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by

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Abstract

Considered is plane rotationally-symmetrical motion of viscous incompressible liquid by inertia in a ring, which boundaries are free. Corresponding initial boundary-value problem for the Navier–Stokes equations was studied in [1] and also earlier in [2], where the case of zero surface tension was considered.

The problem on a rotating ring represents a rich in content and at the same time simple enough object for grounding of approximate methods in theory of viscous flows with free boundaries. An asymptotics of the solution of this problem at large Reynolds numbers was built in [3] on the basis of the scheme suggested in [4]; the closeness of asymptotic and exact solutions was proved at finite time interval.

The analysis of quasi-stationary approximation in the problem on a rotational ring is the aim of the present work. The equations of quasi-stationary approximation for the general problem on motion of isolated volume of viscous incompressible capillary liquid were derived in [5] from the exact equations with the help of expansion by the small parameter of quasi-stationarity equal to the ratio of the Stokes time to the capillary one. The problem contains one more dimensionless parameter proportional to the modulus of conserving angular moment of liquid volume; this parameter can be also considered as a small one. In dependence on the relation between these parameters one can obtain three variants of limit problem: traditional and two new ones. Built in [5] is formal asymptotics of solutions of the problem arising at tending of the quasi-statinarity parameter to zero.

The question of grounding of quasi-stationary approximation was open until recently. Presented in [6] is the ground for the traditional variant of limit problem. Unfortunately the standard model of quasi-stationary approximation is insipid in the problem on a rotational ring.

The problem mentioned above represents the first nontrivial example of such motion of viscous liquid that its topology can be changed with time – a ring turns into a disk in finite period of time in irreversible way [1] when the surface tension is large enough. The usefulness of quasi-stationary approximation for the description of the process of modification of topology of the flow domain in the considered problem in simple terms – by the analysis of the solutions of ordinary autonomous differential equation of the first order – is the remarkable peculiarity.

1 Statement of problem.

Considered is a problem on plane rotationally–symmetrical motion of viscous incompressible capillary liquid in a ring $r_1(t) < r < r_2(t)$, which boundaries are free. Let $v_r = Q(t)/r$, $v_\theta = v(r,t)$ to be the radial and peripherical components of the velocity; $p(r,t)$ is the pressure, ρ is the density of liquid, ν is its viscosity, σ is the coefficient of surface tension, $r_i(0) = r_{i0}$ $(i = 1, 2)$. Functions v_r, v_θ, p must satisfy the Navier–Stokes equations, the initial conditions $v_r(r,0) = Q(0)/r$, $v_\theta = v_0(r)$ $(r_{10} \le r \le r_{20})$ and natural conditions on the free boundaries of the ring $r = r_i(t)$, $i = 1, 2$ at $t > 0$.

Let us determine the time scale $t_0 = \rho\nu r_{10}/\sigma$ (the capillary time) and the length scale, the scale of radial component of velocity and the pressure scale r_{10}, $\sigma/\rho\nu$ and σ/r_{10} respectively; then introduce dimensionless independent variables $\tau = t/t_0$, $x = (r^2 - r_1^2)/r_{10}^2$ and sought for functions $y = r_1^2/r_{10}^2$, $\varphi = \rho\nu Q/\sigma r_{10}$, $u = r_{10}v/Vr$, so that u is the dimensionless angular velocity of a liquid particle and V is the characteristic dimensional peripherical velocity determined below. Functions $y(\tau)$, $\varphi(\tau)$ and $u(x,\tau)$ form the solution of the following problem in a fixed domain:

$$\delta[(y+x)u]_\tau = 4[(y+x)^2 u_x]_x \quad \text{at} \quad \tau > 0, \quad 0 < x < a, \tag{1.1}$$

$$u_x(0,\tau) = u_x(a,\tau) = 0 \quad \text{at} \quad \tau > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{at} \quad 0 \le x \le a, \tag{1.3}$$

$$dy/d\tau = 2\varphi \quad \text{at} \quad \tau > 0, \tag{1.4}$$

$$\delta\frac{d\varphi}{d\tau} = \frac{1}{\ln(1+a/y)}\Big(- \frac{a\varphi(4-\delta\varphi)}{y(y+a)} - \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+a}} + \\ + \frac{\rho V^2 r_{10}}{\sigma}\int_0^a u^2(x,\tau)\,dx\Big) \quad \text{at} \quad \tau > 0, \tag{1.5}$$

$$y(0) = 1, \quad \varphi(0) = \varphi_0. \tag{1.6}$$

Here $\delta = t_s/t_0 = \sigma r_{10}/\rho\nu^2$ is the parameter of quasi-stationarity, $t_s = r_{10}^2/\nu$ is the Stokes time, $a = (r_{20}/r_{10})^2 - 1$ is the relative ring thickness, $u_0(x) = \rho\nu v_0(r_{10}\sqrt{1+x})/\sigma\sqrt{1+x}$, $\varphi_0 = \rho\nu Q(0)/\sigma r_{10}$.

The relations (1.1)–(1.6) coincide with ones derived in [1, 7] to within the notations (there the parameter $\delta^{-1} = \varepsilon^2 = \nu/r_{10}V$, i.e. the inverse Reynolds number, is used instead of δ and the characteristic velocity V is chosen as $\max|v_0(r)|$, $r_{10} \le r \le r_{20}$). The mass integral equivalent to the ring square conservation

$$S = \pi[r_2^2(t) - r_1^2(t)] = \pi(r_{20}^2 - r_{10}^2)$$

was used for derivation of these relations. Furthermore the original problem with free boundary keeps the conservation law of the angular momentum:

$$L = 2\pi\rho \int\limits_{r_1(t)}^{r_2(t)} r^2 v(r,t)dr = 2\pi\rho \int\limits_{r_{10}}^{r_{20}} r^2 v_0(r)dr.$$

So that we can determine the quantity $V = L/\pi\rho r_{10}^3$. If the solution of the problem (1.1)–(1.6) is known then functions $v_r,\ v_\theta,\ r_1,\ r_2 = (S/\pi + r_1^2)^{1/2}$ are expressed explicitly by $u,\ y$ and φ, and pressure p is reconstructed by quadrature.

The previous equality can be rewritten in dimensionless variables as follows

$$2\int\limits_0^a [x + y(\tau)]u(x,\tau)dx = 2\int\limits_0^a (x+1)u_0(x)dx = 1. \tag{1.7}$$

Let us introduce the function

$$w = u - 2/(a(2y + a)). \tag{1.8}$$

The quantity $2/(a(2y + a)) = \omega$ represents the dimensionless istantaneous angular velocity of the ring rotating as a solid body with known area and angular moment. From (1.7) it follows that

$$\int\limits_0^a (x + y)w dx = 0. \tag{1.9}$$

The dimensionless ratio $\alpha = 4\pi^{1/2}L^2/(\rho\sigma S^{5/2})$ can be composed from quantities $L,\ S,\ \rho$ and σ. Parameter α is the unique dimensionless combination of the motion integrals proportional to the square of angular moment and not containing viscosity in the problem on a rotational ring. Further it is suggested that the centrifugal and capillary forces have one and the same order when $\delta \to 0$ in the studied motion. This condition will be satisfied when α will have the order 1.

Passing on to the new sought for function $w(x,\tau)$ in (1.1)–(1.6) ($w(x,\tau)$ is determined by the equality (1.8)) and using the relation (1.9) we obtain:

$$\delta[(y + x)w]_\tau = 4[(y + x)^2 w_x]_x +$$

$$+4\delta(2x - a)(2y + a)^{-2}\varphi \quad \text{at} \quad \tau > 0, \quad 0 < x < a, \tag{1.10}$$

$$w_x(0,\tau) = w_x(a,\tau) = 0 \quad \text{at} \quad \tau > 0, \tag{1.11}$$

$$w(x,0) = w_0(x) \quad \text{at} \quad 0 \le x \le a, \tag{1.12}$$

$$dy/d\tau = 2\varphi \quad \text{at} \tau > 0, \tag{1.13}$$

$$\delta \frac{d\varphi}{d\tau} = \frac{1}{\ln(1+a/y)} \Big[-\frac{a\varphi(4-\delta\varphi)}{y(y+a)} - \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+a}} + \frac{\alpha a^{3/2}}{(2y+a)^2} +$$

$$+ \frac{\alpha a^{5/2}}{4} \int_0^a w^2(x,\tau)dx \Big] \quad \text{at} \quad \tau > 0, \tag{1.14}$$

$$y(0) = 1, \quad \varphi(0) = \varphi_0. \tag{1.15}$$

Here $w_0(x) = -2/(a(2+a)) + u_0(x)$. Function $w_0(x)$ must satisfy the condition

$$\int_0^a (x+1) w_0 dx = 0 \tag{1.16}$$

following from (1.9).

The problem (1.10)–(1.15) represents the subject of our further investigation. Our aim is to build the asymptotics of the solution of this problem for $\delta \to 0$ and its grounding.

2 External expansion. Equations of quasi-stationary approximation.

The asymptotic solution of singulary pertubed problem (1.10)–(1.15) is built for $\delta \to 0$ as a combination of the external and internal expansions. The external expansion is found in the form of formal power series

$$y = y^{(0)} + \delta y^{(1)} + \delta^2 y^{(2)} + \ldots, \quad \varphi = \varphi^{(0)} + \delta\varphi^{(1)} + \delta^2\varphi^{(2)} + \ldots,$$

$$w = \delta w^{(1)} + \delta^2 w^{(2)} + \delta^3 w^{(3)} + \ldots \tag{2.1}$$

The initial conditions for the terms of the expansion of y are: $y^{(0)} = 1$, $y^{(k)} = 0$ at $\tau = 0$ ($k = 1, 2, \ldots$). Initial conditions for the functions $\varphi^{(j)}$ ($j = 0, 1, \ldots$) and $w^{(k)}$ ($k = 1, 2, \ldots$) are not posed. At the same time functions $\varphi^{(j)}$ can be determined explicitely by $y^{(j)}$, $w^{(j-1)}$ (let $w^{(-1)} = w^{(0)} = 0$ from substitution of the expressions (2.1) into (1.14) and equating of terms at common powers of δ. Function $y^{(0)}$ satisfies the nonlinear ordinary differential equation of the first order and functions $y^{(k)}$ ($k = 1, 2, \ldots$) satisfy the linear ordinary differential equations of the first order.

Functions $w^{(j)}$ ($j = 1, 2, \ldots$) can be determined from the solutions of boundary-value problems

$$[(y^{(0)} + x)^2 w_x^{(j)}]_x = R_j(x,\tau), \quad 0 < x < a, \quad \tau > 0, \tag{2.2}$$

$$w_x^{(j)}(0,\tau) = w_x^{(j)}(a,\tau) = 0, \quad \tau > 0, \tag{2.3}$$

where τ is considered as a parameter. The right parts of the equations (2.2) can be expressed by functions $y^{(0)},\dots,y^{(j)};\quad \varphi^{(0)},\dots,\varphi^{(j)};\quad w^{(0)},\dots,w^{(j-1)}$. The solvability condition $\int\limits_0^a R_j(x,\tau)dx = 0$, $\tau > 0$ of the problem (2.2), (2.3) at $j = 1$ follows directly from (1.10). Its fulfillment for $j = 2, 3, \dots$ can be proved by induction. The solution of the problem (2.2), (2.3) is determined to within the additive function τ. This arbitrariness permits to satisfy the solvability condition on the next step of induction.

