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HARMONIC HOPF CONSTRUCTIONS BETWEEN SPHERES II

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1. Introduction and the main result

This paper can be seen as the final remark of the previous paper written by the first author [D]. We consider the existence of harmonic maps between two spheres, via Hopf constructions. Given a non trivial bi-eigenmap f : $S^p \times S^q \longrightarrow S^n$ with bi-eigenvalue (λ, μ) $(\lambda, \mu > 0)$ and a continuous function $\alpha : [\ 0, \frac{\pi}{2}\ ] \longrightarrow [\ 0, \pi\ ]$ with $\alpha(0) = 0$, $\alpha(\frac{\pi}{2}) = \pi$, one defines a map u: $S^{p+q+1} \longrightarrow S^{n+1}$, called the α-Hopf construction on f, by

$$u(\sin t \cdot x, \cos t \cdot y) = (\sin \alpha(t) f(x, y), \cos \alpha(t)),$$

where $x \in S^p, y \in S^q$ and $t \in [0, \frac{\pi}{2}]$. It is known [ER] that u is a harmonic map if and only if α is a solution of the o.d.e.

$$\ddot{\alpha} + (p \cdot \cot t - q \cdot \tan t)\dot{\alpha} - (\frac{\lambda}{\sin^2 t} + \frac{\mu}{\cos^2 t}) \sin \alpha \cdot \cos \alpha = 0 \tag{1.1}$$

with the boundary value condition

$$\lim_{t \longrightarrow 0^+} \alpha(t) = 0, \lim_{t \longrightarrow \frac{\pi}{2} - 0} \alpha(t) = \pi \tag{1.2}$$

Ding [D] proves that , if p22 and q22 , then (1.1)-(1.2) has a solution .In the case that p=q=1 , Eells-Ratto [ER] prove that a (1.1)-(1.2) is solvable if and only if $\lambda = \mu$. In this paper ,we consider the remaining case , i.e. p=1 and $q \geq 2$. In this case , it is proved in [ER] that , a necessary condition for (1.1)-(1.2) to be solvable is that $q\lambda < \mu$. Ding([D] Remark 1.2) conjectured that, it is also a sufficient condition. In this paper, we prove the conjecture.

Theorem 1.1. *If p=1, $q > 1, \lambda \geq 1$ and $\mu > \lambda q$, then the prob.(1.1)-(1.2) has a solution α with $0 < \alpha(t) < \pi$ for $t \in (0, \frac{\pi}{2})$.*

Some applications

An immediate consequence of Thoerem 1.1 is that it can provide non-contractible harmonic maps between spheres. For example, $\phi = e^{i\lambda\theta}$ is an eigenmap with eigenvalue λ^2 and if $\psi : S^q \to S^{2m-1}$ is another eigenmap with eigenvalue μ, then the map $F = \phi\psi : S^1 \times S^q \to S^{2m-1} \subset \mathbb{C}^m$ is a bi-eigenmap with bi-eigenvalue (λ^2, μ).

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If $\mu > \lambda^2 q$, then there exists a harmonic map from S^{q+2} to S^{2m}.

Using orthogonal multiplication and Hopf constructions in the following way, we can obtain more information about the homotopy groups of spheres, which is based on Theorem 1.1. An orthogonal multiplication is a bilinear map $f: \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^n$ such that

$$|f(x,y)| = |x||y|, \ \forall x \in \mathbb{R}^k, \ y \in \mathbb{R}^l.$$

A Hopf construction on f is a map

$$F_f: \ \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{n+1}$$

which sends (x,y) to $(2f(x,y), |x|^2 - |y|^2)$.
Its restriction induces a map

$$\psi: \ S^{k+l-1} \to S^n$$

If $k = l = \frac{q}{2}$, then ψ is an eigenmap with eigenvalue $\mu = 2q$.

Choosing $\phi = e^{i\theta}$, then it is an eigenmap from S^1 to S^1 with eigenvalue 1. Let $g: \ \mathbb{R}^2 \times \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ be an orthogonal multiplication. Then the composition map $g(\phi,\psi): \ S^1 \times S^{q-1} \to S^m$ is a bi-eigenmap with bi-eigenvalue $(1,\mu)$. Then the Hopf construction $[F_{g(\phi,\psi)}]$ is an nontrivial element in $\pi_{q+1}(S^{m+1})$. By Theorem 1.1, we have

Corollary 1.2 If there exist orthogonal multiplications f and g, then $\forall q > 2$, the homotopy class $[F_{g(\phi,\psi)}] \in \pi_{q+1}(S^{m+1})$ has a harmonic representative.

