s_2,\zeta_{12},\zeta_{22}) - \boldsymbol{p}(s_1,\zeta_{11},\zeta_{21})\Big)\cdot \boldsymbol{d}_3(s_1)\Big|\,.$$

Using $\boldsymbol{p}(s_1,\zeta_{11},\zeta_{21})\cdot\boldsymbol{d}_3(s_1) = \boldsymbol{p}(s_1,\zeta_{12},\zeta_{22})\cdot\boldsymbol{d}_3(s_1)$ and deriving (3.15), (3.17) as in the proof of Proposition 3.1, we readily get the estimate

$$\varrho \le 2\tilde{\zeta}\int_{s_1}^{s_2}\sum_{j=1}^{3}(|u_j(s)| + |v_j(s)|)\, ds \;\equiv\; \varrho(s_1,s_2)\,.$$

With $\tilde{\boldsymbol{d}}_i \equiv \boldsymbol{d}_i(s_1)$, $i = 1,2,3$, we further set

$$Q(s_1,s_2) \equiv \{\boldsymbol{q}\in\mathbb{R}^3 \,|\, |(\boldsymbol{q}-\boldsymbol{r}(s_1))\cdot\tilde{\boldsymbol{d}}_1| \le \tilde{d},\ |(\boldsymbol{q}-\boldsymbol{r}(s_1))\cdot\tilde{\boldsymbol{d}}_2| \le \tilde{d},\ |(\boldsymbol{q}-\boldsymbol{r}(s_1))\cdot\tilde{\boldsymbol{d}}_3| \le \varrho(s_1,s_2)\}\,.$$

Obviously

$$\boldsymbol{p}(s,\zeta_1,\zeta_2) \in Q(s_1,s_2) \quad \text{for all } (s,\zeta_1,\zeta_2) \in \Omega_{[s_1,s_2]}$$

(notation in the sense of (3.22)). Thus

$$\text{vol}\,\boldsymbol{p}(\Omega_{[s_1,s_2]}) \le \text{vol}\,Q(s_1,s_2) = \tilde{d}^{\,2}\varrho(s_1,s_2)\,. \tag{3.26}$$

Since $\mathcal{I}_0$ has measure zero, for any $\delta > 0$ there is a sequence of disjoint intervals $(s_{2j-1}, s_{2j}) \subset [0,L]$, $j \in \mathbb{N}$, such that

$$\mathcal{I}_0 \subset \bigcup_{j\in\mathbb{N}} (s_{2j-1}, s_{2j}) \equiv \mathcal{I}_\delta\,, \quad \sum_{j=1}^{\infty} (s_{2j-1} - s_{2j}) \leq \delta\,.$$

By (3.26) we obtain

$$\begin{aligned} \text{vol}\, \boldsymbol{p}(\Omega_{\mathcal{I}_0}) &\leq \sum_{j=1}^{\infty} \text{vol}\, \boldsymbol{p}(\Omega_{(s_{2j-1},s_{2j})}) \\ &\leq \sum_{j=1}^{\infty} \text{vol}\, Q(s_{2j-1}, s_{2j}) \;=\; \tilde{d}^2 \sum_{j=1}^{\infty} \varrho(s_{2j-1}, s_{2j}) \\ &= 2\tilde{\zeta}\tilde{d}^2 \sum_{i=1}^{3} \int_{\mathcal{I}_\delta} |u_i(s)| + |v_i(s)|\, ds\,. \end{aligned} \tag{3.27}$$

Since the measure of $\mathcal{I}_\delta$ is not greater than δ and $\delta > 0$ can be choosen arbitrary small, the right hand side in (3.27) can be made arbitrary small. This verifies (3.23) and the proof is complete. $\diamondsuit$

PROOF of Lemma 3.8. Observe that weak convergence $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{Y}$ just means that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $\mathcal{L}^p$, $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $\mathcal{L}^q$, $\boldsymbol{r}_{0,n} \to \boldsymbol{r}_0$ in $\mathbb{R}^3$, and $\boldsymbol{D}_{0,n} \to \boldsymbol{D}_0$ in $\mathbb{R}^{3\times 3}$ and analogously for the strong convergence $\boldsymbol{w}_n \to \boldsymbol{w}$ in $\boldsymbol{Y}$.

By the convexity of $V(\cdot,\cdot,s)$, the set

$$\boldsymbol{Y}_1 \equiv \{(\boldsymbol{u},\boldsymbol{v},\boldsymbol{r}_0,\boldsymbol{D}_0) \in \boldsymbol{Y} |\; (\boldsymbol{u},\boldsymbol{v}) \;\text{ satisfies (3.2)}\}$$

is convex. Since a strongly convergent sequence in $\mathcal{L}^p$ or $\mathcal{L}^q$ has a subsequence which converges pointwise almost everywhere and since $V(\cdot,\cdot,s)$ is continuous, $\boldsymbol{Y}_1$ is closed with respect to strong convergence and, hence, also with respect to weak convergence. By the closedness of $SO(3)$ in $\mathbb{R}^{3\times 3}$ we finally get that $\boldsymbol{X}_1$ is weakly closed in $\boldsymbol{Y}$.

