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A FORM OF ALEXANDROV-FENCHEL INEQUALITY

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1. Introduction

There are many important integral formulas and integral inequalities for convex bodies (see [5], [16]). Brunn-Minkowski inequality and Alexandrov-Fenchel inequality are among the most important integral inequalities in the theory of convex bodies, and the Minkowski type integral formulas and more general formulas of Chern are very useful in global geometry of convex hypersurfaces. Most of these formulas and inequalities can be stated in the integral forms on S^n with the *convexity* assumption. It seems of interest to establish similar results without the *convexity* assumption. In [17], Trudinger generalized quermassintegral inequalities for k-convex bodies. For a convex body, the polar of the body is also convex. The support function of the convex body corresponds to the gauge function of its polar body. In other words, the geometry of a convex body can be reflected from its polar dual. With this relation, we will introduce a class of domains in $\mathbb{R}^{n+1}$ called k^*-convex (see Definition 4.2) as a natural generalization of convex bodies. We will derive a form of the Alexandrov-Fenchel inequality for this class of domains. k^*-convexity is related to a Hessian equation on S^n by its distance function. It will also be shown that the assumption of *convexity* is not necessary for some of the integral formulas. We will also prove a uniqueness theorem for the Hessian equation, which generalizes the classical Alexandrov-Fenchel-Jensen theorem.

In most cases, our proofs are not far different of those known in the convex case with two exceptions. First, we work directly on the functions and related vector-valued forms on S^n without *convexity* assumptions. Secondly, we make use of hyperbolic polynomial theory instead of Alexandrov's mixed discriminant inequality, which enable us to replace the *convexity* by the more general notions. Our arguments are drawn mainly from three important papers: Chern [8], Cheng-Yau [7] and Garding [10]. In fact, the hyperbolic polynomial theory was already used by

Chern in [8] in the proof the uniqueness theorem and in the proof of Alexandrov-Fenchel inequality for convex bodies in Hörmander [14]. It should be noted that the hyperbolicity of the elementary symmetric functions plays important role in the development of fully nonlinear equations in the work of Caffarelli-Nirenberg-Spruck [6]. So, it is not a coincident the this theory is used here in a crucial way.

2. Integral Formulas for the functions on S^n

Let $e_1, ..., e_n$ is an orthonormal frame on S^n, let $\omega_1, ..., \omega_n$ be the corresponding 1-forms. For each function $u \in C^2(S^n)$, let u_i be the covariant derivative of u with respect to e_i. We define a vector valued function

$$Z = \sum_{i=1}^{n} u_i e_i + u e_{n+1}.$$

where e_{n+1} is the position vector on S^n, that is, the outer normal vector field of S^n. We note that Z is globally defined on S^n. We write the hessian matrix of u with respect to the frame as

$$W = \{u_{ij} + u\delta_{ij}\}.$$

We calculate that,

$$\begin{aligned} u &= Z \cdot e_{n+1}, \\ dZ &= \sum_{i=1}^{n}(du_i e_i + u_i de_i) + du e_{n+1} + u de_{n+1} \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{n} u_{ij}\omega^j - \sum_{j=1}^{n} u_j \omega_j^i) e_i + \sum_{i=1}^{n}(\sum_{\alpha=1}^{n+1} u_i \omega_i^\alpha e_\alpha) \\ &\quad + \sum_{i=1}^{n}(u_i \omega^i) e_{n+1} + u \sum_{i=1}^{n} \omega^i e_i \\ &= \sum_{j=1}^{n}(\sum_{i=1}^{n}(u_{ij} + \delta_{ij} u) e_i)\omega^j. \end{aligned}$$

Let $u^1, ..., u^{n+1} \in C^2(S^n)$, we define $\forall l = 1, ..., n+1$,

$$Z^l = \sum_{i=1}^{n} u_i^l e_i + u^l e_{n+1},$$

and

$$W^l = \{u_{ij}^l + u^l \delta_{ij}\}$$

Set,

$$\Omega(u^1,...,u^{n+1}) = (Z^1, dZ^2, dZ^3, ..., dZ^{n+1}). \tag{2.1}$$

and

$$V(u^1,u^2...,u^{n+1}) = \int_{S^n} \Omega(u^1,...,u^{n+1}). \tag{2.2}$$

We note that

$$\Omega(u^1,...,u^{n+1}) = u^1 S_n(W^2,...,W^{n+1})ds \tag{2.3}$$

where $S_n(W^2,...,W^{n+1})$ is the mixed determinant and ds is the standard area form on S^n. In particular, $\forall 1 \leq k \leq n$, if we set $u^{k+2} = ... = u^{n+1} = 1$, we obtained

$$\Omega(u^1,...,u^{n+1}) = u^1 S_k(W^2,...,W^{k+1})ds \tag{2.4}$$

where $S_k(W^2,...,W^{k+1})$ is the complete polarization of the symmetric function S_k defined for symmetric matrices.

