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SOME ASPECTS OF THE GLOBAL GEOMETRY OF ENTIRE SPACE-LIKE SUBMANIFOLDS

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ABSTRACT. We prove some Bernstein type theorems for entire space-like submanifolds in pseudo-Euclidean space and as a corollary, we give a new proof of the Calabi-Pogorelov theorem for Monge-Ampère equations.

1. INTRODUCTION

The search for Bernstein theorems, i.e. theorems stating that, perhaps under suitable conditions, entire minimal graphs or their higher codimensional analogues in Euclidean space, necessarily are flat and planar, has been a central topic in geometric analysis and led to many important insights in the regularity theory of PDEs. From that perspective, it might look like a curiosity to study the analogous question for space-like entire minimal submanifolds that can be represented by a graph over a linear subspace in pseudo-Euclidean space. It turns out, however, that this situation leads to a rich mathematical structure of its own, with many aspects not shared by their Euclidean counterparts.

The investigations started with a paper of E. Calabi [C1]. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function. If its graph $(x^1, \dots, x^m; f(x^1, \dots, x^m))$ in Minkowski space defines a space-like extremal hypersurface, then f satisfies

$$(1 - |\nabla f|^2) \sum_{i=1}^m \frac{\partial^2 f}{\partial x^{i^2}} + \sum_{i,j=1}^m \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \frac{\partial^2 f}{\partial x^i \partial x^j} = 0, \quad |\nabla f| < 1, \quad (1.1)$$

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E. Calabi raised the Bernstein type problem and proved that (1.1) has only linear entire solutions for $m \leq 4$. Several years later the problem was settled for all m by Cheng-Yau [C-Y].

The issue of space-like hypersurfaces of nonzero constant mean curvature becomes more complicated. The results here depend on conditions about the image under the Gauss map of the hypersurface [Cho-Tr] [X1] [X-Y].

If we wish to study such entire minimal submanifolds of higher codimension, we need to study maps

$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

solving the generalization of (1.1) which reads as

$$\sum_{i,j=1}^m g^{ij} \frac{\partial^2 f^s}{\partial x^i \partial x^j} = 0,$$

where $g_{ij} = \delta_{ij} - \sum_{s=1}^n f_{x^i}^s f_{x^j}^s$ and (g^{ij}) is the inverse matrix of (g_{ij}) . We let M be the graph of f . A natural approach is to investigate the Gauss map

$$\gamma : M \rightarrow \mathbf{G}_{m,n}^n,$$

associating to each point in M its tangent space, considered as an element of the pseudo-Grassmannian $\mathbf{G}_{m,n}^n$. In contrast to its dual and Euclidean counterpart $\mathbf{G}_{m,n}$, $\mathbf{G}_{m,n}^n$ is a symmetric space of noncompact type.

Therefore, in particular, it carries a complete Riemannian metric of nonpositive curvature. This is much better adapted to the geometry of the Gauss map as a harmonic map than the (partly) positive curvature of $\mathbf{G}_{m,n}$, and this can be considered as the reason why one has stronger Bernstein theorems in the pseudo-Euclidean than in the Euclidean case. Related to this fact is that for a space-like submanifold in pseudo-Euclidean space with parallel mean curvature, its Ricci curvature is bounded from below. This is a main advantage compared with the Euclidean space as the ambient space. By Cheng-Yau's method we have two estimates for the squared norm of the second fundamental form. One is in terms of the mean curvature, and the other estimate is in terms of its mean curvature and the image diameter of its Gauss map.

In the present situation, however, an entire solution not necessarily defines a complete manifold as in the ambient Euclidean case. The completeness argument thus becomes a key issue. Fortunately, Cheng-Yau's method allows an extension to the higher codimensional case. We also obtain a gradient estimate for the pseudo-distance on a space-like m -submanifold in pseudo-Euclidean space \mathbb{R}_n^{m+n} with index n .

All the geometric conclusions in the paper stem from those estimates.

We obtain a general Bernstein type theorem stating that any space-like submanifold in pseudo-Euclidean space with parallel mean curvature that is closed w.r.t. the Euclidean topology and whose Gauss map is bounded is necessarily planar.