Let now $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{Y}$ where $\boldsymbol{w}_n \in \boldsymbol{X}_2$. Clearly the left-hand side in (3.6) is weakly continuous. We show that

$$\limsup_{n\to\infty}\; \text{vol}\, \boldsymbol{p}[\boldsymbol{w}_n](\Omega) \leq \text{vol}\, \boldsymbol{p}[\boldsymbol{w}](\Omega) \tag{3.28}$$

which then establishes that $\boldsymbol{w}$ satisfies (3.6). Since $SO(3)$ is closed in $\mathbb{R}^{3\times 3}$, this implies that also $\boldsymbol{X}_2$ is weakly closed in $\boldsymbol{Y}$.

By the continuity of $\boldsymbol{p}[\boldsymbol{w}](\cdot)$ the set $\boldsymbol{p}[\boldsymbol{w}](\Omega)$ is compact. For the δ-neighborhoods $U_\delta(\boldsymbol{p}[\boldsymbol{w}](\Omega))$, $\delta > 0$, we thus have that

$$\lim_{\delta\to 0}\; \text{vol}\, U_\delta(\boldsymbol{p}[\boldsymbol{w}](\Omega)) = \text{vol}\, \boldsymbol{p}[\boldsymbol{w}](\Omega)\,. \tag{3.29}$$

In Gonzalez et al. [7, Prop. 3.4] it is shown that

$$\boldsymbol{r}[\boldsymbol{w}_n] \to \boldsymbol{r}[\boldsymbol{w}]\,, \quad \boldsymbol{d}_i[\boldsymbol{w}_n] \to \boldsymbol{d}_i[\boldsymbol{w}]\,, \;\; i = 1,2,3, \quad \text{in } \mathcal{C}([0,L])\,. \tag{3.30}$$

Hence, for any $\delta > 0$ there is $n_\delta \in \mathbb{N}$ such that

$$\boldsymbol{p}[\boldsymbol{w}_n](\Omega) \subset U_\delta(\boldsymbol{p}[\boldsymbol{w}](\Omega)) \quad \text{for all } n > n_\delta\,.$$

But this combined with (3.29) implies (3.28). $\diamondsuit$

4 Rigid obstacles and topological constraints

4.1 Formulation of the results

The deformation of an elastic body is often restricted by a rigid obstacle, i.e., the rod cannot occupy the points in space occupied by the obstacle. It seems reasonable to assume that a rigid obstacle can be identified with the closure $\mathcal{O}$ of an open set in $\mathbb{R}^3$ (in Schuricht [8] the case where $\operatorname{int}\mathcal{O} = \emptyset$ is discussed for planar deformations). Admissible deformations of the rod must then satisfy

$$\boldsymbol{p}(s,\zeta_1,\zeta_2) \notin \operatorname{int}\mathcal{O} \quad \text{for all } (s,\zeta_1,\zeta_2) \in \Omega.$$

As a special case we allow that $\mathcal{O} = \emptyset$ which describes the case without obstacle. Again we ask whether this constraint provides a weakly closed subset in $\boldsymbol{Y}$.

Lemma 4.1 *The set*

$$\boldsymbol{X}_{\mathcal{O}} \equiv \{\boldsymbol{w} \in \boldsymbol{X} \mid \boldsymbol{p}[\boldsymbol{w}](s,\zeta_1,\zeta_2) \notin \operatorname{int}\mathcal{O} \text{ for all } (s,\zeta_1,\zeta_2) \in \Omega\} \tag{4.2}$$

is weakly closed in $\boldsymbol{Y}$.

The proof is given at the end of this section.

Let us now study rods where the ends are glued together. More precisely we consider

$$\begin{aligned}\boldsymbol{X}_{\mathcal{C}} \;\equiv\; \{\boldsymbol{w} \in \boldsymbol{X} \mid\, & \boldsymbol{r}[\boldsymbol{w}](0) = \boldsymbol{r}[\boldsymbol{w}](L),\ \boldsymbol{d}_3[\boldsymbol{w}](0) = \boldsymbol{d}_3[\boldsymbol{w}](L),\\ & \boldsymbol{d}_1[\boldsymbol{w}](L) = \Big(\alpha_1\boldsymbol{d}_1[\boldsymbol{w}](0) + \alpha_2\boldsymbol{d}_2[\boldsymbol{w}](0)\Big) \times \boldsymbol{d}_3[\boldsymbol{w}](0)\}\end{aligned} \tag{4.3}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1^2 + \alpha_2^2 = 1$ are given. Besides the coincidence of the end points of the base curve these conditions ensure that the orientation of the cross-sections is the same and that the rotation angle between $\boldsymbol{d}_1(0)$ and $\boldsymbol{d}_1(L)$ is fixed.