Let us pass to the constructing of the main terms of asymptotic expansions (2.1). After substitution of these expansions into (1.14) and passage to the limit when $\delta \to 0$ we shall obtain

$$-\frac{4a\varphi^{(0)}}{y^{(0)}(y^{(0)}+a)} - \frac{1}{\sqrt{y^{(0)}}} + \frac{\alpha a^{3/2}}{(2y^{(0)}+a)^2} = 0.$$

So we conclude that

$$\varphi^{(0)} = -\frac{Y(Y+a)}{4a}G(Y), \tag{2.4}$$

where $Y = y^{(0)}$,

$$G(Y) = \frac{1}{\sqrt{Y}} + \frac{1}{\sqrt{Y+a}} - \frac{\alpha a^{3/2}}{(2Y+a)^2}.$$

As it follows from (1.13), (1.15) and (2.1), (2.4) the function $Y(\tau)$ is a solution of the Cauchy problem

$$\frac{dY}{d\tau} = -\frac{Y(Y+a)}{2a}G(Y) \quad \text{at} \quad \tau > 0, \tag{2.5}$$

$$Y(0) = 1. \tag{2.6}$$

And finally the function $w^{(1)}$ is determined as a solution of the problem (2.2), (2.3) with $R_1 = -(2x-a)(2y^{(0)}+a)^{-2}\varphi^{(0)}$, satisfying the condition $\int\limits_0^a (x+y^{(0)})w^{(1)}dx = 0$ (here $\varphi^{(0)}$ is expressed by $y^{(0)} = Y$ by formula (2.4)). This solution has the form

$$w^{(1)} = -\frac{Y(Y+a)}{a(2Y+a)^2}G(Y)\Big[x + (2Y+a)\log\left(\frac{Y+a}{Y+x}\right) - \frac{Y(Y+a)}{Y+x} - \frac{a(6Y+7a)}{6(2Y+a)} + \frac{Y^2}{a}\log\left(1+\frac{a}{Y}\right)\Big]. \tag{2.7}$$

The existence of constant solutions determined by the condition $G(Y) = 0$ is sufficient for the further investigation.The last equation is equivalent to the system

$$\alpha a^{7/2}\omega^2 = 4\Big(\frac{1}{\sqrt{Y}} + \frac{1}{\sqrt{Y+a}}\Big), \quad \frac{a\omega}{2} = \frac{1}{2Y+a}. \tag{2.8}$$

Its solution Y, ω describes the rotation of a capillary ring as a rotation of a solid body with dimensionless angular velocity ω at predetermined angular moment and area of a ring. Stationary solution of the system (1.10), (1.13), (1.14) where $\varphi = 0$, $w = 0$ corresponds to it. The analysis of system (2.8) was fulfilled in [1, 7]. It was found out that this system has no real solutions at $\alpha < \alpha^* \approx 5.89$ and has one solution $Y^* = \kappa a$, ω^* at $\alpha = \alpha^*$ ($\kappa = const \approx 0.121$) and two solutions $Y_i(\alpha, a)$, $\omega_i(\alpha, a)$ $(i = 1, 2)$ at $\alpha > \alpha_*$, moreover $0 < Y_1(\alpha, a) < Y^* < Y_2(\alpha, a)$ and $\lim Y_1 = 0$, $\lim Y_2 = \infty$ when $\alpha \to \infty$. If$\alpha > \alpha^*$ then function $G(Y)$ is positive at the intervals $(0, Y_1)$, (Y_2, ∞) and negative at (Y_1, Y_2).

If the inequalities

$$\alpha > \alpha^*, \quad Y_1(\alpha, a) < 1 \tag{2.9}$$

are fulfilled then the solution $Y(\tau)$ of the Cauchy problem (2.5), (2.6) is positive for all $\tau > 0$ and $Y \to Y_2(\alpha, a)$ when $\tau \to \infty$ with the exponential velocity. (Fulfillment of the condition $a < a^* \approx 8.26$ is sufficient for the fulfillment of the second inequality (2.9).) Here the function Y is growing monotonically if $1 < Y_2(\alpha, a)$ and diminishing in the opposite case. If $Y_1(\alpha, a) = 1$ and $Y_2(\alpha, a) = 1$ then the solution of the problem (2.5), (2.6) is $Y = 1$. It is unstable in the first case and stable in the second one.

If we change the second inequality in (2.9) on the opposite one or if $\alpha < \alpha^*$ then function Y is diminishing monotonically with the growth of τ and vanishes at some $\tau^* > 0$ by the law

$$Y = (\tau^* - \tau)^2/16 + O(\tau^* - \tau)^3 \quad \text{at} \quad \tau \nearrow \tau^*, \tag{2.10}$$

and for $\tau > \tau^*$ it should be continued with zero. (The necessity of such continuation of function Y is caused by the fact that it is the main term of the external expansion of the function $y(\tau) = r_1^2/r_{10}^2$ at $\delta \to 0$.) At the same time functions $\varphi^{(0)}$ and $w^{(1)}$ determined in formulae (2.4), (2.7) are continued with zero too.

Substitution of expressions (2.1) for y, φ in the equation (1.14) and retention of the terms with the order of δ in it lead to the relation

$$\varphi^{(1)} = -\Big[\frac{Y+a}{8a\sqrt{Y}} + \frac{Y}{8a\sqrt{Y+a}} + \frac{\alpha a^{5/2}}{4(2Y+a)^3}\Big]y^{(1)} + \frac{1}{4}(\varphi^{(0)})^2 -$$

$$-\frac{1}{4a}\log\left(1+\frac{a}{Y}\right)\frac{d\varphi^{(0)}}{d\tau}. \tag{2.11}$$

One can derive the equation for $y^{(1)}$ from (2.11) and (1.13)

$$\frac{dy^{(1)}}{d\tau} = -\Big[\frac{Y+a}{4a\sqrt{Y}}+\frac{Y}{4a\sqrt{Y+a}}+\frac{\alpha a^{5/2}}{2(2Y+a)^2}\Big]y^{(1)}+\frac{1}{2}(\varphi^{(0)})^2-$$

$$-\frac{1}{2a}\log\left(1+\frac{a}{Y}\right)\frac{d\varphi^{(0)}}{d\tau}. \tag{2.12}$$

This equation must be solved with the initial condition

$$y^{(1)}(0)=0 \tag{2.13}$$

(here the function $\varphi^{(0)}$ is determined by the equality (2.4)). Further as in the case of the study of the main terms of the external expansion we must distinguish the two cases: the solution $Y(\tau)$ of the problem (2.5), (2.6) is positive for all $\tau > 0$ (case A); $Y > 0$ for $0 \le \tau < \tau^* < \infty$, $Y = 0$ for $\tau \ge \tau^*$ (case B).

In the case A the solution $y_1(\tau)$ of the problem (2.12), (2.13) is regular for all $\tau > 0$ and $y_1 \to$ *const* when $\tau \to \infty$ in the exponential way. The case B is more complicated. Using the relation (2.10) we obtain the asymptotics of the solution of mentioned in the form

$$y^{(1)} = \frac{\tau^*-\tau}{32a}\log^2(\tau^*-\tau)+O[(\tau^*-\tau)\log(\tau^*-\tau)] \tag{2.14}$$

when $\tau \nearrow \tau^*$. We suggest that $y^{(1)}(\tau)=0$ for $\tau \ge \tau^*$ on the basis of (2.14) so that the inclusion $y^{(1)} \in C^\beta[0,\infty)$ will be valid with any $\beta \in (0,1)$. However the function $\varphi^{(1)}(\tau)$ has a logarithmic singularity when $\tau \nearrow \tau^*$. Setting $\varphi^{(1)} = 0$ for $\tau > \tau^*$ we obtain the discontinuous function $\varphi^{(1)}(\tau)$ determined for all $\tau > 0$ although $\varphi^{(1)} \in L^q(0,\infty)$ with any $q > 1$. At the same time this variant of continuation of the function $\varphi^{(1)}$ to the domain $\tau > \tau^*$ is the unique possible variant so far as this function is the element of the expansion of the radial component of velocity of a ring by the parameter δ and this component turns to zero after transformation of a ring into a disk.

We see that in the case B the expansion (2.1) becomes invalid for τ close to τ^*. It turns out that if we pass to the new independent variable y instead of τ and new sought for function $\Phi[y(\tau)]=\varphi(\tau)$ and $W[x,y(\tau)]=w(x,\tau)$ in the relations (1.10)–(1.15) then the process of building of the external expansion can be regularized. It follows from the fact that function φ is negative at the interval $[\tau_0, \tau^*)$ if $\tau^* - \tau_0$ is small enough so that $dy/d\tau = 2\varphi < 0$ [1]. As a result we pass to the problem:

$$\delta\Phi[(y+x)W]_y = 2[(y+x)^2 W_x]_x + 2\delta(2x-a)(2y+a)^{-2}\Phi$$

$$\text{at} \quad y < y_0, \quad 0 < x < a, \tag{2.15}$$

$$W_x(0,y) = W_x(a,y) = 0 \quad \text{at} \quad y < y_0, \tag{2.16}$$

$$W(x,y_0) = W_0(x) \quad \text{at} \quad 0 \le x \le a, \tag{2.17}$$

$$\delta\frac{d\Phi}{dy} = \frac{1}{2\log(1+a/y)\Phi}\Big[-\frac{a\Phi(4-\delta\Phi)}{y(y+a)} - \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+a}} + \frac{\alpha a^{3/2}}{(2y+a)^2} +$$

$$+\frac{\alpha a^{5/2}}{4}\int_0^a W^2(x,y)dx\Big] \quad \text{at} \quad y < y_0, \tag{2.18}$$

$$\Phi(y_0) = \Phi_0 \tag{2.19}$$

(here $y_0 = y(\tau_0) > 0$, $\Phi_0 = \varphi(\tau_0) < 0$).

Equations (2.15), (2.18) permit the existence of formal solutions in the form

$$\Phi = \Phi^{(0)} + \delta\Phi^{(1)} + \delta^2\Phi^{(2)} + \ldots, \quad W = \delta W^{(1)} + \delta^2 W^{(2)} + \delta^3 W^{(3)} + \ldots. \tag{2.20}$$

Functions $W^{(k)}$, $k = 1, 2, \ldots$ satisfy the boundary conditions (2.16). The process of calculation of the terms of the expansion (2.20) is very similar with one discussed in the beginning of this subsection so we shall omit it. Let us note that the expression for $\Phi^{(0)}$ coincides with (2.4) after substitution of y instead of Y (now y is the independent variable). Analogously the function $W^{(1)}$ is determined by (2.7) where y is substituted instead of Y. However the essential distinctions arise among the next terms of the expansion (2.1) and its analog (2.20).

The expression for $\Phi^{(1)}(y)$:

$$\Phi^{(1)} = -\frac{y(y+a)}{4a}\frac{d}{dy}\{\log(1+a/y)[\Phi^{(0)}(y)]^2\}. \tag{2.21}$$

In order to obtain the dependence $\Phi^{(1)}[y(\tau)]$ we must solve the Cauchy problem

$$\frac{dy}{d\tau} = 2\Phi^{(0)}(y) + 2\delta\Phi^{(1)}(y) \quad \text{at} \quad \tau > \tau_0,$$

$$y(\tau_0) = y_0.$$

From this fact and (2.21) it follows that function $y(\tau)$ does not permit the expansion in the asymptotic series (2.1) at τ close to τ^*.

From the other side the smoothness of functions $W^{(k)}(x,y)$, $\Phi^{(k)}(y)$ is not worsening near the values $x=0,\ y=0$ it is even increasing with the growth of k. In particular

$$\Phi^{(0)} = -\sqrt{y}/4 + O(y), \tag{2.22}$$

$$\Phi^{(1)} = -y\log(1/y)/64 + O(y),$$

$$\Phi^{(2)} = -3y^{3/2}\log^2(1/y)/1028 + O[y^{3/2}\log(1/y)],$$

when $y \to 0$. Moreover using the induction one can show that $\Phi^{(k)} \in C^{(k+\beta)/2}[0,y_0]$, $W^{(k)} \in C^{(k+\beta-1)/2}([0,a]\times[0,y_0])$ for $k=1,2,\ldots$ with any $\beta \in (0,1)$.

3 Internal expansion. The matching conditions.

Built above formal solution (2.1) of the equations (1.10), (1.13), (1.14) is not satisfying the initial conditions (1.12) and the second condition (1.15). In order to compensate the arising discrepancies we must search for the components φ, w of the solution of the problem (1.10)–(1.15) in the form

$$\varphi = \varphi^{(0)} + \psi^{(0)} - \chi^{(0)} + \delta(\varphi^{(1)} + \psi^{(1)} - \chi^{(1)}) + \ldots,$$

$$w = v^{(0)} + \delta(w^{(1)} + v^{(1)} - z^{(1)}) + \delta^2(w^{(2)} + v^{(2)} - z^{(2)}) + \ldots,$$

where functions $v^{(0}$, $v^{(k)}$, $\psi^{(k)}$, $k=1,2,\ldots$ (the elements of the internal expansion) depend on "rapid time" $\eta = \tau/\delta$ (functions $v^{(0)}$, $v^{(k)}, \ldots$ depend on x too); quantities $\chi^{(k)}$, $k=1,2,\ldots$ are constant and $z^{(k)}$ are functions of x.