Apparently, Hitchin [H] was the first to observe the connection between Lagrangian minimal graphs in pseudo-Euclidean space and Monge-Ampère equations. Namely, if $F : \mathbb{R}^m \rightarrow \mathbb{R}$ satisfies the Monge-Ampère equation

$$\det \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right) = \text{const.} \quad (1.2)$$

then the graph of its gradient defines a special (i.e. minimal) Lagrangian submanifold of \mathbb{R}_m^{2m} . This submanifold is space-like precisely if F is convex. Therefore, we may apply our Bernstein theorem to obtain a new proof of the famous theorem of Calabi [C2] (dimension ≤ 4) and Pogorelov [P] (any dimension) that the only entire convex solutions are quadratic polynomials which is a fundamental result in affine differential geometry. There may exist further connections with Lagrangian geometry related to the mirror symmetry conjecture. The starting point is McLeans construction [M] of the moduli space M of special Lagrangian submanifolds of a Calabi-Yau manifold. McLean constructed a natural Riemannian metric on this moduli space M . The key result for us now is Hitchin's [H] construction of a natural embedding of M as a Lagrangian submanifold of pseudo-Euclidean space so that its space-like metric is precisely McLean's metric.

Now, this embedding of M in general is not minimal, but if it is, in view of our results, this has strong geometric consequences for the space of special Lagrangian submanifolds of the original Calabi-Yau manifold. We therefore believe that exploring these connections between Lagrangian geometry and Bernstein theorems in pseudo-Euclidean spaces is a rewarding topic for future research.

2. ESTIMATES OF THE SECOND FUNDAMENTAL FORM

Let \mathbb{R}_n^{m+n} be an $(m+n)$ -dimensional pseudo-Euclidean space of index n . Let M be a space-like oriented m -submanifold in \mathbb{R}_n^{m+n} . Choose a local Lorentzian frame field $\{e_i, e_s\}$ along M with dual frame field $\{\omega_i, \omega_s\}$, such that the e_i are tangent vectors to M . We agree with the following range of indices

$$A, B, C, \dots = 1, \dots, m+n;$$

$$i, j, k \dots = 1, \dots, m; \quad s, t, \dots = m+1, \dots, m+n.$$

The induced Riemannian metric of M is given by $ds_M^2 = \sum_i \omega_i^2$ and the induced structure equations of M are

$$\begin{aligned} d\omega_i &= \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} &= 0, \\ d\omega_{ij} &= \omega_{ik} \wedge \omega_{kj} - \omega_{is} \wedge \omega_{sj}, \\ \Omega_{ij} &= d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l. \end{aligned}$$

By Cartan's lemma we have

$$\omega_{si} = h_{sij}\omega_j,$$

where the h_{sij} are the components of the second fundamental form of M in \mathbb{R}_n^{m+n} . The mean curvature vector of M in \mathbb{R}_n^{m+n} is defined by

$$H = \frac{1}{m}h_{sii}e_s.$$

We also have the Gauss equation

$$R_{ijkl} = -(h_{sik}h_{sjl} - h_{sil}h_{sjk}), \quad (2.1)$$

and the Ricci curvature

$$R_{ij} = R_{kikj} = -(h_{skk}h_{sij} - h_{ski}h_{skj}),$$

from which it follows that

$$\text{Ric}_M \geq -\frac{1}{4}m^2|H|^2. \quad (2.2)$$

There is an induced connection on the normal bundle NM in \mathbb{R}_n^{m+n} . We have

$$\begin{aligned} d\omega_{st} &= -\omega_{sr} \wedge \omega_{rt} + \Omega_{st}, \\ \Omega_{st} &= -\frac{1}{2}R_{stij}\omega_i \wedge \omega_j, \\ R_{stij} &= (h_{ski}h_{tkj} - h_{skj}h_{tki}). \end{aligned} \quad (2.3)$$

The covariant derivative of h_{sij} is given by

$$h_{sijk}\omega_k = dh_{sij} + h_{slj}\omega_{li} + h_{sil}\omega_{lj} - h_{tij}\omega_{ts}. \quad (2.4)$$

It is easily seen that $h_{sijk} = h_{sikj}$, so h_{sijk} is symmetric in i, j, k . If

$$DH = \frac{1}{m}h_{sijk}\omega_k e_s \equiv 0, \quad (2.5)$$

then M is called a space-like submanifold with parallel mean curvature. In this case we have [X2]

$$\frac{1}{2}\Delta S \geq \sum h_{sijk}^2 - m|H|S^{\frac{3}{2}} + \frac{1}{n}S^2, \quad (2.6)$$

where S is the squared norm of the second fundamental form.