Before we glue together the ends of the rod in the above mentioned way we have a lot of freedom. We can, e.g., form the rod into some knot. Then, if we prevent self-penetration, the type of the knot cannot change during deformation. This means that the set of all globally injective deformations has different components which represent the knot type as topological constraint. For a precise mathematical formulation of such a restriction we need the notion of isotopy class for closed curves. Let $\boldsymbol{r}_1$, $\boldsymbol{r}_2 : [0,L] \to \mathbb{R}^3$ be two continuous curves with $\boldsymbol{r}_i(0) = \boldsymbol{r}_i(L)$, $i = 1,2$. The curves $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are called *isotopic* ($\boldsymbol{r}_1 \simeq \boldsymbol{r}_2$) if there are open neighborhoods N_1 of $K_1 \equiv \boldsymbol{r}_1([0,L])$,

N_2 of $K_2 \equiv \boldsymbol{r}_2([0,L])$ and a continuous mapping $\Phi : N_1 \times [0,1] \to \mathbb{R}^3$ such that $\Phi(N_1, \tau)$ is homeomorphic to N_1 for all $\tau \in [0,1]$ and

$$\Phi(\boldsymbol{q}, 0) = \boldsymbol{q} \text{ for all } \boldsymbol{q} \in N_1, \quad \Phi(N_1, 1) = N_2, \quad \Phi(K_1, 1) = K_2.$$

Note that isotopy does not depend on the special parametrization of the curves and sometimes we also write $K_1 \simeq K_2$ instead of $\boldsymbol{r}_1 \simeq \boldsymbol{r}_2$. The next lemma shows, roughly speaking, that isotopy classes provide weakly closed subsets in $\boldsymbol{Y}$ as long as deformations are globally injective. To avoid technicalities we assume that there is some $\varepsilon_0 > 0$ such that

$$B_0 \equiv \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 | \; \zeta_1^2 + \zeta_2^2 < \varepsilon_0^2\} \subset \mathcal{A}(s) \quad \text{for all } s \in [0, L]. \tag{4.4}$$

This means that the base curve lies always in the interior of the rod. Otherwise isotopy has to be considered for curves in the interior of the rod different from the base curve. The proof of the lemma can be found at the end of this section.

Lemma 4.5 *Let W satisfy* (3.3) *and let $E_s(\tilde{\boldsymbol{w}}) \leq \tilde{c}$ for some $\tilde{c} \in \mathbb{R}$ and some $\tilde{\boldsymbol{w}} \in \boldsymbol{X}_\mathcal{C}$ satisfying* (3.2) *and* (3.6)*. Then*

$$\boldsymbol{X}_\mathcal{K} \equiv \{\boldsymbol{w} \in \boldsymbol{X}_\mathcal{C} | \; \boldsymbol{r}[\boldsymbol{w}] \simeq \boldsymbol{r}[\tilde{\boldsymbol{w}}], \; \boldsymbol{w} \textit{ satisfies } (3.2), \; (3.6), \; E_s(\boldsymbol{w}) \leq \tilde{c}\}$$

is weakly closed in $\boldsymbol{Y}$.

If we take some rod and glue together its ends such that it belongs to $\boldsymbol{X}_\mathcal{K}$ we realize that there is still some freedom in doing that, since we can rotate the terminal cross-sections around the normal axis $\boldsymbol{d}_3(0)$ against each other and after each full rotation we meet the same boundary conditions as fixed in $\boldsymbol{X}_\mathcal{C}$. This means that $\boldsymbol{X}_\mathcal{K}$ still has infinitely many components. We can characterized these components by homotopy classes in $SO(3)$. Two continuous mappings $D_1, D_2 : [0,L] \to SO(3)$ with $D_1(0) = D_2(0)$ and $D_1(L) = D_2(L)$ are called *homotopic* ($D_1 \sim D_2$), if there is a continuous mapping $\Psi : [0,L] \times [0,1] \to SO(3)$ such that

$$\Psi(\cdot, 0) = D_1(\cdot) \text{ and } \Psi(\cdot, 1) = D_2(\cdot) \quad \text{on } [0, L],$$

$$\Psi(0, \cdot) = D_1(0) \text{ and } \Psi(L, \cdot) = D_1(L) \quad \text{on } [0, 1].$$

Again we have to ask whether the set of all rods with $\boldsymbol{D}(\cdot)$ belonging to the same homotopy class form a weakly closed set in $\boldsymbol{Y}$.

Lemma 4.6 *Let $\boldsymbol{D} : [0, L] \to SO(3)$ be a continuous curve. Then the set*

$$\boldsymbol{X}_\mathcal{L} \equiv \{\boldsymbol{w} \in \boldsymbol{X} | \; \boldsymbol{D}[\boldsymbol{w}] \sim \boldsymbol{D}\}$$

is weakly closed in $\boldsymbol{Y}$.

If we consider closed rods, i.e., $\boldsymbol{w} \in \boldsymbol{X}_\mathcal{C}$, the significance of $\boldsymbol{X}_c L$ is to fix the linking number of the two curves $\boldsymbol{r}(\cdot)$ and $\boldsymbol{r}(\cdot) + \varepsilon \boldsymbol{d}_1(\cdot)$ where $\varepsilon > 0$ is so small that $(\varepsilon, 0) \in \mathcal{A}(s)$ for all $s \in [0, L]$. This linking number is a topological invariant and obviously removes the freedom of full rotations we had discussed above. The proof of the lemma, which can be also found in Gonzalez et al. [7, Lemma 3.7], is sketched at the end of this section for completeness.

