Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On a Hessian equation on S^n

by

Pengfei Guan and Xi-Nan Ma

Preprint no.: 43 2001

ON A HESSIAN EQUATION ON S^n

PENGFEI GUAN AND XI-NAN MA

1. Introduction

We consider the following Hessian equation on S^n:

$$S_k(\{u_{ij} + u\delta_{ij}\}) = \varphi \quad \text{on} \quad S^n, \tag{1.1}$$

where S_k is the k-th elementary symmetric function and u_{ij} the covariant derivatives of u with respect to orthonormal frames on S^n. The equation is associated to Christoffel-Minkowski problem, we refer [7] for its geometric connection. Driven by geometric application, the existence of *convex* solution of (1.1) was considered in [7]. In this paper, we establish the existence result for equation (1.1) in its natural solution class (k-convex functions).

We recall that a function $u \in C^2(S^n)$ is k-convex if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in S^n$, where

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, ..., S_k(\lambda) > 0\}.$$

A function u is called an admissible solution of (1.1) if u is k-convex and satisfies (1.1). It has been shown in [1] (see also [9], [10] and [13]) that k-convex functions are the natural class of functions where equation (1.1) is defined and elliptic.

We note that in order that equation (1.1) to be solvable, it is necessary that

$$\int_{S^n} x_i \varphi(x) dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{1.2}$$

In fact, $\forall v \in C^2(S^n)$,

$$\int_{S^n} x_i S_k(v_{ij}(x) + v(x)\delta_{ij}) dx = 0, \qquad \forall i = 1, 2, ..., n+1.$$

Research of the first author was supported in part by NSERC Grant OGP-0046732. Research of the second author was supported by Foundation for University Key Teacher by the Ministry of Eduction of China and NSFC No.10001011.

To see this, we set $u^t = 1 + tv$. For $t > 0$ small, u^t is a supporting function of some smooth strictly convex hypersurface, and

$$S_n(u^t_{ij} + u^t\delta_{ij}) = \sum_{k=1}^{n} \frac{n!}{i!(n-i)!} S_k\, t^k.$$

Here, we write $S_k = S_k(v_{ij} + v\delta_{ij})$. By Stokes theorem,

$$\int_{S^n} x_j S_n(u^t_{ij}(x) + u^t(x)\delta_{ij})dx = 0.$$

It follows that

$$\int_{S^n} x_j S_k dx = 0, \quad \forall\, 1 \le j \le n+1, \;\; 1 \le k \le n$$

The following is our main result.

Theorem 1.1. (Existence Theorem) *Let $\varphi(x) \in C^{1,1}(S^n)$ be a positive function, suppose φ satisfies (1.2), then equation (1.1) has a solution. More precisely, there exists a $C^{3,\alpha}$ ($\forall 0 < \alpha < 1$) k-convex function u of (1.1) with estimates:*

$$||u||_{C^{3,\alpha}}(S^n) \le C, \tag{1.3}$$

with the constant C depending only on $n, l, \alpha, \min\varphi$, and $||\varphi||_{C^{l,1}}(S^n)$.

Furthermore, if $\varphi(x) \in C^{l,\gamma}(S^n)$ ($l \ge 2, \gamma > 0$), then M is $C^{2+l,\gamma}$. If φ is analytic, M is analytic.

Acknowledgement: The work was completed while the first author was visiting Max-Planck Institute for Mathematical Sciences in Leipzig. He would like to thank Professor J. Jost for the invitation and MPI for the hospitality.

2. Integral Formulas and A Uniqueness Result

We first prove some integral formulas. These formulas were proved in [3] for support functions of convex bodies. Here we validate them for any C^2 functions on S^n.

We introduce some notations. For any $n \times n$ symmetric matrice $W_1, ..., W_k$, let $S_k(W_1, ..., W_k)$ be the complete polarization of S_k. Let u and $\tilde{u}$ are two C^2 functions on S^n. Let W and $\tilde{W}$ are the corresponding Hessian matrice of u and $\tilde{u}$ respectively. Following Chern's notations in [3], we let $P_{rs} = S_{r+s}(W, ..., W, \tilde{W}, ...\tilde{W})$ where W appears r times and $\tilde{W}$ appears s times. So, P_{rs} is a polynomial in W_{ij}, $\tilde{W}_{ij}$, homogeneous of degrees r and s respectively.