The space-like m -planes in \mathbb{R}_n^{m+n} form the pseudo-Grassmannian $\mathbf{G}_{m,n}^n$. It is a symmetric space of noncompact type which is the noncompact dual space of the

Grassmannian manifold $\mathbf{G}_{m,n}$. The canonical Riemannian metric on $\mathbf{G}_{m,n}^n$ is given by

$$ds_G^2 = \sum_{s,i} (\omega_{si})^2.$$

Let 0 be the origin of \mathbb{R}_n^{m+n} . Let $SO^0(m+n, n)$ denote the identity component of the Lorentzian group $O(m+n, n)$. $SO^0(m+n, n)$ can be viewed as the manifold consisting of all Lorentzian frames $(0; e_i, e_s)$, and $SO^0(m+n, n)/SO(m) \times SO(n)$ can be viewed as $\mathbf{G}_{m,n}^n$. Let $P = \{(x; e_1, \dots, e_m); x \in M, e_i \in T_x M\}$ be the principal bundle of orthonormal tangent frames over M , $Q = \{(x; e_{m+1}, \dots, e_{m+n}); x \in M, e_s \in N_x M\}$ be the principal bundle of orthonormal normal frames over M , then $\bar{\pi} : P \oplus Q \rightarrow M$ is the projection with fiber $SO(m) \times SO(n)$, $i : P \oplus Q \hookrightarrow SO^0(m+n, n)$ is the natural inclusion.

We define the generalized Gauss map $\gamma : M \rightarrow \mathbf{G}_{m,n}^n$ by

$$\gamma(x) = T_x M \in \mathbf{G}_{m,n}^n$$

via parallel translation in \mathbb{R}_n^{m+n} for $\forall x \in M$. Thus, the following commutative diagram holds

$$\begin{array}{ccc} P \oplus Q & \xrightarrow{i} & SO^0(m+n, n) \\ \bar{\pi} \downarrow & & \downarrow \pi \\ M & \xrightarrow{\gamma} & \mathbf{G}_{m,n}^n \end{array}$$

With respect to the canonical metric ds_G^2 of $\mathbf{G}_{m,n}^n$ the Levi-Civita connection is given by

$$\omega_{(si)(tj)} = \delta_{st}\omega_{ij} - \delta_{ij}\omega_{st}.$$

Using the above diagram, we have

$$\gamma^* \omega_{si} = h_{sij} \omega_j. \quad (2.7)$$

Let r, \tilde{r} be the respective distance functions on M and $\mathbf{G}_{m,n}^n$ relative to fixed points $x_0 \in M, \tilde{x}_0 \in \mathbf{G}_{m,n}^n$. Let B_a be a closed geodesic ball of radius a and centered at x_0 . Define the maximum modulus of the Gauss map $\gamma : M \rightarrow \mathbf{G}_{m,n}^n$ on B_a by

$$\mu(\gamma, a) \stackrel{def.}{=} \max \{\tilde{r}(\gamma(x)); x \in B_a \subset M\}.$$

For a fixed positive number a , choose $b \geq \mu^2(\gamma, a)$. Define $f : B_a \rightarrow \mathbb{R}$ by

$$f = \frac{(a^2 - r^2)^2 S}{(b - h \circ \gamma)^2},$$

where $h = \tilde{r}^2$. By applying the maximum principle to f we can derive an estimate for S , the squared norm of the second fundamental form of M in \mathbb{R}_n^{m+n} , in terms

of the mean curvature and the image diameter of the Gauss map. In fact, we have [X2] that for any $x \in B_a$

$$S(x) \leq k \left(\frac{(8\mu a + m a^2 |H|)^2 \mu^4}{(2 + \frac{1}{n} \mu^2)^2 (a^2 - r^2)^2} + \frac{(2(m+4)a^2 + m(m-1)|H|a^3)\mu^2}{(2 + \frac{1}{n} \mu^2)(a^2 - r^2)^2} \right), \quad (2.8)$$

where k is an absolute constant. In what follows k may be different in different inequalities.

Consider the auxiliary function

$$f = (a^2 - r^2)^2 S$$

on a geodesic ball B_a of radius a and centered at $x_0 \in M$. By a similar method we can obtain an estimate in terms only of the mean curvature of M in \mathbb{R}_n^{m+n} :

$$S(x) \leq k \frac{m^2 n^2 |H|^2 a^4 + mn(m-1)|H|a^3 + 2n(m+4)a^2}{(a^2 - r^2)^2}, \quad (2.9)$$

for all $x \in B_a \subset M$.

3. COMPLETENESS

In this section we generalize the argument of Cheng-Yau [C-Y] to higher codimension with some technical modifications.