4.2 Proofs

PROOF of Lemma 4.1. Let $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}_0, \boldsymbol{D}_0)$ in $\boldsymbol{Y}$ with $\boldsymbol{w}_n \in \boldsymbol{X}_{\mathcal{O}}$. As in the proof of Lemma 3.8 we get $\boldsymbol{D}_0 \in SO(3)$ and (3.30) which readily verifies the assertion, since the complement of $\operatorname{int}\mathcal{O}$ is closed in $\mathbb{R}^3$. $\diamondsuit$

PROOF of Lemma 4.5. Let $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{Y}$ with $\boldsymbol{w}_n \in \boldsymbol{X}_{\mathcal{K}}$. We have $E_{\mathrm{s}}(\boldsymbol{w}) \leq \tilde{c}$ by the weak lower semicontinuity of $E_{\mathrm{s}}(\cdot)$ (cf. Section 2). Lemma 3.8 implies that $\boldsymbol{w} \in \boldsymbol{X}$ and $\boldsymbol{w}$ satisfies (3.2), (3.6). We get (3.30) as in the proof of Lemma 3.8 and, thus, $\boldsymbol{w} \in \boldsymbol{X}_{\mathcal{C}}$. We will show below that $\boldsymbol{r}[\boldsymbol{w}_n] \simeq \boldsymbol{r}[\boldsymbol{w}]$ for some sufficiently large $n \in \mathbb{N}$. Hence $\boldsymbol{r}[\boldsymbol{w}] \simeq \boldsymbol{r}[\tilde{\boldsymbol{w}}]$ by $\boldsymbol{w}_n \in \boldsymbol{X}_{\mathcal{K}}$ and the assertion follows.

Since we consider closed rods (i.e., belonging to $\boldsymbol{X}_{\mathcal{C}}$), the parameter set Ω can be replaced with a parameter set

$$\Omega_0 \equiv \{(s, \zeta_1, \zeta_2) |\ s \in S_L,\ (\zeta_1, \zeta_2) \in \mathcal{A}(s)\}$$

where S_L is a circle with perimeter L and the points on S_L are identified according to some arc length parametrization. Note that $(s, 0, 0) \in \operatorname{int}\Omega_0$ for all $s \in S_L$ by (4.4). Recalling the proof of Proposition 3.4 we readily see that $\boldsymbol{p}[\boldsymbol{w}]$ is an open mapping even on $\operatorname{int}\Omega_0$ and, therefore, the compact curve $K_0 \equiv \boldsymbol{r}[\boldsymbol{w}](S_L)$ lies in the open set $\boldsymbol{p}[\boldsymbol{w}](\operatorname{int}\Omega_0)$. Thus we can find some open β-neighborhood $N_\beta(K_0) \subset \boldsymbol{p}[\boldsymbol{w}](\operatorname{int}\Omega_0)$, $\beta > 0$. For some subdivision $0 = s_0 < s_1 < \ldots < s_{N+1} = L$ of S_L (note that s_0 and s_{N+1} in fact coincide), we consider the piecewise affine closed curve K_1 consisting of the straight pieces connecting $\boldsymbol{r}[\boldsymbol{w}](s_i)$ with $\boldsymbol{r}[\boldsymbol{w}](s_{i+1})$, $i = 0, \ldots, N$. The subdivision can be supposed to be so fine that

$$K_1 \subset N_{\beta/4}(K_0). \tag{4.7}$$

Let us again recall the arguments of the proof of Proposition 3.1 for a special situation. We fix $(\sigma_0, 0, 0) \in \operatorname{int}\Omega_0$ and choose $U(\sigma_0) = U_{\varepsilon,\delta}$ such that (3.10), (3.11), (3.12) are satisfied. Then we arbitrarily choose $(\sigma_1, 0, 0)$, $(\sigma_2, 0, 0) \in U(\sigma_0)$, $\sigma_1 < \sigma_2$ and, by the same arguments leading to (3.16), we can show that

$$\Big(\boldsymbol{r}[\boldsymbol{w}](\sigma_2) - \boldsymbol{r}[\boldsymbol{w}](\sigma_1)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](\sigma_1) > 0 \tag{4.8}$$

and, similarly,

$$\Big(\boldsymbol{r}[\boldsymbol{w}](\sigma_1) - \boldsymbol{r}[\boldsymbol{w}](\sigma_2)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](\sigma_2) < 0\,. \tag{4.9}$$

By the compactness of $\Omega_0' \equiv \{(s, 0, 0) \in \Omega_0 |\ s \in S_L\}$ there are finitely many of the open sets $U(s_0)$, $s_0 \in S_L$, which already cover Ω_0'. Without loss of generality we can now assume that the subdivision of S_L is so fine that consecutive points $(s_i, 0, 0)$, $(s_{i+1}, 0, 0)$, $i = 0, \ldots, N$, always belong to the same set of the finite covering of Ω_0'. By (4.8), (4.9) we then have that

$$\Big(\boldsymbol{r}[\boldsymbol{w}](s_{i+1}) - \boldsymbol{r}[\boldsymbol{w}](s_i)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](s_i) > 0 \quad \text{for } i = 0, \ldots, N\,, \tag{4.10}$$

$$\Big(\boldsymbol{r}[\boldsymbol{w}](s_i) - \boldsymbol{r}[\boldsymbol{w}](s_{i+1})\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](s_{i+1}) < 0 \quad \text{for } i = 0, \ldots, N\,. \tag{4.11}$$

