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Translating solutions to the
second boundary value problem for
curvature flows

by

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TRANSLATING SOLUTIONS TO THE SECOND BOUNDARY VALUE PROBLEM FOR CURVATURE FLOWS

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ABSTRACT. We consider the flow of strictly convex hypersurfaces driven by curvature functions subject to the second boundary condition and show that they converge to translating solutions. We also discuss translating solutions for Hessian equations.

1. INTRODUCTION

We consider the parabolic initial value problem describing the evolution of a hypersurface in $\mathbb{R}^{n+1}$

$$\begin{cases} \dot{X} = -(\log F - \log f)\nu, \\ \nu(M) = \nu(M_0), \\ M|_{t=0} = M_0, \end{cases} \tag{1.1}$$

where X is the embedding vector of a smooth strictly convex hypersurface with boundary, $M = \operatorname{graph} u|_\Omega$, $u : \overline{\Omega} \to \mathbb{R}$, $\dot{X}$ is its total time derivative and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth strictly convex domain. The strictly convex hypersurface M evolves such that its velocity in direction of the upwards pointing unit normal vector ν is determined by a given smooth positive function $f : \overline{\Omega} \to \mathbb{R}$ and a curvature function F of the class $\left(\tilde{K}^\star\right)$ defined below. We remark that this class of curvature functions contains especially the Gauß curvature. The curvature function F is evaluated at the vector $(\kappa_i(X))$ the components of which are the principal curvatures of M at $X \in M$, f is evaluated at X where the $(n+1)$-th component of X is ignored. The image of the normal of M, $\nu(M)$, coincides with the image of the normal of the smooth strictly convex hypersurface $M_0 = \operatorname{graph} u_0|_\Omega$ we start with. We will assume that the closure of $\nu(M_0)$ is a geodesically strictly convex subset of the unit sphere S^n contained in $S^n \cap \{x^{n+1} > 0\}$.

From the definition of the unit normal ν of M it follows that prescribing $\nu(M) = \nu(M_0)$ is equivalent to prescribing $Du(\Omega) = Du_0(\Omega) =: \Omega^*$, where Ω^* is a strictly convex subset of $\mathbb{R}^n$. Thus we consider a flow equation subject to a second boundary value condition.

To formulate our main theorem we introduce – referring also to the defini-
tions in Section 2.2 –

$$v = \sqrt{1 + |Du|^2} \quad \text{and} \quad \delta_0 := \sup_{\kappa \in \Gamma_+} \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \kappa_i} \cdot \kappa_i$$

where Γ_+ is the positive cone in $\mathbb{R}^n$. If F is a positive homogeneous function,
then δ_0 equals the degree of F. We assume the following inequality

$$v \cdot (\log F - \log f) \leq \delta_0 \quad \text{for } t = 0, \tag{1.2}$$

i. e. the initial velocity is not too large. This restriction (1.2) for the initial
velocity is unattractive. At the moment, however, we need this assumption
to prove uniform a priori estimates for the principal curvatures of M during
the evolution.

Under the assumptions stated above, we obtain the following main theorem.

Theorem 1.1. *The initial value problem (1.1) admits a convex solution*
$M(t) = \text{graph } u(t)|_{\Omega}$ *that exists for all times $t \geq 0$ and converges smoothly*
to a translating solution $M^{\infty} = \text{graph } u^{\infty}$ of the flow equation

$$\begin{cases} \dot{X} = -(\log F - \log f)\nu, \\ \nu(M) = \nu(M_0), \end{cases} \tag{1.3}$$

i. e. there exists $v^{\infty} \in \mathbb{R}$ such that $u^{\infty}(x,t) = u^{\infty}(x,0) + v^{\infty} \cdot t$. Up to
additive constants, the translating solution is independent of the choice of
M_0, *but depends on $\nu(M_0)$, F, f and Ω. The function u is smooth for $t > 0$*
and u, Du, D^2u, $\dot{u}$ are continuous up to $t = 0$.

We will also consider flow equations for Hessian equations in Section B.
There we do not need a condition like (1.2) for the initial velocity.

We mention some similar papers. In [1] the authors study translating solu-
tions for the mean curvature flow whereas flow equations are considered in
[3, 5, 7] to prove existence for elliptic problems. Flows with boundary con-
ditions are studied in [8, 10, 14]. Elliptic Hessian equations with Neumann
and oblique boundary conditions are solved in [13, 18], the second boundary
value problem is considered in [17, 19] for Hessian and in [16] for curvature
equations.

Some techniques used in [14, 16] are useful for our proof of the a priori
estimates, although we found a different notation more appropriate for our
C^2-a priori estimates. For the proof of the convergence to a stationary
solution we can adapt a proof of [1]. We do not explicitly cite these papers
each time we use them in the following.

Our paper is organized as follows. In Section 2 we describe our differential-
geometric notations, introduce a class of curvature functions, rewrite our

evolution equation in non-parametric form, define the Legendre transformation, state some properties of the curvature functions introduced, and describe the effects of compatibility conditions to solution for short time intervals. We show that our boundary condition is strictly oblique and prove lower order estimates in Section 3. In Section 4 we derive geometric evolution equations needed for the C^2-a priori estimates proved in Section 5. Then, we prove our main theorem in Section 6 and conclude with some remarks on convergence to hypersurfaces of prescribed curvature in Section A and on translating solutions for Hessian equations in Section B.

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2. Preliminaries

2.1. **Geometric notations.** The notation used is very similar to [7]. Having fixed coordinate systems in $\mathbb{R}^{n+1}$ and M, we use Greek indices running from 0 to n to denote components of geometric quantities defined in $\mathbb{R}^{n+1}$ and Latin indices starting at 1 for quantities related to the hypersurface M. Lower and upper indices refer to covariant and contravariant transformation properties, respectively. We use the Einstein summation convention. Covariant derivatives are indicated by (additional) indices, sometimes preceded by a semicolon for greater clarity, whereas a comma indicates partial derivatives, so we have

$$X_{ij}^\alpha = X_{,ij}^\alpha - \Gamma_{ij}^k X_k^\alpha,$$

where $\left(\Gamma_{ij}^k\right)$ denotes the Christoffel symbols of M. These derivatives of the embedding vector M are related to the second fundamental form (h_{ij}) and to the upwards pointing unit normal (ν^α) of M by the Gauß formula

$$X_{ij}^\alpha = -h_{ij}\nu^\alpha.$$

Using $M = \operatorname{graph} u$, partial derivatives and lifting the indices with respect to the Kronecker-delta, we see that ν is given by

$$(\nu^\alpha) = \frac{1}{v}\left(1, -u^i\right), \quad v = \sqrt{1 + u^i u_i}.$$

Covariant derivatives of ν can be expressed by the Weingarten equation

$$\nu_i^\alpha = h_i^k X_k^\alpha,$$

where we lifted the index of the second fundamental form with respect to $\left(g^{ij}\right)$, the inverse of the induced metric (g_{ij}) of M,

$$g_{ij} = \delta_{ij} + u_i u_j, \quad g^{ij} = \delta^{ij} - \frac{u^i u^j}{v^2}.$$

If not stated otherwise we will lift and lower indices with respect to the induced metric when we use covariant derivatives and with respect to the

Kronecker-delta if we use partial derivatives. The Codazzi equation – together with the symmetry of the second fundamental form – states that $h_{ij;\,k}$ is unchanged under permutations of the indices. The Gauß equation gives the Riemannian curvature tensor (R_{ijkl}) of M

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

used in the Ricci identity which we mention only for the second fundamental form

$$h_{ik;lj} = h_{ik;jl} + h_k^a R_{ailj} + h_i^a R_{aklj}.$$

From the 0-th component of the Gauß formula we obtain

$$\frac{1}{v}h_{ij} = -u_{ij},$$

so M is strictly convex if $(-u_{ij})$ is positive definite. Calculating $u_{ij} = u_{,ij} - \Gamma_{ij}^k u_k = \frac{1}{v^2}u_{,ij}$, we see that the convexity of M is equivalent to the concavity of u. Of course, the function u is called concave if $u(\cdot, t)$ is concave for all t.

In what follows we rewrite our evolution equation as follows

$$\dot{X} = -(\log F - \log f)\nu \equiv -\left(\hat{F} - \hat{f}\right)\nu.$$

Sometimes it will be convenient to work with indices that indicate partial derivatives. We will point out this in the respective sections. In contrast to the lifting of indices as mentioned above, (u^{ij}) denotes the inverse of (u_{ij}). We also wish to introduce the abbreviation $u_\nu = u_i \nu^i$ for a vector ν. The letter c is used to denote constants. These constants are positive estimated quantities that may change its value from line to line. Inequalities remain valid if a constant c on the "right-hand" side is enlarged.

2.2. Curvature functions.

We introduce some classes of curvature functions similar to [5, 15]. A slightly different class of curvature functions is considered in [16]. Our choice of the class of curvature functions used in our main theorem is not the most general choice possible. Instead we preferred a choice that corresponds to the examples of curvature functions we know for which such a theorem holds.