Lemma 2.1. *Suppose u and $\tilde{u}$ are two C^2 functions on S^n, then the following identities hold.*

$$\int_{S^n} uS_k(W) = \int_{S^n} S_{k+1}(W), \quad \forall 1 \le k < n, \tag{2.1}$$

And $\forall 1 \le k \le n$,

$$\int_{S^n} [uP_{0k} - \tilde{u}P_{1,k-1}] = 0, \tag{2.2}$$

$$\int_{S^n} [uP_{k-1,1} - \tilde{u}P_{k0}] = 0, \tag{2.3}$$

and,

$$2\int_{S^n} u(P_{0k} - P_{k-1,1}) = \int_{S^n} \{\tilde{u}(P_{1,k-1} - P_{k0}) - u(P_{k-1,1} - P_{0k})\}. \tag{2.4}$$

Proof. Identity (2.4) follows from (2.2) and (2.3), also (2.3) follows from (2.2) by symmetry. We only need to prove indentities (2.1) and (2.2)in the lemma. We adapt the argument in [2] making the computation directly on S^n. Let $e_1, ..., e_n$ is an orthonormal frame on S^n, let $\omega_1, ..., \omega_n$ be the corresponding 1-forms. For each function $u \in C^2(S^n)$, let u_i be the covariant derivative of u with respect to e_i. We define a vector valued function

$$Z = \sum_{i=1}^{n} u_i e_i + u e_{n+1}.$$

where e_{n+1} is the position vector on S^n, the outer normal vector field of S^n. We note that Z is globally defined on S^n.

We calculate that,

$$\begin{aligned}
u &= Z \cdot e_{n+1}, \\
dZ &= \sum_{i=1}^{n} (du_i e_i + u_i de_i) + du e_{n+1} + u de_{n+1} \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{n} u_{ij}\omega^j - \sum_{j=1}^{n} u_j \omega_j^i) e_i + \sum_{i=1}^{n} (\sum_{\alpha=1}^{n+1} u_i \omega_i^\alpha e_\alpha) \\
&\quad + \sum_{i=1}^{n} (u_i \omega^i) e_{n+1} + u \sum_{i=1}^{n} \omega^i e_i \\
&= \sum_{j=1}^{n} (\sum_{i=1}^{n} (u_{ij} + \delta_{ij} u) e_i) \omega^j.
\end{aligned}$$

Let $\tilde{u} \in C^2(S^n)$, similarly we define

$$\tilde{Z} = \sum_{i=1}^{n} \tilde{u}_i e_i + u^l e_{n+1},$$

and

$$\tilde{W} = \{\tilde{u}_{ij} + \tilde{u}\delta_{ij}\}$$

Set,

$$\Omega_{r,s} = (Z, dZ, \cdots, dZ, d\tilde{Z}, \cdots, d\tilde{Z}, I, \cdots, I),$$
$$\tilde{\Omega}_{s,r} = (\tilde{Z}, d\tilde{Z}, \cdots, d\tilde{Z}, dZ, \cdots, dZ, I, \cdots, I),$$

where dZ appears $r-1$ times and $d\tilde{Z}$ appears s times in $\Omega_{r,s}$, dZ appears r times and $d\tilde{Z}$ appears $s-1$ times in $\tilde{\Omega}_{r,s}$, and I the identity matrix which appears $n-r-s+1$ times both in Ω and $\tilde{\Omega}$.

We note that

$$\Omega_{r,s} = uS_{r+s}(W, \cdots, W, \tilde{W}, \cdots, \tilde{W})d\sigma \tag{2.5}$$

where W appears r times and $\tilde{W}$ appears s times, $d\sigma$ is the standard area form on S^n, and S_{r+s} is the complete polarization of the symmetric function S_k defined for symmetric matrices. Also, similar identity holds for $\tilde{\Omega}_{r,s}$

We make an assumption that $u, \tilde{u} \in C^3(S^n)$. Set,

$$\omega = (Z, \tilde{Z}, dZ, \cdots, dZd\tilde{Z}, \cdots, d\tilde{Z}, I, \cdots, I),$$

where dZ appears $r-1$ times, $d\tilde{Z}$ appears $s-1$ times. We have

$$d\omega = \Omega_{r,s} - \tilde{\Omega}_{s,r}.$$

Now, (2.2) and (2.3) follow from Stokes theorem and (2.5). The identities are valid for C^2 functions by approximation. The identity (2.1) can be proved in a similar way replacing ω by

$$\omega^* = (Z, Z, dZ, \cdots, dZ, I, \cdots, I),$$

with dZ appears k times. □

Corollary 2.2. *If u is a nonnegative k-convex function, and u satisfies*

$$\frac{S_k(W)}{S_l(W)} = c, \tag{2.6}$$

for some $0 \leq l < k$ and constant $c > 0$, then u is a constant upto to linear functions.