Let us now show that a straight segment $K_1^i \equiv [\boldsymbol{r}[\boldsymbol{w}](s_i), \boldsymbol{r}[\boldsymbol{w}](s_{i+1})]$ of K_1 (i=0,1,...,N) can intersect a cross-section $\mathcal{S}(s) \equiv \boldsymbol{p}[\boldsymbol{w}](s, \mathcal{A}(s))$, $s \in S_L$, at most once. Otherwise, if some straight segment K_1^i contains at least two points of some $\mathcal{S}(\tilde{s})$, then $K_1^i \subset \mathcal{S}(\tilde{s})$ by (4.7). But this contradicts the global injectivity of $\boldsymbol{p}[\boldsymbol{w}]$ on int Ω_0 and the fact that K_1^i contains both $\boldsymbol{r}[\boldsymbol{w}](s_i)$ and $\boldsymbol{r}[\boldsymbol{w}](s_{i+1})$.

Next we show that the straight segment K_1^i intersects exactly all cross-sections $\mathcal{S}(s)$ with $s \in [s_i, s_{i+1}]$. Let $\tau \to \boldsymbol{r}_1(\tau)$ be the arc length parametrization of K_1 such that $K_1^i = \boldsymbol{r}_1([\tau_i, \tau_{i+1}])$. Since $\boldsymbol{p}[\boldsymbol{w}](\cdot)$ is an open mapping, the inverse $\boldsymbol{q} \to (\breve{s}(\boldsymbol{q}), \breve{\zeta}_1(\boldsymbol{q}), \breve{\zeta}_2(\boldsymbol{q}))$ is continuous. Hence $\tau \to \breve{s}(\boldsymbol{r}_1(\tau))$ is continuous with $s_i = \breve{s}(\boldsymbol{r}_1(\tau_i))$, $s_{i+1} = \breve{s}(\boldsymbol{r}_1(\tau_{i+1}))$, i.e., K_1^i intersects at least all $\mathcal{S}(s)$ with $s \in [s_i, s_{i+1}]$. Suppose now that $\tilde{s} \equiv \breve{s}(\boldsymbol{q}(\tilde{\tau})) \notin [s_i, s_{i+1}]$ for some $\tilde{\tau} \in [\tau_i, \tau_{i+1}]$. By continuity arguments we can assume that $\tilde{s}$ is arbitrarily close to either s_i or s_{i+1}. Let first $\tilde{s} < s_i$. By the same arguments as in the proof of Proposition 3.1 leading to (3.16) we can derive that

$$\Big(\boldsymbol{p}[\boldsymbol{w}](\breve{s}(\boldsymbol{r}_1(\tilde{\tau})), \breve{\zeta}_1(\boldsymbol{r}_1(\tilde{\tau})), \breve{\zeta}_2(\boldsymbol{r}_1(\tilde{\tau}))) - \boldsymbol{r}[\boldsymbol{w}](s_i)\Big) \cdot \boldsymbol{d}_3(s_i) < 0$$

which is obviously the same as

$$\Big(\boldsymbol{r}_1(\tilde{\tau}) - \boldsymbol{r}[\boldsymbol{w}](s_i)\Big) \cdot \boldsymbol{d}_3(s_i) < 0\,. \tag{4.12}$$

On the other hand, by (4.10) we have that

$$(\boldsymbol{q} - \boldsymbol{r}[\boldsymbol{w}](s_i)) \cdot \boldsymbol{d}_3(s_i) > 0$$

for all $\boldsymbol{q} \in K_1^i$, since K_1^i is a straight segment. But this contradicts (4.12). Analogously we can argue if $\tilde{s} > s_{i+1}$ by using (4.11). Hence the straight segment K_1^i intersects exactly all cross-sections $\mathcal{S}(s)$ with $s \in [s_i, s_{i+1}]$, $i = 0, \ldots, N$ and we can conclude that the curve K_1 intersects each cross-section $\mathcal{S}(s)$, $s \in S_L$, exactly once.