Let M be a space-like submanifold in pseudo-Euclidean space \mathbb{R}_n^{m+n} with index n . Let $X = (x_1, \dots, x_m; y_1, \dots, y_n)$ be the position vector of M . Define the pseudo-distance function on M by $\langle X, X \rangle = \sum_i x_i^2 - \sum_s y_s^2$. It is non-negative because M is space-like.

Proposition 3.1. *If M is closed with respect to the Euclidean topology, then when $0 \in M$, $z = \langle X, X \rangle$ is a proper function on M .*

proof. Let $\bar{c} = \inf\{c; \text{the set where } \langle X, X \rangle \leq c \text{ is compact}\}$. Then we will show $\bar{c} = \infty$.

Let $\mathbb{R}_1^2 \subset \mathbb{R}_n^{m+n}$ be a Minkowski plane. Since $0 \in \mathbb{R}_1^2 \cap M$ and M is space-like, M meets \mathbb{R}_1^2 transversally. It follows that there are positive constants $\varepsilon_1, \varepsilon_2$ and ε_3 such that for $(x, y) \in \mathbb{R}_1^2 \cap M$ and $\sum_i x_i^2 = \varepsilon_1$, we have $\varepsilon_2 \geq \langle X, X \rangle \geq \varepsilon_3$.

Suppose $\bar{c} < \infty$. By the assumption that M is closed with respect to the Euclidean topology we have a sequence of points $(x_1^\alpha, \dots, x_n^\alpha; y_1^\alpha, \dots, y_m^\alpha)$ in M such that

$$\begin{aligned} \sum_i (x_i^\alpha)^2 &\rightarrow \infty, \\ \sum_s (y_s^\alpha)^2 &\rightarrow \infty, \\ \sum_i (x_i^\alpha)^2 - \sum_s (y_s^\alpha)^2 &< \bar{c}. \end{aligned}$$

Choose α sufficiently large, such that

$$\sqrt{\sum_i (x_i^\alpha)^2} > \varepsilon_1^{\frac{1}{2}} \varepsilon_3^{-1} (\bar{c} + 2\varepsilon_2). \quad (3.1)$$

By an action of $SO(m) \times SO(n)$ we have new coordinates of \mathbb{R}_n^{m+n} such that the point $(x_1^\alpha, \dots, x_m^\alpha; y_1^\alpha, \dots, y_n^\alpha)$ becomes $(\sqrt{\sum_i (x_i^\alpha)^2}, 0, \dots, 0; \sqrt{\sum_s (y_s^\alpha)^2}, 0, \dots, 0)$ in the new coordinates. For simplicity it is denoted by $(x_1^\alpha, 0, \dots, 0; y_1^\alpha, 0, \dots, 0)$ with $y_1^\alpha > 0$ and (3.1) becomes

$$x_1^\alpha > \varepsilon_1^{\frac{1}{2}} \varepsilon_3^{-1} (\bar{c} + 2\varepsilon_2). \quad (3.2)$$

Let P^α be the Minkowski 2-plane spanned by the x_1 -axis and the y_1 -axis. By the previous argument P^α intersects M in a point $(x_1^0, 0, \dots, 0; y_1^0, 0, \dots, 0)$ with $(x_1^0)^2 = \varepsilon_1$ and $\varepsilon_2 \geq (x_1^0)^2 - (y_1^0)^2 \geq \varepsilon_3$.

Since M is space-like, the point $(x_1^\alpha, 0, \dots, 0; y_1^\alpha, 0, \dots, 0)$ can not lie in the light cone of $(x_1^0, 0, \dots, 0; y_1^0, 0, \dots, 0)$. Therefore,

$$\bar{c} + (x_1^0)^2 - (y_1^0)^2 \geq 2x_1^\alpha(x_1^0 - y_1^0) + 2(x_1^\alpha - y_1^\alpha)y_1^0, \quad (3.3)$$

and

$$\bar{c}(x_1^\alpha + y_1^\alpha)^{-1} \geq x_1^\alpha - y_1^\alpha \geq 0. \quad (3.4)$$

From $x_1^0 = \sqrt{\varepsilon_1} > |y_1^0|$ and $(x_1^0)^2 - (y_1^0)^2 \geq \varepsilon_3$ we have

$$x_1^0 - y_1^0 > 2^{-1} \varepsilon_1^{-\frac{1}{2}} \varepsilon_3. \quad (3.5)$$