Lemma 2.1. *V is a symmetric multilinear form on $(C^2(S^n))^{n+1}$.*

Proof. The multilinearility follows directly from the definition. Also, by the definition, for any permutation σ of $\{2,...,n+1\}$,

$$\Omega(u^1,u^2...,u^{n+1}) = \Omega(u^1,u^{\sigma(2)}...,u^{\sigma(n+1)}),$$

so $V(u^1,u^2...,u^{n+1}) = V(u^1,u^{\sigma(2)}...,u^{\sigma(n+1)})$. To see V is a symmetric form, we only need to show

$$V(u^1,u^2,u^3...,u^{n+1}) = V(u^2,u^1,u^3...,u^{n+1}). \tag{2.5}$$

We first assume $u^i \in C^3(S^n), \forall i$. Let,

$$\omega(u^1,...,u^{n+1}) = (Z^1, Z^2, dZ^3, ..., dZ^{n+1}),$$

we have

$$d\omega(u^1,...,u^{n+1}) = -\Omega(u^2,u^1,u^3...,u^{n+1}) + \Omega(u^1,u^2,u^3...,u^{n+1}),$$

Now, (2.5) follows from Stokes theorem. The identity (2.5) is valid for C^2 function by approximation. □

We remark that if $u^1,...,u^{n+1}$ are the support functions of some convex bodies $K_1,...,K_{n+1}$ respectively, then $V(u^1,u^2...,u^{n+1})$ is exactly the Minkowski mixed volume $V(K_1,...,K_{n+1})$.

The following is a direct corollary of the lemma. If u is a support function of a convex body, it is well known as Minkowski type integral.

Corollary 2.2. *For any function $u \in C^2(S^n)$, $W = \{u_{ij}+\delta_{ij}u\}$. For any $1 \le k < n$, we have the Minkowski type integral formulas.*

$$\int_{S^n} uS_k(W)\,ds = \int_{S^n} S_{k+1}(W)\,ds, \tag{2.6}$$

where ds is the standard area element on S^n.

For any $n \times n$ symmetric matrice $W_1, ..., W_k$, let $S_k(W_1, ..., W_k)$ be the complete polarization of S_k. Let u and $\tilde{u}$ are two C^2 functions on S^n. Let W and $\tilde{W}$ are the corresponding Hessian matrice of u and $\tilde{u}$ respectively. Following Chern's notations in [8], we let $P_{rs} = S_{r+s}(W, ..., W, \tilde{W}, ...\tilde{W})$ where W appears r times and $\tilde{W}$ appears s times. So, P_{rs} is a polynomial in W_{ij}, $\tilde{W}_{ij}$, homogeneous of degrees r and s respectively. The following dual generalization of Chern's formulas is another corollary of Lemma 2.1.

Corollary 2.3. *Suppose u and $\tilde{u}$ are two C^2 functions on S^n, then the following identities hold.*

$$\int_{S^n} [uP_{0k} - \tilde{u}P_{1,k-1}]dx = 0, \tag{2.7}$$

$$\int_{S^n} [uP_{k-1,1} - \tilde{u}P_{k0}]dx = 0, \tag{2.8}$$

and,

$$\begin{aligned} &2\int_{S^n} u(P_{0k} - P_{k-1,1})dx \\ &= \int_{S^n} \{\tilde{u}(P_{1,k-1} - P_{k0}) - u(P_{k-1,1} - P_{0k})\}dx = 0. \end{aligned} \tag{2.9}$$

3. k-CONVEX FUNCTIONS ON S^n

Now, we consider functions satisfying the following equation,

$$S_k(W) = \varphi \quad \text{on} \quad S^n. \tag{3.1}$$

Definition 3.1. For $1 \le k \le n$, let Γ_k is a convex cone in $\mathbb{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, ..., S_k(\lambda) > 0\}$$

Suppose $u \in C^2(S^n)$, we say u is k-convex, if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in S^n$. u is convex on S^n if W is semi-positive definite on S^n. Furthermore, u is called an admissible solution of (3.1), if u is k-convex and satisfies (3.1).