Now the complex multiplication gives the Hopf's fibration $\psi_1: \ S^3 \to S^2$, with the above $q = 4, \mu = 8$. The quaternion multiplication gives the eigenmap $\psi_2: \ S^7 \to S^4$ with $q = 8$ and $\mu = 16$. Also from octonion multiplication we get the eigenmap $\psi_3: \ S^{15} \to S^8$ with $q = 16$ and $\mu = 32$. It is known (see [ER]) that there exist orthogonal multiplications $g_1: \ \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^4$, $g_2: \ \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}^6$ and $g_3: \ \mathbb{R}^2 \times \mathbb{R}^9 \to \mathbb{R}^{10}$. Therefore, according to Corollary 1.2 $[F_{g_1(\phi,\psi_1)}] \in \pi_5(S^4)$, $[F_{g_2(\phi,\psi_2)}] \in \pi_9(S^6)$ and $[F_{g_3(\phi,\psi_3)}] \in \pi_{17}(S^{10})$ are non trivial classes and have harmonic representatives respectively.

In the following sections, we will focus on the proof of Theorm 1.1.

Recall that ([D]) for each $s \in (0, \frac{\pi}{2})$. There exists a unique β_s which is the minimizer of the functional

$$J_s(\alpha) = \int_0^s (\dot{\alpha}^2 + Q \cdot \sin^2 \alpha) f dt$$

over the Hilbert space

$$X_s = \{\alpha \in H^1_{\text{loc}}(0,s) : \int_0^s (\dot{\alpha}^2 + \alpha^2) f dt < \infty \text{ and } \alpha(s) = \frac{\pi}{2}\}.$$

Here

$$Q(t) = \frac{\lambda}{\sin^2 t} + \frac{\mu}{\cos^2 t}, \ f(t) = \sin t \cdot \cos^q t.$$

Since β_s is the minimizer of J_s, it satisfies (1.1) in $(0,s)$. Moreover, $\beta_s(t) \to 0$ as $t \to 0$ if and only if $J_s(\beta_s) < J_s(\frac{\pi}{2})$. In the present case, we always have $J_s(\frac{\pi}{2}) = +\infty$, it follows that

$$\lim_{t\to 0}\beta_s(t) = 0. \tag{1.3}$$

Similarly, we may define

$$J_s^*(\alpha) = \int_s^{\frac{\pi}{2}} (\dot\alpha^2 + Q\cdot\sin^2\alpha) f\,dt$$

over the Hilbert space

$$X_s^* = \{\alpha \in H^1_{\mathrm{loc}}(s,\frac{\pi}{2}) : \int_s^{\frac{\pi}{2}} (\dot\alpha^2 + \alpha^2) f\,dt < \infty \text{ and } \alpha(s) = \frac{\pi}{2}\}.$$

¿From [D], we know that there exists a unique $\beta_s^* \in X_s^*$ which is the minimizer of J_s^*, satisfies (1.1) in $(s, \frac{\pi}{2})$ and β_s^* satisfies

$$\lim_{t\to\frac{\pi}{2}}\beta_s^*(t) = \pi \tag{1.4}$$

if and only if

$$J_s^*(\beta_s^*) = \inf_{X_s^*} J_s^* := c_s^* < J_s^*(\frac{\pi}{2}).$$

Since one can easily show that c_s^* is uniformly bounded (by using test functions) for $s \in (0, \frac{\pi}{2})$, while $J_s^*(\frac{\pi}{2}) \to +\infty$ as $s \to 0$, we see that 1.4 holds true for $s > 0$ small. Now we define

$$\alpha_s(t) = \begin{cases} \beta_s(t) & \text{if } t \in (0, s] \\ \beta_s^*(t) & \text{if } t \in (s, \frac{\pi}{2}). \end{cases}$$

It is known ([D]) that $\alpha_s : (0, \frac{\pi}{2}) \to X$ is a continuous curve in the space

$$X = \{\alpha \in H^1_{\mathrm{loc}}(0,\frac{\pi}{2}) : \int_0^{\frac{\pi}{2}} (\dot\alpha^2 + \alpha^2) f\,dt < \infty\}.$$