Substituting (3.4) and (3.5) into (3.3) gives

$$\bar{c} + \varepsilon_2 \geq \varepsilon_1^{-\frac{1}{2}} \varepsilon_3 x_1^\alpha - \bar{c} \varepsilon_1^{\frac{1}{2}} (x_1^\alpha)^{-1},$$

namely,

$$\varepsilon_1^{-\frac{1}{2}} \varepsilon_3 (x_1^\alpha)^2 - (\bar{c} + \varepsilon_2) x_1^\alpha - \bar{c} \varepsilon_1^{\frac{1}{2}} \leq 0.$$

It follows that

$$x_1^\alpha < \varepsilon_1^{\frac{1}{2}} \varepsilon_3^{-1} (\bar{c} + 2\varepsilon_2),$$

which contradicts (3.2) and the proof is complete. \square

Now let us study $X : M \rightarrow \mathbb{R}_n^{m+n}$ being a space-like submanifold with parallel mean curvature. Choose a Lorentzian frame field $\{e_i, e_s\}$ along M , such that e_i are tangent vectors to M with $\nabla_{e_i} e_j = 0$ at the considered point. We need also carefully choose the normal vectors. This is the main technical point to generalize Cheng-Yau's proof to higher codimension.

Let $\bar{X} = X - \langle X, e_i \rangle e_i$. At a point, say q , choose

$$e_{m+1} = \frac{\bar{X}}{|\bar{X}|},$$

then choose other normal vectors e_{m+2}, \dots, e_{m+n} , so that they are all mutually orthogonal and then expand them around the point q to form a local normal frame field.

Let $z = \langle X, X \rangle$ be the pseudo-distance function on M . Then

$$z_i \stackrel{\text{def.}}{=} e_i(X) = 2 \langle X, e_i \rangle, \quad (3.6)$$

$$z_{ij} \stackrel{\text{def.}}{=} \text{Hess}(z)(e_i, e_j) = 2(\delta_{ij} - \langle X, e_s \rangle h_{sij}), \quad (3.7)$$

$$\Delta z = 2m - 2m \langle X, e_s \rangle H_s. \quad (3.8)$$

On the compact set $\{z \leq k\}$ in M for some k define a function

$$f = \frac{|\nabla z|^2}{(z+1)^2} \exp\left(\frac{-c}{k-z}\right), \quad (3.9)$$

where c will be chosen later. It attains its maximum at a point q . Then,

$$\nabla f(q) = 0,$$

$$\Delta f(q) \leq 0.$$

By computations we have that at q

$$2z_j z_{ij} - g |\nabla z|^2 z_i = 0, \quad (3.10)$$

$$2 \sum_{ij} z_{ij}^2 + 2z_j z_{iji} - g' |\nabla z|^4 - g(2z_{ij} z_i z_j + |\nabla z|^2 \Delta z) \leq 0, \quad (3.11)$$

where

$$g = \frac{2}{z+1} + \frac{c}{(k-z)^2}.$$

By the choice of the normal vectors e_s at the point q (3.7) reduces to

$$z_{ij} = 2\delta_{ij} - 2 \langle X, e_{m+1} \rangle h_{m+1ij} \quad (3.12)$$

It follows that

$$h_{m+1ij} = \frac{2\delta_{ij} - z_{ij}}{2 \langle X, e_{m+1} \rangle}, \quad (3.13)$$

$$\Delta z = 2m - 2m \langle X, e_{m+1} \rangle H_{m+1}, \quad (3.14)$$

and (3.6) means

$$z = \frac{1}{4}|\nabla z|^2 - \langle X, e_{m+1} \rangle^2. \quad (3.15)$$

By Schwarz inequality (see Lemma 2 in [Y]), we have

$$\sum_{ij} z_{ij}^2 \geq \frac{2m-1}{2m-2} \sum_i \left(\sum_j z_{ij} z_j \right)^2 |\nabla z|^{-2} - \frac{1}{m-1} (\Delta z)^2. \quad (3.16)$$

Substituting (3.10), (3.14) and (3.15) into (3.16) yields

$$\sum_{ij} z_{ij}^2 \geq \frac{2m-1}{8m-8} g^2 |\nabla z|^4 - \frac{1}{m-1} \left(2m - mH_{m+1} (|\nabla z|^2 - 4z)^{\frac{1}{2}} \right)^2. \quad (3.17)$$