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^\infty(\Gamma_+) \cap C^0(\overline{\Gamma}_+)$ a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0;$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices $Sym^+(n)$, for, let $(h_{ij}) \in Sym^+(n)$ with eigenvalues κ_i, $1 \le i \le n$, then define F on $Sym^+(n)$ by

$$F(h_{ij}) = F(\kappa_i).$$

We have $F \in C^\infty(Sym^+) \cap C^0\left(\overline{Sym^+}\right)$. If we define

$$F^{ij} = \frac{\partial F}{\partial h_{ij}},$$

then we get in an appropriate coordinate system

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}\left|\xi^i\right|^2 \quad \forall\, \xi \in \mathbb{R}^n,$$

and F^{ij} is diagonal, if h_{ij} is diagonal. We define furthermore

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}.$$

Definition 2.1. A curvature function F is said to be of the class (K), if

$$F \in C^\infty(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right), \tag{2.1}$$

$$F \text{ is symmetric,} \tag{2.2}$$

$$F \text{ is positive homogeneous of degree } d_0 > 0,$$

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+, \tag{2.3}$$

$$F|_{\partial\Gamma_+} = 0, \tag{2.4}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} \leq F^{-1}\left(F^{ij}\eta_{ij}\right)^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym,$$

where $\left(\tilde{h}^{ij}\right)$ denotes the inverse of (h_{ij}), or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \leq -\hat{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym,$$

where F is evaluated at (h_{ij}).

If F satisfies

$$\exists\, \varepsilon_0 > 0: \quad \varepsilon_0 FH \equiv \varepsilon_0 F \operatorname{tr} h_i^j \leq F^{ij}h_{ik}h_j^k$$

for any $(h_{ij}) \in Sym^+$, where the index is lifted by means of the Kronecker-Delta, then we indicate this by using an additional star, $F \in (K^\star)$.

The class of curvature functions F which fulfill, instead of the homogeneity condition, the following weaker assumption

$$\exists\, \delta_0 > 0: \quad 0 < \frac{1}{\delta_0}F \leq \sum_i F_i\kappa_i \leq \delta_0 F \tag{2.5}$$

is denoted by an additional tilde, $F \in \left(\tilde{K}\right)$ or $F \in \left(\tilde{K}^\star\right)$.

A curvature function F which satisfies for any $\varepsilon > 0$

$$F(\varepsilon, \ldots, \varepsilon, R) \to +\infty, \quad \text{as } R \to +\infty,$$

or equivalently

$$F(1, \dots, 1, R) \to +\infty, \quad \text{as } R \to +\infty,$$

in the homogeneous case, a condition similar to an assumption in [2], is said to be of the class (CNS).

Example 2.2. We mention examples of curvature functions of the class $\left(\tilde{K}^\star \right)$ as given in [5, 15].

Let H_k be the k-th elementary symmetric polynomials,

$$H_k(\kappa_i) \quad := \sum_{1 \le i_1 < \dots < i_k \le n} \kappa_{i_1} \cdot \dots \cdot \kappa_{i_k}, \quad 1 \le k \le n, \quad (2.6)$$

$$\sigma_k \quad := \quad (H_k)^{\frac{1}{k}}$$

the respective curvature functions homogeneous of degree 1 and define furthermore

$$\tilde{\sigma}_k(\kappa_i) := \frac{1}{\sigma_k \left(\kappa_i^{-1} \right)} \equiv (S_{n,n-k})^{\frac{1}{k}}.$$

The functions $S_{n,k}$ belong to the class (K) for $1 \le k \le n-1$ and H_n belongs to the class $(K^\star)$.

Furthermore, see [5],

$$F := H_n^{a_0} \cdot \prod_{i=1}^{N} F_{(i)}^{a_i}, \quad a_i > 0, \quad (2.7)$$

belongs to the class $\left(\tilde{K}^\star \right)$ provided $F_{(i)} \in (\tilde{K})$, and we may even allow $F_{(i)} \ne 0$ on $\partial \Gamma_+$.

An additional construction gives inhomogeneous examples [15]. Let F be as in (2.7), $\eta \in C^\infty (\mathbb{R}_{\ge 0})$ and $c_\eta > 0$ such that

$$0 < \frac{1}{c_\eta} \le \eta \le c_\eta, \quad \eta' \le 0,$$

then

$$\tilde{F}(\kappa_i) := F \left(\exp \left(\int_1^{\kappa_i} \frac{\eta(\tau)}{\tau} \, d\tau \right) \right)$$

belongs to the class $\left(\tilde{K}^\star \right)$.

The considerations above remain applicable if we evaluate F in what follows at the eigenvalues (κ_i) of the second fundamental form (h_{ij}) with respect to the metric (g_{ij}), i. e. κ is an eigenvalue if there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ such that

$$\kappa \cdot g_{ij} \xi^j = h_{ij} \xi^j.$$

Then

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}$$

is a symmetric covariant tensor of second order.

2.3. Non-parametric flow equation.

Our boundary condition guarantees that we can represent our solution as graph u. Now we will derive a parabolic evolution equation for u equivalent to

$$\dot{X} = -\left(\hat{F} - \hat{f}\right)\nu.$$

Therefore we choose local coordinates (x^i) of $\mathbb{R}^n$ and obtain

$$\frac{d}{dt}X^0 = \frac{d}{dt}u\left(X^i(x,t),t\right) = \frac{\partial u}{\partial t} + u_i\dot{X}^i = -\left(\hat{F} - \hat{f}\right)\frac{1}{v},$$

$$\frac{d}{dt}X^i = \left(\hat{F} - \hat{f}\right)\frac{u^i}{v}, \quad 1 \le i \le n,$$

where we used the definition of ν and (1.1). Combining these equations yields

$$\frac{d}{dt}u(x,t) = \frac{\partial u}{\partial t} = -v\left(\hat{F} - \hat{f}\right).$$

2.4. Legendre transformation.

In this section we use indices to denote partial derivatives and ignore our convention that upper and lower indices correspond to contravariant and covariant quantities, respectively.

The Legendre-transformation of $u : \Omega \times [0,T) \to \mathbb{R}$, $u^* : \Omega^* \times [0,T) \to \mathbb{R}$, is defined by

$$u^*(y,t) := x^i u_i(x,t) - u(x,t) \equiv x^i y_i - u, \quad y^i = u^i(x,t).$$

We look for an evolution equation for u^*. From the definition of u^* we get

$$\dot{u}^* = -\dot{u}, \quad \frac{\partial u^*}{\partial y^k} = x_k, \quad \frac{\partial^2 u^*}{\partial y^k \partial y^l} = \left((D^2 u)^{-1}\right)_{kl} \equiv u^{kl},$$

where y is considered as a time independent variable. We use $\left(\sqrt{g}_i^j\right)$ and $\left(\sqrt{g^{-1}}_i^j\right)$ to denote the square roots of (g_{ij}) and (g^{ij}), respectively, which are positive definite symmetric matrices such that $\sqrt{g}_i^j \sqrt{g}_j^k \delta_{kl} = g_{il}$ and $\sqrt{g^{-1}}_j^i \sqrt{g^{-1}}_k^j \delta^{kl} = g^{il}$, explicitly

$$\sqrt{g}_i^j = \delta_i^j + \frac{u_i u^j}{1+v}, \quad \sqrt{g^{-1}}_i^j = \delta_i^j - \frac{u_i u^j}{v(1+v)}.$$

Then, following [9], the principal curvatures κ_i, $1 \le i \le k$, are the eigenvalues of the matrix (a_{ij}), where

$$a_{ij} = -\sqrt{g^{-1}}_i^k \frac{u_{kl}}{v}\sqrt{g^{-1}}_j^l.$$

We may consider $\sqrt{g}$ as a function of y and set

$$a_{ij}^* = -v\sqrt{g}_k^i u^{kl} \sqrt{g}_l^j = -\sqrt{1+|y|^2}\sqrt{g}_i^k(y)u_{kl}^*\sqrt{g}_j^l(y).$$

Then the eigenvalues of a_{ij}^* are given by $\frac{1}{\kappa_i}$, $1 \le i \le n$. We set for $\kappa \in \Gamma_+$

$$F^*(\kappa_i) := \frac{1}{F\left(\frac{1}{\kappa_i}\right)}$$

and

$$f^* = \frac{1}{f}.$$

Thus we obtain the following evolution equation for u^*

$$\begin{cases} \dot{u}^* = -\sqrt{1+|y|^2}\left(\log F^*(a_{ij}^*) - \log f^*(Du^*)\right) & \text{in } \Omega^*, \\ Du^*(\Omega^*) = \Omega, \end{cases}$$

where F^* is evaluated at the eigenvalues of a_{ij}^*. For later use we differentiate this flow equation using the index k for derivatives with respect to y^k

$$\begin{aligned} \dot{u}_k^* = & -\frac{y_k}{\sqrt{1+|y|^2}}\left(\hat{F}^*(a_{ij}^*) - \hat{f}^*(Du^*)\right) & (2.8) \\ & -\sqrt{1+|y|^2}\left(\hat{F}_{u_{ij}^*}^* u_{ijk}^* + \hat{F}_{y^k}^* - \hat{f}_{q_i}^* u_{ik}^*\right). \end{aligned}$$

We compute $\hat{F}_{u_{ij}^*}^*$ and $\hat{F}_{y^k}^*$ explicitly,

$$\hat{F}_{u_{ij}^*}^* = -\hat{F}_{a_{kl}^*}^* \sqrt{1+|y|^2}\sqrt{g}_k^i \sqrt{g}_l^j$$

and

$$\hat{F}_{y^k}^* = -\hat{F}_{a_{ij}^*}^* \left(\left(\sqrt{1+|y|^2}\sqrt{g}_i^a\right)_k u_{ab}^*\sqrt{g}_j^b + \sqrt{1+|y|^2}\sqrt{g}_i^a u_{ab}^*\sqrt{g}_{j,k}^b\right).$$

2.5. **Properties of curvature functions.** Important properties of the class $\left(\tilde{K}^\star\right)$ for the a priori estimates of the second derivatives of u at the boundary are stated in the following lemmata.

