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Classification of solutions of a
toda system in R^2

by

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CLASSIFICATION OF SOLUTIONS OF A TODA SYSTEM IN $\mathbb{R}^2$

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ABSTRACT. In this paper, we consider solutions of the following (open) Toda system (Toda lattice) for $SU(N+1)$

$$-\frac{1}{2}\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2,$$

for $i = 1, 2, \cdots, N$, where $K = (a_{ij})_{N \times N}$ is the Cartan matrix for $SU(N+1)$. We show that any solution $u = (u_1, u_2, \cdots, u_N)$ with

$$\int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \cdots, N,$$

can be obtained from a rational curve in $\mathbb{C}P^N$.

1. INTRODUCTION

Let $N > 0$ be an integer. The 2-dimensional (open) Toda system (Toda lattice) for $SU(N+1)$ is the following system

$$(1.1) \qquad -\frac{1}{2}\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2,$$

for $i = 1, 2, \cdots, N$, where $K = (a_{ij})_{N \times N}$ is the Cartan matrix for $SU(N+1)$ given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(Here the factor $\frac{1}{2}$ comes from $\frac{1}{2}\Delta u = u_{z\bar{z}}$.) System (1.1) is a very natural generalization of the Liouville equation

$$(1.2) \qquad -\Delta u = 2e^u,$$

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which is completely integrable, known from Liouville [26]. Roughly speaking, any solution of (1.2) in a simply connected domain arises from a holomorphic function. System (1.1) is also completely integrable. All solutions of (1.1) in a simply connected domain arise from N holomorphic functions, see [21, 23, 24, 14]. However, it is difficult to determine the precise form of these holomorphic functions, when we require additional (or boundary) conditions for (1.2) or (1.1).

Recently, Chen-Li [6] classified all solutions of (1.2) in $\mathbb{R}^2$ with

$$(1.3) \qquad \int_{\mathbb{R}^2} e^u < \infty.$$

Their result is very useful for 2-dimensional problems, especially for the study of the Moser-Trudinger inequality and the mean field equation, see [12, 27]. To obtain their classification, they used an energy inequality of Ding [11] and the method of moving plane to show that any solution has a rotational symmetry. Other proofs of the classification result were given by Chou-Wan [10] by using the complete integrability mentioned above and complex analysis and by Chanillo-Kiessling [5] by using an isoperimetric inequality and a global Pohozaev identity. The latter was applied to classify solutions of a class of Liouville type systems with non-negative coefficients in [5]. See also the work of Chipot-Shafrir-Wolansky [9]. The methods used by Chen-Li and Chanillo-Kiessling rely on the maximum principle. Hence, it is difficult (or impossible) to apply their method to study the similar problem for System (1.1). We believe the method of Chou-Wan can be applied to study (1.1). In fact, we notice that a similar method was used by Bryant [3] in his study of pseudo-metrics. We believe that his method can be adapted to classify (1.1) by using the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$ instead of that for holomorphic functions. For the Nevanlinna theory for holomorphic curves into $\mathbb{C}P^N$, see for example [16].

System (1.1) has a very close relationship with a few geometric objects, holomorphic curves into $\mathbb{C}P^N$, flat $SU(N+1)$ connections and harmonic sequences, see for instance, [4, 22, 18, 17, 1, 15, 8]. To classify solutions of (1.1), it is natural to seek the help of differential geometry. Here, with such a help, we classify system (1.1) with

$$(1.4) \qquad \int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \cdots, N.$$

Theorem 1.1. *Any C^2 solution $u = (u_1, u_2, \cdots, u_N)$ of (1.1) and (1.4) has the following form*

$$(1.5) \qquad u_i(z) = \sum_{j=1}^{N} a_{ij} \log \|\Lambda_j(f)\|^2$$

for some rational curve in $\mathbb{C}P^N$. For the definition of $\Lambda_k(f)$, see section 3 below.