Noting the Ricci formula, the Gauss equation (2.1) and

$$(\Delta z)_i = m h_{sij} z_j H_s,$$

we have

$$z_j z_{iji} = h_{ski} h_{skj} z_i z_j \geq h_{m+1 ki} h_{m+1 kj} z_i z_j.$$

At the point q we can use (3.13), and the above expression becomes

$$z_j z_{iji} \geq 4 - 2g |\nabla z|^2 + \frac{1}{4} g^2 |\nabla z|^4. \quad (3.18)$$

Substituting (3.10), (3.13), (3.14), (3.17) and (3.18) into (3.11), we have

$$\begin{aligned} 0 \geq & \left(\frac{1}{4(m-1)} g^2 - g' \right) |\nabla z|^4 - g \left(4 + 2m + m |H_{m+1}| (|\nabla z|^2 - 4z)^{\frac{1}{2}} \right) |\nabla z|^2 \\ & + 8 - \frac{2}{m-1} \left(2m + m |H_{m+1}| (|\nabla z|^2 - 4z)^{\frac{1}{2}} \right)^2. \end{aligned} \quad (3.19)$$

The coefficient of $|\nabla z|^4$ is

$$\frac{1}{4(m-1)} \left(\frac{4}{(z+1)^2} + \frac{c^2}{(k-z)^4} + \frac{4c}{(z+1)(k-z)^2} \right) + \frac{2}{(z+1)^2} - \frac{2c}{(k-z)^3}. \quad (3.20)$$

Choose $c = 8(m-1)k$ and so that

$$(3.20) \geq \frac{2}{(z+1)^2}.$$

Hence, at the point q

$$\frac{2}{(z+1)^2} |\nabla z|^4 \leq g(4 + 2m + m |H| |\nabla z|) |\nabla z|^2 + \frac{2}{m-1} (2m + m |H| |\nabla z|)^2.$$

This means that

$$\begin{aligned} f^2 \leq & \frac{16m^2}{(m-1)(z+1)^2} \exp\left(\frac{-2c}{k-z}\right) + \frac{16m^2}{(m-1)(z+1)} \exp\left(\frac{-3c}{2(k-z)}\right) |H| f^{\frac{1}{2}} \\ & + 4(m+2)g \exp\left(\frac{-c}{k-z}\right) f + \frac{4m^2}{m-1} \exp\left(\frac{-c}{k-z}\right) |H|^2 f \\ & + 2mg(z+1) \exp\left(\frac{-c}{2(k-z)}\right) |H| f^{\frac{3}{2}}. \end{aligned}$$

We then can find a constant P depending only on m so that

$$f^2 \leq \frac{1}{4}P \left((z+1)^{-2} + |H|(z+1)^{-1} f^{\frac{1}{2}} + (1+|H|^2)f + |H|f^{\frac{3}{2}} \right).$$

It follows that

$$\sup_{z \leq k} f \leq \max \left\{ P^{\frac{1}{2}} \sup_{z \leq k} (z+1)^{-1}, P^{\frac{2}{3}} |H|^{\frac{2}{3}} \sup_{z \leq k} (z+1)^{-\frac{2}{3}}, P(1+|H|^2), P^2 |H|^2 \right\}.$$

Now, we arrive at the following conclusion.

Proposition 3.2. *Let M be a space-like submanifold in pseudo-Euclidean space \mathbb{R}_n^{m+n} of index n with parallel mean curvature. Let z be the pseudo-distance function on M . If for some $k > 0$, the set $\{z \leq k\}$ is compact, then there is constant b depending only on the dimension m and the norm of the mean curvature $|H|$, such that for all $x \in M$ with $z(x) \leq \frac{k}{2}$,*

$$|\nabla z| \leq b(z+1). \quad (3.21)$$

□

Without loss of generality we assume that $0 \in M$. If M is closed with respect to the Euclidean topology, then z is a proper function on M by Proposition 3.1 and (3.21) is valid for any k . Let $\gamma : [0, r] \rightarrow M$ be a geodesic on M issuing from the origin 0 . Integrating (3.21) gives

$$z(\gamma(r)) + 1 \leq \exp(br),$$

which forces M to be complete. In summary we have

Theorem 3.3. *Let M be a space-like submanifold in the pseudo-Euclidean space \mathbb{R}_n^{m+n} . Assume that M is closed with respect to the Euclidean topology and its mean curvature is parallel. Then M is complete with respect to the induced metric from the ambient space.*

□

4. BERNSTEIN TYPE THEOREMS

We are now in a position to prove some theorems.