The next is a uniqueness theorem which generalizes Alexandrov-Fenchel-Jessen theorem ([2], [9] and [8]) to k-convex case.

Theorem 3.2. *Suppose u and $\tilde{u}$ are two C^2 k-convex functions on S^n satisfying (3.1). If $S_k(W) = S_k(\tilde{W})$, and if one of u and $\tilde{u}$ is nonnegative, then $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$ on S^n.*

Proof of Theorem 3.2. We follow the same lines as in [8]. We may assume u is nonnegative. Since $S_k(W)$ is positive, we conclude that u is positive almost everywhere on S^n. By the hyperbolicity of S_k, $\forall W^i \in \Gamma_k, i = 1, ..., k$,

$$S_k(W^1, ..., W^k) \geq S_k(W^1) \cdots S_k(W^k), \tag{3.2}$$

with the equality holds if and only if the k matrices are pairwise proportional.

Suppose $S_k(W) = S_k(\tilde{W})$ on S^n, where $W = \{u_{ij} + \delta_{ij} u\}$ and $\tilde{W} = \{\tilde{u}_{ij} + \delta_{ij}\tilde{u}\}$ The left hand side of the integral formula (2.9) in Corollary 2.3 is non-positive.The same is therefore true of the right hand side of (2.9). The latter is ansti-symmetric on the two function u and $\tilde{u}$, and hence must be zero. It follows that $P_{k-1,1} = P_{0k}$ by (3.2). Again, the equality gives that W and $\tilde{W}$ are proportional. Since $S_k(W) = S_k(\tilde{W})$, we conclude that $W = \tilde{W}$ at each point of S^n, that is, $u - \tilde{u} \in Span\{x_1, ..., x_{n+1}\}$. □

The following is an infinitesimal version of Theorem 3.2, which we will use in our proof of the generalized Alexandrov-Fenchel inequality.

Proposition 3.3. *For any C^2 function u, let L_u be the linearized operator of the Hessian operator $S_k(\{u_{ij} + \delta_{ij} u\})$. Then L_u is self-adjoint. If in addition, u is nonnegative admissible solution of (3.1), the kernel of L_u is $Span\{x_1, ..., x_{n+1}\}$.*

The above proposition is a special case of the following result.

Proposition 3.4. *$\forall u^2, ..., u^k \in C^2(S^n)$ fixed, define*

$$L(v) = \Omega(1, v, u^2 ..., u^k, 1, ..., 1), \tag{3.3}$$

then, L is self-adjoint. If in addition, $u^2, ..., u^k$ are k-convex, and at least one of them is nonnegative, the kernel of L is $Span\{x_1, ..., x_{n+1}\}$.

Proof of Proposition 3.4. First if $u \in C^3$, the linearized operator L_u of S_k is self-adjoint (see, e.g., [15]). Let $t_2, ..., t_k$ be real numbers, let $u_t = \sum_{l=2}^k t_l u^l$, the operator L in (3.3) is the coefficient of the linearized operator L_{u_t} of $t_2 \cdots t_k$. Since L_{u_t} is self-adjoint for all $t = (t_2, ..., t_k)$, we conclude that L is self-adjoint,if $u^l \in C^3, \forall 2 \leq l \leq k$. By approximation, the same conclusion is true for C^2 functions.

To compute the kernel, we may assume u^2 is nonnegative. Since u^2 is k-convex, it is positive almost everywhere. Suppose v is in kernel of L, i.e.,

$$L(v) = 0. \tag{3.4}$$

Simple calculation shows that

$$\Omega(1, v, v, u^3, ..., u^k, 1, ..., 1) = S_k(A, A, W^3, ..., W^k)ds,$$

where $A = \{v_{ij} + \delta_{ij}v\}$ and $W^l = \{u^l_{ij} + \delta_{ij}u^l\}$.

We *claim* that, if (3.4) holds, then

$$S_k(A, A, W^3, ..., W^k) \le 0, \tag{3.5}$$

with equality if and only if $A = 0$, i.e., $v \in Span\{x_1, ..., x_{n+1}\}$.