The problem of determination of $v^{(0)}$ is obtained in the following way. Expressions (3.1) are substituted into the equation (1.10) where we pass to the "rapid time" and then put $y=1$, $\delta=0$. As a result we derive the equation

$$[(x+1)v^{(0)}]_\eta = 4[(x+1)^2 v_x^{(0)}]_x, \tag{3.2}$$

that must be solved in the semistrip $S_a = \{x,\eta:\ 0<x<a,\ \eta>0\}$ with the boundary conditions

$$v_x^{(0)}(0,\eta) = v_x^{(0)}(a,\eta) = 0, \quad \eta > 0 \tag{3.3}$$

and the initial condition

$$V^{(0)}(x,0) = w_0(x), \quad 0 \le x \le a. \tag{3.4}$$

The solution of the problem (3.2)–(3.4) is representable by the Fourier series

$$v^{(0)} = \frac{1}{\sqrt{x+1}} \sum_{i=1}^{\infty} c_i^{(0)} exp(-\lambda_i^2 \eta) f_i(\sqrt{x+1}), \tag{3.5}$$

where $f_i(r)$ is the solution of the equation

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \Big(\lambda_i^2 - \frac{1}{r^2}\Big) f = 0,$$

satisfying the conditions

$$\frac{df}{dr} - \frac{f}{r} = 0 \quad \text{at} \quad r = 1, \quad r = \sqrt{1+a}$$

(i.e. the linear combination of the Bessel functions of the first and second kinds of argument $\lambda_i r$); $\lambda_i > 0, \ (i = 1, 2, \ldots)$ are the roots of the equation

$$J_2(\lambda_i \sqrt{1+a}) Y_2(\lambda_i) - J_2(\lambda_i) Y_2(\lambda_i \sqrt{1+a}) = 0,$$

and $c_i^{(0)}$ - constants,

$$c_i^{(0)} = \Big(\int_0^a \sqrt{x+1} f_i^2(\sqrt{x+1}) dx \Big)^{-1} \int_0^a \sqrt{x+1} w_0(x) f_i(\sqrt{x+1}) dx.$$

The absence of any term not depending on η in the right part of (3.5) is quaranteed by the condition (1.16). This property is well coordinated with the fact that the external expansion of the function $w(x,\tau)$ begins from the term with the order of δ. Further we suggest that function $w_0 x$ belongs to the Hölder class $C^{2+\beta}[0,a], \ 0 < \beta < 1$ and satisfies the compatibility conditions $w_0^{'}(0) = w_0^{'}(a) = 0$. This fact ensures the convergence of the series (3.5) in metric of the space $C^{2+\beta, 1+\beta/2}(\bar{S}_a)$.

Function $\psi^{(0)}(\eta)$ is determined as a solution of the linear Cauchy problem:

$$\frac{d\psi^{(0)}}{d\eta} = \frac{1}{\log(1+a)} \Big\{ -\frac{4a\psi^{(0)}}{1+a} - 1 - \frac{1}{\sqrt{1+a}} + \\ + \frac{\alpha a^{3/2}}{4} \int_0^a [v^{(0)}(x,\eta)]^2 dx \Big\} \quad \text{at} \quad \eta > 0, \tag{3.6}$$

$$\psi^{(0)}(0) = \varphi_0. \tag{3.7}$$

In accordance with (3.5)–(3.7)

$$\lim_{\eta \to \infty} \psi^{(0)} = \frac{1+a}{4a} \Big[1 + \frac{1}{\sqrt{1+a}} - \frac{\alpha a^{3/2}}{(2+a)^2} \Big],$$

moreover this convergence to the limit has an exponential character; this limit is denoted by $\chi^{(0)}$. So taking into account this fact and (2.4) one can see that

$$\lim_{\eta\to\infty}\psi^{(0)} = \chi(0) = \lim_{\tau\to 0}\varphi(0).$$

In order to obtain the equation for the function $v^{(1)}(x,\eta)$ one must substitute the expansions (3.1) and $y = y^{(0)} + \delta y^{(1)} + \ldots$ into the equation(1.10) and keep the terms with zero and first orders on δ in the resulting equality, then pass to the limit when $\tau \to 0$ in the expansion of the function $y(\tau)$. Taking into account the equality (3.2) and the fact that $y^{(0)}(0) = 1,\ y^{(1)}(0) = 0$ we derive the equation

$$[(x+1)v^{(1)}]_\eta = 4[(x+1)^2 v_x^{(1)}]_x + 4(2x-a)(2+a)^{-2}\psi^{(0)}(\eta). \tag{3.8}$$

This equation must be solved in the semistrip S_a with homogeneous boundary and initial conditions

$$v_x^{(1)}(0,\eta) = v_x^{(1)}(a,\eta) = 0, \quad \eta > 0, \tag{3.9}$$

$$v^{(1)}(x,0) = 0, \quad 0 \le x \le a. \tag{3.10}$$

The solution of the problem (3.8)–(3.10) is expressed by the Fourier series analogous with (3.5). The following representation takes place at $\eta \to \infty$

$$v^{(1)} = z^{(1)}(x) + O[\exp(-\gamma\eta)] \tag{3.11}$$

(with some $\gamma > 0$, uniformly in x, $0 \le x \le a$). Here

$$z^{(1)} = -\frac{1+a}{a(2+a)}\left[1 + \frac{1}{\sqrt{1+a}} - \frac{\alpha a^{5/2}}{(2+a)^2}\right]\left[x + (2+a)\log\left(\frac{1+a}{1+x}\right) - \right.$$
$$\left. -\frac{1+a}{1+x} - \frac{a(6+7a)}{6(2+a)} + \frac{1}{a}\log(1+a)\right]. \tag{3.12}$$

In accordance with (3.12) the comparison of (3.11) with (2.7) shows that

$$\lim_{\eta\to\infty} v^{(1)} = z^{(1)} = \lim_{\tau\to 0} w^{(1)}, \quad 0 \le x \le a.$$

The Cauchy problem for determination of function $\psi^{(1)}(\eta)$ is derived in the same way; it has the following form:

$$\frac{d\psi^{(1)}}{d\eta} = \frac{1}{\log(1+a)}\left[-\frac{4a}{1+a}(\psi^{(1)} - \chi^{(1)}) + \frac{a}{1+a}(\psi^{(0)} - \chi^{(0)})^2 + \right.$$
$$\left. + \frac{\alpha a^{5/2}}{2}\int_0^a v^{(0)}(x,\eta)v^{(1)}(x,\eta)\,dx\right], \quad \eta > 0; \quad \psi^{(1)}(0) = 0,$$

where $\chi^{(1)}$ is a constant not determined at this step. In accordance with exponential decay of functions $\psi^{(0)}(\eta) - \chi^{(0)}$, $v^{(0)}(x,\eta)$ when $\eta \to \infty$ and represenation (3.11) for the function $v^{(1)}(x,\eta)$ we have: $\psi^{(1)}(\eta) \to \chi^{(1)}$ when $\eta \to \infty$. Now let $\chi^{(1)} = \varphi^{(1)}(0)$; the necessary compatibility condition is derived

$$\lim_{\eta\to\infty} \psi^{(1)} = \chi^{(1)} = \lim_{\tau\to 0} \varphi^{(1)}.$$

The process of construction of functions $v^{(k)}(x,\eta)$, $z^{(k)}(x)$, $\psi^{(k)}(\eta)$ and determination of constants $\chi^{(k)}$ can be continued upto any natural N. As a result all members of the first formal power series (3.1) are determined in terms of solutions of the linear Cauchy problems for ordinary differential equations of the first order, and the members of the second series (3.1) – by means of solving the second initial boundary value problem for the linear parabolic equations of the (3.8) type.

4 Justification of the asymptotic expansion (case A).

Let us define N-approximate solution of the problem (1.10) – (1.15) ($N = 1, 2, \ldots$) by formulae

$$w_N = v^{(0)}(x, \frac{\tau}{\delta}) + \sum_{k=1}^{N} \delta^k [w^{(k)}(x,\tau) + v^{(k)}(x, \frac{\tau}{\delta}) - z^{(k)}(x)]$$

$$y_N = \sum_{i=0}^{N} \delta^i y^{(i)}(\tau), \tag{4.1}$$

$$\varphi_N = \sum_{i=0}^{N} \delta[\varphi^{(i)}(\tau) + \psi^{(i)}(\frac{\tau}{\delta}) - \chi^{(i)}].$$

Our aim is to obtain some estimates of closeness of N-approximate solution to the exact solution of the problem (1.10) – (1.15) when $\delta \to 0$. One must distinguish the two cases, A and B (see Section 2). Case A is considered in this section: function $y^{(0)} = Y(\tau)$ determined as the solution of the Cauchy problem (2.5), (2.6) is positive for all $\tau > 0$. In this case functions w_N, y_N, φ_N are determined for $0 \le \tau \le T$, $0 \le x \le a$ for any $T > 0$ and the following inclusions take place: $y_N \in C^2[0,T]$, $\varphi_N \in C^2[0,T]$, $w_N \in C_0^{2+\beta,1+\beta/2}(\bar{Q}_T)$, where Q_T – the rectangle $\{x,\tau :\ 0 < x < a,\ 0 < \tau < T\}$ and $C_0^{2+\beta,1+\beta/2}(\bar{Q}_T)$ – the subspace of $C^{2+\beta,1+\beta/2}(\bar{Q}_T)$ generated by functions $u(x,\tau)$ satisfying the conditions $u_x(0,\tau) = u_x(a,\tau) = 0$ for $0 \le \tau \le T$, $\int_0^a (x+1)u(x,0)dx = 0$.

Let us rewrite the equalities (1.10)–(1.15) in the operator form

$$A(f) = 0, \tag{4.2}$$

where $f = \{w, y, \varphi\}$ is the totality of the unknown functions and A – 6-component operator–function defined by the following relations:

$$A_1 = \delta[(y+x)w]_\tau - 4[(y+x)^2 w_x]_x - 4\delta(2x-a)(2y+a)^{-2}\varphi,$$

$$A_2 = \frac{dy}{d\tau} - 2\varphi,$$

$$A_3 = \delta\frac{d\varphi}{d\tau} - \frac{1}{\log(1+a/y)}\Big[-\frac{a\varphi(4-\delta\varphi)}{y(y+a)} - \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+a}} +$$

$$+\frac{\alpha a^{3/2}}{(2y+a)^2} + \frac{\alpha a^{5/2}}{4}\int_0^a w^2(x,\tau)dx\Big],$$

$$A_4 = w(x,0) - w_0(x),$$

$$A_5 = y(0) - 1,$$

$$A_6 = \varphi(0) - \varphi_0.$$

Let us define two Banach spaces: $B_1 = C_0^{2+\beta,1+\beta/2}(\bar{Q}_T) \times C^2[0,T] \times C^2[0,T]$ and $B_2 = C^{\beta,\beta/2}(\bar{Q}_T) \times C^1[0,T] \times C^1[0,T] \times C_0^{2+\beta}[0,a] \times \Re \times \Re$, where the subspace $C_0^{2+\beta}[0,a]$ of the space of functions $z(x) \in C^{2+\beta}[0,a]$ is separated by the conditions $z'(0) = z'(a) = 0$, $\int_0^a (x+1)z(x)dx = 0$. The norms (in the spaces B_1 and B_2) are determined as sums of norms of corresponding elements. For example, if $f = \{w, y, \varphi\} \in B_1$ then

$$||f||_{B_1} = ||w||_{C^{2+\beta,1+\beta/2}(\bar{Q}_T)} + ||y||_{C^2[0,T]} + ||\varphi||_{C^2[0,T]}.$$

Let us introduce the notation $f_N = \{w_N, y_N, \varphi_N\}$. Operator A is defined in some ball $\bar{\Omega}_\gamma : ||f - f_N||_{B_1} \le \gamma < 1$ of the space B_1 and acts into the space B_2. This operator is differentiable by Frechét in this ball and its Frechét derivative A'_g satisfies the Lipschitz condition:

$$||A'_g - A'_h||_{B_2} \le K||g-h||_{B_1} \quad \text{for} \quad g, h \in \bar{\Omega}_\gamma, \tag{4.3}$$

where the quantity $K > 0$ depends on the parameter δ. It is essential for the further investigation that if the inequality $0 < \delta \le \delta_0$ is fulfilled then the estimate (4.3) takes place with constant K not depanding on δ (the former notation K is kept). This fact follows directly from the definition of operator A.