Moreover, there exists a constant $C > 0$ such that

$$J_s(\alpha_s) \le C \text{ for } s \in (0, \frac{\pi}{2}). \tag{1.5}$$

Note that $\alpha_s(t)$ satisfies (1.1) in $(0,s)\cup(s,\frac{\pi}{2})$, hence it is smooth there. However, the first derivative $\dot\alpha_s(t)$ may have a jump at $t = s$, i.e. in general

$$\dot\alpha_s(s-0) \ne \dot\alpha_s(s+0).$$

By the fact that α_s is continuous in X and it satisfies (1.1) in $(0,s)\cup(s,\frac{\pi}{2})$, one can show that both $\dot\alpha_s(s-0)$ and $\dot\alpha_s(s+0)$ are continuous in $(0,\frac{\pi}{2})$. So we may define a continuous function

$$l(s) := \dot\alpha_s(s+0) - \dot\alpha_s(s-0).$$

It is clear that α_s is a solution to (1.1)-(1.2) if and only if $l(s) = 0$.

We notice that the arguments in [G] imply $l(s) < 0$ for $s \in (0, \frac{\pi}{2})$ with $\frac{\pi}{2} - s$ small enough. Therefore Theorem 1.1 is a consequence of the following Proposition.

Proposition 1.2. *Under the assumptions of Theorem 1.1, we have*

$$l(s) < 0$$

for s sufficiently small.

The remaining part of this paper will be devoted to the proof of the Proposition.

2. Blow-up analysis

We set

$$\gamma_s(t) := \alpha_s(st)\ t \in (0, \frac{\pi}{s}).$$

It is clear that

$$\gamma_s(1) = \frac{\pi}{2}\ \dot{\gamma}_s = s\dot{\alpha}_s,\ \text{and}\ \ddot{\gamma}_s = s^2\ddot{\alpha}_s.$$

By the equation (1.1), we have

$$\ddot{\gamma}_s + s(\cot(st) - q\tan(st))\dot{\gamma}_s - s^2(\frac{\lambda}{\sin^2(st)} + \frac{\mu}{\cos^2(st)})\sin\gamma_s\cos\gamma_s = 0 \quad (2.1)$$

in $(0,1) \cup (1, \frac{\pi}{s})$. It is clear that

$$s(\cot(st) - q\tan(st)) \to \frac{1}{t},$$

$$s^2(\frac{\lambda}{\sin^2(st)} + \frac{\mu}{\cos^2(st)}) \to \frac{\lambda}{t^2},$$

as $s \to 0$.

Given $\epsilon > 0$ small, there is $\delta = \delta(\epsilon) > 0$ such that, if $s < \delta$ the equation (2.1) is well defined in $(\epsilon, \epsilon^{-1})$ with uniformly bounded coefficients and uniformly bounded nonlinearity. Therefore, by elliptic estimates up to boundary (Noting that γ_s satisfies the boundary conditions $\gamma_s(\epsilon) \in (0, \frac{\pi}{2})$, $\gamma_s(1) = \frac{\pi}{2}$, and $\gamma_s(\frac{1}{\epsilon}) \in (\frac{\pi}{2}, \pi)$.), we may get the estimates

$$\|\gamma_s\|_{C^k([\epsilon,1])} \leq C(k, \epsilon),$$

and

$$\|\gamma_s\|_{C^k([1,\epsilon^{-1}])} \leq C(k, \epsilon).$$

It follows that there exists a sequence $s_i \to 0$ such that $\gamma_{s_i} \to \phi$ in $C^2([\epsilon, 1])$ and $C^2([1, \epsilon^{-1}])$, where ϕ is C^2 on $[\epsilon, 1]$ and $[1, \epsilon^{-1}]$, continuous on $[\epsilon, \epsilon^{-1}]$, and satisfies the limit equation

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} - \frac{\lambda}{t^2}\sin\phi\cos\phi = 0 \quad (2.2)$$

in $(\epsilon, 1) \cup (1, \epsilon^{-1})$ with $\phi(1) = \frac{\pi}{2}$.

Using a diagonal subsequence argument, we may deduce the existence of a sequence $s_i \to 0$ such that $\gamma_{s_i} \to \phi$ in $C^2([\epsilon, 1])$ and $C^2([1, \epsilon^{-1}])$ for any $\epsilon > 0$, where ϕ satisfies (2.2) in $(0,1) \cup (1, \infty)$ with $\phi(1) = \frac{\pi}{2}$.