We note that,

$$\begin{aligned} 0 &= \int_{S^n} vL(v) = \int_{S^n} \Omega(v, v, u^2, u^3, ..., u^k, 1, ..., 1) \\ &= V(v, v, u^2, u^3, ..., u^k, 1, ..., 1) = V(u^2, v, v, u^3, ..., u^k, 1, ..., 1) \\ &= \int_{S^n} u^2\Omega(1, v, v, u^3, ..., u^k, 1, ..., 1) \\ &= \int_{S^n} u^2 S_k(A, A, W^3, ..., W^k)ds. \end{aligned}$$

If the *claim* is true, we will conclude that v is in $Span\{x_1, ..., x_{n+1}\}$ since u^2 is positive almost everywhere.

To prove the claim, we make use of the result of Garding [10] result on hyperbolicity of S_k in the cone Γ_k in [10] (see also [14]). Since u^l is k-convex, $W^l \in \Gamma_k, \forall 2 \le l \le k$. For $W^3, ..., W^k$ fixed, the polarization $S_k(B, B, W^3, ..., W^k)$ is also hyperbolic and complete for $B \in \Gamma_k$. Let $W_t = W^2 + tA$, we have

$$\begin{aligned} &S_k(W_t, W_t, W^3, ...W^k) = S_k(W^2, W^2, W^3, ..., W^k) \\ &+2tS_k(A, W^2, W^3, ..., W^k) + t^2 S_k(A, A, W^3, ..., W^k). \end{aligned}$$

Since

$$S_k(W^2, W^2, W^3.., W^k) > 0,$$

and

$$S_k(A, W^2, ..., W^k) = 0.$$

By the hyperbolicity, $S_k(W_t, W_t, W^3, ..., W^k)$ has only real roots in t variable, so (3.5) must be true. If in addition, $S_k(A, A, W, ..., W) = 0$, we would have

$$S_k(W_t, W_t, W, ..., W) = S_k(W, ..., W),$$

for all $t \in \mathbb{R}$. By the completeness, $A = 0$. The *claim* is proved. □

4. k^*-convex bodies and Alexandrov-Fenchel inequality

For any $n \geq k \geq 1$ fixed, set $u^{k+2} = ... = u^{n+1} = 1$ we define $\forall u^1, ..., u^{k+1} \in C^2(S^n)$,

$$V_{k+1}(u^1, u^2, ..., u^{k+1}) = V(u^1, u^2, ..., u^{n+1}). \tag{4.1}$$

Now we state a form of Alexandrov-Fenchel inequality for positive k-convex functions.

Theorem 4.1. *If $u^1, ..., u^k$ are k-convex, and u_1 positive, and at least one of u^l is nonnegative on S^n (for $2 \leq l \leq k$), then $\forall v \in C^2(S^n)$,*

$$V_{k+1}^2(v, u^1, ..., u^k) \geq V_{k+1}(u^1, u^1, u^2, ..., u^k) V_{k+1}(v, v, u^2, ..., u^k), \tag{4.2}$$

the equality holds if and only if $v = au^1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, ..., a_{n+1}$.

Our proof of the theorem follows the similar arguments of Alexandrov's second proof of Alexandrov-Fenchel inequality in [2] (see also [14]), which in turn is adapted from Hilbert's proof of the Brunn-Minkowski inequality when $n = 3$. Instead of using Alexandrov's inequality for mixed discriminants in his original proof, we will make use of the hyperbolicity of the elementary symmetric functions as in [14]. This replacement enable us to drop the *convexity* assumption. We sketch here some key steps.

Proof.

Statement: If

$$V_{k+1}(v, u^1, u^2, ..., u^k) = 0, \quad \text{for some} \quad v \in C^2(S^n), \tag{4.3}$$

then

$$V_{k+1}(v, v, u^2, ..., u^k) \leq 0, \tag{4.4}$$

with equality if and only if $v = \sum_{i=1}^{n+1} a_i x_i$.

The theorem follows directly from above statement. The proof of the **Statement** will be reduced to an eigenvalue problem for certain elliptic differential operators.