Any rational curve in $\mathbb{C}P^N$ can be transformed to

$$\phi_0(z) = [1, z, \cdots, \sqrt{\binom{N}{k}} z^k, \cdots, z^N],$$

by a holomorphic isometry, which is an element of $PSL(N+1, \mathbb{C})$. Hence the space of solutions of (1.1) and (1.4) is equivalent to $PSL(N+1, \mathbb{C})/PSU(N+1)$. The dimension of the solution space is $N^2 + 2N$.

Theorem 1.1 can be restated in a geometric way as follows:

Theorem 1.2. *Any totally unramified holomorphic map ϕ from $\mathbb{C}$ to $\mathbb{C}P^N$ satisfying the finite energy condition (3.11) below can be compactified to a rational curve.*

Theorem 1.2 is a generalization of the following well-known result: Any totally unramified compact curve in $\mathbb{C}P^N$ is rational.

When $N = 1$, Theorem 1.1 is just the classification result of Chen-Li.

The Toda system is of great interest not only in geometry, but also in mathematical physics. One of our motivations to study this system is the non-abelian Chern-Simons Higgs model, in which non-topological solutions are solutions of a perturbed Toda system. See [14, 19, 33, 30, 28].

2. ANALYTIC ASPECTS OF THE TODA SYSTEM

In this section, we analyze the asymptotic behavior of solutions of (1.1)-(1.4) and obtain a global Rellich-Pohozaev identity. Since some results were presented in our previous work [20], we only give an outline of ideas. Similar methods were used in [6, 7, 5].

Let $I = \{1, 2, \cdots, N\}$. First, we have

Lemma 2.1. *Let u be a solution of (1.1)-(1.4). Then*

$$u_i(z) = -\gamma_i \log |z| + a_i + O(|z|^{-1}) \quad \text{for } |z| \text{ near } \infty,$$

where $a_i \in \mathbb{R}$ are some constants and γ_i are given by

$$\gamma_i = \frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j}.$$

Proof. First, one shows that

$$(2.1) \qquad\qquad \max_{i \in I} \sup_{z \in \mathbb{R}^2} u_i(z) < \infty,$$

see [20]. Set

$$v_i(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\log|x - y| - \log(|y| + 1) \right) \sum_{j=1}^{N} e^{u_j}(y) dy,$$

for $i \in I$. (Note again that (1.1) has a factor $\frac{1}{2}$.) The potential analysis implies

$$(2.2) \qquad\qquad -\gamma_i \log|z| - C \leq v_i(z) \leq -\gamma_i \log|z| + C,$$

for some constant $C > 0$, see [7]. Clearly $u_i - v_i$ is a harmonic function. Hence (2.1) and (2.2) imply that $u_i - v_i = c_i$ for some constant c_i. That is, u has the following representation formula

$$u_i(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\log|x - y| - \log(|y| + 1) \right) \sum_{j=1}^{N} e^{u_j}(y) dy + c_i.$$

The above results and (1.4) imply

$$(2.3) \qquad\qquad \gamma_i > 2, \quad i \in I.$$

Furthermore, we can show that

$$u_i = \gamma_i \log|z| + a_i + O(|z|^{-1}).$$

See, for example, [31]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Next, we have a global Rellich-Pohozaev identity for our system (1.1). Such an identity was obtained in [5] for a Liouville type system with nonnegative entries. Similar arguments work for our case, see [20]. Here we give another proof that is similar to the spirit of proof of the Main Theorems.

Proposition 2.2. *Let u be a solution of (1.1) and (1.4). Then we have*

$$(2.4) \qquad\qquad \sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j \gamma_k) = 0,$$

where the matrix (a^{ij}) is the inverse of the Cartan matrix (a_{ij}).