Proof. We deal with the case $l > 0$. The case $l = 0$ will be proved in the next theorem.

By (2.1), we have,

$$\begin{aligned}
&\int_{S^n} [S_{k-1}(W)\frac{S_l(W)}{S_k(W)} - S_{l-1}(W)]u \\
&= \int_{S^n} [S_l(W) - S_{l-1}(W)u] + \int_{S^n} [S_{k-1}(W)\frac{S_l(W)}{S_k(W)}u - S_k(W)\frac{S_l(W)}{S_k(W)}] \\
&= \int_{S^n} [S_l(W) - S_{l-1}(W)u] + c\int_{S^n} [S_{k-1}(W)u - S_k(W)] \\
&= 0.
\end{aligned}$$

Since u is nonnegative and satisfying (2.6), u is positive almost everywhere. On the other hand, by Newton's inequality

$$S_{k-1}(W)\frac{S_l(W)}{S_k(W)} - S_{l-1}(W) \le 0.$$

We conclude that

$$S_{k-1}(W)\frac{S_l(W)}{S_k(W)} - S_{l-1}(W) = 0.$$

As W is in Γ_k, by the equality case in Newton's inequality, W is a constant multipler of the identity matrix. That is, u is a constant modulus a linear function. $\square$

The following uniqueness theorem generalizes Alexandrov-Fenchel-Jessen theorem to k-convex case.

Theorem 2.3. *Suppose u and $\tilde{u}$ are two C^2 k-convex functions on S^n satisfying (1.1). If $S_k(W) = S_k(\tilde{W})$, and if one of u and $\tilde{u}$ is nonnegative, then $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$ on S^n. In particular, admissible solution to (1.1) with $\varphi =$ constant > 0 is unique upto to linear functions.*

Proof of Theorem 2.3. We follow the same lines as in [3]. We may assume u is nonnegative. Since $S_k(W)$ is positive, we conclude that u is positive almost everywhere on S^n. By Garding's theory of hyperbolic polynomals [4], S_k is hyperbolic, and $\forall W^i \in \Gamma_k, i = 1, ..., k$,

$$S_k(W^1, ..., W^k) \ge S_k(W^1) \cdots S_k(W^k), \tag{2.7}$$

with the equality holds if and only if the k matrices are pairwise proportional.

Suppose $S_k(W) = S_k(\tilde{W})$ on S^n, where $W = \{u_{ij} + \delta_{ij}u\}$ and $\tilde{W} = \{\tilde{u}_{ij} + \delta_{ij}\tilde{u}\}$ The left hand side of the integral formula (2.4) in Corollary 2.1 is non-positive.The same is therefore true of the right hand side of (2.4). The latter is ansti-symmetric on the two function u and $\tilde{u}$, and hence must be zero. It follows that $P_{k-1,1} = P_{0k}$ by (2.7). Again, the equality gives that W and $\tilde{W}$ are proportional. Since $S_k(W) = S_k(\tilde{W})$, we conclude that $W = \tilde{W}$ at each point of S^n, that is, $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$. □

We remark that the uniqueness of solutions to (1.1) is closely related to the dimension of the kernel of the linearized operator of the equation. The identification of the kernel of the linearized operator is crucial in the proof of Alexandrov-Fenchel inequalities for k-convex functions, we refer [8] for this direction of study.

3. Regularity estimates and Existence

We establish the a priori estimates for admissible solutions of equation (1.1) and prove Theorem 1.1 in this section.