Further the positive quantities not depending on δ (however their dependence on δ_0 and T is possible) are denoted by $C_k (k = 1, 2, \ldots)$. The following inequality

$$||A(f_N)||_{B_2} \le C_1\delta^{N+1} \tag{4.4}$$

takes place with $\delta \in (0, \delta_0]$ and any natural N by definition of the approximate solution (the process of its construction was described above). It will be proved below that the exact solution of the equation (4.2) exists in the vicinity of the approximate solution of this equation for any $T > 0$ and sufficiently small $\delta > 0$.

Proposition 1. Let the following assumptions to be fulfilled:

$$\text{i) } w_0 x \in C^{2+\beta}[0, a],\ 0 < \beta < 1; \quad w_0'(0) = w_0'(a) = 0$$

$$\int_0^a (x+1) w_0(x) dx = 0;$$

ii) the solution $Y(\tau)$ of the Cauchy problem (2.5), (2.6) is positive for all $\tau > 0$.

Then one can find such $\delta_0 > 0$ that for $\delta \in (0, \delta_0]$ the equation (4.2) has the unique solution $f^* \in \bar{\Omega}_\gamma$ and the following inequality is valid

$$||f^* - f_N||_{B_1} \leq C_2 \delta^{N+1}. \tag{4.5}$$

Proof. The proof is grounded on examination of the applicability conditions of the Kantorovich theorem about the convergence of the Newton method to the equation (4.2). It is comfortable to use the variant of this theorem stated in [8]. Let us remind you this formulation.

Let $A(f)$ to be the operator defined in the convex domain D of the Banach space B_1 acting into the Banach space B_2 and differentiable by Frechét. Let us assume that the Frechét derivative A_f' of the operator A satisfies the Lipschitz condition in the domain D with constant K and that for some $f_0 \in D$ operator A_{f_0}' has bounded inverse operator $(A_{f_0}')^{-1}$ as a linear operator acting from B_2 to B_1.If the following inequality is fulfilled

$$K||A(f_0)||_{B_2}||(A_{f_0}')^{-1}||^2 = \varepsilon < 1/2, \tag{4.6}$$

then the equation (4.2) has the solution f^* unique in some ball of the space B_1 with center in the point f_0, moreover

$$||f^* - f_0||_{B_1} \leq [1 - (1 - 2\varepsilon)^{1/2}] K^{-1} ||A_{f_0}'||. \tag{4.7}$$

It was noted above that constant K can be chosen not depending on $\delta \in (0, \delta_0]$ in the inequality (4.3) in the conditions of the Proposition 1. If we choose the N-approximate solution $f_N = \{w_N, y_N, \varphi_N\}$ of the problem (1.10)–(1.15) as f_0 then the estimate (4.4) is valid. So now one must estimate from above the norm of operator $(A_{f_N}')^{-1}$ in order to apply the

Kantorovich theorem. It is evident that this norm grows indefinitely when $\delta \to 0$. If we will manage to prove that this growth has the power nature

$$||(A'_{f_N})^{-1}|| \le C_3 \delta^{-m} \quad \text{for} \quad \delta \to 0 \tag{4.8}$$

with fixed $m > 0$ then it will be possible to guarantee the fulfillment of inequality (4.6) for $\delta_0 < 1$ and sufficiently large N as it follows from (4.4) and (4.8).

Let us consider the linear equation

$$A'_{f_N}(g) = b, \tag{4.9}$$

where $g = \{\omega(x,\tau), q(\tau), \xi(\tau)\} \in B_1$ and $b = \{h(x,\tau), z(\tau), \chi(\tau), l(x), c_1, c_2\} \in B_2$ (c_1 and c_2 are constants). Starting from the definition of the operator A we can rewrite the equation (4.9) in the form of the dependent system of one parabolic equation and two ordinary equations with the initial and boundary conditions. As a result we obtain the following problem

$$\begin{gathered}\delta[(y_N + x)\omega + w_N q]_\tau - 4[(y_N + x)^2\omega_x + 2(y_N + x)w_N q]_x + \\ +16\delta(2x - a)(2y_N + a)^{-3}\varphi_N q - \\ -4\delta(2x - a)(2y_N + a)^{-2}\xi = h(x,\tau), \quad (x,\tau) \in Q_T,\end{gathered} \tag{4.10}$$

$$\frac{dq}{d\tau} - 2\xi = z(\tau), \quad \tau \in (0,T), \tag{4.11}$$

$$\begin{gathered}\delta\frac{d\xi}{d\tau} - \frac{aq}{y_N(y_N + a)\log^2(1 + a/y_N)}\Big[-\frac{a\varphi_N(4 - \delta\varphi_N)}{y_N(y_N + a)} - \frac{1}{\sqrt{y_N}} - \frac{1}{\sqrt{y_N + a}} + \\ +\frac{\alpha a^{3/2}}{(2y_N + a)^2} + \frac{\alpha a^{5/2}}{4}\int_0^a w_N^2(x,\tau)dx\Big] - \frac{1}{\log(1 + a/y_N)}\Big\{-\frac{2a(2 - \delta\varphi_N)\xi}{y_N(y_N + a)} + \\ +\frac{q}{2}\Big[\frac{1}{y_N^{3/2}} + \frac{1}{(y_N + a)^{3/2}}\Big] - \frac{4\alpha a^{3/2} q}{(2y_N + a)^3} + \\ +\frac{\alpha a^{5/2}}{2}\int_0^a w_N(x,\tau)\omega(x,\tau)dx\Big\} = \chi(\tau), \quad \tau \in (0,T),\end{gathered} \tag{4.12}$$

$$\omega(x,0) = l(x), \quad 0 \le x \le a, \tag{4.13}$$

$$q(0) = c_1, \tag{4.14}$$

$$\xi(0) = c_2, \tag{4.15}$$

$$\omega_x(0,\tau) = \omega_x(a,\tau) = 0, \quad 0 \le \tau \le T. \tag{4.16}$$

The univalent solvability of the problem (4.10) – (4.16) is obvious under the conditions of the Proposition 1. The proof of the inequality (4.8) is equivalent to the obtaining of estimate of its solution in the form

$$||\omega||_{C^{2+\beta,1+\beta/2}(\bar{Q}_T)} + ||q||_{C^2[0,T]} + ||\xi||_{C^2[0,T]} \le C_3\delta^{-m}(||h||_{C^{\beta,\beta/2}(\bar{Q}_T)} +$$

$$+||z||_{C^1[0,T]} + +||\chi||_{C^1[0,T]} + ||l||_{C^{2+\beta}[0,a]} + |c_1| + c_2|) \qquad (4.17)$$

when $\delta \to 0$. The obtaining of this estimate is grounded on the passage to the "rapid time" $\eta = \tau/\delta$ in the equations (4.10) – (4.12).

Let us introduce the notations $\hat{\omega}(x,\eta) = \omega(x,\delta\eta)$, $\hat{q}(\eta) = q(\delta\eta)$, $\hat{\xi}(\eta) = \xi(\delta\eta)$, $\hat{h}(x,\eta) = h(x,\delta\eta)$, $\hat{z}(x,\eta) = z(\delta\eta)$, $\hat{\chi}(\eta) = \chi(\delta\eta)$. $\Pi_{T/\delta}$ denotes the rectangle $0 < x < a$, $0 < \eta < T/\delta$ and $\mathrm{I}_{T/\delta}$ - the interval $0 < \eta < T/\delta$. Problem (4.10) – (4.16) takes the following form in these notations:

$$[(y_N + x)\hat{\omega}]_\eta - 4[(y_N+x)^2\hat{\omega}_x]_x + \Lambda_1(x,\eta)\hat{q} + \Lambda_2(x,\eta)\hat{\xi} = \qquad (4.18)$$

$$= \hat{h}(x,\eta) - \delta w_N \hat{z}(\eta), \quad (x,\eta) \in \Pi_{T/\delta},$$

$$\frac{d\hat{q}}{d\eta} - 2\delta\hat{\xi} = \delta\hat{z}(\eta), \quad \eta \in \mathrm{I}_{T/\delta}, \qquad (4.19)$$

$$\frac{d\hat{\xi}}{d\eta} + \mu_1(\eta)\hat{\xi} + \mu_2(\eta)\hat{q} + \int_0^a \mu(x,\eta)\hat{\omega}(x,\eta)dx = \hat{\chi}(\eta), \quad \eta \in \mathrm{I}_{T/\delta}, \qquad (4.20)$$

$$\hat{\omega}(x,0) = l(x), \quad 0 \le x \le a, \qquad (4.21)$$

$$\hat{q}(0) = c_1, \qquad (4.22)$$

$$\hat{\xi}(0) = c_2, \qquad (4.23)$$

$$\hat{\omega}_x(0,\eta) = \hat{\omega}_x(a,\eta) = 0, \eta \in \bar{\mathrm{I}}_{T/\delta}. \qquad (4.24)$$

Here the following notations were used

$$\Lambda_1 = w_{N,\eta} - 8[(y_N+x)w_{N,x}]_x + 16\delta(2x-a)(2y_N+a)^{-2}\varphi_N,$$

$$\Lambda_2 = 2\delta w_N - 4\delta(2x-a)(2y_N+a)^{-2},$$

$$\mu_1 = \frac{2a(2-\delta\varphi_N)}{y_N(y_N+a)\log(1+a/y_N)}, \quad \mu = -\frac{\alpha a^{5/2} w_N(x,\delta\eta)}{2\log(1+a/y_N)},$$

$$\mu_2 = \frac{a}{y_N(y_N+a)\log^2(1+a/y_N)}\Big[\frac{a\varphi_N(4-\delta\varphi_N)}{y_N(y_N+a)} + \frac{1}{\sqrt{y_N}} +$$

$$+\frac{1}{\sqrt{y_N+a}} - \frac{\alpha a^{3/2}}{(2y_N+a)^2} - \frac{\alpha a^{5/2}}{4}\int_0^a w_N^2(x,\delta\eta)dx\Big] +$$

$$+\frac{1}{\log(1+a/y_N)}\Big[\frac{4\alpha a^{3/2}}{(2y_N+a)^3}--\frac{1}{2y_N^{3/2}}-\frac{1}{2(y_N+a)^{3/2}}\Big].$$

Equation (4.11) was used for derivation of the equation (4.18) from (4.10).

Let us list the characteristics of coefficients and right parts of the equations (4.18) – (4.20) essential for the further consideration. The inequalities

$$0<m\le y_N(\delta\eta)\le M<\infty,\quad \eta\in\bar{\mathrm{I}}_{T/\delta}$$

are fulfilled in the conditions of Proposition 1. This fact guarantees both the uniform parabolicity of the equation (4.18) in the rectangle $\bar{\Pi}_{T/\delta}$ and the boundedness both of functions $\mu_k(\eta)$ and their derivatives $d\mu_k/d\eta$, $k=1,2$ on the interval $\bar{\mathrm{I}}_{T/\delta}$. Using the definition (4.1) of function w_N and representation (3.5) of the function v_0 we derive the inequality

$$|\mu(x,\eta)|+|\mu_\eta(x,\eta)|\le C_4\exp(-C_5\eta)+\delta C_6,\quad (x,\eta)\in\bar{\Pi}_{T/\delta},\tag{4.25}$$

where one can put any positive number less then λ_1 instead of C_5.