Proof. Set

$$f = \sum_{j,k=1}^{N} a^{jk} \left\{ (u_k)_{zz} - \frac{1}{2}(u_j)_z \cdot (u_k)_z \right\}.$$

We can check that f is a holomorphic function as follows:

$$\begin{aligned}
f_{\bar{z}} &= \tfrac{1}{2}\sum_{j,k=1}^{N} a^{jk}\{(\Delta u_k)_z - (u_j)_z \cdot \Delta u_k\} \\
&= -\sum_{j,k,l=1}^{N} a^{jk}\{a_{kl}e^{u_l}(u_l)_z - a_{kl}(u_j)_z e^{u_l}\} \\
&= -\sum_{j=1}^{N}(e^{u_j}(u_j)_z - e^{u_j}(u_j)_z) \\
&= 0.
\end{aligned}$$

In the first equality we have used the symmetry of the matrix (a^{ij}). Using Lemma 2.1, we have the following expansion of f near infinity

$$\frac{1}{8}\frac{1}{z^2}\sum_{j,k-1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) + \frac{c_{-3}}{z^3} + \cdots.$$

Hence, f is a constant (zero, in fact) and

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j\gamma_k) = 0.$$

$\square$

3. Geometric aspects of the Toda system

In this section, we recall some relations between the Toda system and various geometric objects, flat connections, holomorphic curves into $\mathbb{C}P^N$ and harmonic sequences. Furthermore, we relate the mild singularities of solutions of the Toda system with the holonomy of the corresponding flat connections.

3.1. From solutions of Toda systems to flat connections. Let Ω be a simply connected domain and $u = (u_1, u_2, \cdots, u_N)$ a solution of (1.1) on Ω. Define $w_0, w_1, w_2, \cdots, w_N$ by the following relations

$$(3.1) \qquad u_i = 2w_i - 2w_0 \quad \text{for } i \in I \text{ and } \sum_{i=0}^{N} w_i = 0.$$

It is easy to check that $w_0, w_1, \cdots, w_N$ satisfies

$$(3.2) \qquad \begin{cases}
-\Delta w_0 = 2(w_0)_{z\bar{z}} &= e^{w_1 - w_0} \\
-\Delta w_1 = 2(w_1)_{z\bar{z}} &= -e^{w_1 - w_0} + e^{w_2 - w_1} \\
\cdots & \cdots \cdots \\
-\Delta w_N = 2(w_N)_{z\bar{z}} &= -e^{w_N - w_{N-1}}.
\end{cases}$$

It is well-known that (3.2) is equivalent to an integrability condition

$$(3.3) \qquad \mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}]$$

of the following two equations

$$(3.4) \qquad \phi^{-1} \cdot \phi_z = \mathcal{U}$$

and

$$(3.5) \qquad \phi^{-1} \cdot \phi_{\bar{z}} = \mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \ddots & \\ & & & (w_N)_z \end{pmatrix} + \begin{pmatrix} 0 & e^{w_1 - w_0} & & \\ & 0 & & \\ & & \ddots & e^{w_N - w_{N-1}} \\ & & & 0 \end{pmatrix}$$

and

$$\mathcal{V} = - \begin{pmatrix} (w_0)_{\bar{z}} & & & \\ & (w_1)_{\bar{z}} & & \\ & & \ddots & \\ & & & (w_N)_{\bar{z}} \end{pmatrix} - \begin{pmatrix} 0 & & & \\ e^{w_1 - w_0} & 0 & & \\ & & \ddots & \\ & & e^{w_N - w_{N-1}} & 0 \end{pmatrix}$$

Hence, from a solution of (1.1) (or equivalently (3.2)) we first get a one-form $\alpha = \mathcal{U}dz + \mathcal{V}d\bar{z}$. Then, with the help of the Frobenius Theorem, we obtain a map $\phi : \Omega \to SU(N+1)$ such that

$$\alpha = \phi^{-1} \cdot d\phi.$$

It is clear that α (or $d+\alpha$) is a flat $SU(N+1)$ connection on the trivial bundle $\Omega \times \mathbb{C}^{N+1} \to \Omega$, i.e., α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

which is equivalent to the integrability condition (3.3), hence (3.2).

Lemma 3.1. *The map ϕ is determined upto an element of $SU(N+1)$. That is, any two $\phi_1, \phi_2 : \Omega \to SU(N+1)$ with $\phi_1^{-1}d\phi_1 = \phi_2^{-1}d\phi_2 = \alpha$ satisfy*

$$\phi_1 = g \cdot \phi_2,$$

for some element $g \in SU(N+1)$.

We call ϕ and α a *Toda map* and *Toda form* respectively.