Consider the function ϕ on $[0, 1]$, we have

$$C \geq \int_0^s fQ\sin^2\alpha_s dt = \int_0^1 sf(st)Q(st)\sin^2\gamma_s dt.$$

Since the integrand of the last integral converges point wise to $\frac{\lambda}{t}\sin^2\phi$, by Fatou's lemma we have

$$\int_0^1 \frac{\lambda}{t}\sin^2\phi dt \leq C.$$

It follows that $\phi \not\equiv \frac{\pi}{2}$. The analysis in [D] then shows that ϕ is the unique solution of the problem

$$\begin{cases} \ddot{\phi} + \frac{1}{t}\dot{\phi} - \frac{\lambda}{t^2}\sin\phi\cos\phi = 0 \\ \phi(0+) = 0,\ \phi(1) = \frac{\pi}{2}. \end{cases}$$

This problem is explicitly solvable, and we get

$$\phi(t) = \arccos(\frac{1-t^a}{1+t^a}) \tag{2.3}$$

where $a = 2\sqrt{\lambda}$. One can similarly show that on $[1,\infty)$, (2.3) is also the right expression for ϕ.

It is useful to note that ϕ is a special solution of the problem

$$\begin{cases} \ddot{\phi} + \frac{1}{t}\dot{\phi} - \frac{\lambda}{t^2}\sin\phi\cos\phi = 0\ t \in (0,\infty) \\ \phi(0+) = 0,\ \phi(\infty) = \pi. \end{cases} \tag{2.4}$$

All solutions of (2.4) are given by

$$\phi_s(t) = \arccos(\frac{s^a - t^a}{s^a + t^a}),\ s \in (0,\infty).$$

3. Comparison

For comparison purposes we need to introduce a family of functions which are solutions to the equation

$$\begin{cases} \ddot{\psi} + \cot t\dot{\psi} - \frac{\lambda}{\sin^2 t\cos^2 t}\sin\psi\cos\psi = 0 \text{ in } (0,\frac{\pi}{2}) \\ \psi(0) = 0,\ \psi(\frac{\pi}{2}) = \pi. \end{cases} \tag{3.1}$$

The solution can be explicitly given by

$$\psi_s(t) = 2\arctan(\cot^{\frac{a}{2}} s \cdot \tan^{\frac{a}{2}} t),\ s \in (0,\frac{\pi}{2}) \tag{3.2}$$

where, as before, $a = 2\sqrt{\lambda}$.

Lemma 3.1. *Let $s \in (0,\frac{\pi}{2})$ and ψ be the solution of (3.1). Assume that for some $t_0 > s$, $\alpha_s(t_0) > \psi(t_0) > \max\{\theta, \frac{3\pi}{4}\}$. Then $\alpha_s(t) \geq \psi(t)$, for all $t \in (t_0,\frac{\pi}{2})$. Here*

$$\theta = \arccos(\frac{-\lambda(q-1)}{\mu-\lambda}) \in (\frac{\pi}{2},\pi).$$

Proof. It will be convenient to write the equation (1.1) as

$$(f\dot{\alpha})^{\cdot} - fQ\sin\alpha\cos\alpha = 0. \tag{3.3}$$

We first show that, under the assumptions of the lemma, ψ satisfies

$$(f\dot{\psi})^{\cdot} - fQ\sin\psi\cos\psi > 0 \text{ in } (t_0,\frac{\pi}{2}). \tag{3.4}$$

Note that ψ as given in (3.2) satisfies

$$\dot{\psi} = \frac{\sqrt{\lambda}\sin\psi}{\sin t\cos t}. \tag{3.5}$$

Using this we have

$$\begin{aligned}(f\dot\psi)^\cdot &= \sqrt{\lambda}(\cos^{q-1} t \sin\psi)^\cdot \\ &= \lambda \sin^{-1} t \cos^{q-2} t \sin\psi \cos\psi - \lambda(q-1)\sin t \cos^{q-2} t \sin\psi.\end{aligned}$$

Then it is easy to derive that

$$(f\dot\psi)^\cdot - fQ\sin\psi\cos\psi = \sin t\cos^{q-2} t((\lambda-\mu)\cos\psi - \lambda(q-1))\sin\psi.$$

¿From (3.5) we see that

$$\psi(t) > \psi(t_0) > \theta \text{ for } t \in (t_0, \frac{\pi}{2}).$$

Hence

$$-\cos\psi > \frac{\lambda(q-1)}{\mu-\lambda}.$$

It is then clear that (3.4) holds true.