Let us denote the quantity $\max\limits_{\bar{\mathrm{I}}_{T/\delta}}|\varphi_N(\delta\eta)|$ as C_7 and choose δ_0 less then $2C_7^{-1}$. Then the estimate

$$\mu_1(\eta)\ge C_8>0,\quad \eta\in\bar{\mathrm{I}}_{T/\delta}\tag{4.26}$$

is valid for any $\delta\in(0,\delta_0]$. Let us introduce the notations

$$\Lambda_4=w_{N,\eta}-8[(y_N+x)w_{N,x}]_x,\quad \Lambda_3=\Lambda_1-\Lambda_4.$$

The inequalities

$$||\hat{h}||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})}+||w_N\hat{z}||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})}+||\Lambda_4||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})}\le C_9,\tag{4.27}$$

$$||\Lambda_k||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})}\le\delta C_{10},\quad k=2,3,\tag{4.28}$$

$$||\Lambda_4(x,\eta)|\le C_{11}\exp(-C_5\eta)+\delta C_{12},\quad (x,\eta)\in\bar{\Pi}_{T/\delta}\tag{4.29}$$

follow from the assumptions about the functions $\hat{h}$, $\hat{z}$ and the characteristics of functions w_N, y_N, φ_N. Note that constants $C_4,\dots,C_{12}$ used in inequalities (4.25) – (4.29) do not depend on δ.

Now our aim is to obtain the estimate of $\max\limits_{\bar{\Pi}_{T/\delta}}|\hat{\omega}(x,\eta)|$ uniform with respect to δ. With this aim we represent $\hat{\omega}$ in the form of the sum $\hat{\omega}_1+\hat{\omega}_2+\hat{\omega}_3$, where the functions $\hat{\omega}_k$ satisfy the relations

$$(y_N+x)\hat{\omega}_{1,\eta}-4(y_N+x)^2\hat{\omega}_{1,xx}-8(y_N+x)\hat{\omega}_{1,x}+\zeta\hat{\omega}_1=$$

$$= -\delta w_N \hat{z} - \Lambda_3 \hat{q} - \Lambda_2 \hat{\xi}, \quad (x,\eta) \in \Pi_{T/\delta}, \tag{4.30}$$

$$\hat{\omega}_1(x,0) = l(x), \quad x \in [0,a], \tag{4.31}$$

$$\hat{\omega}_{1,x}(0,\eta) = \hat{\omega}_{1,x}(a,\eta) = 0, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}, \tag{4.32}$$

$$[(y_N + x)\hat{\omega}_2]_\eta - 4[(y_N + x)^2 \hat{\omega}_{2,x}]_x = \hat{h}, \quad (x,\eta) \in \Pi_{T/\delta}, \tag{4.33}$$

$$\hat{\omega}_2(x,0) = 0, \quad x \in [0,a], \tag{4.34}$$

$$\hat{\omega}_{2,x}(0,\eta) = \hat{\omega}_{2,x}(a,\eta) = 0, \quad \eta \in \bar{\mathrm{I}}_{t/\delta}, \tag{4.35}$$

$$[(y_N + x)\hat{\omega}_3]_\eta - 4[(y_n + x)^2 \hat{\omega}_{3.x}]_x = -\Lambda_4 \hat{q}, \quad (x,\eta) \in \Pi_{T/\delta}, \tag{4.36}$$

$$\hat{\omega}_3(x,0) = 0, \quad x \in [0,a], \tag{4.37}$$

$$\hat{\omega}_{3,x}(0,\eta) = \hat{\omega}_{3,x}(a,\eta) = 0, \quad \eta \in \bar{\mathrm{I}}_{T/\delta} \tag{4.38}$$

Here the notation $\zeta(\eta) = dy_N(\delta\eta)/d\eta$ was introduced. Note that $dy_N(\tau)/d\tau = 2\varphi_N(\tau) + O(\delta^{N+1})$ when $\delta \to 0$ in accordance with definition of y_N so that

$$|\zeta(\eta)| \le C_{13}\delta, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}. \tag{4.39}$$

Let us pass to the new sought for function

$$s_1 = \hat{\omega}_1 \exp[-\delta C_{14}\eta + \delta C_{15} x(a-x)]$$

in the relations (4.30) – (4.32). This function is the solution of the following problem

$$(y_N + x)s_{1,\eta} - 4(y_N + x)^2 s_{1,xx} - [8(y_N + x) + 8\delta C_{15}(y_N + x)^2(2x - a)]s_{1,x} +$$

$$+\{\zeta + \delta[C_{14}(y_N + x) + \delta C_{15}^2 (y_N + x)^2 (2x - a)^2 - 8C_{15}(y_N + x)(y_N + 3x - a)]\}s_1 =$$

$$= -(\delta w_N \hat{z} + \Lambda_3 \hat{q} + \Lambda_2 \hat{\xi}) \exp[-\delta C_{14}\eta + \delta C_{15} x(a-x)], \quad (x,\eta) \in \Pi_{T/\delta}, \tag{4.41}$$

$$s_1(x,0) = l(x) \exp[\delta C_{15} x(a-x)], \quad x \in [0,a], \tag{4.42}$$

$$s_{1,x}(0,\eta) - \delta C_{15} a s_1(0,\eta) = 0, \quad \eta \in \bar{I}_{T/\delta}, \tag{4.43}$$

$$s_{1,x}(a,\eta) + \delta C_{15} z s_1(a,\eta) = 0, \quad \eta \in \bar{\mathrm{I}}_{T/\delta} \tag{4.44}$$

Constants C_{14} and C_{15} can be chosen in the following form $C_{14} = (2 + C_{13})/m$, $C_{15} = 1/8m(2M + a)$. We obtain the lower bound for the coefficient j at the function s_1 in the equation (4.41)

$$j(x,\eta) \ge \delta, \quad (x,\eta) \in \bar{\Pi}_{T/\delta}, \tag{4.45}$$

on the basis of inequality (4.39). This estimate permits to apply the maximum principle [9] to the solution s_1 of the problem (4.41) – (4.44). Using the estimates (4.27), (4.28) and (4.25) we obtain

$$\max_{\bar{\Pi}_{T/\delta}} |s_1| \le C_{16} \max_{[0,a]} |l| + C_{17} \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{z}| + C_{18} (\max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}| + \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{\xi}|). \tag{4.46}$$

The estimate of $\max\limits_{\bar{\Pi}_{T/\delta}} |\hat{\omega}_1|$ by $\max\limits_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}|$ and $\max\limits_{\bar{\mathrm{I}}_{T/\delta}} |\hat{\xi}|$ follows from the above formula and (4.40).

The estimate of $\max\limits_{\bar{\Pi}_{T/\delta}} |\hat{\omega}_2|$ is grounded on the integral identity

$$\frac{1}{2}\frac{d}{d\eta}\int_0^a (y_N + x)\hat{\omega}_2^2 dx + \int_0^a [4(y_N+x)^2 \hat{\omega}_{2,x}^2 +$$

$$+\zeta\hat{\omega}_2^2] dx = \int_0^a \hat{h}\hat{\omega}_2 dx, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}. \tag{4.47}$$

The solution of the problem (4.33) – (4.35) satisfies this identity. The inclusion $h(x,\tau) \in C_0^{\beta,\beta/2}(\bar{Q}_T)$ implies the fulfillment of the equality

$$\int_0^a \hat{h}(x,\eta) dx = 0, \quad \eta \in \bar{\mathrm{I}}_{T/\delta},$$

so that the problem (4.33) – (4.35) possess the conservation law

$$\int_0^a (y_N + x)\hat{\omega}_2 dx = 0, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}. \tag{4.48}$$

The smooth function $\hat{\omega}_2(x,\eta)$ satisfying the relation (4.48) for any $\eta \in \bar{\mathrm{I}}_{T/\delta}$ is subjected to the inequality

$$\int_0^a \hat{\omega}_2^2 dx \le a^2 \int_0^a \hat{\omega}_{2,x}^2 dx, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}.$$

Using this inequality, the estimate (4.39) and choosing δ_0 less then $2C_{13}^{-1}(m/a+1)^2$ we obtain from (4.47)

$$\frac{1}{2}\frac{d}{d\eta}\int_0^a (y_N + x)\hat{\omega}_2^2 dx + 2\Big(\frac{m}{a}\Big)^2 \int_0^a \hat{\omega}_2^2 dx \le \max_{\bar{\Pi}_{T/\delta}} |\hat{h}| \int_0^a |\hat{\omega}_2| dx, \quad \eta \in \bar{\mathrm{I}}_{T/\delta},$$

indepedently on $\delta \in (0, \delta_0]$. So the following conclusion

$$\Big(\int_0^a \hat{\omega}_2^2(x,\eta) dx\Big)^{1/2} \le C_{19} \max_{\bar{\Pi}_{T/\delta}} |\hat{h}|, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}$$

can be made from the above inequality and (4.47). The existence of estimate (4.49) together with uniform boundedness and continuity of coefficients and the right part of the equation (4.33) in the domain $\bar{\Pi}_{T/\delta}$ gives possibility to apply the results of L^q–theory of linear parabolic equations of the second order [9] to the problem (4.33) – (4.35). This fact leads to the inequality

$$\max_{\bar{\Pi}_{T/\delta}} |\hat{\omega}_2| \le C_{20} \max_{\bar{\Pi}_{T/\delta}} |\hat{h}|. \qquad (4.50)$$

The identity analogous to (4.47) is valid for the solution $\hat{\omega}_3$ of the problem (4.36) – (4.38); it has the form

$$\frac{1}{2}\frac{d}{d\eta}\int_0^a (y_N+x)\hat{\omega}_3^2 dx + \int_0^a [4(y_N+x)^2\hat{\omega}_{3,x}^2 + \zeta\hat{\omega}_3^2]dx = -\hat{q}\int_0^a \Lambda_4\hat{\omega}_3 dx, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}.$$

According to the inequalities (4.29), (4.39) the last identity leads to the inequality

$$\frac{1}{2}\frac{d}{d\eta}\int_0^a (y_N+x)\hat{\omega}_3^2 dx - C_{13}\delta\int_0^a \hat{\omega}_3^2 dx \le$$

$$\le [C_{11}\exp(-C_5\eta) + \delta C_{12}] \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}| \int_0^a |\hat{\omega}_3| dx, \quad \eta \in \bar{\mathrm{I}}_{T/\delta}. \qquad (4.51)$$

Integration of the differential inequality (4.51) with the initial condition (4.37) leads to the estimate

$$\left(\int_0^a \hat{\omega}_3^2(x,\eta)dx\right)^{1/2} \le C_{21} \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}|, \quad \eta \in \bar{\mathrm{I}}_{T/\delta},$$

and together with this estimate to the estimate

$$\max_{\bar{\Pi}_{T/\delta}} |\hat{\omega}_3| \le C_{22} \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}|. \qquad (4.52)$$

The inequality

$$\max_{\bar{\Pi}_{T/\delta}} |\hat{\omega}| \le C_{23}(\max_{\bar{\Pi}_{T/\delta}} |\hat{h}| + \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{z}| + \max_{[0,a]} |l| + \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}| + \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{\xi}|). \qquad (4.53)$$

follows from (4.40), (4.46), (4.50) and (4.52).

Let us pass to derivation of a priori estimate of functions $\hat{q}(\eta)$ and $\hat{\xi}(\eta)$. The number $\delta_0 > 0$ is chosen less then $\min[2C_7^{-1}, C_8(C_6C_{23}a)^{-1}, 2C_{13}^{-1}(m/a+1)^2]$. Then we obtain

$$\frac{d\hat{\xi}}{d\eta} + C_{24}\hat{\xi} \le \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{\chi}| + C_{25} \max_{\bar{\mathrm{I}}_{T/\delta}} |\hat{q}| + C_4C_{23}a\exp(-C_5\eta) \max_{[0,\eta]} |\hat{\xi}| +$$

$$+C_{25}(\max_{\bar{\Pi}_{T/\delta}}|\hat{h}| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{z}| + \max_{[0,a]}|l|). \qquad (4.54)$$

on the basis of the equation (4.20) and inequalities (4.25), (4.53). Taking into account the fact that the positive constants entering the inequality (4.54) do not depend on $\delta \in (0, \delta_0]$ and integrating this inequality with the initial condition (4.23) we obtain

$$\max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\xi}| \leq C_{26}(\max_{\bar{\Pi}_{T/\delta}}|\hat{h}| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{z}| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\chi}| + \max_{[0,a]}|l| + |c_2| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{q}|). \qquad (4.55)$$

So a priori estimate

$$\max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{q}| \leq C_{27}(\max_{\bar{\Pi}_{T/\delta}}|\hat{h}| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{z}| + \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\chi}| + \max_{[0,a]}|l| + |c_1| + |c_2|) \qquad (4.56)$$

follows from the above inequality and (4.19), (4.22). A priori estimate analogous to (4.56) of quantities $\max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\xi}|$ and $\max_{\bar{\Pi}_{T/\delta}}|\hat{\omega}|$ follows from the above estimate in view of (4.53) and (4.55).