By the previous results we can construct neighborhoods $M^0_{\beta/4}$ of K_0 and $M^1_{\beta/4}$ of K_1 by taking all circles in $\mathcal{S}(s)$, $s \in S_L$, with radius $\beta/4$ and centers on K_0 and K_1, respectively. By continuously translating theses circles within the cross-sections $\mathcal{S}(s)$ we readily verify that

$$\boldsymbol{r}[\boldsymbol{w}] \simeq \boldsymbol{r}_1\,. \tag{4.13}$$

As in the proof of Lemma 3.8 we get (3.30), i.e.,

$$\boldsymbol{r}[\boldsymbol{w}_n] \to \boldsymbol{r}[\boldsymbol{w}], \quad \boldsymbol{d}_3[\boldsymbol{w}_n] \to \boldsymbol{d}_3[\boldsymbol{w}] \quad \text{uniformly on } [0,L] \text{ as } n \to \infty\,. \tag{4.14}$$

For sufficiently large $n \in \mathbb{N}$ we have $r[\boldsymbol{w}_n](S_L) \subset M^0_{\beta/4}$ and we can find parameters $t_0^n < t_1^n < \ldots < t_{N+1}^n$ such that for all large $n \in \mathbb{N}$

$$\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) \in \mathcal{S}(s_i) \quad \text{for } i = 0, \ldots, N+1\,.$$

By the compactness of S_L we have, at least for a subsequence, that

$$\lim_{n\to\infty} t_i^n = t_i \in S_L \quad \text{for } i = 0, \ldots, N+1\,.$$

By

$$|\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) - \boldsymbol{r}[\boldsymbol{w}](t_i)| \le |\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) - \boldsymbol{r}[\boldsymbol{w}](t_i^n)| + |\boldsymbol{r}[\boldsymbol{w}](t_i^n) - \boldsymbol{r}[\boldsymbol{w}](t_i)|\,, \tag{4.15}$$

by (4.14), and by the continuity of $\boldsymbol{r}[\boldsymbol{w}](\cdot)$, we conclude that $\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) \to \boldsymbol{r}[\boldsymbol{w}](t_i)$. On the other hand, for each $\rho > 0$ there is $n(\rho) \in \mathbb{N}$ such that $\boldsymbol{r}[\boldsymbol{w}_n](S_L) \subset M_\varrho^0$ for all $n > n(\varrho)$ where M_ϱ^0 is a neighborhood of K_0 which is built as $M_{\beta/4}^0$ above but whith circles having radius $\varrho > 0$. Hence we also get $\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) \to \boldsymbol{r}[\boldsymbol{w}](s_i)$ which implies $t_i = s_i$ for $i = 0, \dots, N+1$ by the injectivity of $\boldsymbol{r}[\boldsymbol{w}](\cdot)$. Using an estimate like (4.15) also for $\boldsymbol{d}_3$, we thus have that

$$\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) \to \boldsymbol{r}[\boldsymbol{w}](s_i), \quad \boldsymbol{d}_3[\boldsymbol{w}_n](t_i^n) \to \boldsymbol{d}_3[\boldsymbol{w}](s_i) \quad \text{as } n \to \infty\,. \tag{4.16}$$

Thus we can fix $n \in \mathbb{N}$ so large that $K_3 \equiv \boldsymbol{r}[\boldsymbol{w}_n](S_L) \subset M_{\beta/4}^0$ and such that, by (4.10), (4.11),

$$\Big(\boldsymbol{r}[\boldsymbol{w}_n](t_{i+1}^n) - \boldsymbol{r}[\boldsymbol{w}_n](t_i^n)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](s_i) > 0\,, \ \Big(\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) - \boldsymbol{r}[\boldsymbol{w}_n](t_{i+1}^n)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}](s_{i+1}) < 0\,, \tag{4.17}$$

$$\Big(\boldsymbol{r}[\boldsymbol{w}_n](t_{i+1}^n) - \boldsymbol{r}[\boldsymbol{w}_n](t_i^n)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}_n](t_i^n) > 0\,, \ \Big(\boldsymbol{r}[\boldsymbol{w}_n](t_i^n) - \boldsymbol{r}[\boldsymbol{w}_n](t_{i+1}^n)\Big) \cdot \boldsymbol{d}_3[\boldsymbol{w}_n](t_{i+1}^n) < 0 \tag{4.18}$$

for $i = 0, \dots, N$.

By K_2 we now denote the piecewise affine closed curve connecting the points $\boldsymbol{r}[\boldsymbol{w}_n](t_0^n)$, $\boldsymbol{r}[\boldsymbol{w}_n](t_1^n), \dots, \boldsymbol{r}[\boldsymbol{w}_n](t_{N+1}^n)$. By $K_3 \subset M_{\beta/4}^0$, we have $K_2 \subset M_{\beta/2}^0$. Using the same arguments as above and by (4.17) we can show that K_2 intersects each cross-section $\mathcal{S}(s)$, $s \in S_L$, exactly once and that

$$K_1 \simeq K_2\,.$$

Finally, again by the same arguments as before but based on the deformed cross-sections of $\boldsymbol{p}[\boldsymbol{w}_n]$ and by (4.18), we can show that

$$K_2 \simeq K_3\,.$$

Take, however, $M_{\beta/2}^0$ instead of an analogue to $N_{\beta/4}(K_0)$ for these arguments and note that $M_{\beta/2}^0$ is an open neighborhood of K_3. Furthermore we have that $K_2 \subset M_{\beta/2}^0$ and, by (4.4), (4.16) and by choosing $\beta < \varepsilon_0/2$ at the beginning, we can assume that $M_{\beta/2}^0 \subset \boldsymbol{p}[\boldsymbol{w}_n](\operatorname{int}\Omega_0)$ for $n \in \mathbb{N}$ large enough.