Lemma 2.3. *Let $F \in \left(\tilde{K}^\star\right)$, then for fixed $\varepsilon > 0$*

$$F(\varepsilon, \dots, \varepsilon, R) \to \infty \quad \text{as } R \to \infty,$$

i. e. $\left(\tilde{K}^\star\right) \subset \left(\tilde{K}\right) \cap (CNS)$, moreover, when $F \in \left(\tilde{K}\right) \cap (CNS)$, $0 < \frac{1}{c} \le F \le c$, and

$$0 < \lambda_1 \le \dots \le \lambda_n,$$

then the following three conditions are equivalent

$$\lambda_1 \to 0, \quad \lambda_n \to \infty, \quad \text{and} \quad \operatorname{tr} F^{ij} \to \infty.$$

Proof. We refer to [15].   □

For the dual functions we have a similar lemma.

Lemma 2.4. *Let* $F \in \left(\tilde{K}^{\star} \right)$,

$$0 < \lambda_1 \leq \ldots \leq \lambda_n,$$

and $0 < \frac{1}{c} \leq F \leq c$. *Then the following three conditions are equivalent*

$$\lambda_1 \to 0, \quad \lambda_n \to \infty, \quad and \quad \mathrm{tr}\, F^{*ij} \to \infty.$$

Proof. We have $F_1 \geq \ldots \geq F_n > 0$, see [6, 17], so we get in view of the definition of F^*

$$F_i^*(\lambda_1, \ldots, \lambda_n) \geq \frac{F_i \left(\frac{1}{\lambda_i} \right)}{(F^*)^2} \cdot \frac{1}{\lambda_i^2}.$$

Thus $F_1^* \to \infty$ as $\lambda_1 \to 0$ gives the result as $(F^*)^{-1} = F$, and $\lambda_n \to \infty$ forces $\lambda_1 \to 0$ in view of Lemma 2.3. To get $\mathrm{tr}\, F^{*ij} \to \infty$, λ has to leave any compact subset of Γ_+. $\qquad\square$

Lemma 2.5. *Let* $F \in \left(\tilde{K} \right) \cap (CNS)$. *Then* F^* *as defined above satisfies* (2.1), (2.2), (2.3), (2.4) *and* $F^* \in (CNS)$. *For* $F = (S_{n,k})^{\frac{1}{n-k}}$, $1 \leq k \leq n-1$, *and obviously, see Lemma 2.3, also for* $F \in \left(\tilde{K} \right) \cap (CNS)$ *we have for any* $\varepsilon > 0$

$$\sum_i F_i \lambda_i^2 \leq (c(\varepsilon) + \varepsilon \cdot |\lambda|) \cdot \sum_i F_i. \tag{2.9}$$

Proof. See [17]. $\qquad\square$

2.6. Shorttime existence and compatibility conditions.
In this section we use partial derivatives. In the introduction, we have rewritten our boundary condition $\nu(M) = \nu(M_0)$ equivalently as $Du(\Omega) = \Omega^*$. Now, we take a smooth strictly concave function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h = 0$ and $|\nabla h| = 1$ on $\partial\Omega^*$. In what follows we use h_{p_k} instead of h_k as h will be evaluated by using the gradient of a function. For smooth strictly convex functions u, our boundary condition is equivalent to $h(Du) = 0$ on $\partial\Omega$.

We will derive compatibility conditions fulfilled by a smooth solution $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ and show then how compatibility conditions affect the regularity of u at $t = 0$. We take a solution u, smooth up to $t = 0$, and compute time derivatives of our boundary condition,

$$\left(\frac{d}{dt} \right)^m h(Du) \bigg|_{t=0} = 0 \quad \text{on } \partial\Omega, \quad m \in \mathbb{N}.$$

For fixed m, we call this equation the compatibility condition of order m. For $m = 0$ we get back our boundary condition. In the case $m \geq 1$ we can substitute time derivatives of u, Du, $\ldots$, inductively by using $\dot{u} = -v \left(\hat{F} - \hat{f} \right)$

and derivatives of this equation. Thus we can express compatibility conditions of any order so that they contain only spatial derivatives of u at $t = 0$. These necessary conditions for smoothness of a solution of (1.1) at $t = 0$ are also sufficient for smoothness, more precisely, let $M_0 = \operatorname{graph} u_0$ satisfy the compatibility conditions of m-th order for $0 \le m \le m_0$. Later-on we will prove that our boundary condition is strictly oblique, so we deduce from Theorem 5.3, p. 320 [11], and the implicit function theorem, see also [4], that there exists a solution of our initial value problem (1.1) on a maximal time interval $[0, T)$, $T > 0$. This solution is smooth for $t \in (0, T)$ and has continuous derivatives up to $2(m_0 + 1)$-th order at $t = 0$, where time derivatives have to be counted twice.

As usual, longtime existence follows from shorttime existence and uniform a priori estimates as follows. Assume that we have - as long as a solution exists - estimates for the function u of the form

$$\|u\|_{C^k\left(\overline{\Omega} \times \left(\frac{t}{2}, t\right)\right)} \le c_k \cdot (1 + t), \quad k \in \mathbb{N},$$

where it would be sufficient to have a locally bounded function – defined for $t > 0$ – of t on the right-hand side. We assume that $[0, T)$ is the maximal time interval where our solution exists. Then the a priori estimates guarantee that we can extend our solution to $[0, T]$. As we have a smooth solution, it satisfies the compatibility conditions of any order at $t = T$, so applying our considerations above, we get a solution on a time interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$, which is smooth for $t \in (0, T + \varepsilon)$ contradicting the maximality of T.

3. Obliqueness and lower order estimates

In this section we use indices to denote partial derivatives and ν is the inner unit normal vector to $\partial\Omega$.

3.1. Strict obliqueness.

Lemma 3.1. *As long as a solution of (1.1) exists, our boundary condition is strictly oblique, i. e.*

$$\langle \nu(x), \nu^*(Du(x, t)) \rangle > 0, \quad x \in \partial\Omega, \tag{3.1}$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively.

Proof. To prove (3.1) we use

$$\nu^i(x) \cdot \nu_i^*(Du(x, t)) = \nu^i \cdot h_{p_i}(Du(x, t)).$$

As $h(Du)$ is positive in Ω and vanishes on $\partial\Omega$, we get on $\partial\Omega$ for τ orthogonal to ν

$$h_{p_k} u_{k\tau} = 0, \quad h_{p_k} u_{k\nu} \ge 0. \tag{3.2}$$

Thus we see from

$$h_{p_k}\nu^k = h_{p_k}u_{ki}u^{ij}\nu_j = h_{p_k}u_{k\nu}\cdot u^{\nu\nu} \geq 0 \qquad (3.3)$$

that the quantity whose positivity we wish to show is at least nonnegative.