On account of the equations (4.19) and (4.20) the similar estimates are valid for the quantities $\max_{\bar{\mathrm{I}}_{T/\delta}}|d\hat{\xi}/d\eta|$ and $\max_{\bar{\mathrm{I}}_{T/\delta}}|d\hat{q}/d\eta|$. The existence of these estimates guarantees the belonging of the "free term" $\gamma = -\Lambda_1\hat{q} - \Lambda_2\hat{\xi} + \hat{h} - \delta w_N\hat{z}$ of the equation (4.18) to the class $C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})$, moreover

$$||\gamma||_{C^{\beta,\beta/2}((\bar{\Pi}_{T/\delta})} \leq C_{28}(||\hat{h}||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})} + ||\hat{z}||_{C^1(\bar{\mathrm{I}}_{T/\delta})} +$$

$$+ \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\chi}| + \max_{[0,a]}|l| + |c_1| + |c_2|). \qquad (4.57)$$

We may apply the results of general theory of linear parabolic equations in Hölder spaces [9] to the problem (4.18), (4.21), (4.24) in view of the estimate (4.57). It follows from these results that the solution $\hat{\omega}$ of mentioned problem belongs to the class $C^{2+\beta,1+\beta/2}(\bar{\Pi}_{T/\delta})$ and the inequality

$$||\hat{\omega}||_{C^{2+\beta,1+\beta/2}((\bar{\Pi}_{T/\delta})} \leq C_{29}(||\hat{h}||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})} + ||\hat{z}||_{C^1(\bar{\mathrm{I}}_{T/\delta})} +$$

$$+ \max_{\bar{\mathrm{I}}_{T/\delta}}|\hat{\chi}| + ||l||_{C^{2+\beta}[0,a]} + |c_1| + |c_2|) \qquad (4.58)$$

is valid.

The last step is to obtain the estimates of norms $||\hat{\xi}||_{C^2(\bar{\mathrm{I}}_{T/\delta})}$ and $||\hat{q}||_{C^2(\bar{\mathrm{I}}_{T/\delta})}$. In order to obtain the first estimate we differentiate the equation (4.20) and use the inequality (4.25) with already existing estimates $||\hat{\omega}||_{C^{2+\beta,1+\beta/2}((\bar{\Pi}_{T/\delta})}$, $||\hat{\xi}||_{C^1(\bar{\mathrm{I}}_{T/\delta})}$ and $||\hat{q}||_{C^1(\bar{\mathrm{I}}_{T/\delta})}$. Function $d\hat{\xi}/d\eta$ satisfies the linear equation of the first order with the coefficient μ_1 permitting the estimate (4.26) and the right part module-bounded by some constant on the interval $\bar{\mathrm{I}}_{T/\delta}$ independently on $\delta \in (0, \delta_0]$. So the demanded inequality is derived

$$||\hat{\xi}||_{C^2((\bar{\mathrm{I}}_{T/\delta})} \leq C_{30}(||\hat{h}||_{C^{\beta,\beta/2}(\bar{\Pi}_{T/\delta})} + ||\hat{z}||_{C^1(\bar{\mathrm{I}}_{T/\delta})} +$$
$$+ ||\chi||_{C^1(\bar{\mathrm{I}}_{T/\delta})} + ||l||_{C^{2+\beta}[0,a]} + |c_1| + |c_2|). \quad (4.59)$$

The estimate of the norm $||\hat{q}||_{C^2(\bar{\mathrm{I}}_{T/\delta})}$ analogous to (4.59) is obtained directly from the equation (4.19). Returning to the "slow" time $\tau = \delta\eta$ in this inequality and in inequalities (4.58), (4.59) we obtain the demanded estimate (4.17) with $m = 2$ equipotent with the estimate (4.8).

Now let us choose $N = 4$ and fix it. Then it follows from (4.4), (4.8) that

$$\varepsilon = K||A(f_0)||_{B_2}||(A'_{f_0})^{-1}||^2 \leq KC_1C_3^2\delta^{N-3}, \quad \delta \in (0, \delta_0]. \quad (4.60)$$

We can achieve the fulfillment of inequality $\varepsilon < 1/2$ with $\delta \in (0, \delta_0]$ guaranteeing the solvability of the equation (4.2) by decreasing (if it is necessary) the quantity δ_0. We must derive the inequality (4.5) in order to complete the proof of Proposition 1. Let N to be an arbitrary natural number. Choosing the N^*-approximate ($N^* = N + 4$) solution of the problem (1.10) – (1.15) as f_0 and using the triangle inequality, (4.7) and (4.60) we obtain

$$||f^* - f_N||_{B_1} \leq ||f^* - f_{N^*}|| + ||f_N - f_{N^*}|| \leq C_{31}\delta^{N^*-3} + C_{32}\delta^{N+1} \leq C_2\delta^{N+1},$$

if $\delta \in (0, \delta_0]$. Proposition 1 is proved.

5 Justification of the asymptotic expansion (case B).

In this case one can find such τ^* that solution $Y(\tau)$ of the problem (2.5), (2.6) is positive for $\tau \in [0, \tau^*)$ but $Y = 0$ for $\tau \geq \tau^*$. As it was already mentioned in Section 2, here it is appropriate to pass to the new independent variable y instead of τ and new sought for functions $\Phi[y(\tau)] = \varphi(t)$, $W[x, y(\tau)] = w(x, \tau)$. As a result the problem (1.10) – (1.15) takes the form of relations (2.15) – (2.19). Note that $y_0 = y(\tau_0)$ in these relations and $\tau_0 \in (0, \tau^*)$ is chosen in a special way so that $dy/d\tau = 2\varphi < 0$ when $\tau_0 \leq \tau < \tau^*$. We may assume that the asymptotic solution $f_N = \{w_N, y_N, \varphi_N\}$ of the problem (1.10) – (1.15) is already built on the interval $0 \leq \tau \leq \tau_0$ and its closeness to the exact solution $f^* = \{w, y, \varphi\}$ is defined by inequality (4.5). In particular this inequality gives the relations

$$||W(x, y_0) - W_N(x, y_0)||_{C^{2+\beta}[0,a]} \leq C_{33}\delta^{N+1},$$

$$|\Phi(y_0) - \Phi_N(y_0)| \le C_{34}\delta^{N+1}, \quad \delta \in (0, \delta_0], \tag{5.1}$$

where

$$W_N = \sum_{k=1}^{N} \delta^k W^{(k)}(x, y), \quad \Phi_N = \sum_{i=0}^{N} \delta^i \Phi^{(i)}(y), \tag{5.2}$$

and the terms of the external expansion $W^{(k)}$, $\Phi^{(i)}$ are defined by the algorithm suggested in subsection 2.

The terms depending on quick evolutionary variable are absent in representation (5.2) of N-approximate solution $\{W_N, \Phi_N\}$ of the problem (2.15) – (2.19) in contrast to (4.1). We can explain it by the fact that functions $v^{(k)}(x, \tau/\delta) - z^{(k)}$, $\psi^{(i)}(\tau/\delta) - \chi^{(i)}$ used in definition (4.1) satisfy the inequalities

$$||v^{(k)}(x, \tau/\delta) - z^{(k)}(x)||_{C^{2+\beta}[0,a]} \le C_{36}\exp(-C_{35}/\delta),$$

$$|\psi^{(i)}(\tau/\delta) - \chi^{(i)}| \le C_{37}\exp(-C_{35}/\delta)$$

when $0 < \tau_0 \le \tau < \tau^*$, $0 < \delta \le \delta_0$. The estimate of the "approximate conservation law"

$$|\int_0^a (y + x)W_N(x, y)dx| \le C_{38}\delta^{N+1}, \quad y \in (0, y_0], \quad \delta \in (0, \delta_0] \tag{5.3}$$

is the other important property of the approximate solution of the problem (2.15) – (2.19). The identity

$$\int_0^a (y + x)W(x, y)dx = 0, \quad y \in (0, y_0], \tag{5.4}$$

equipotent with (1.9) is used for the proof of (5.3). The exact solution of the problem (2.15) – (2.19), the first estimate (5.1) and the equality

$$\frac{d}{dy}\int_0^a (y + x)W^{(k)}(x, y)dx = 0, \quad y \in (0, y_0], \quad k = 1, 2, \ldots \tag{5.5}$$

satisfy this identity. The equality (5.5) is proved by induction starting from the definition of functions $W^{(k)}$. For $k = 1$ (5.5) follows directly from the relations

$$\Phi^{(0)}[(y + x)W^{(1)}]_y = 2[(y + x)^2 W_x^{(2)}]_x - 2(2x - a)(2y + a)^{-2}\Phi^{(1)},$$

$$0 < x < a, \quad 0 < y < y_0,$$

$$W_x^{(1)}(0, y) = W_x^{(1)}(a, y) = 0, \quad 0 < y \le y_0$$

(function $W^{(1)}$ satisfies these relations).

The equality (5.5) means practically that

$$\int_0^a (y+x)W_N(x,y)dx = \kappa(\delta),$$

where the function κ not depending on y supposes the estimate $|\kappa| \leq C_{38}\delta^{N+1}$ when $\delta \to 0$. This fact permits to introduce the new function $\check{W}_N(x,y)$ satisfying the exact equality

$$\int_0^a (y+x)\check{W}_N(x,y)dx = 0, \tag{5.6}$$

and

$$W_N = \check{W}_N + \frac{2\kappa(\delta)}{a(a+2y)} \tag{5.7}$$

It is essential that both functions $\check{W}_N$ and W_N satisfy the homogeneous boundary conditions (2.16) and an addition with the order δ^{N+1} arises in the initial condition for $\check{W}_N$ so that inequality (5.1) remains valid (may be with some new constant) after the change of W_N on $\check{W}_N$. The order of discrepancy is not changed at substitution of the pair $\check{W}_N$, Φ_N instead of W_N, Φ_N in the equalities (2.15), (2.18); the corresponding norms of each of them are estimated from above by quantity proportional to δ^{N+1} when $\delta \to 0$.

Further it is suggested that for the solution W,Φ of the problem (2.15)–(2.19) the inequality $\Phi(y) < 0$ is fulfilled if $0 < y \leq y_0$. As it was noted before for $\alpha < \alpha^* \approx 5.89$ this inequality can be provided by choice of sufficiently small y_0. In this case the following estimate can be written:

$$-\frac{1}{4}\sqrt{y}\Big[1 + \frac{C_{39}}{\sqrt{\ln{(1+a/y)}}}\Big] \leq \Phi \leq -\frac{1}{4}\sqrt{y},\ 0 \leq y \leq y_0, \tag{5.8}$$

where the positive constant C_{39} is independent of $\delta \in (0, \delta_0]$. Estimate (5.8) was derived in [1]. Estimate written below is valid for the element Φ_N of the approximate solution of current problem:

$$-\frac{1}{4}\sqrt{y}\Big[1 + C_{40}\sqrt{y}\ln\frac{1}{y}\Big] \leq \Phi_N \leq -\frac{1}{4}\sqrt{y}\Big[1 + C_{41}\sqrt{y}\ln\frac{1}{y}\Big],$$

$$0 \leq y \leq y_0 < 1, \tag{5.9}$$

constants $C_{40} > C_{41} > 0$ are independent of δ. Estimate (5.9) follows from inequalities (2.22), inclusions $\Phi^{(k)} \in C^{(k+\beta)/2}[0, y_0]$, $0 < \beta < 1$ and recursion relations of (2.21) type for $\Phi^{(k)}$ with $k = 1, 2, \ldots$. Further we'll suppose that $y_0 < 1$ without loss of generality.

Let us inroduce the notations $\Pi_\varepsilon = \{x, y : \ 0 < x < a, \ \varepsilon < y < y_0\}$, $I_\varepsilon = \{y : \ \varepsilon < y < y_0\}$. Our aim is to prove the proximity of functions Φ and Φ_N on the interval I_0 and functions W and $\tilde{W}_N$ in the rectangle Π_0.