Then, we consider the function $u = \alpha_s - \psi$. We have

$$u(t_0) > 0 \text{ and } \lim_{t\to\frac{\pi}{2}} u(t) = 0.$$

If the lemma is false, we can find $t_1 \in (t_0, \frac{\pi}{2})$ where u achieves a negative local minimum, i.e.

$$u(t_1) < 0,\ \dot u(t_1) = 0, \text{ and } \ddot u(t_1) \geq 0.$$

However, using (3.3) and (3.4), we get, at $t = t_1$,

$$\begin{aligned}f(t_1)\ddot u(t_1) &< f(t_1)Q(t_1)\left(\frac{\sin 2\alpha_s - \sin 2\psi}{2\alpha_s - 2\psi}\right)u(t_1) \\ &= f(t_1)Q(t_1)\cos\xi u(t_1)\end{aligned}$$

where $\xi = 2(r\alpha_s(t_1) + (1-r)\psi(t_1))$ for some $r \in (0,1)$. Since $\alpha_s(t_1),\ \psi(t_1) \in (\frac{3\pi}{4}, \pi)$, we see that $\xi \in (\frac{3\pi}{2}, 2\pi)$. So, $\cos\xi > 0$ and we arrive at $\ddot u(t_1) < 0$, a contradiction! This proves the lemma. Q.E.D.

4. Proof of the proposition

Consider the function α_s which satisfies (3.3) in $(0,s)\cup(s,\frac{\pi}{2})$. Multiplying the equation (3.3) by $f\dot\alpha_s$ and integrating over $(0,s)$ and $(s,\frac{\pi}{2})$ respectively we get

$$f^2(s)\dot\alpha_s^2(s-0) = \int_0^s f^2 Q\frac{\partial}{\partial t}(\sin^2\alpha_s)dt$$

$$-f^2(s)\dot\alpha_s^2(s+0) = \int_s^{\frac{\pi}{2}} f^2 Q\frac{\partial}{\partial t}(\sin^2\alpha_s)dt.$$

By integration by parts, we obtain

$$f(s)(\dot\alpha_s^2(s+0) - \dot\alpha_s^2(s-0)) = \int_0^{\frac{\pi}{2}} (f^2Q)'\sin^2\alpha_s dt.$$

Since we [D] have $\dot{\alpha}_s(s+0) > 0$ and $\dot{\alpha}_s(s-0) > 0$, we can see that $l(s) = \dot{\alpha}_s(s+0) - \dot{\alpha}_s(s-0) > 0$, if and only if the integral on the right hand side of the last identity is positive. Denote this integral by I_s, then we have

$$\begin{aligned} I_s &= 2(\mu - \lambda q)\int_0^{\frac{\pi}{2}} \sin t \cos^{2q-1} t \sin^2 \alpha_s dt - 2\mu(q-1)\int_0^{\frac{\pi}{2}} \sin^3 t \cos^{2q-3} t \sin^2 \alpha_s dt \\ &:= 2(\mu - \lambda q) I_s^1 - 2\mu(q-1) I_s^2. \end{aligned}$$

To estimate I_s^1, we let $\gamma_s(t) = \alpha_s(st)$ and have

$$\begin{aligned} I_s^1 &= s\int_0^{\frac{\pi}{2s}} \sin st \cos^{2q-1} st \sin^2 \gamma_s dt \\ &= s^2 \int_0^{\frac{\pi}{2s}} (s^{-1} \sin st) \cos^{2q-1} st \sin^2 \gamma_s dt \end{aligned}$$

Note that the integrand of the last integral converges point wise to the function $t \sin^2 \phi(t)$ as $s \to 0$, where ϕ is given by (2.3). By Fatou's lemma, we have

$$\liminf_{s\to 0} s^{-2} I_s^1 \geq \int_0^\infty t \sin^2 \phi(t) dt = \int_0^\infty \frac{4t^{a+1}}{(1+t^a)^2} dt,$$

where $a = 2\sqrt{\lambda} \geq 1$. It follows that if $\lambda > 1$,

$$I_s^1 \geq A(\lambda) s^2 \text{ for } s \text{ sufficiently small where } A(\lambda) > 0, \tag{4.1}$$

and if $\lambda = 1$,

$$I_s^1 \geq B(s) s^2 \text{ where } B(s) \to \infty \text{ as } s \to 0. \tag{4.2}$$

To estimate I_s^2, we need use the comparison lemma. First, fix an arbitrary large number $R > 0$, we have

$$\alpha_s(sR) = \gamma_s(R) \to \phi(R) = \arccos\left(\frac{1-R^a}{1+R^a}\right) \text{ as } s \to 0,$$

i.e.