We compute in view of (3.2) and (3.3) on $\partial\Omega$

$$
\begin{aligned}
\left(h_{p_k}\nu^k\right)^2 &= u^{\nu\nu}h_{p_k}u_{k\nu}u^{\nu\nu}u_{\nu l}h_{p_l} \\
&= u^{\nu\nu}h_{p_k}u_{ki}u^{ij}u_{jl}h_{p_l} \\
&= u^{\nu\nu}u_{kl}h_{p_k}h_{p_l},
\end{aligned}
$$

so we deduce the positivity of the quantity considered. $\qquad\square$

3.2. $\dot{u}$- and C^0-estimates. We define the function

$$r := (\dot{u})^2$$

and consider $F(\kappa_i)$ as a function of (Du, D^2u), $F = F(Du, D^2u)$. Calculations similar to those in Section 2.4 show that $\left(F_{u_{ij}}\right)$ is negative definite. We get

$$\ddot{u} = -v\left(\hat{F}_{u_{ij}}\dot{u}_{ij} + \hat{F}_{p_i}\dot{u}_i\right) - \frac{u^i\dot{u}_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

These preparations allow to deduce the following parabolic evolution equation for r

$$\dot{r} + v\hat{F}_{u_{ij}}r_{ij} = 2v\hat{F}_{u_{ij}}\dot{u}_i\dot{u}_j - v\hat{F}_{p_i}r_i - \frac{u^i r_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

Lemma 3.2. *As long as a smooth solution of (1.1) exists, we obtain the estimate*

$$\min\left\{\min_{t=0}\dot{u}, 0\right\} \leq \dot{u} \leq \max\left\{\max_{t=0}\dot{u}, 0\right\}.$$

Proof. If $(\dot{u})^2$ admits a local maximum at $x \in \partial\Omega$ for some positive time, we differentiate our boundary condition and get there

$$h_{p_k}\dot{u}_k = 0.$$

As h_{p_k} is strictly oblique, this contradicts the Hopf maximum principle unless $\dot{u}$ is constant. On the other hand, the evolution equation for r implies

$$\dot{r} + v\hat{F}_{u_{ij}}r_{ij} \leq -v\hat{F}_{p_i}r_i - \frac{u^i r_i}{\sqrt{1+|Du|^2}}\left(\hat{F}-\hat{f}\right).$$

This excludes an increasing local maximum of $(\dot{u})^2$ in $\Omega \times (0,T)$ and we get the claimed inequality. $\qquad\square$

Corollary 3.3. *As long as a smooth solution of (1.1) exists, we get*

$$\|u(\cdot, t)\|_{C^0(\overline{\Omega})} \leq \|u(\cdot, 0)\|_{C^0(\overline{\Omega})} + t\cdot\|\dot{u}(\cdot, 0)\|_{C^0(\overline{\Omega})}$$

and $\hat{F}$ is uniformly a priori bounded.

Remark 3.4. Applying the strong maximum principle to $\dot{u}$ we deduce that either $v \cdot \left(\hat{F} - \hat{f} \right) < \delta_0$ uniformly provided $t > \varepsilon$ for fixed $\varepsilon > 0$ or $v \cdot \left(\hat{F} - \hat{f} \right) = \delta_0$ everywhere independent of t, because (1.2) tells us that $v \cdot \left(\hat{F} - \hat{f} \right) \leq \delta_0$ for $t = 0$. Thus we may assume in view of the shorttime existence – by restricting to $t > \varepsilon$ – that $v \cdot \left(\hat{F} - \hat{f} \right) < \delta_0$ uniformly during the flow as the trivial case $v \cdot \left(\hat{F} - \hat{f} \right) \equiv \delta_0$ corresponds to a translating solution and needs no further considerations. This difference between $v \cdot \left(\hat{F} - \hat{f} \right)$ and δ_0 will be used in the proof of the C^2-a priori estimates, we set

$$0 < \delta_\varepsilon := \inf_{t > \varepsilon} \left\{ \delta_0 - \left(\hat{F} - \hat{f} \right) \right\}. \tag{3.4}$$

3.3. **Strict obliqueness estimates.** The purpose of this section is to quantify the strict obliqueness of our boundary condition.

Lemma 3.5. *For a smooth solution of (1.1) we have the strict obliqueness estimate*

$$\langle \nu(x), \nu^*(Du(x,t)) \rangle \geq \frac{1}{c} > 0, \quad x \in \partial\Omega,$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively. The positive lower bound is independent of time.

Proof. We fix a time interval $(0, T]$, where a smooth solution of our flow (1.1) exists and prove that there exists a positive lower bound for $h_{p_k}\nu^k$ for $(x, t) \in \partial\Omega \times [0, T]$ which is independent of T. To establish this positive lower bound, we choose $(x_0, t_0) \in \partial\Omega \times [0, T]$ such that $h_{p_k}\nu^k$ is minimal there. As we have a positive lower bound for $h_{p_k}\nu^k$ on $\partial\Omega \times \{0\}$, we may assume that $t_0 > 0$. Further on, we may assume that $\nu(x_0) = e_n$ and extend ν smoothly to a tubular neighborhood of $\partial\Omega$ such that in the matrix sense

$$D_k\nu^l \equiv \nu_k^l \leq -\frac{1}{c_1}\delta_k^l \tag{3.5}$$

there for a positive constant c_1. For a positive constant A to be chosen below we define

$$w = h_{p_k}\nu^k + Ah(Du).$$

The function $w|_{\partial\Omega \times (0,T]}$ attains its minimum over $\partial\Omega \times (0, T]$ in (x_0, t_0), so we deduce there

$$0 = w_r = h_{p_np_k}u_{kr} + h_{p_k}\nu_r^k + Ah_{p_k}u_{kr}, \quad 1 \leq r \leq n - 1, \tag{3.6}$$

$$0 \geq \dot{w}. \tag{3.7}$$

We assume for a moment that there holds

$$w_n(x_0, t_0) \geq -c(A), \tag{3.8}$$

show that this estimate yields a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$ and prove (3.8) afterwards. Then the lemma follows from the calculations in the proof of Lemma 3.1 and from a positive lower bound for $u^{\nu\nu}$.

We rewrite (3.8) as

$$h_{p_n p_l}u_{ln} + h_{p_k}\nu_n^k + Ah_{p_k}u_{kn} \geq -c(A).$$

Multiplying this with h_{p_n} and adding (3.6) multiplied with h_{p_r} we obtain at (x_0, t_0)

$$Au_{kl}h_{p_k}h_{p_l} \geq -c(A)h_{p_n} - h_{p_k}\nu_l^k h_{p_l} - h_{p_k}h_{p_n p_l}u_{lk}.$$

Using (3.2), the concavity of h and (3.5), we get there

$$Au_{kl}h_{p_k}h_{p_l} \geq -c(A)h_{p_n} + \frac{1}{c_1}$$

as $|\nabla h| = 1$ on $\partial\Omega^*$. We may assume that the right-hand side of the inequality above is positive as otherwise $h_{p_n} = h_{p_k}\nu^k$ is bounded from below. Thus we deduce a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$.

We now sketch the proof of (3.8). As for a similar proof with more details we refer to [14]. We differentiate our flow equation $\dot{u} = -v\left(\hat{F} - \hat{f}\right)$ and obtain

$$\dot{u}_k = -v\left(\hat{F}_{u_{ij}}u_{ijk} + \hat{F}_{p_i}u_{ik} - \hat{f}_k\right) - \left(\hat{F} - \hat{f}\right)v_{p_i}u_{ik}. \tag{3.9}$$

This is a motivation to introduce the following linear parabolic differential operator L by

$$L\tilde{w} := -\dot{\tilde{w}} - v\hat{F}_{u_{ij}}\tilde{w}_{ij} - v\hat{F}_{p_i}\tilde{w}_i - \left(\hat{F} - \hat{f}\right)v_{p_i}\tilde{w}_i.$$

We remark that the chain rule and (2.5) show that $\hat{F}_{p_i}$ is bounded, the chain rule and (2.3) give a positive lower bound for $-\operatorname{tr}\hat{F}_{u_{ij}} \equiv -\hat{F}_{u_{ij}}\delta_{ij}$. Direct calculations give for A sufficiently large

$$
\begin{aligned}
Lw \quad &\leq \quad -v\hat{F}_{u_{ij}}u_{li}u_{mj}\nu^k h_{p_k p_l p_m} - Av\hat{F}_{u_{ij}}h_{p_k p_l}u_{ki}u_{lj} - c(A)\cdot\operatorname{tr}\hat{F}_{u_{ij}} \\
&\leq \quad -c(A)\cdot\operatorname{tr}\hat{F}_{u_{ij}}.
\end{aligned}
$$

As Ω is strictly convex, there exist $\mu \gg 1$ and $\varepsilon > 0$ such that for $\vartheta := d - \mu d^2$, where $d = \operatorname{dist}(\cdot, \partial\Omega)$, we have near $\partial\Omega$ in view of Lemma 2.3

$$L\vartheta \leq +\varepsilon\cdot\operatorname{tr}\hat{F}_{u_{ij}}. \tag{3.10}$$

The proof of this inequality is omitted here as it is carried out in [14] and in Lemma 5.4 in similar situations. We consider ϑ only in $\Omega_\delta := \Omega \cap B_\delta(x_0)$, where $\delta > 0$ is chosen so small that ϑ is smooth and nonnegative there and the above inequality holds. As w is bounded and attains its minimum over $\partial\Omega \times [0, T]$ in (x_0, t_0) we find $C \gg B \gg 1$ such that the function

$$\Theta := C\cdot\vartheta + B\cdot|x - x_0|^2 + w - w(x_0, t_0)$$

satisfies

$$\begin{cases} \Theta & \geq & 0 \quad \text{on } (\partial\Omega_\delta \times [0,T]) \cup (\Omega_\delta \times \{0\}), \\ L\Theta & \leq & 0 \quad \text{in } \Omega_\delta \times [0,T]. \end{cases}$$

Thus the maximum principle gives

$$(C \cdot \vartheta + w)_n (x_0, t_0) \geq 0$$

as the function $C \cdot \vartheta + B \cdot |x - x_0|^2 + w - w(x_0, t_0)$ vanishes in (x_0, t_0). This shows Inequality (3.8).