We conclude that $K_0 \simeq K_3$ which just means that $\boldsymbol{r}[\boldsymbol{w}] \simeq \boldsymbol{r}[\boldsymbol{w}_n]$ and the proof is complete. $\diamondsuit$

Proof of Lemma 4.6. Each element $Q \in SO(3)$ can be represented by a vector $\boldsymbol{q}(Q) \in \mathbb{R}^3$. Here $\boldsymbol{q}(Q)$ describes the direction of the rotation axis and the length $|\boldsymbol{q}(Q)|$ gives the rotation angle $\varphi(Q) \in [0, \pi]$. In a neighborhood of the identity in $SO(3)$ the mapping $Q \to \boldsymbol{q}(Q)$ and its inversion $\boldsymbol{q} \to Q(\boldsymbol{q})$ are uniquely defined and continuous. Note that $Q(\boldsymbol{q}(Q_0)) = Q_0$, $\boldsymbol{q}(id) = \boldsymbol{0}$, $Q(\boldsymbol{0}) = id$. Furthermore the mapping $A \to A^{-1}$ is continuous in $\mathbb{R}^{3\times3}$ near $A = id$. Let now $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ in $\boldsymbol{X}$ for $\boldsymbol{w}_n \in \boldsymbol{X}_{\mathcal{L}}$. As in the proof of Lemma 3.8 we get (3.30), i.e., $\boldsymbol{D}[\boldsymbol{w}_n] \to \boldsymbol{D}[\boldsymbol{w}]$ in $\mathcal{C}([0, L])$ which readily implies that $\boldsymbol{w} \in \boldsymbol{X}$. Furthermore $\boldsymbol{D}[\boldsymbol{w}](s)\boldsymbol{D}[\boldsymbol{w}_n](s)^{-1}$ is continuous in s and uniformly close to the identity for $n \in \mathbb{N}$ large. With

$$\Psi(s, \tau) \equiv Q(\tau\boldsymbol{q}(\boldsymbol{D}[\boldsymbol{w}](s)\boldsymbol{D}[\boldsymbol{w}_n](s)^{-1})\boldsymbol{D}[\boldsymbol{w}_n](s) \quad \text{for } s \in [0, L],\ \tau \in [0, 1]$$

we readily verify that $\boldsymbol{D}[\boldsymbol{w}_n] \sim \boldsymbol{D}[\boldsymbol{w}]$ for sufficiently large $n \in \mathbb{N}$ and thus also $\boldsymbol{D}[\boldsymbol{w}] \sim \boldsymbol{D}$, i.e., $\boldsymbol{D}[\boldsymbol{w}] \in \boldsymbol{X}_{\mathcal{L}}$. $\diamondsuit$

5 Existence of solutions

In this section we verify the existence of an energy minimizing equilibrium state for two general situations. First we look for globally injective deformations possibly subjected to a rigid obstacle and then we consider closed rods restricted by topological constraints.

Let us study variational problems where we calim to minimize the energy

$$E(\boldsymbol{w}) \equiv \int_0^L W(\boldsymbol{u}(s), \boldsymbol{v}(s), s)\, ds - \int_\Omega \boldsymbol{p}[\boldsymbol{w}](s, \zeta_1, \zeta_2) \cdot d\mathfrak{f}$$

with respect to further side conditions. To get coercivity of $E(\cdot)$ in $\boldsymbol{Y}$ we impose the usual growth condition

$$W(\boldsymbol{u}, \boldsymbol{v}, s) \geq c(|\boldsymbol{u}|^p + |\boldsymbol{v}|^q) + \gamma(s) \quad \text{for all } (\boldsymbol{u}, \boldsymbol{v}, s) \tag{5.1}$$

where $\gamma \in \mathcal{L}^1([0, L])$ is given and $1 < p, q < \infty$ are the constants identifying the space $\boldsymbol{X} = \boldsymbol{X}^{p,q}$ (cf. (2.2)). Recall also the general assumptions for W which we imposed in Section 2.

To avoid unnecessary technicalities we fix $\boldsymbol{r}(0)$ at the origin and thus consider the general variational problem:

$$E(\boldsymbol{w}) \to \text{Min!}, \quad \boldsymbol{w} \in \boldsymbol{X}, \tag{5.2}$$

$$\boldsymbol{r}[\boldsymbol{w}](0) = \boldsymbol{0}, \quad \boldsymbol{w} \in \boldsymbol{X}_{\mathcal{O}}, \tag{5.3}$$

$$\boldsymbol{w} \text{ satisfies (3.2) and (3.6).} \tag{5.4}$$

Theorem 5.5 *Let W satisfy* (3.3) *and* (5.1). *If there is at least one admissible $\boldsymbol{w}_0$ for the variational problem* (5.2) – (5.4) *with finite energy $E(\boldsymbol{w}_0) < \infty$, then the variational problem has a minimizer.*

Observe that the theorem remains true in the case without obstacle, since we have included the case $\mathcal{O} = \emptyset$. Instead of the boundary condition $\boldsymbol{r}(0) = \boldsymbol{0}$ we can fix the position of any other point of the rod or it is even sufficient to impose much weaker conditions as, e.g., done in Schuricht [8].