3.2. **Holonomy of a singular connection.** Now consider an $SU(N+1)$ connection α on the punctured disk D^*. We can define its holonomy as in [29]. When $\Omega = D^*$ is not simply connected, we cannot apply the Frobenius theorem directly and have to consider the *holonomy*. Let (r, θ) be the polar coordinates. Write α as $\alpha = \alpha_r dr + \alpha_\theta d\theta$. α_r and α_θ are $su(N+1)$-valued. For any given $r \in (0, 1)$, the following initial value problem,

$$\frac{d\phi_r}{d\theta} + \alpha_\theta \phi_r = 0, \quad \phi_r(0) = Id,$$

has a unique solution $\phi_r(\theta) \in SU(N+1)$. Here Id is the identity matrix.

Lemma 3.2. *If α is a flat connection on D^*, then $\phi_r(2\pi)$ is independent of r.*

Let h_α denote $\phi_r(2\pi)$. h_α is called the *holonomy* of α.

Remark. Here, we use a slightly different definition of holonomy. The usual holonomy is defined by the conjugacy class of h_α, which is invariant under gauge transformations.

Proposition 3.3. *Let $u = (u_1, u_2, \cdots, u_N)$ be a solution of (2.1) with*
$$u_i(z) = -\mu_i \log|z| + O(1), \quad near\ 0.$$
If $\mu_i < 2$ for $i \in I$, then the corresponding flat connection α has holonomy
$$h_\alpha = \begin{pmatrix} e^{2\pi i \beta_0} & & & \\ & e^{2\pi i \beta_1} & & \\ & & \ddots & \\ & & & e^{2\pi i \beta_N} \end{pmatrix},$$
where $\beta_0, \beta_1, \cdots, \beta_N$ are determined by
$$(3.6) \qquad \beta_i - \beta_0 = \frac{1}{2}\mu_i \quad (i \in I) \quad and \quad \sum_{j=0}^{N} \beta_j = 0.$$

Proof. Define w_i by (3.1). From the assumption, we have
$$w_i = -\beta_i \log|z| + O(1), \quad near\ 0.$$
A direct computation shows that
$$\mathcal{U} = \frac{1}{2z} \begin{pmatrix} -\beta_0 & & & \\ & -\beta_1 & & \\ & & \ddots & \\ & & & -\beta_N \end{pmatrix} + o(\frac{1}{|z|}),$$
where $o(\frac{1}{|z|})$ means that a matrix (b_{ij}) with entries satisfying $|z|b_{ij} \to 0$ as $|z| \to 0$. Here, we have used the condition that $\mu_i < 2$ for any $i \in I$. Similarly,
$$\mathcal{V} = \frac{1}{2\bar{z}} \begin{pmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \ddots & \\ & & & \beta_N \end{pmatrix} + o(\frac{1}{|z|}).$$
Hence,
$$\alpha_\theta = \sqrt{-1} \begin{pmatrix} \beta_0 & & & \\ & \beta_1 & & \\ & & \ddots & \\ & & & \beta_N \end{pmatrix} + o(1).$$
Now it is easy to compute the holonomy of α. $\qquad\qquad \square$

3.3. From solutions of (1.1) to holomorphic curves.

When we have a Toda map $\phi : \Omega \to SU(N+1)$ from a solution of the Toda system, we can get a harmonic sequence as follows. First, define $N+1$ $\mathbb{C}^{N+1}$-valued functions $\hat{f}_0, \hat{f}_1, \cdots \hat{f}_N$ by

$$(\hat{f}_0, \hat{f}_1, \cdots \hat{f}_N) = \phi \cdot \begin{pmatrix} e^{w_0} & & & \\ & e^{w_1} & & \\ & & \ddots & \\ & & & e^{w_N} \end{pmatrix}.$$

Let f_i denote the map into $\mathbb{C}P^N$ obtained from $\hat{f}_i$. It is easy to check that f_i is a harmonic map and satisfies

$$(3.7) \qquad \begin{aligned} (\hat{f}_k)_z &= \hat{f} + a_k \hat{f}_k, \\ (\hat{f}_k)_{\bar{z}} &= b_k \hat{f}_{k-1}, \end{aligned}$$

where

$$a_k = (\log|\hat{f}_k|^2)_z = (e^{2w_k})_z \text{ and } b_{k-1} = -|\hat{f}_k|^2/|\hat{f}_{k-1}|^2 = -w^{2(w_k - w_{k-1})}.$$