Similar to the argument above we extend ν^* smoothly to a tubular neighborhood of $\partial\Omega^*$ such that $\nu_i^{*k} \leq -\frac{1}{c}\delta_i^k$ in the matrix sense and take h^* as a smooth strictly concave function such that $\{h^* = 0\} = \partial\Omega$ and $|Dh^*| = 1$ on $\partial\Omega$. We define

$$w^* = h_{q_k}^*(Du^*)\nu^{*k} + Ah^*(Du^*)$$

and in view of (2.8) we define furthermore

$$L^*\tilde{w} := -\dot{\tilde{w}} - v\hat{F}_{u_{ij}^*}^* \tilde{w}_{ij} - v\hat{f}_{q_i}^* \tilde{w}_i.$$

As before we obtain that $w^*|_{\partial\Omega \times [0,T]}$ is positive. We fix $T > 0$ and assume that $w^*|_{\partial\Omega \times [0,T]}$ attains its minimum in (x_0, t_0). As we wish to establish a positive lower bound for w^* we may assume that $t_0 > 0$. Direct calculations and Lemma 2.4, which implies a positive lower bound for $-\text{tr}\,\hat{F}_{u_{ij}^*}^*$, give for A sufficiently large

$$L^*w^* \leq c_k(1 + A)\left|\hat{F}_{y^k}^*\right| - \frac{1}{2}Av\hat{F}_{u_{ij}^*}^* h_{q_k q_l}^* u_{ki}^* u_{lj}^* - c(A) \cdot \text{tr}\,\hat{F}_{u_{ij}^*}^*. \qquad (3.11)$$

Then

$$\hat{F}_{a_{ij}^*}^* = -\frac{1}{v}\hat{F}_{u_{rs}^*}^* \sqrt{g^{-1}}_i^r \sqrt{g^{-1}}_j^s$$

and Young's inequality imply for any $\varepsilon > 0$

$$\left|\hat{F}_{y^k}^*\right| \leq -\varepsilon\hat{F}_{u_{rs}^*}^* u_{ri}^* u_{sj}^* \delta^{ij} - \frac{c}{\varepsilon}\text{tr}\,\hat{F}_{u_{rs}^*}^*.$$

Combining this with (3.11) gives

$$L^*w^* \leq c(A) \cdot \left|\text{tr}\,\hat{F}_{u_{ij}^*}^*\right|.$$

Now we can proceed as above, use Lemma 2.4 and get at (x_0, t_0) an inequality of the form

$$Au_{kl}^* h_{q_k}^* h_{q_l}^* \geq -c(A)h_{q_k}^* \nu^{*k} - \nu_k^{*l} h_{q_k}^* h_{q_l}^*. \qquad (3.12)$$

Since $h_{q_k}^* \nu^{*k} = \langle \nu^*, \nu \rangle$, we may assume again that this quantity is small. The second term on the right-hand side is bounded below by a positive constant in view of the convexity of Ω^* and $|Dh^*| = 1$ on $\partial\Omega^*$, so we deduce $u_{kl}^* h_{q_k}^* h_{q_l}^* \geq \frac{1}{c} > 0$. Using $u_{kl}^* = u^{kl}$ and $h_{q_k}^* = \nu^k$ we obtain a positive lower bound for $u^{\nu\nu}$ completing the strict obliqueness estimate. $\qquad \square$

4. Geometric evolution equations

In this section we describe how geometric quantities evolve during the flow. Proofs for these results in a similar notation can be found in [7]. We start with the evolution equation for the metric

$$\dot{g}_{ij} = -2\left(\hat{F} - \hat{f}\right)h_{ij}$$

and for the unit normal of the hypersurface M

$$\dot{\nu}^\alpha = g^{ij}\left(\hat{F} - \hat{f}\right)_i X_j^\alpha.$$

For the second fundamental form of M we state the evolution equation both for the mixed and for the covariant form

$$
\begin{aligned}
\dot{h}_i^j &= \left(\hat{F} - \hat{f}\right)_i{}^j + \left(\hat{F} - \hat{f}\right)h_i^k h_k^j, \\
\dot{h}_{ij} &= \left(\hat{F} - \hat{f}\right)_{ij} - \left(\hat{F} - \hat{f}\right)h_i^k h_{kj}.
\end{aligned}
$$

Applying the chain rule to the term $\hat{F} - \hat{f}$ there and interchanging covariant derivatives by means of the Codazzi equations and Ricci identities gives

$$
\begin{aligned}
\dot{h}_i^j - \hat{F}^{kl}h_{i;kl}^j &= \hat{F}^{kl}h_{kr}h_l^r h_i^j - \hat{F}^{kl}h_{kl}h_i^r h_r^j + \left(\hat{F} - \hat{f}\right)h_i^k h_k^j \qquad (4.1) \\
&\quad + \hat{F}^{kl,rs}h_{kl;i}h_{rs;}{}^j - \hat{f}_{\alpha\beta}X_i^\alpha X_k^\beta g^{kj} + \hat{f}_\alpha \nu^\alpha h_i^j.
\end{aligned}
$$

For the scalar product $\tilde{v}$ of ν and $e_{n+1} = e_0$, i. e. for $\tilde{v} \equiv \langle \nu, \eta \rangle = \nu^\alpha \eta_\alpha$, where $(\eta_\alpha) = (1, 0, \dots, 0)$, we get the evolution equation

$$\dot{\tilde{v}} - \hat{F}^{ij}\tilde{v}_{ij} = \hat{F}^{ij}h_i^k h_{kj}\tilde{v} - \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij}$$

and for $v = \tilde{v}^{-1}$ we get thus

$$\dot{v} - \hat{F}^{ij}v_{ij} = -v\hat{F}^{ij}h_i^k h_{kj} + v^2 \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij} - 2\frac{1}{v}\hat{F}^{ij}v_i v_j.$$

For these two evolution equations we assumed Euclidean coordinates in $\mathbb{R}^{n+1}$, so derivatives of η_α vanish. In the following we will always assume that we have chosen Euclidean coordinates in $\mathbb{R}^{n+1}$. As a direct consequence of the evolution equations obtained so far we get for the mean curvature, $H \equiv h_{ij}g^{ij} = h_i^i$,

$$
\begin{aligned}
\dot{H} - \hat{F}^{kl}H_{kl} &= \hat{F}^{kl}h_{kr}h_l^r H - \hat{F}^{kl}h_{kl}h_i^r h_r^i + \left(\hat{F} - \hat{f}\right)h_i^k h_k^i \\
&\quad + \hat{F}^{kl,rs}h_{kl;i}h_{rs;j}g^{ij} - \hat{f}_{\alpha\beta}X_i^\alpha X_j^\beta g^{ij} + \hat{f}_\alpha \nu^\alpha H.
\end{aligned}
$$

The right-hand side of (4.1) is a tensor with a covariant index i and a contravariant index j. Thus we can multiply this equation with vector fields and the result at a fixed point depends only on the value of these vector fields there but especially not on any derivatives. We deduce that the same is true for both terms on the left-hand side.

Taking any non-vanishing vector field $\tilde{\xi} : \mathbb{R}^n \to \mathbb{R}^n$ we can project $\tilde{\xi}(x)$ to the tangential hyperplane to M at $X = (u(x,t), x)$ and normalize it such that the result $\xi(x)$ satisfies $g_{ij} \xi^i \xi^j = 1$. We set $\xi_i = g_{ij} \xi^j$. In view of the above considerations we get directly an evolution equation for $h_i^j \xi_j \xi^i$. For simplicity we set $h_1^1 := h_i^j \xi_j \xi^i$ and consider h_1^1 as a scalar function. Thus we get for $W := h_1^1 \cdot v$ the following evolution equation

$$
\begin{aligned}
\dot{W} - \hat{F}^{ij} W_{ij} + 2\frac{1}{v}\hat{F}^{ij} v_j W_i \; = \; & v\left(\left(\hat{F} - \hat{f} \right) - \hat{F}^{ij} h_{ij} \right) h_1^k h_k^1 \qquad\qquad (4.2) \\
& + h_1^1 \left(v^2 \hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij} + v \hat{f}_\alpha \nu^\alpha \right) \\
& + v\hat{F}^{kl,rs} h_{kl;1} h_{rs;}{}^1 - v\hat{f}_{\alpha\beta} X_1^\alpha X_k^\beta g^{k1}.
\end{aligned}
$$

For a similar combination, $W := \log(H \cdot v)$, we get

$$
\begin{aligned}
\dot{W} - \hat{F}^{ij} W_{ij} - \hat{F}^{ij} W_i \left(W_j - 2\frac{v_j}{v} \right) = \qquad\qquad\qquad (4.3) \\
= \; \frac{1}{H}\left(\left(\hat{F} - \hat{f} \right) - \hat{F}^{ij} h_{ij} \right) h_k^l h_l^k + \frac{1}{H}\hat{F}^{kl,rs} h_{kl;i} h_{rs;j} g^{ij} \\
- \frac{1}{H}\hat{f}_{\alpha\beta} X_i^\alpha X_j^\beta g^{ij} + \hat{f}_\alpha \nu^\alpha + v\hat{f}_\beta X_i^\beta X_j^\alpha \eta_\alpha g^{ij}.
\end{aligned}
$$

5. C^2-estimates

Making T slightly smaller we may assume the existence of a solution to our flow equation (1.1) on the compact time interval $[0, T]$. This is no restriction as the a priori estimates obtained will not depend on T. We remark, however, that our estimate depends on the constant chosen in (3.4), see also the beginning of the proof of Lemma 5.2.