We note that for any solution $u(x)$ of (1.1), $u(x) + l(x)$ is also a solution of the equation for any linear function $l(x) = \sum_{i=1}^{n+1} a_i x_i$. We will confine ourselves to solutions satisfying the following orthogonality condition

$$\int_{S^n} x_i u \, dx = 0, \qquad \forall i = 1, 2, ..., n+1. \tag{3.1}$$

When u is convex, it is a support function of some convex body Ω. Condition (3.1) implies that the Steiner point of Ω coincides with the origin.

In the case of $k = 1$, the equation (1.1) is a linear elliptic equation on sphere. A priori estimates for solution u satisfies (3.1) in this case follows from standard linear elliptic theory. Therefore, we will restrict ourselves to the case $k \geq 2$. The equation (1.1) will be a uniformly elliptic once C^2 estimates are established for u (see [1]). By Evans-Krylov theorem and Schauder theory, one may obtain higher derivative estimates for u. Therefore, we only need to get C^2 estimates for u.

Proposition 3.1. *Suppose G is a homogeneous function of degree 1, monotone, concave on Γ_k and $\sum_{i=1}^{n} \frac{\partial G}{\partial W_{ii}} \geq 1 > 0$. Suppose $u \in C^4(S^n)$, and $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\} \in \Gamma_k, \forall x \in S^n$. Suppose*

$$G(W) = \tilde{\varphi} \quad on \quad S^n. \tag{3.2}$$

Then,

$$0 < H \leq \max_{x \in S^n} (n\tilde{\varphi}(x) - \triangle\tilde{\varphi}(x)), \tag{3.3}$$

where

$$H := trace(u_{ij} + \delta_{ij}u) = \triangle u + nu. \tag{3.4}$$

Proof. The positivity of H follows from Newton-Maclaurin inequality. Assume the maximum value of H is attained at a point $x_0 \in S^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal, so $W = \{u_{ij} + \delta_{ij}u\}$ is also diagonal at x_0.

For the standard metric on S^n, we have,

$$H_{ii} = \triangle W_{ii} - nW_{ii} + H.$$

By the the monotonicity assumption, $\{G^{ij}\}$ is positive definite. Since $\{H_{ij}\} \le 0$, it follows that at x_0,

$$0 \ge G^{ii}H_{ii} = G^{ii}(\triangle W_{ii}) - nG^{ii}W_{ii} + H\sum_{i}^{n} G^{ii}. \tag{3.5}$$

As G is homogeneous of degree one, we have

$$G^{ii}W_{ii} = \tilde{\varphi}. \tag{3.6}$$

Next we apply the Laplace operator to equation (3.2), we obtain

$$\begin{aligned} &G^{ij}W_{ijk} = \nabla_k\tilde{\varphi}, \\ &G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\triangle W_{ij} = \triangle\tilde{\varphi}. \end{aligned}$$

By the concavity of G, at x_0 we have

$$G^{ii}\triangle(W_{ii}) \ge \triangle\tilde{\varphi}. \tag{3.7}$$

Combining (3.6), (3.7) and (3.5), we see that

$$0 \ge \triangle\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^{n} G^{ii}.$$

As $\sum_{i=1}^{n} \frac{\partial G}{\partial W_{ii}} \ge 1$, it follows that $H \le n\tilde{\varphi} - \triangle\tilde{\varphi}$. □

In order to obtain a C^2 bound, we need a C^0 bound for u. In the case of Minkowski problem, such crucial C^0 bound was established by Cheng-Yau in [2] in case $k = n$ and for general k with *convexity* assumption in [7]. Their estimates assume the convexity. Here, we use the a priori bounds in Proposition 3.1 to get a C^0 bound for k-convex solutions. The similar argument was also used in [6].

Lemma 3.2. *For any C^2 k-convex function v, there is a constant C depending only on n and $\max_{S^n}(n(\triangle v + nv) + |v|)$ such that,*

$$||v||_{C^2} \leq C. \tag{3.8}$$

Proof. At any point $x \in S^n$, we may assume the matrix $(v_{ij} + \delta_{ij} v)$ is diagonal. Let λ_i is a eigenvalue of that matrix, as $k \geq 2$, we have

$$v_{ii} + v \leq max_i \lambda_i \leq \triangle v + nv.$$

In turn,

$$v_{ii} \leq (\triangle v + nv) - v, \quad \forall i. \tag{3.9}$$

It follows from (3.9) that, for any $i = 1, ..., n$,

$$v_{ii} = (\triangle v + nv) - nv - \sum_{k \neq i} v_{kk} \geq -(n-2)(\triangle v + nv) - v.$$