Here we assume that $\hat{f}_{-1} = \hat{f}_{N+2} = 0$. Hence, f_0 is a holomorphic map and f_{N+1} is an anti-holomorphic map into $\mathbb{C}P^N$. In fact, (3.7) is the Frenet frame of the holomorphic map f_0, see [18] or below. Furthermore, f_0 is unramified in Ω. For the definition of the ramification index, see [18] or below.

3.4. From a curve to a solution of the Toda system.

From a non-degenerate (i.e. not contained in a proper projective subspace of $\mathbb{C}P^N$) holomorphic curve f_0 into $\mathbb{C}P^N$, one can get a family of associated curves into various Grassmannians as follows. Lift f_0 locally to $\mathbb{C}^{N+1}$ and denote the lift by $v = (v_0, v_1, \cdots, v_N)$. Hence, $f_0 = [v_0, v_1, \cdots, v_N]$. The k-th associated curve of f_0 is defined by

$$\begin{aligned} f_k : \Omega &\to G(k+1, n+1) \subset \mathbb{C}P^{N_k} \\ f_k(z) &= [\Lambda_k], \end{aligned}$$

where

$$\Lambda_k = v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k)}(z).$$

See for example [18]. Here $N_k = \begin{pmatrix} N+1 \\ k+1 \end{pmatrix}.$

Let ω_k be the Fubini-Study metric on $\mathbb{C}P^{N_k}$. The well-known (infinitesimal) Plücker formula is

$$(3.8) \qquad f_k^*(\omega_k) = \frac{\sqrt{-1}}{2} \frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4} dz \wedge \bar{z},$$

which implies

$$(3.9) \qquad \frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k\|^2 = \frac{\|\Lambda_{k-1}\|^2 \cdot \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4}, \qquad \text{for } k = 1, \cdots, N,$$

where $\|\Lambda_0\|^2 = 1$ and $\|\Lambda_N\| = \det(f, f', \cdot, f^{(N)})$. By choosing the normalization $\|\Lambda_N\| = 1$ (we can do this when we lift f), we can identify (3.9) with the Toda system (1.1) as follows. By setting

$$v_k = \log \|\Lambda_k\|^2,$$

system (3.9) becomes

$$(3.10) \qquad -\frac{1}{2}\Delta v_i = \exp\{\sum_{j=1}^{N} a_{ij} v_j\}.$$

Clearly, (3.10) is equivalent to (1.1) by setting

$$u_i = \sum_{j=1}^{N} a_{ij} v_j.$$

For any curve $f : \Omega \to \mathbb{C}P^n$, the *ramification index* $\beta(z_0)$ at $z_0 \in \Omega$ is defined by the unique real number such that

$$f^* \omega = \frac{\sqrt{-1}}{2} |z - z_0|^{2\beta(z_0)} \cdot h(z) \cdot dz \wedge d\bar{z}$$

with h C^∞ and non-zero at z_0, where ω is the Kähler form of the Fubini-Study metric on $\mathbb{C}P^n$. For other definitions, see [18]. f is *unramified* if for any $z_0 \in \Omega$ the ramification index $\beta(z_0)$ vanishes. Hence, a solution of the Toda system (1.1) corresponds to an unramified holomorphic curve.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic curve and f_k its k-th associated curves. The finite energy condition is defined by

$$(3.11) \qquad \int_{\mathbb{R}^2} f^*(\omega_k) < \infty, \qquad \text{for any } k \in I.$$

(3.11) means that the area of the k-th associated curve is bounded.