Lemma 5.1. *For a solution of our flow equation (1.1), we have the following bounds for partial derivatives of u on $\partial\Omega$,*

$$ u_{\tau\beta} = 0 \quad and \quad |u_{\beta\beta}| \leq (c(\varepsilon) + \varepsilon \cdot M) \quad for\ any\ \varepsilon > 0, $$

where τ denotes a vector tangential to $\partial\Omega$, β^k is an abbreviation for h_{p_k}, and

$$ M := \sup_{\overline{\Omega} \times [0,T]} |D^2 u|. $$

Proof. We use indices to denote partial derivatives and differentiate the boundary condition $h(Du) = 0$ on $\partial\Omega$ tangentially to obtain

$$ u_{\tau\beta} = 0 \quad \text{on } \partial\Omega. $$

To prove our second assertion, we apply the linear operator L defined by

$$ L\tilde{w} := -\dot{\tilde{w}} - v\hat{F}_{u_{ij}} \tilde{w}_{ij} - v\hat{F}_{p_i} \tilde{w}_i - \left(\hat{F} - \hat{f} \right) v_{p_i} \tilde{w}_i $$

to $w := h(Du)$ and obtain using (3.9), (2.5) or Lemma 2.5

$$|Lw| \leq (c(\varepsilon) + \varepsilon \cdot M) \cdot \left| \operatorname{tr} \hat{F}_{u_{ij}} \right|$$

for any $\varepsilon > 0$. Applying the barrier used near Equation (3.10) we obtain the claimed estimate for $u_{\beta\beta}$. □

Lemma 5.2 (Interior estimates). *For a solution of our flow equation (1.1), we can estimates the second derivatives of u in $\Omega \times [0, T]$ compared to those at the parabolic boundary, more precisely*

$$
\begin{aligned}
\sup_{\Omega \times [0,T]} |D^2 u| &\leq c \cdot \left(1 + \sup_{(\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})} |D^2 u| \right) \\
&\leq c \cdot \left(1 + \sup_{\partial\Omega \times [0,T]} |D^2 u| \right).
\end{aligned}
$$

Proof. In view of the shorttime existence we fix $\varepsilon > 0$ sufficiently small such that $|D^2 u|(\cdot, \tau)$, $0 \leq \tau \leq \varepsilon$, is a priori bounded by a constant depending only on u_0 and use (3.4), i. e. we exclude especially the trivial flow by translation. Using $u(\cdot, \cdot + \varepsilon)$ instead of $u(\cdot, \cdot)$ we may assume again that our flow is defined on a time interval of the form $[0, T]$ and that (3.4) holds.

We consider the quantity $W := \log(H \cdot v)$ and take $(x_0, t_0) \in \overline{\Omega} \times [0, T]$ such that $W(x_0, t_0) \geq W(x, t)$ for $(x, t) \in \overline{\Omega} \times [0, T]$. As our claim is obvious if (x_0, t_0) belongs to the parabolic boundary, we may assume that $(x_0, t_0) \in \Omega \times (0, T]$. Further on we may assume $H \geq 1$ there. Equation (4.3) implies

$$0 \leq -\delta_\varepsilon \frac{1}{H} h_k^l h_l^k + c \quad \text{at } (x_0, t_0).$$

Thus H is bounded there. This yields the assertion of our lemma. □

It remains to bound the second derivatives of u on $\partial\Omega \times (0, T]$. The next lemma reduces this estimate to an estimate for tangential directions.

Lemma 5.3. *For a solution of our flow equation (1.1) we have*

$$\sup_{\overline{\Omega} \times [0,T]} |D^2 u| \leq c \cdot \left(1 + \sup_\tau u_{\tau\tau} \right),$$

where τ runs through all directions, i. e. vectors with $|\tau| = 1$, tangential to $\partial\Omega$.

Proof. We consider a fixed point in $\partial\Omega \times [0, T]$. Let ξ be any direction in $\mathbb{R}^n$. This direction can be decomposed as

$$\xi = a\tau + b\beta,$$

where τ is a tangential direction to $\partial\Omega$ at the fixed point and $\beta^k = h_{p_k}(Du)$ there. The constants a, $b \in \mathbb{R}$ are uniformly a priori bounded due to our strict obliqueness estimates. Using Lemma 5.1 we get

$$u_{\xi\xi} = a^2 u_{\tau\tau} + 2ab u_{\tau\beta} + b^2 u_{\beta\beta} \le c \cdot \left(\sup_\tau u_{\tau\tau} + c(\varepsilon) + \varepsilon \cdot M \right)$$

with M as in the cited lemma. Fixing $\varepsilon > 0$ sufficiently small and using Lemma 5.2 twice gives the claimed estimate. $\qquad\square$

In the next lemma we bound the second tangential derivatives of u on $\partial\Omega \times [0, T]$.

Lemma 5.4. *For a solution of our flow equation (1.1), the second derivatives of u are a priori bounded,*

$$\sup_{\overline{\Omega} \times [0,T]} |D^2 u| \le c.$$

Proof. Here we use covariant derivatives. We proceed as in the proof of Lemma (5.2) and may assume that (3.4) holds. We may furthermore assume that

$$(x, t, \xi) \mapsto v \cdot \frac{h_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j}(x, t),$$

where $(x, t) \in \partial\Omega \times [0, T]$ and ξ runs through vectors tangentially to $M(t)$ and $\partial\Omega \times \mathbb{R}$, attains its maximum in (x_0, t_0, ξ_0) with $t_0 > 0$. Here we identified (x, t) and $(\{x\} \times \mathbb{R}) \cap M(t)$. Constructing a vector field ξ as in Section 4 near x_0 we get

$$v \cdot h_1^1(x_0, t_0) \ge v \cdot h_1^1(x, t), \quad (x, t) \in \partial\Omega \times [0, T].$$

We may assume that this inequality holds also for $(x, t) \in \overline{\Omega} \times \{0\}$ as otherwise the estimate claimed in this lemma is obvious. Setting $W := v \cdot h_1^1$ we obtain, see (4.2),

$$\dot{W} - \hat{F}^{ij}W_{ij} + 2\frac{1}{v}\hat{F}^{ij}v_j W_i \le c,$$

where we used (3.4). We remark that (2.5) yields that $\left| 2\frac{1}{v}\hat{F}^{ij}v_j \right|$ is a priori bounded. We may assume that $W(x_0, t_0) \ge 1$ and set $C_\delta := B_\delta(x_0) \times \mathbb{R}$. The a priori estimates obtained so far imply for

$$Lw := \dot{w} - \hat{F}^{ij}w_{ij} + 2\frac{1}{v}\hat{F}^{ij}v_j w_i$$

that for small $\delta > 0$ and

$$\tilde{W} := \frac{W}{W(x_0, t_0)} - 1$$

we get

$$
\begin{cases}
L\tilde{W} & \leq & c & \text{on } M \cap C_\delta, \\
\tilde{W} & \leq & c & \text{on } M \cap C_\delta, \\
\tilde{W} & \leq & 0 & \text{on } \partial M \cap C_\delta, \\
\tilde{W} & \leq & 0 & \text{on } M(0), \\
\tilde{W} & = & 0 & \text{at } (x_0, t_0).
\end{cases}
$$

We start to construct a barrier which will be used to obtain the claimed estimate. The main part of this barrier function consists of

$$
\vartheta := -d + \mu d^2,
$$

where $d = \mathrm{dist}\,(\cdot, \partial\Omega \times \mathbb{R})$ is the Euclidean distance to the cylinder over $\partial\Omega$ and $\mu \gg 1$ will be fixed later-on. Direct computations yield on $M \cap C_\delta$

$$
\begin{aligned}
L\vartheta \quad \leq \quad & \hat{F}^{ij} d_{\alpha\beta} X_i^\alpha X_j^\beta - 2\mu \hat{F}^{ij} d_\alpha X_i^\alpha d_\beta X_j^\beta \\
& + c \cdot (1 + \mu \cdot \delta) + c \cdot \mu \cdot \delta \cdot \mathrm{tr}\,\hat{F}^{ij}, \quad \mathrm{tr}\,\hat{F}^{ij} \equiv \hat{F}^{ij} g_{ij}.
\end{aligned}
$$