PROOF . Let $\boldsymbol{w}_n = (\boldsymbol{u}_n, \boldsymbol{v}_n, \boldsymbol{r}_{0,n}, \boldsymbol{D}_{0,n}) \in \boldsymbol{X}$ be a minimizing sequence of the variational problem. Since $\boldsymbol{r}_{0,n} = \boldsymbol{0}$, since all $\boldsymbol{D}_{0,n}$ belong to the bounded set $SO(3)$, since E_{p} is linear, and by (5.1), the sequence $\boldsymbol{w}_n$ is bounded in the reflexive space $\boldsymbol{Y}$. Hence there is a weakly convergent subsequence (denoted the same way)

$$\boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \in \boldsymbol{Y}.$$

In Section 2 we have seen that E_{s} is weakly lower semicontinuous. E_{p} is obviously a linear continuous functional on $\boldsymbol{Y}$ and thus also weakly lower semicontinuous. Therefore $E(\boldsymbol{w}) \leq E(\bar{\boldsymbol{w}})$

for all admissible $\bar{\boldsymbol{w}}$. The assertion of the theorem is verified if we can show that $\boldsymbol{w}$ is admissible. But this readily follows from Lemma 3.8 and Lemma 4.1. $\diamondsuit$

Let us now consider the variational problem with topological constraints. We assume that a given closed continuous curve $\tilde{\boldsymbol{r}}$ represents some prescribed knot type and that a given continuous curve $\tilde{\boldsymbol{D}}$ in $SO(3)$ identifies some prescribed homotopy classs for the frames $\boldsymbol{D}(\cdot)$. Furthermore we choose some closed set $\mathcal{O} \subset \mathbb{R}^3$ (which is the closure of an open set) and numbers $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1^2 + \alpha_2^2 = 1$ in order to fix the sets $\boldsymbol{X}_{\mathcal{O}}$ and $\boldsymbol{X}_{\mathcal{C}}$ defined in (4.2) and (4.3), respectively. Thus we can study the variational problem

$$E(\boldsymbol{w}) \to \text{Min!}, \quad \boldsymbol{w} \in \boldsymbol{X}, \tag{5.6}$$

$$\boldsymbol{w} \in \boldsymbol{X}_{\mathcal{C}}, \quad \boldsymbol{w} \in \boldsymbol{X}_{\mathcal{O}}, \quad \boldsymbol{r}[\boldsymbol{w}] \simeq \tilde{\boldsymbol{r}}, \quad \boldsymbol{D}[\boldsymbol{w}] \sim \tilde{\boldsymbol{D}}, \tag{5.7}$$

$$\boldsymbol{w} \text{ satisfies (3.2) and (3.6).} \tag{5.8}$$

Theorem 5.9 *Let W satisfy* (3.3) *and* (5.1). *If there is at least one admissible $\boldsymbol{w}_0$ for the variational problem* (5.6) – (5.8) *with finite energy $E(\boldsymbol{W}_0) < \infty$, then the variational problem has a minimizer.*

A similar result for rods with homogeneous circular cross-sections has been proved by Gonzalez et al. [7, Theorem 4.1] based on a nonlocal geometric side condition instead of (3.2) and (3.6). The results are identical for unshearable rods. In the case of shearable rods, however, the geometric condition in [7] can model global injectivity and self-contact only approximatly while our results are exact also in that case even with non-homogeneous and non-circular cross-sections. The disadvantage of condition (3.6) is that it seems to be unsuitable for the derivation of the Euler-Lagrange equation as necessary optimality condition. With respect to that question the nonlocal geometric condition used in [7] seems to be more powerful, since the Euler-Lagrange equation could be derived at least for unshearable inextensible rods (cf. Schuricht & v.d. Mosel [11]).

Proof . We argue as in the proof of Theorem 5.5 and still have to verify that $\boldsymbol{w} \in \boldsymbol{X}_{\mathcal{C}}$, $\boldsymbol{r}[\boldsymbol{w}] \simeq \tilde{\boldsymbol{r}}$, and $\boldsymbol{D}[\boldsymbol{w}] \sim \tilde{\boldsymbol{D}}$ where $\boldsymbol{w}$ is the weak limit of the minimizing sequence.

As in the proof of Lemma 3.8 we get (3.30) which readily implies that $\boldsymbol{w} \in \boldsymbol{X}_{\mathcal{C}}$. Lemma 4.6 gives that $\boldsymbol{D}[\boldsymbol{w}] \sim \tilde{\boldsymbol{D}}$. Now note that $E_{\rm s}(\boldsymbol{w}_n)$ has to be bounded for the minimizing sequence $\boldsymbol{w}_n$ by (5.1) and the linearity of $E_{\rm p}(\cdot)$. Hence Lemma 4.5 provides that $\boldsymbol{r}[\boldsymbol{w}] \simeq \tilde{\boldsymbol{r}}$ and the proof is complete. $\diamondsuit$

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