Theorem 4.1. *Let M be a space-like m -submanifold in pseudo-Euclidean space \mathbb{R}_n^{m+n} with index n . Assume that*

- (1) M is closed with respect to the Euclidean topology;
- (2) M has parallel mean curvature;
- (3) the image under the Gauss map from M into $\mathbf{G}_{m,n}^n$ is bounded.

Then M has to be a linear subspace.

Proof. Let $\tilde{x}_0 \in \mathbf{G}_{m,n}^n$ and R be a positive number and large enough such that the image under the Gauss map $\gamma : M \rightarrow \mathbf{G}_{m,n}^n$ is contained in the geodesic ball $B_R(\tilde{x}_0)$. Since the mean curvature is parallel, the Gauss map is harmonic, and so we have a harmonic map $\gamma : M \rightarrow B_R(\tilde{x}_0) \subset \mathbf{G}_{m,n}^n$. On the other hand, from (2.2) we know that the Ricci curvature of M is bounded from below. So, we can use the maximum principle to conclude that (see Thm 3.10 in [X3]) for the energy density of γ ,

$$\inf e(\gamma) = 0.$$

From (2.7)

$$\inf S = 0. \tag{4.1}$$

But, on the other hand, by the Schwarz inequality and the assumption of parallel mean curvature

$$const. = |H|^2 \leq \frac{1}{m} S. \tag{4.2}$$

(4.1) and (4.2) force that $H = 0$. Now, we use the estimate (2.8) and obtain

$$S(x) \leq k \left(\frac{32a^2R^6}{(2 + \frac{1}{n}R^2)^2(a^2 - r^2)^2} + \frac{(m + 4)a^2R^2}{(2 + \frac{1}{n}R^2)(a^2 - r^2)^2} \right). \tag{4.3}$$

It is valid on a geodesic ball $B_a(x_0) \subset M$. By Theorem 3.3 M is complete and we can fix x and let a tend to infinity in (4.3). Thus, $S(x) = 0$ for all $x \in M$. The proof is complete. \square

Now, we study the following special case. Let \mathbb{R}_m^{2m} be the pseudo-Euclidean $2m$ space with index m . Let $(x, y) = (x^1, \dots, x^m; y^1, \dots, y^m)$ be null coordinates; this means that the indefinite metric is defined by

$$ds^2 = \frac{1}{2} \sum_i dx^i dy^i. \tag{4.4}$$

Let F be a smooth convex function. We consider the graph M of ∇F , defined by

$$(x^1, \dots, x^m; \frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^m}).$$

The induced Riemannian metric on M is defined by

$$ds^2 = \frac{\partial^2 F}{\partial x^i \partial x^j} dx^i dx^j. \quad (4.5)$$

The underlying Euclidean space \mathbb{R}^{2m} of \mathbb{R}_m^{2m} has the usual complex structure. It is easily seen that M is a Lagrangian submanifold in \mathbb{R}^{2m} . Let us derive the condition on F for M being an extremal submanifold in \mathbb{R}_m^{2m} .

Choose a tangent frame field (e_1, \dots, e_m) along M , where

$$e_i = \frac{\partial}{\partial x^i} + \frac{\partial^2 F}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j}.$$

Obviously,

$$\langle e_i, e_j \rangle = \frac{\partial^2 F}{\partial x^i \partial x^j}.$$

Let (n_1, \dots, n_m) be the normal frame field of M in \mathbb{R}^{2m} defined by

$$n_i = \frac{\partial}{\partial x^i} - \frac{\partial^2 F}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j}$$

with

$$\langle n_i, n_j \rangle = -\frac{\partial^2 F}{\partial x^i \partial x^j}.$$

Thus, M is space-like precisely if F is convex. By direct computations

$$\begin{aligned} \nabla_{e_i} e_j &= \nabla_{\frac{\partial}{\partial x^i} + \frac{\partial^2 F}{\partial x^i \partial x^k} \frac{\partial}{\partial y^k}} \left(\frac{\partial}{\partial x^i} + \frac{\partial^2 F}{\partial x^i \partial x^l} \frac{\partial}{\partial y^l} \right) \\ &= \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^l} \frac{\partial}{\partial y^l} \\ &= \frac{1}{2} \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^l} g^{lk} e_k - \frac{1}{2} \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^l} g^{lk} n_k, \end{aligned}$$

where g^{ij} denotes the elements of the inverse matrix of $(g_{ij}) \stackrel{def.}{=} \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right)$. It follows that the second fundamental form of M in \mathbb{R}_m^{2m}

$$B_{ij} \stackrel{def.}{=} (\nabla_{e_i} e_j)^\perp = -\frac{1}{2} \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^l} g^{lk} n_k, \quad (4.6)$$

and the mean curvature vector

$$H \stackrel{def.}{=} \frac{1}{m} \sum_i B_{ii} = -\frac{1}{2mg} \frac{\partial g}{\partial x^l} g^{lk} n_k, \quad (4.7)$$

where $g = \det(g_{ij})$.