We use that in an Euclidean coordinate system $d_{\alpha\beta}$ is equal to the respective partial derivatives and the strict convexity of $\partial\Omega$ to get for a positive constant $\varepsilon > 0$ that depends only on the principal curvatures of $\partial\Omega$

$$
\begin{aligned}
L\vartheta \quad \leq \quad & -2\varepsilon \cdot \mathrm{tr}\,\hat{F}^{ij} - \mu \hat{F}^{ij} d_\alpha X_i^\alpha d_\beta X_j^\beta \\
& + c \cdot (1 + \mu \cdot \delta) + c \cdot \mu \cdot \delta \cdot \mathrm{tr}\,\hat{F}^{ij}.
\end{aligned}
$$

By virtue of Lemma 2.3 we can fix μ sufficiently large and then δ sufficiently small to control the third term on the right-hand side. Fixing $\delta > 0$ even smaller if necessary, we can absorb the fourth term and get

$$
L\vartheta \leq -\varepsilon \cdot \mathrm{tr}\,\hat{F}^{ij}.
$$

Further on we may assume that δ is so small that

$$
\vartheta \leq 0 \quad \text{on } \partial C_\delta.
$$

As a barrier function we choose

$$
\Theta := A\vartheta - B \cdot |x - x_0|^2 + \tilde{W},
$$

where $|x - x_0|$ denotes the Euclidean distance for points in Ω and is evaluated on M by projecting $\Omega \times \mathbb{R}$ orthogonally to Ω. We fix $B \gg 1$ to obtain an appropriate behavior on the boundary and then A sufficiently large to obtain an appropriate sign in the differential inequality, more precisely

$$
\begin{cases}
L\Theta & \leq & 0 & \text{on } M \cap C_\delta, \\
\Theta & \leq & 0 & \text{on } \partial(M \cap C_\delta), \\
\Theta & \leq & 0 & \text{on } M(0), \\
\Theta & = & 0 & \text{at } (x_0, t_0).
\end{cases}
$$

Thus the maximum principle implies that $\Theta \leq 0$ in $M \cap C_\delta$. We consider Θ as being defined on $\overline{\Omega} \times [0, T]$, use partial derivatives and get

$$
\Theta_\beta \equiv h_{p_k} \Theta_k \leq 0 \quad \text{at } (x_0, t_0)
$$

and thus by direct computations and Lemma 5.1

$$u_{11\beta} \leq c \cdot (c(\varepsilon) + \varepsilon \cdot M) \cdot u_{11} \quad \text{at } (x_0, t_0). \tag{5.1}$$

We differentiate the boundary condition twice and get at (x_0, t_0)

$$h_{p_k p_l} u_{k1} u_{l1} + h_{p_k} u_{k11} + h_{p_k} u_{kn} \omega_{11} = 0,$$

where ω is a function such that locally $\partial\Omega = \text{graph } \omega|_{\mathbb{R}^{n-1}}$ and $D\omega = 0$ at the point corresponding to x_0. The index n corresponds to a direction orthogonal to $\partial\Omega$. Combining this with the Inequality (5.1) and with the Lemmata 5.1 and 5.3 we get at (x_0, t_0)

$$-h_{p_k p_l} u_{k1} u_{l1} \leq c \cdot \left(c(\varepsilon) + \varepsilon \cdot (u_{11})^2 \right).$$

As h is strictly concave we can estimate the left-hand side from below by $\inf_k (-h_{p_k p_k}) \cdot (u_{11})^2 > 0$, thus fixing $\varepsilon > 0$ sufficiently small bounds u_{11} and the claimed estimate follows. $\qquad\square$

6. Proof of the main theorem

6.1. **Longtime existence.** Here and in the following we may restrict our considerations to time intervals starting at $\varepsilon > 0$ instead of 0. Thus we may ignore questions concerning compatibility conditions and smoothness at $t = 0$. We get uniform C^2-estimates for the partial derivatives of u and a positive lower bound for F and conclude that the flow operator is uniformly parabolic and concave. So we can apply the results of chapter 14 in [12] to obtain uniform $C^{2,\alpha}$-estimates for u, with a small positive constant α. Then standard Schauder estimates [11, 12] imply uniform bounds in C^k for all $k \geq 1$. It follows from the considerations concerning shorttime existence that a smooth solution of (1.1) exists for all $t \geq 0$.

6.2. **Convergence to a translating solution.** We finish the proof of our Main Theorem 1.1 by showing that our solution that exists for all positive times converges to a translation solution. In this section we use partial derivatives.

A similar proof can be found in [1], where the existence of a translating solution is established differently. In our situation, however, the existence of a translating solution is in general not obvious.

We fix $t_0 > 0$ and establish a boundary value problem fulfilled by

$$w(x, t) := u(x, t) - u(x, t + t_0).$$

By the mean value theorem we find a positive definite matrix (a^{ij}) and a vector field (b^i) – both depending on x and t – such that

$$\dot{w} = a^{ij} w_{ij} + b^i w_i \quad \text{in } \overline{\Omega} \times (0, \infty).$$

The boundary value condition for w is derived as follows. For any function $h : \mathbb{R}^n \to \mathbb{R}$ with $\{h = 0\} = \partial\Omega^*$ and for any smooth strictly convex function $u : \overline{\Omega} \to \mathbb{R}$, the boundary condition $Du(\Omega) = \Omega^*$ is equivalent to $h(Du) = 0$ on $\partial\Omega$. We have proved that $h_{p_k}(Du)\nu^k$ is uniformly bounded from below by a positive constant on $\partial\Omega$, if $|\nabla h| = 1$ on $\partial\Omega^*$ and ∇h points inside Ω^* there. Here we use $h = \min\{\operatorname{dist}(\cdot, \partial\Omega^*), \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. We could also mollify h slightly near $\{\operatorname{dist}(\cdot, \partial\Omega^*) = \varepsilon\}$ to obtain a smooth function h. We get

$$
\begin{aligned}
0 &= h(Du(x,t)) - h(Du(x, t+t_0)) \\
&= \int_0^1 h_{p_k}(\tau Du(x,t) + (1-\tau)Du(x, t+t_0))\, d\tau \cdot w_k \equiv \beta^k w_k,
\end{aligned}
$$

so for $\varepsilon > 0$ sufficiently small, β^k is almost equal to

$$
\sigma \cdot h_{p_k}(Du(x,t)) + (1-\sigma) \cdot h_{p_k}(Du(x, t+t_0)),
$$

for some $\sigma \in [0,1]$. Since we have uniform obliqueness estimates during the evolution, it is possible to fix $\varepsilon > 0$ sufficiently small, depending only on the obliqueness estimates and on the domain Ω^*, such that β as defined above is a uniformly strictly oblique vector field, i. e. $\beta^k \nu_k \geq \frac{1}{c} > 0$.

The strong maximum principle implies that

$$
\operatorname{osc}(t) := \max_{x \in \overline{\Omega}} w(x,t) - \min_{x \in \overline{\Omega}} w(x,t)
$$

is a strictly decreasing function or w is constant. Next, we will exclude the case when $\operatorname{osc}(t)$ is strictly decreasing but tends to a positive constant $\varepsilon > 0$ as $t \to \infty$. For any sequence $t_n \to \infty$ we find – in view of our a priori estimates – a subsequence (again denoted by) t_n such that for $x_0 \in \overline{\Omega}$ fixed

$$
u(x, t+t_n) - u(x_0, t_n) \text{ and } u(x, t+t_0+t_n) - u(x_0, t_0 + t_n),
$$

$(x, t) \in \overline{\Omega} \times [-t_n, \infty)$, converge locally uniformly in any C^k-norm and their limits u^∞ and $u^{t_0, \infty}$ satisfy our flow equation in $\overline{\Omega} \times \mathbb{R}$. We define $\tilde{w} := u^\infty - u^{t_0, \infty}$ and observe – as the oscillation is monotone –, that the oscillation of $\tilde{w}$ is $\varepsilon > 0$. This, however, is impossible, as the strong maximum principle shows that a positive oscillation is strictly decreasing. So we obtain that

$$
u(x,t) - u(x, t+t_0) \to -v^\infty \cdot t_0 \quad \text{as } t \to \infty, \tag{6.1}
$$

uniformly in $x \in \overline{\Omega}$. As we will see later-on, the constant v^∞ has been introduced such that it equals the velocity of any translating solution. For an arbitrary sequence $t_n \to \infty$, we consider

$$
u(x, t+t_n) - u(x_0, t_n), \quad (x,t) \in \overline{\Omega} \times [-t_n, \infty).
$$

In view of our a priori estimates we may extract a not relabeled subsequence $t_n \to \infty$ such that

$$
u(x, t+t_n) - u(x_0, t_n) \to u^0(x,t) \tag{6.2}
$$

locally uniformly in $\overline{\Omega} \times \mathbb{R}$ in any C^k-norm as $n \to \infty$. Equations (6.1) and (6.2) give

$$u^0(x, t + t_0) = u^0(x, t) + v^\infty \cdot t_0 \text{ for } (x, t) \in \overline{\Omega} \times \mathbb{R}.$$

The function u^0 is again a solution to our flow equation. We repeat the argument given above with (u^0, t_1), $t_1 > 0$, instead of (u, t_0) and obtain a solution u^1 of our flow equation satisfying

$$u^1(x, t + t_i) = u^1(x, t) + v^\infty \cdot t_i \quad \text{for } (x, t) \in \overline{\Omega} \times \mathbb{R}, \quad i \in \{0, 1\},$$

where it is easy to see that v^∞ is the same constant as above. We get a smooth function $u^\infty : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, that satisfies our flow equation and

$$u^\infty(x, t + \tau) = u^\infty(x, \tau) + c_\infty \cdot \tau \text{ for } (x, t) \in \overline{\Omega} \times \mathbb{R}$$

for any $\tau > 0$. To see this we can either take t_0 and t_1 as incommensurable positive numbers or we iterate the above argument for appropriate $t_k > 0$, $k \in \mathbb{N}$, and consider a diagonal sequence. Thus we have established the existence of a translating solution to our flow equation. We remark that this solution is – up to additive constants – the only translating solution of our flow equation. This follows from the strong maximum principle applied to the difference of two translating solutions similar as at the beginning of this section.