Thus at x, as $\triangle v + nv \geq 0$,

$$|v_{ii}|_{C^0} \leq (n(\triangle v + nv) + |v|).$$

we obtain

$$|\nabla^2 v|_{C^0} \leq C(\max_{x \in S^n}(n(\triangle v + nv) + |v|)). \tag{3.10}$$

By interpolation, $|\nabla v|_{C^0}$ can be bounded by $|v|_{C^0}$ and $|\nabla^2 v|_{C^0}$. The lemma is proved. □

Now we establish the C^0-estimate. The proof is based on a rescaling argument.

Proposition 3.3. *If u is a k-convex function satisfying equation (3.2) and (3.1), then there exist a positive constant C depending only on $n, k, \min_{S^n} \tilde{\varphi}, ||\tilde{\varphi}||_{C^{1,1}}$, such that,*

$$||u||_{C^0} \leq C. \tag{3.11}$$

Proof. Suppose (3.11) is false, then $\exists u^l (l = 1, 2, ...)$ satisfying (3.1), there is a constant $\tilde{C}$ independent of l, and $G(\{u^l_{ij} + \delta_{ij} u^l\}) = \tilde{\varphi}^l$, with φ^l satisfies

$$||\tilde{\varphi}^l||_{C^2} \leq \tilde{C}, \quad ||\frac{1}{\tilde{\varphi}^l}||_{C^0} \leq \tilde{C}, \quad ||u^l||_{L^\infty} \geq l.$$

Let $v^l = \frac{u^l}{||u^l||_{L^\infty}}$, then

$$||v^l||_{L^\infty} = 1, \tag{3.12}$$

By Proposition 3.1, we have

$$0 \le H^l := \triangle u^l + nu^l \le C, \tag{3.13}$$

where the constant C independent of l. From (3.13) v^l satisfies the following estimates

$$0 \le \triangle v^l + nv^l \le \frac{C}{||u^l||_{L^\infty}} \longrightarrow 0. \tag{3.14}$$

On the other hand, by Lemma 3.2, (3.12) and (3.14), we have

$$||v^l||_{C^2} \le C.$$

Hence, there exists a subsequence $\{v^{l_i}\}$ and a function $v \in C^{1,\alpha}(S^n)$ satisfying (3.1) such that

$$v^{l_i} \longrightarrow v \quad \text{in} \quad C^{1,\alpha}(S^n), \quad \text{with} \quad ||v||_{L^\infty} = 1. \tag{3.15}$$

Combining (3.14) and (3.15), in the distribution sense we have

$$\triangle v + nv = 0, \qquad \text{on} \quad S^n.$$

By linear elliptic theory, v is in fact smooth. Since v satisfied (3.1), we conclude that, $v \equiv 0$ on S^n. This is a contradiction to (3.15). □

Now, C^2 a priori bounds follows from Lemma 3.2, Proposition 3.1 and Proposition 3.3. By Evans-Krylov theorem and Schauder theory (e.g, see [5]), we have the following a priori estimates.

Theorem 3.4. *For each integer $l \ge 1$ and $0 < \alpha < 1$, there exist a constant C depending only on $n, l, \alpha, \min \tilde{\varphi}$, and $||\tilde{\varphi}||_{C^{l,1}}(S^n)$ such that*

$$||u||_{C^{l+1,\alpha}}(S^n) \le C, \tag{3.16}$$

for all k-convex solution of (3.2) satisfying the condition (3.1). In particular, the estimate (3.16) is true for any admissible solution of (1.1) and (3.1) with $\tilde{\varphi} = \varphi^{\frac{1}{k}}$.

Proof of Theorem 1.1. For each fixed $0 < \varphi \in C^\infty(S^n)$ with φ satisfying (3.1), and for $0 \le t \le 1$, we define

$$T_t(u) = S_k(\{u_{ij} + u\delta_{ij}\}) - t\varphi - (1-t). \tag{3.17}$$

For $\alpha > 0$, $l \ge 0$ integer, we set,

$$\mathcal{A}^{l,\alpha} = \{u \in C^{l,\alpha}(S^n) : u \quad \text{satisfying (3.1).}\}$$

T_t is a nonlinear differential operator which maps $\mathcal{A}^{l+2,\alpha}$ to $\mathcal{A}^{l,\alpha}$.