New notations $U_N = W - \tilde{W}_N$ and $\Psi_N = \Phi - \Phi_N$ are introduced here. Function $\tilde{W}_N$ satisfies the equation (2.15) (in accordance with construction of the approximate solution) if Φ is replaced by Φ_N and the right part is added by residual denoted by Z_N. If Ψ_N is known then function $U_N(x, y)$ is determined as the solution of the linear initial boundary value problem:

$$\delta\Phi[(y + x)U_N]_y = 2[(y + x)^2 U_{N,x}]_x +$$

$$+\delta\Psi_N\{(2y + a)^{-2}(2x - a) - [(y + x)\tilde{W}_N]_y\} - Z_N, \ (x, y) \in \Pi_0, \quad (5.10)$$

$$U_{N,x}(0, y) = U_{N,x}(a, y) = 0, \ y \in I_0, \quad (5.11)$$

$$U_N(x, y_0) = U^0(x) \equiv W_0(x) - W_N(x, y_0), \ 0 \le x \le a. \quad (5.12)$$

Analogously in the case of given U_N function $\Psi_N(y)$ is the solution of the linear Cauchy problem:

$$\delta\frac{d\Psi_N}{dy} = \frac{\delta a \Psi_N}{2y(y + a)\ln(1 + a/y)} + \frac{\Psi_N}{2\Phi\Phi_N \ln(1 + a/y)}\Big[\frac{1}{\sqrt{y}} +$$

$$+\frac{1}{\sqrt{y + a}} - \frac{\alpha a^{3/2}}{(2y + a)^2} - \frac{\alpha a^{5/2}}{4}\int_0^a W^2 dx\Big] +$$

$$+\frac{\alpha a^{5/2}}{2\Phi_N \ln(1 + a/y)}\int_0^a (W + W_N)U_N dx + \Omega_N, \ y \in I_0, \quad (5.13)$$

$$\Psi_N(y_0) = \Psi^0 \equiv \Phi_0 - \Phi_N(y_0). \quad (5.14)$$

Here denoted by Ω_N is the residual obtained by substitution of the approximate solution Φ_N, W_N to the equation (2.18).

Functions $Z_N(x, y), \ \Omega_N(y)$ with $N = 2, 3, \ldots$ satisfy the following inequalities that will be necessary for us in future:

$$|Z_N| \le C_{42}\delta^{N+1}\sqrt{y}, \ (x, y) \in \bar{\Pi}_0, \quad (5.15)$$

$$|\Omega_N| \le C_{43}\delta^{N+1}, \ y \in \bar{I}_0. \quad (5.16)$$

If $N = 1$ then the right parts of these inequalities will be added with factor $\ln(1/y)$. Used for obtaining of the estimates (5.15), (5.16) are the explicit expressions of $Z_N, \ \Omega_N$ in terms of $\Phi^{(0)}, \ldots, \Phi^{(N)}$; $W^{(1)} \ldots, W^{(N)}$, the smoothness properties of these functions expressed by the inclussions $\Phi^{(k)} \in C^{(k+\beta)/2}[0, y_0]$, $W^{(k)} \in C^{(k+\beta-1)/2}(\bar{\Pi}_0)$, and the fact that $\Phi^{(k)}(0) = 0$, $W^{(k)}(x, 0) = 0$ for $x \in [0, a]$ and any $k = 1, 2, \ldots$.

It will be convenient for further considerations to distinguish the singularities of functions Φ and Φ_N for $y \to 0$ and to introduce the new functions $\Gamma$and Γ_N by relations

$$\Phi = -\frac{1}{4}\sqrt{y}(1 + \Gamma),\ \Phi_N = -\frac{1}{4}\sqrt{y}(1 + \Gamma_N). \tag{5.17}$$

As it follows from inequalities (5.8), (5.9) functions Γ and Γ_N are nonnegative for $y \in \bar{I}_0$ and $\Gamma \to 0$, $\Gamma_N \to 0$ when $y \to 0$. Let us denote the difference $\Gamma - \Gamma_N$ by Δ_N. It proves to be that it is easier to obtain the estimate of function Δ_N then the direct estimate of function $\Psi_N = \Phi - \Phi_N$.

In consequence of (5.13), (5.17) function Δ_N satisfies the equation

$$\frac{d\Delta_N}{dy} + q\Delta_N = \omega_N,\ y \in I_0, \tag{5.18}$$

where

$$q = \frac{1}{\delta \ln(1 + a/y)(1 + \Gamma)(1 + \Gamma_N)}\Big[-\frac{8a}{y^{3/2}(y + a)} - \frac{16}{y\sqrt{y + a}} - \frac{16}{\sqrt{y}(y + a)} +$$

$$+ \frac{16\alpha a^{3/2}}{y(2y + a)^2} + \frac{\delta}{2y} + \frac{\delta a}{y(y + a)\ln(1 + a/y)} - \frac{4\alpha a^{5/2}}{y}\int_0^a W^2 dx\Big], \tag{5.19}$$

$$\omega_N = -\frac{4\Omega_N}{\delta\sqrt{y}} - \frac{2\alpha a^{5/2}}{\delta y \ln(1 + a/y)(1 + \Gamma)}\int_0^a (W + W_N)U_N dx. \tag{5.20}$$

Then the initial condition

$$\Delta_N(y_0) = \Delta^0 \equiv -4\Psi^0/\sqrt{y_0}. \tag{5.21}$$

following from (5.14), (5.17) is joined to the equation (5.18). The equation (5.10) is rewritten in new terms as:

$$\delta\Phi[(y + x)U_N]_y = 2[(y + x)^2 U_{N,x}]_x - Z_N + c_N\Delta_N,\ (x, y) \in \Pi_0, \tag{5.22}$$

where

$$c_N = -\frac{\delta\sqrt{y}}{4}\{(2y + a)^{-2}(2x - a) - [(y + x)\tilde{W}_N]_y\}. \tag{5.23}$$

The correctness of inequalities

$$|c_N| \le C_{44}\delta\sqrt{y},\ (x, y) \in \bar{\Pi}_0 \tag{5.24}$$

for any $N = 2, 3, \dots$ is defined from representation (2.7) for any $w^{(1)}(x, \tau) = W^{(1)}(x, y)$, recursion relations for functions $W^{(k)}$, $k = 2, 3, \dots$ and the induction.

The equation (5.22) degenerates on the part $y = 0$, $0 \le x \le a$ of the boundary of the domain Π_0; the equation (5.18) degenerates in the endpoint $y = 0$ of the interval I_0. Hence we will consider the problem (5.22), (5.11), (5.12) in the contracted domain Π_ε with $\varepsilon > 0$; analogously the problem (5.18), (5.21) will be considered on the interval I_ε. At the same time the obtained estimates of functions U_N, Δ_N will be independent on ε. This fact will permit to pass to the limit for $\varepsilon \to 0$. We will use the following statements for obtaining of these estimates.

Lemma: Let $u(x, y)$ to be the classical solution of the problem

$$\delta\Phi[(y + x)u]_y = 2[(y + x)^2 u_x]_x + h, \; (x, y) \in \Pi_0, \tag{5.25}$$

$$u_x(0, y) = u_x(a, y) = 0, \; 0 < y \le y_0, \tag{5.26}$$

$$u(x, y_0) = u_0(x), \; 0 \le x \le a, \tag{5.27}$$

where $\Phi(y)$, $u_0(x)$, $h(x, y)$ are the functions continuous respectively on the segments $[0, y_0]$, $[0, a]$ and in the restangle $\bar{\Pi}_0$. We assume the fulfillment of inequalities

$$\Phi < 0 \text{ when } 0 < y \le y_0, \tag{5.28}$$

$$|h| \le C_{45}\sqrt{y}, \; (x, y) \in \bar{\Pi}_0. \tag{5.29}$$

Then the estimate

$$|u| \le C_{46}\sqrt{y}\left[\ln\frac{y + x}{y} + \frac{y}{y + x}\right], \; (x, y) \in \bar{\Pi}_0 \tag{5.30}$$

is valid for any $\delta \in (0, \delta_0]$, where the constant $C_{46} > 0$ depends only on a, y_0, C_{45} and $C_{47} = \max|u_0(x)|$, $0 \le x \le a$.

Proof: Let us consider the problem (5.25)–(5.27) in the restangle Π_ε when $\varepsilon > 0$ and pass from x to the new independent variable $\xi = x/y$. The domain $\Sigma_\varepsilon = \{\xi, y : \; 0 < \xi < a/y, \; \varepsilon < y < y_0\}$ is the image of Π_ε on the plane ξ, y. We introduce the new sought for function $z(\xi, y)$ with the help of relation

$$u(x, y) = \sqrt{y}\left[\ln(1 + \xi) + \frac{1}{1 + \xi}\right]z(\xi, y) \tag{5.31}$$

In view of (5.25)–(5.27) this function satisfied the equation

$$\delta y\Phi[(1 + \xi)\ln(1 + \xi) + 1]Z_y = 2(1 + \xi)[(1 + \xi)\ln(1 + \xi) + 1]z_{\xi,\xi} +$$

$$+2\{2(1+\xi)\ln(1+\xi)+3+\xi+\frac{\delta}{2}\Phi\xi[(1+\xi)\ln(1+\xi)+1]\}z_\xi+$$

$$+\{2-\frac{\delta\Phi}{2}(3+\xi)[\ln(1+\xi)+\frac{1}{1+\xi}]\}z+\frac{1}{\sqrt{y}}h(\xi y,y),\ (\xi,y)\in\Sigma_\varepsilon, \quad (5.32)$$

the boundary conditions

$$z_\xi(0,y)=0,\ \xi\le y\le y_0, \quad (5.33)$$

$$\Big[\ln(\frac{a+y}{y})+\frac{y}{a+y}\Big]z_\xi\Big(\frac{a}{y},y\Big)+\frac{ay}{(a+y)^2}z\Big(\frac{a}{y},y\Big)=0,\ \varepsilon\le y\le y_0 \quad (5.34)$$

and the initial condition

$$z(\xi,y_0)=y_0^{-1/2}\Big[\ln(1+\xi)+\frac{1}{1+\xi}\Big]^{-1}u_0(y_0\xi),\ 0\le\xi\le\frac{a}{y_0}. \quad (5.35)$$

As it follows from (5.28) the equation (5.32) is parabolic in the domain Σ_ε for any $\varepsilon>0$. (Here we take into account the fact that the problem (5.33) –(5.35) is solved in the direction of y diminution with respect to this equation.) The coefficient at z has the "required" sign in this equation permitting to apply the Maximum principle [9, 10]; moreover this coefficient is not less then 2 when $(\xi,y)\in\Sigma_\varepsilon$ independently of ε. Therefore function $|z|$ can't exceed $C_{45}/2$ (where C_{45} is the constant from the inequality (5.29)) in inner points of the domain Σ_ε or on the part of its boundary $0<\xi<a/\varepsilon,\ y=\varepsilon$. As it follows from Theorem 14, Chapter II [10], function z can't reach its positive maximum or negative minimum on side boundaries of the domainΣ_ε so far as it satisfies the homogeneous Neumann condition (5.33) on the left boundary and homogeneous condition of the third genus with positive constant at z (5.34) on the right boundary. So it follows that

$$|z(\xi,y)|\le\max\Big(\frac{C_{45}}{2},C_{47}C_{48}\Big),\ (\xi,y)\in\bar\Sigma_\varepsilon, \quad (5.36)$$

where $C_{47}=\max\limits_{[0,a]}|u_0(x)|$ and $C_{48}(a,y_0)>0$ – the maximum value of coefficient at u_0 in the equality (5.35) on the interval $[0,a/y_0]$.

Returning to the function $u(x,y)$ by formula (5.31) and using the inequality (5.36) we obtain the estimate (5.30) in the domain $\bar\Pi_\varepsilon$. So far as constants appearing in the right part of (5.36) don't depend on ε (note that the inequality (5.29) is assumed to be fulfilled in the whole domain $\bar\Pi_0$) the desired eatimate (5.30) is valid for $(x,y)\in\bar\Pi_0$ too. Lemma is proved.