Finally, we show that u converges to a translating solution. As above we get a linear parabolic differential equation for $W := u - u^\infty$,

$$\begin{cases} \dot{W} = a^{ij} W_{ij} + b^i W_i & \text{in } \overline{\Omega} \times (0, \infty), \\ 0 = \beta^k W_k & \text{on } \partial\Omega \times [0, \infty), \end{cases}$$

with a strictly oblique vector field β. Then we get that the oscillation of W tends to zero, thus $u - u^\infty$ tends to a constant c_∞ as $t \to \infty$. We can use interpolation inequalities of the form

$$\|Dw\|_{C^0}^2 \le c(\Omega) \cdot \|w\|_{C^0} \cdot \left(\|D^2 w\|_{C^0} + \|Dw\|_{C^0} \right)$$

for $w = W - c_\infty$ and its derivatives and get smooth convergence of u to a translating solution. This finishes the proof that any solution of our flow equation (1.1) exists for all positive times and tends eventually smoothly to a translating solution.

Appendix A. Prescribed curvature

If we assume in contrast to the assumptions above that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is smooth, positive and satisfies $f_z > 0$ we can prove as in [14] that our flow converges to a hypersurface of prescribed curvature provided that either

$$\frac{f_z}{f} \ge c_f > 0$$

or both the first two compatibility conditions for u_0 and

$$\dot{u} \leq 0 \quad \text{for } t = 0$$

are fulfilled. (The derivative of f with respect to the second argument is denoted by f_z.)

The a priori estimates obtained in the sections above guarantee that we get a solution for all positive times with estimates as above. Convergence to a hypersurface of prescribed curvature follows then similar to [14].

APPENDIX B. HESSIAN FLOWS

B.1. **Second boundary value problems.** We consider the second boundary value problem for non-parametric logarithmic Hessian flows

$$\begin{cases} \dot{u} = \hat{F}(D^2 u) - \log g(x, Du) & \text{in } \Omega \times [0, T), \\ Du(\Omega) = \Omega^*, \end{cases} \tag{B.1}$$

on a maximal time interval $[0, T)$, $T > 0$, $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$. The Hessian function F belongs to the class $\left(\tilde{K}^* \right)$ or equals $S_{n,k}$, $1 \leq k \leq n - 1$. We assume that Ω, $\Omega^* \subset \mathbb{R}^n$, $n \geq 2$, are strictly convex domains, $u_0 : \overline{\Omega} \to \mathbb{R}$ is a strictly convex function, $Du_0(\Omega) = \Omega^*$, and $g : \overline{\Omega} \times \overline{\Omega^*} \to \mathbb{R}$ is a smooth function. As initial condition for u we take

$$u|_{t=0} = u_0.$$

It is known [14] that this initial value problem has a smooth solution $u : \overline{\Omega} \times (0, \infty)$ and u, $\dot{u}$, Du, and $D^2 u$ are continuous up to $t = 0$. For $t \in [\varepsilon, \infty)$, $\varepsilon > 0$, we have uniform bounds for all C^k-norms besides for $|u|$ that may increase as follows

$$\|u(\cdot, t)\|_{C^0} \leq \|u(\cdot, 0)\|_{C^0} + t \cdot \|\dot{u}(\cdot, 0)\|_{C^0}.$$

These estimates are not stated explicitly in [14], but follow immediately from the calculations there. For the longtime behavior of solutions we have the following result.

Theorem B.1. *Under the assumptions stated above, u converges smoothly to a translating solution u^∞ with velocity v^∞, i. e. $u^\infty(x, t) = u^\infty(x, 0) + v^\infty \cdot t$, of (B.1) as $t \to \infty$. The translating solution u^∞ is independent – up to additive constants – of the choice of u_0. If $F(D^2 u) = \det D^2 u$ and $g(x, p) = \frac{g_1(x)}{g_2(p)}$ with smooth positive functions g_1 and g_2, then v^∞ is given by*

$$v^\infty := \log \int\limits_{\Omega^*} g_2(p) \, dp - \log \int\limits_{\Omega} g_1(x) \, dx.$$

Proof. It follows from [14] that a solution to our initial value problem exists for all positive times. Furthermore we get bounds for the C^k-norms as in the proof of our Main Theorem 1.1 and thus longtime existence of solutions. As above we conclude that our solutions converge to translating solutions, that are unique up to additive constants.

It remains to compute the velocity of a translating solution in the special case mentioned above. Let u be a translating solution. We get

$$v^\infty = \log \det D^2 u - \log \frac{g_1(x)}{g_2(Du)}$$

or equivalently

$$g_1(x) \cdot e^{v^\infty} = \det D^2 u \cdot g_2(Du).$$

We integrate this equation over Ω and get

$$e^{v^\infty} \cdot \int_\Omega g_1(x)\, dx = \int_\Omega \det D^2 u \cdot g_2(Du)\, dx = \int_{\Omega^*} g_2(p)\, dp,$$

where we have used the transformation rule. This implies that v^∞ is as claimed. $\qquad\square$

Remark B.2. In the special case of Theorem B.1 when $g(x,p) = \frac{g_1(x)}{g_2(p)}$ and $F(D^2 u) = \det D^2 u$, it is possible to obtain the translating solution directly as in Theorem 2 [19]. The second boundary value problem

$$\begin{cases} \det D^2 u_\varepsilon = e^{\varepsilon u_\varepsilon + v^\infty} \cdot \frac{g_1(x)}{g_2(Du)} & \text{in } \Omega, \\ Du_\varepsilon(\Omega) = \Omega^*, \end{cases}$$

is known [19, 14] to have a solution u_ε for $0 < \varepsilon < 1$. We integrate this equation and use the transformation formula for integrals

$$e^{v^\infty} \cdot \int_\Omega e^{\varepsilon u_\varepsilon} \cdot g_1(x)\, dx = \int_{\Omega^*} g_2(p)\, dp,$$

so we infer from the definition of v^∞

$$\int_\Omega e^{\varepsilon u_\varepsilon} \cdot g_1(x)\, dx = \int_\Omega g_1(x)\, dx,$$

and deduce that u_ε is zero somewhere in Ω. Now, uniform C^k-a priori estimates follow from the proofs in [19, 14]. We let $\varepsilon \to 0$, extract a suitable subsequence of u_ε and obtain a solution u of

$$\begin{cases} v^\infty = \log \det D^2 u - \log g(x) & \text{in } \Omega, \\ Du(\Omega) = \Omega^*. \end{cases} \tag{B.2}$$

Then we define

$$u^\infty(x,t) := u(x) + v^\infty \cdot t, \quad (x,t) \in \overline{\Omega} \times \mathbb{R},$$

and get a solution u^∞ of

$$\begin{cases} \dot{u}^\infty = \log \det D^2 u^\infty - \log g(x) & \text{in } \Omega \times \mathbb{R}, \\ Du^\infty(\Omega) = \Omega^* \end{cases} \tag{B.3}$$

that moves by translation with velocity v^∞.

B.2. Neumann and oblique boundary value problems. In [14] we considered flows for Hessian equations subject to Neumann and oblique boundary conditions, e. g. flow equations of the form

$$\dot{u} = \log \det u_{ij} - \log f(x, u, Du) \quad \text{in } \Omega$$

for boundary conditions of the form

$$u_\nu = \varphi(x, u) \quad \text{or} \quad u_\beta = \varphi(x, u) \quad \text{on } \partial\Omega,$$

where ν is the inner unit normal vector to Ω and β is a vector that is C^1-close to ν. For more details we refer to [14].

If we assume in contrast to the cited paper that both f and φ depend only on x, then both for Neumann and oblique boundary conditions with β C^1-close to ν, it follows immediately from the techniques used here and in [14], that any solution converges smoothly to a translating solution of the respective flow equation.

We remark that it is also true for the remaining flow equations in [14] that solutions converge smoothly to translating solutions if f and φ are as assumed above.

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