4. Proof of Main Theorems

Now we start to prove our main Theorems.
Proof of Theorem 1.1. Let

$$v_i(z) = u_i\left(\frac{\bar{z}}{|z|^2}\right) - 4 \log |z|, \quad i \in I.$$

$v = (v_1, v_2, \cdots, v_N)$ satisfies (1.1) on $\mathbb{R}^2 / \{0\}$. Applying Lemma 2.1, we have

$$v_i(z) = (\gamma_i - 4) \log |z| + O(1) \quad \text{near } 0.$$

Hence, using (2.3) Proposition 3.3 implies that the holonomy of the corresponding Toda form of v is

$$h_\alpha = \begin{pmatrix} e^{2\pi i \beta_0} & & & \\ & e^{2\pi i \beta_1} & & \\ & & \cdots & \\ & & & e^{2\pi i \beta_N} \end{pmatrix},$$

where $\beta_0, \beta_1, \cdots, \beta_N$ are determined by

$$\beta_i - \beta_0 = \frac{1}{2}(\gamma_i - 4) \quad \text{and} \quad \sum_{j=0}^{N} \beta_j = 0.$$

Now we know that the holonomy is trivial, i.e., h_α is the identity matrix, which clearly implies

$$\beta_i = 1 \mod \mathbb{Z} \quad \text{for } i = 0, 1, \cdots, N.$$

Hence, we have

$$\gamma_i = 2 \mod \mathbb{Z} \quad \text{for any } i \in I,$$

which, together with (2.3), implies that $\gamma_i \geq 4$ for any $i \in I$. Thus, $4\gamma_k - \gamma_j \gamma_k \leq 0$ for any $j, k \in I$. On the other hand, one can check that the matrix (a^{ij}) admits only positive entries. In fact, a direct computation shows that

$$a^{ij} = \frac{i(N + 2 - j)}{N + 2}, \quad \text{for } i, j \leq \{\frac{N + 2}{2}\},$$

where $\{b\}$ means the least integer larger than or equal to b. Other entries are determined by an obvious symmetry. Altogether, we obtain

$$\sum_{j,k=1}^{N} a^{jk}(4\gamma_k - \gamma_j \gamma_k) \leq 0,$$

and the equality holds if and only if $\gamma_i = 4$ for any $i \in I$. Applying the global Rellich-Pohozaev identity (2.4), we have

$$\gamma_i = 4 \quad \text{for any } i \in I.$$

Hence, v_i is bounded near 0. The elliptic theory implies that v_i is smooth. From the discussions presented in Section 3, it follows that

the corresponding holomorphic curve f can be viewed as an unramified map from $\mathbb{S}^2$ to $\mathbb{C}P^N$, hence this curve is a rational curve, namely

$$f = [1, z, \cdots, \sqrt{\binom{N}{k}} z^k, \cdots, z^N],$$

up to a holomorphic isometry, an element in $PSL(N+1, \mathbb{C})$. Now we can investigate any solution of (1.1) and (1.4) as in subsection 3.4. From a holomorphic curve f, we get the k-th associated curves f_k and Λ_k. The solution of (1.1) $u = (u_1, u_2, \cdots, u_N)$ is given by

$$u_i = \sum_{j=1}^{N} a_{ij} \log \|\Lambda_j\|^2.$$

This completes the proof.

Proof of Theorem 1.2. From such a holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$, we get a solution u of (1.1). It is clear that the condition (3.11) implies that u satisfies (1.4). As in the proof of Theorem 1.1, f can be extended to a curve from $\mathbb{S}^2$ to $\mathbb{C}P^N$, which is totally unramified. Hence, it is a rational curve.

Corollary 4.1. *The space of solutions of (1.1) and (1.4) is $PSL(N+1, \mathbb{C})/PSU(N+1)$.*

Proof. It follows from Theorem 1.1 and Lemma 3.1. □

Corollary 4.2. *Any solution $u = (u_1, u_2, \cdots, u_N)$ of (1.1) and (1.4) satisfies*

$$\frac{1}{\pi} \sum_{j=1}^{N} a_{ij} \int_{\mathbb{R}^2} e^{u_j} = 4.$$

In particular, if $N = 2$, then

$$2 \int_{\mathbb{R}^2} e^{u_1} = 2 \int_{\mathbb{R}^2} e^{u_2} = 8\pi.$$

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