Proposition 2. Let us suppose that $\alpha < \alpha^*$ and $y_0 > 0$ is sufficiently small. Then such $\delta_0 > 0$ can be found that for $\delta \in (0, \delta_0]$ and $N = 1, 2, \ldots$ the following inequalities are fulfilled

$$|U_N| \le C_{49}\delta^{N+1}\sqrt{y}\Big[\ln\left(\frac{y+x}{y}\right) + \frac{y}{y+x}\Big],\ 0 \le x \le a,\ 0 \le y \le y_0, \quad (5.37)$$

$$|\Psi_N| \le C_{50}\delta^{N+1}\sqrt{y},\ 0 \le y \le y_0, \quad (5.38)$$

where $U_N = W - \tilde{W}_N$, $\Psi_N = \Phi - \Phi_N$, (W, Φ) is the exact solution and $(\tilde{W}_N, \Phi_N)$ – the approximate solution of the problem (2.15)–(2.19).

Proof: At first we consider the case $N \ge 2$. Let us assume that function $\Delta_N(y)$ used in the equation (5.22) is continuous on the interval $[\varepsilon, y_0]$ and then estimate the free term of this equation with the help of inequalities (5.15), (5.24):

$$\max_{\bar{\Pi}_\varepsilon} |c_N\Delta_N - Z_N| \le C_{44}\delta\sqrt{y}\max_{\bar{I}_\varepsilon}|\Delta_N| + C_{42}\delta^{N+1}\sqrt{y}. \quad (5.39)$$

Our aim is to obtain the estimate of $\max|U_N|$ uniform with respect to ε in the domain $\bar{\Pi}_\varepsilon$ in terms of input data of the problem (5.22), (5.11), (5.12) and the estimate of $\max_{\bar{I}_\varepsilon}|\Delta_N|$. In view of the estimate (5.39) and the inequality (5.8) we may use the Lemma statement and conclude that

$$|U_N| \le C_{51}\delta\max_{[\varepsilon, y_0]}|\Delta_N| + C_{52}\delta^{N+1},\ (x, y) \in \bar{\Pi}_\varepsilon. \quad (5.40)$$

(The boundedness of the right part of the inequality (5.30) in the domain $\bar{\Pi}_0$ is sufficient for us on this step.) As it follows from the inequality (5.8) the Lemma statement is applicable to the solution of the problem (2.15)–(2.17); this fact leads to inequality

$$|W| \le C_{53}\delta,\ (x, y) \in \bar{\Pi}_0. \quad (5.41)$$

The analogous inequality for the function W_N is evident:

$$|W_N| \le C_{54}\delta,\ (x, y) \in \bar{\Pi}_0. \quad (5.42)$$

These inequalities permit to estimate the integrals used in the relations (5.19), (5.20):

$$\int_0^a W^2(x, y)dx \le C_{55}\delta^2,\ 0 \le y \le y_0, \quad (5.43)$$

$$|\int_0^a (W + W_N)U_N dx| \le C_{56}\delta^2\max_{[\varepsilon, y_0]}|\Delta_N| + C_{57}\delta^{N+2},\ \varepsilon \le y \le y_0, \quad (5.44)$$

moreover constants $C_{56},\ C_{57}$ are independent of ε.

Here we use the representation of the solution of the Cauchy problem (5.18), (5.21) in the form

$$\Delta_N(y) = \Delta^0 \exp\,[\int\limits_y^{y_0} q(\eta)d\eta] - \int\limits_y^{y_0} \exp\,[\int\limits_y^{\eta} q(\zeta)d\zeta]\omega_N(\eta)d\eta. \tag{5.45}$$

Starting from the equality (5.19) and taking into account (5.41) we obtain the estimate

$$y^{3/2}\ln{(1/y)}q(y) \leq -C_{58}/\delta,\ 0 \leq y \leq y_0. \tag{5.46}$$

The inequalities (5.16), (5.44) used for estimating of $|\omega_N(y)|$ give

$$|\omega_N| \leq \frac{4C_{43}}{\sqrt{y}}\delta^N + \frac{C_{59}}{y\ln{(1/y)}}(\delta \max_{[\varepsilon,y_0]} |\Delta_N| + C_{60}\delta^{N+1}),\ \varepsilon \leq y \leq y_0 \tag{5.47}$$

with constants $C_{59},\ C_{60}$ independent of ε. The following estimate is derived from the representation (5.45) with the help of the inequalities(5.46), (5.47) and the fact that $\Delta^0 = O(\delta^{N+1})$ when $\delta \to 0$ on the basis of (5.14), (5.21):

$$|\Delta_N(y)| \leq C_{61}\delta \max_{[\varepsilon,y_0]} |\Delta_N| + C_{62}\delta^{N+1},\ \varepsilon \leq y \leq y_0,$$

where $C_{61},\ C_{62}$ are independent of ε. So the inequality

$$|\Delta_N|(y)| \leq C_{63}\delta^{N+1},\ \varepsilon \leq y \leq y_0 \tag{5.48}$$

is obtained if $0 < \delta \leq \delta_0 < C_{61}^{-1}$ (further it is assumed that this restriction on quantity δ_0 is fulfilled). Hence quantity C_{63} is independent of ε the inequality (5.48) is valid in the limit $\varepsilon = 0$ too.

The inequality (5.38) for $N = 2, 3, \ldots$ follows directly from the estimate (5.48) and definition $\Psi_N = -\sqrt{y}\Delta_N/4$. Using the estimate (5.48) with $\varepsilon = 0$ and the inequality (5.39) with C_{44} and C_{42} independent of ε we obtain the estimate of free term of the equation (5.22) in the form

$$|c_N\Delta_N - Z_N| \leq C_{64}\sqrt{y},\ (x, y) \in \bar{\Pi}_0.$$

This fact permits to apply the Lemma statement to the solution of the problem (5.22), (5.11), (5.12). Formula (5.12) and estimate $|U_N(x, y_0)| = O(\delta^{N+1})$ when $\delta \to 0$ give us second desired inequality (5.37) if $N \geq 2$.

Now it is necessary to obtain the estimates (5.37), (5.38) for $N = 1$. Starting from representation $\Psi_1 = \delta^2\Phi^{(2)} - \Psi_2$, taking into account the third equality (5.22) and using the triangle inequality and the inequality (5.38) for $N = 2$ we conclude that $|\Psi_1| \leq const\delta^2\sqrt{y},\ 0 \leq y \leq y_0$. The analogous reasoning proves the correctness of the estimate (5.37) for the case $N = 1$; here we take into account the fact that $|W^{(2)}| \leq const\sqrt{y}[\ln(1 + x/y) + y(y + x)^{-1}]$ when $(x, y) \in \bar{\Pi}_0$. The proof of Proposition 2 is complete.

Proposition 2 means that differences U_N, Ψ_N between the exact and approximate solutions of the problem (2.15) – (2.19) have the same smoothness with respect to variables x and y as the main terms of the asymptotic expansions (2.20). It follows from (2.7) that $W^{(1)}(x, y) \in C^{\beta/2}(\bar{\Pi}_0)$ where β is an arbitrary number from the interval $(0, 1)$, but in the case of $\beta = 1$ this inclusion loses its validity. According to (2.22) $\Phi^{(0)} \in C^{1/2}(\bar{I}_0)$ and here the Hőlder exponent can't be substituted by any larger value. These concepts show that it is impossible to wait for smallness of more strong norms of functions U_N, Ψ_N (when $\delta \to 0$) considered in the whole rectangle $\bar{\Pi}_0$ or on the whole interval $\bar{I}_0$ respectively. However we can obtain the corresponding estimates in subdomains of $\bar{\Pi}_0$ and $\bar{I}_0$. With this aim we must deviate from the degeneration line of the equations(2.15), (2.18) $y = 0$ on the distance with the order δ^n, $n > 0$. Let us show that for any $\delta \in (0, \delta_0]$ and $N = 1, 2\ldots$ the following inequalities are valid:

$$\left|\frac{d\Psi_N}{dy}\right| \leq C_{65}\delta^N, \ \gamma\delta^2 \leq y \leq y_0, \tag{5.49}$$

$$\int_0^a U^2_{N,x}(x, y)dx \leq C_{66}\delta^{2(N+1)}, \ \gamma\delta^2 \leq y \leq y_0, \tag{5.50}$$

where $\gamma > 0$ is constant independent of δ.

In order to prove the inequality (5.49) we must start from representation

$$\frac{d\Psi_N}{dy} = -\frac{1}{4}\sqrt{y}\frac{d\Delta_N}{dy} + \frac{\Delta_N}{8\sqrt{y}}, \tag{5.51}$$

following from (5.17) and definition of functions $\Psi_N = \Phi - \Phi_N$, $\Delta_N = \Gamma - \Gamma_N$. In view of (5.38) the second term in the right part of (5.51) already has the necessary order when $y \in [\gamma\delta^2, y_0]$. For estimating of the first term we use the inequality

$$\left|\frac{d\Delta_{N+k}}{dy}\right| \leq \frac{C_{67}\Delta_{N+k}}{\delta y^{3/2}} + \frac{C_{67}\delta^{N+k}}{y}, \ \gamma\delta^2 \leq y \leq y_0 \tag{5.52}$$

where N and k are arbitrary natural numbers. This inequality follows from the relations (5.18)–(5.20) and the estimates (5.47), (5.48). Let us choose $k = 4$ and consider the obvious representation

$$\frac{d\Psi_N}{dy} = \frac{d\Psi_{N+4}}{dy} + \sum_{i=N+1}^{N+4} \delta^i \frac{d\Phi^{(i)}}{dy}.$$

Using the triangle inequality, the relation (5.51) where N is replaced by $N+4$, the estimate (5.52) and taking into account the fact that functions $d\Phi^{(i)}/dy$ are bounded on the interval $[\gamma\delta^2, y_0]$ for any $i = 2, 3\dots$ we obtain the first of required estimates (5.49).

In order to obtain the inequality (5.50) we choose some natural number $M > N$ and consider the initial boundary value problem (5.22), (5.11), (5.12) for functions $U_M(x, y)$ (number N must be replaced by M in all mentioned relations). In view of the estimates (5.15), (5.24) and (5.48) the free term of the equation (5.22) for U_N satisfies the inequality

$$|c_M \Delta_M - Z_M| \leq C_{69}\delta^{M+1}, \ 0 \leq x \leq a, \ \gamma\delta^2 \leq y \leq y_0. \tag{5.53}$$

The estimate

$$|U_{M,x}(x, y_0)| \leq C_{70}\delta^{M+1}, \ 0 \leq x \leq a. \tag{5.54}$$

is valid in the "initial moment" $y = y_0$ in accordance with (5.12) and the statement of Proposition 1.

Here we use the identity

$$\delta\Phi \int\limits_0^a \{[(x+y)U_M]_y\}^2 dx = -\frac{d}{dy}\int\limits_0^a (x+y)^3 U_{M,x}^2 dx +$$

$$+\int\limits_0^a (x+y)^2 U_{M,x}^2 dx + \int\limits_0^a (c_M\Delta_M - Z_M)[(x+y)U_M]_y dx.$$

Solution U_M of the equation (5.22) with boundary conditions (5.11) satisfies this identity. With the aim of estimating of the last term of this identity we use the Cauchy–Bunyakovsky inequality and the inequality (5.53), integrate the obtained inequality from the current value y to y_0 and then use the inequality (5.54) and throw off wittingly positive terms. As the final result we come to the inequality

$$\int\limits_0^a (x+y)^3 U_{M,x}^2(x, y) dx \leq C_{71}\delta^{2M+1} \tag{5.55}$$

where $\delta \in (0, \delta_0]$ uniformly with respect to y, $\gamma\delta^2 \leq y \leq y_0$. Then we use the representation

$$U_N = U_M + \sum_{k=N+1}^{M} \delta^k W^{(k)}.$$

Using the triangle inequality for estimating of the norm $||U_{N,x}(\cdot, y)||_{L^2(0,a)}$, taking into account the smoothness of function $W^{(k)}(x, y)$ (independent of δ) in the domain $\bar{\Pi}_{\gamma\delta^2}$, using the inequality (5.55) and choosing $M = N+4$ we obtain the estimate (5.50).

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