For $R > 0$ fixed, let

$$\mathcal{O} = \{w \quad k\text{-convex}, w \in \mathcal{A}^{l,\alpha} : \|w\|_{C^{l,\alpha}(S^n)} < R\}.$$

For R large enough, by our a priori estimates Theorem 3.4, $T_t(u) = 0$ has no solution on $\partial\mathcal{O}$. Therefore, the degree of T_t is well-defined (e.g., [11]). As degree is a homotopic invariant,

$$\deg(T_0, \mathcal{O}, 0) = \deg(T_1, \mathcal{O}, 0).$$

At $t = 0$, by Theorem 2.3, $u = 1$ is the unique solution of (1.1) in $\mathcal{O}$. We may compute the degree using formula

$$\deg(T_0, \mathcal{O}, 0) = \sum_{\lambda_j > 0} (-1)^{\beta_j},$$

where λ_j are the eigenvalues of the linearized operator of T_0 and β_j its mutiplicity. A simple calculation shows that the linearized operator of T_0 at $u = 1$ is

$$L = \triangle + n.$$

Therefore,

$$\deg(T_1, \mathcal{O}, 0) = \deg(T_0, \mathcal{O}, 0) = -1.$$

That is, there is a k-convex solution of equation (1.1). The regularity and estimates of the solution follows directly from Theorem 3.4. □

Remark 3.5. There are still several important issues to be addressed for the equation (1.1). First is the uniqueness question. We only have a uniqueness result for nonnegative solutions of (1.1). We temper to conjecture that the same uniqueness result is true for any admissible solution of (1.1). Of course, this is already known in the cases $k = 1$ and $k = n$. The second outstanding question is when there is a positive solution of equation (1.1). A positive solution of (1.1) is related to problem of prescribing polar area function (see [8]). Finally, a sufficient condition was given in [7] for the existence of *convex* solution to equation (1.1) (see also [12]). A *convex* solution is a solution to Christoffel-Minkowski problem, one would like to obtain necessary and sufficient conditions for the existence of *convex* solutions.

References

[1] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian* Acta Math. **155**, (1985),261 - 301.

[2] S.Y. Cheng and S.T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem* Comm. Pure Appl. Math. **29**, (1976), 495-516.
[3] S.S. Chern, *Integral formulas for hypersurfaces in euclidean space and their applications to uniqueness theorems*, J. Math. Mech. **8**, (1959), 947-955.
[4] L. Garding, *An inequality for hyperbolic polynomials*, J. Math. Mech, **8**, (1959), 957-965.
[5] D. Gilbarg and N.S. Trudinger, *Elliptic Partical Differential Equations of Second Order*, 2nd edition, Grundlehren 224, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.
[6] B. Guan and P. Guan, *Hypersurfaces with prescribed curvatures*, Preprint, (2000).
[7] P. Guan and X. Ma, *Cheristoffel-Minkowski problem: convexity of solutions of a Hessian Equation*, Preprint, Issac Newton Institute, University of Cambridge, (2001).
[8] P. Guan, X.Ma, N. Trudinger and X. Zhu, *A form of Alexandrov-Fenchel inequality*, Preprint, Max-Planck Institute for Mathematics in Sciences, Leipzig, (2001).
[9] N. Ivochkina, *Solution of the curvature problem for equation of curvature of order m*, Soviet Math. Dokl., **37**, (1988), 322-325.
[10] N.V. Krylov, *On the general notion of fully nonlinear second-order elliptic equations*, Trans. Amer. Math. Soc., **347** (1995), no. 3, 857–895.
[11] Y.Y. Li, *Degree theory for second order nonlinear elliptic operators and its applications*, Comm. PDE, **14** (1989), no. 11, 1541-1578.
[12] A.V. Pogorelov, *The Minkowski multidimensional problem*, Wiley, New York, (1978).
[13] N.S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math., **175** (1995), 151–164.

Department of Mathematics, McMaster University, Hamilton, On. L8S 4K1, Canada.
E-mail address: guan@math.mcmaster.ca

Department of mathematics, East China Normal University, Shanghai, 200062, China.
E-mail address: xnma@math.ecnu.edu.cn