Theorem 4.2. *Let M be a space-like extremal m -submanifold in \mathbb{R}_n^{m+n} . If M is closed with respect to the Euclidean topology, then M has to be a linear subspace. In particular, when such an M is defined by the graph $(x; \nabla F)$ of the gradient ∇F of a smooth function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ in null coordinates $(x; y)$ in \mathbb{R}_m^{2m} , then F has to be a quadratic polynomial.*

Proof. By Theorem 3.3 M is complete. On the other hand, substituting $H = 0$ into (2.9) yields

$$S(x) = k \frac{2n(m+4)a^2}{(a^2 - r^2)^2}. \quad (4.8)$$

We fix x and let a go to infinity in (4.8). Hence, $S(x) = 0$ for any $x \in M$. We complete the proof of the theorem. \square

(4.7) reveals that $H = 0$ is equivalent to the Monge - Ampère equation

$$\det \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right) = \text{const.} \quad (4.9)$$

We thus obtain an alternative proof of the following famous result shown by Calabi (for $m \leq 5$) [C2] and Pogorelov [P] (for all dimensions).

Corollary 4.3. *The only entire convex solutions to (4.9) are quadratic polynomials.*

Remarks. 1. In [A] and [I], some results were proved for complete space-like submanifolds. It was shown in [A] that completeness implies that the manifolds are closed with respect to the Euclidean topology. Therefore, the present results are generalizations of their results.

2. A convex solution to the Monge - Ampère equation (4.9) represents an improper affine hypersphere in affine differential geometry. The present discussion shows the close relationship between Lagrangian extremal submanifolds and affine hypersurfaces.

5. FINAL REMARKS

Let L be a compact special Lagrangian submanifold of a Calabi - Yau manifold Y . Let M be the moduli space of the special Lagrangian submanifolds near L . McLean [M] showed that M is a smooth manifold of dimension $\beta(L) = \dim(H^1(L, \mathbb{R}))$. He also defined a natural Riemannian metric on M . Hitchin [H] then studied this moduli space M . He showed that there is natural embedding of the moduli space M as a Lagrangian submanifold in the product $H^1(L, \mathbb{R}) \times H^{m-1}(L, \mathbb{R})$ (where $m = \dim L$) of two dual vector spaces and that McLean's metric is the metric induced by the ambient pseudo-Euclidian metric. He also showed that as a Lagrangian submanifold M is defined locally by graph of the gradient of a function F . So, we are in the situation studied in the previous section.

Therefore, the curvature properties of the moduli space M can be determined by our previous calculations. From the Gauss equation and (4.6) we obtain the Riemannian curvature, the Ricci curvature and the scalar curvature of the moduli space M with respect to McLean's metric as follows.

$$R_{ijkl} = -\frac{1}{4} g^{st} \frac{\partial^3 F}{\partial x^s \partial x^i \partial x^k} \frac{\partial^3 F}{\partial x^t \partial x^j \partial x^l} + \frac{1}{4} g^{st} \frac{\partial^3 F}{\partial x^s \partial x^i \partial x^l} \frac{\partial^3 F}{\partial x^t \partial x^j \partial x^k},$$

$$R_{ik} = -\frac{1}{4g} g^{st} \frac{\partial^3 F}{\partial x^s \partial x^i \partial x^k} \frac{\partial g}{\partial x^t} + \frac{1}{4} g^{st} g^{jl} \frac{\partial^3 F}{\partial x^s \partial x^i \partial x^l} \frac{\partial^3 F}{\partial x^t \partial x^j \partial x^k},$$

and

$$R = -\frac{1}{4} g^{st} \frac{\partial \ln g}{\partial x^s} \frac{\partial \ln g}{\partial x^t} + \frac{1}{4} g^{st} g^{jl} g^{ik} \frac{\partial^3 F}{\partial x^s \partial x^i \partial x^l} \frac{\partial^3 F}{\partial x^t \partial x^j \partial x^k}.$$

It is interesting to observe that when the moduli space M not only is Lagrangian, but also is special, in this case by (4.9) the Ricci curvature of the moduli space is nonnegative.

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