$$\cos \alpha_s(sR) \to -1 + \frac{2}{1+R^a}. \tag{4.3}$$

Next, let $d > 1$, we consider a solution of (3.1) as in (3.2),

$$\psi_{ds}(t) = 2\arctan(\cot^{\frac{a}{2}} ds \cdot \tan^{\frac{a}{2}} t).$$

By a straightforward computation, we get

$$\cos \psi_{ds}(Rs) = -1 + \frac{2\tan^a(ds)}{\tan^a(ds) + \tan^a(Rs)}.$$

Using the fact that

$$\tan t = t + O(t^2) \text{ near } t = 0,$$

we get

$$\cos \psi_{ds}(Rs) = -1 + \frac{2}{1+(R/d)^a} + O(R^2 s^2). \tag{4.4}$$

¿From (4.3 and (4.4), we can see that, if we choose R large enough, s small enough,

$$\alpha_s(Rs) > \psi_{ds}(Rs) \geq \pi - \epsilon,$$

where $\epsilon > 0$ can be chosen as small as we need. By the comparison lemma, for $t \in (Rs, \frac{\pi}{2})$,

$$\alpha_s(Rs) \geq \psi_{ds}(t),$$

and consequently,

$$\sin^2 \alpha_s(t) \leq \sin^2 \psi_{ds}(t).$$

For simplicity, set $\epsilon = \tan(ds)$. We have

$$\sin^2 \psi_{ds} = \frac{4\epsilon^a \tan^a t}{\epsilon^a + \tan^a}.$$

Then,

$$\begin{aligned} A_s &:= \int_{Rs}^{\frac{\pi}{2}} \sin^3 t \cos^{2q-3} t \sin^2 \alpha_s dt \\ &\leq 4\epsilon^a \int_{Rs}^{\frac{\pi}{2}} \frac{\tan t}{\epsilon^a + \tan^a t} \sin^3 t \cos^{2q-3} t dt \\ &:= 4\epsilon^a J. \end{aligned}$$

Making the transformation $u = \tan t$, we get

$$J = \int_{\tan(Rs)}^{\infty} \frac{u^{a+3} du}{(\epsilon^a + u^a)^2 (1+u^2)^{\frac{b+5}{2}}},$$

where $b = 2q - 3 > -1$.

$$\begin{aligned} J &\leq \int_{\tan(Rs)}^{1} \frac{u^{a+3} du}{(\epsilon^a + u^a)^2} + \int_1^{\infty} \frac{du}{u^{a+b+2}} \\ &:= J_1 + J_2. \end{aligned}$$

Since $a + b + 2 > 3$, we have $J_2 < \infty$. Letting $u = \epsilon x$ in J_1, we get

$$J_1 = \int_{\tan(Rs)/\epsilon}^{1/\epsilon} \frac{\epsilon^{4-a} x^{a+3} dx}{(1+x^a)^2} \leq \epsilon^{4-a} \int_{\tan(Rs)/\epsilon}^{1/\epsilon} x^{3-a} dx.$$

Noting that $\epsilon = \tan(ds)$ and $\tan(Rs)/\epsilon = R/d + O(R^2 s^2)$, we see that

$$\begin{aligned} J_1 &\leq C\epsilon^{4-a}/R^{a-2} \text{ if } a > 2, \\ J_1 &\leq C\epsilon^{4-a} |\log \tan(Rs)| \text{ if } a = 2. \end{aligned}$$

Thus,

$$A_s = \int_{Rs}^{\frac{\pi}{2}} \sin^3 t \cos^{2q-3} t \sin^2 \alpha_s dt \leq C_1 \epsilon^a + C_2 \epsilon^{4-\delta},$$

where $\delta > 0$ can be arbitrarily small. Since $\epsilon = \tan(ds)$, we can rewrite the estimate as

$$A_s = O(s^a) + O(s^{4-\delta}).$$

Finally, we consider

$$B_s = \int_0^{Rs} \sin^3 t \cos^{2q-3} t \sin^2 \alpha_s dt.$$

It is obvious that, with Rs sufficiently small,

$$B_s = \tan^2(Rs)\int_0^{Rs} \sin t \cos^{2q-1} t \sin^2 \alpha_s dt = o(1)I_s^1.$$

Therefore,

$$I_s^2 = o(1)I_s^1 + O(s^{2\sqrt{\lambda}}) + O(s^{4-\delta}). \tag{4.5}$$

Combining (4.5) with (4.1) and (4.2), we obtain,

$$I_s = 2(\mu - \lambda q)I_s^1 - 2\mu(q-1)I_s^2 > 0$$

for $s > 0$ sufficiently small. This proves the proposition. Q.E.D.

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