Fisrt, for $u^2, ..., u^k \in \Gamma_k$ fixed, we set

$$L(v) = \Omega(1, v, u^2 ..., u^k, 1, ..., 1).$$

By Garding [10] $L(v) > 0$ if v is k-convex. We claim that L is an elliptic differential operator with negative principal symbol. This can be done following the same line as in [14]. The principal symbol of L at the co-tangent vector $\theta = (\theta_1, ..., \theta_n)$ is obtained when A is replaced by $-\theta \otimes \theta$ in

$$S_k(A, W^2, ..., W^k).$$

So it is equal to

$$-S_k(\theta \otimes \theta, W^2, ..., W^k).$$

Since S_k is hyperbolic with respect to the positive cone Γ_k, and $\theta \otimes \theta$ is semi-positive definite and is not a 0 matrix if θ not 0. By the complete hyperbolicity,

$$-S_k(\theta \otimes \theta, W^2, ..., W^k) < 0.$$

We now use continuity method to finish the job. For $0 \leq t \leq 1$, let $u_t^i = t + (1-t)u^i$, and set

$$\rho_t = \frac{\Omega(1, u_t^1, u_t^2 ..., u_t^k, 1, ..., 1)}{u_t^1},$$

We examine the eigenvalue problem:

$$L_t(v) = \lambda \rho_t v. \tag{4.5}$$

If for we set $Q_t(u, v) = \int_{S^n} u L_t(v)$, the eigenvalue problem (4.5) is corresponding to the quadratic form Q_t with respect to the inner-product $< u, v >_{\rho_t} = \int_{S^n} u v \rho_t$.

We want to show **Claim:** $\lambda = 1$ is the only positive eigenvalue of multiplicity 1 with eigenfunction u_t^1, and $\lambda = 0$ is the eigenvalue of multiplicity $n + 1$ with eigenspace $Span\{x_1, ..., x_{n+1}\}$ for the eigenvalue problem of (4.5).

We note that u_t^1 is an eigenfunction corresponding to the eigenvalue $\lambda = 1$. If the *Claim* is true, (4.3) implies that v is orthogonal to eigenspace corresponding to $\lambda = 1$ with respect to the inner product $< ., . >_{\rho_1}$. If the claim is true, **Statement** follows from the standard spectral theory of self-adjoint elliptic operators.

We now prove the *Claim.* When $t = 0$, the problem can be reduced to the following simple form by straightforward calculations:

$$\Delta v + n v = n \lambda v.$$

The eigenvectors of Δ are the spherical harmonics of degree $\nu = 0, 1, ...$, with the corresponding eigenvalues $-\nu(\nu + n - 1)$. $\nu = 0$ corresponds to $\lambda = 1$ and $\nu = 1$ corresponds to $\lambda = 0$ in the eigenvalue problem (4.5) respectively in this special case. And $\lambda < 0$ when $\nu > 1$. It is well known that spherical harmonics of degree 0 are constants, and spherical harmonics of degree 1 are linear functions, i.e.,

$Span\{x_1, ..., x_{n+1}\}$. Therefore, the **Claim** is true for $t = 0$. For arbitrary t, since 1 is an eigenvalue of the problem (4.5) with eigenfunction u_t^1, by the theory of elliptic equations, we only need to prove that 0 is the eigenvalue of multiplicity $n + 1$. It's obvious that $x_1, ..., x_{n+1}$ are the eigenfunctions of L corresponding to the eigenvalue 0. The theorem now follows from Proposition 3.4. □

Now, we consider a class of domains which will be named k^*-convex. They can be viewed as a generalization of convex bodies via polar dual. Let D be a star-shaped bounded domain in $\mathbb{R}^{n+1}$ with C^2 boundary. The distance function of D is defined as,

$$u(x) = \min\{\lambda | x \in \lambda D\}, \quad \forall x \in S^n. \tag{4.6}$$

When D is convex, the distance function is also called the gauge function of D.

Definition 4.2. Let D be a star-shaped bounded domain in $\mathbb{R}^{n+1}$ with C^2 boundary. We say D is k^*-convex if its distance function u is k-convex on S^n. We day D is polar centrized if its distance function u satisfies

$$\int_{S^n} x_j u(x) ds = 0, \quad \forall j = 1, 2, ..., n + 1.$$

If $D_1, ..., D_{k+1}$ are k^*-convex bodies, let $u_1, ..., u_{k+1}$ are the corresponding distance functions, and $W_1, ..., W_{k+1}$ be the corresponding hessians of the gauge functions respectively. For $0 \leq l \leq k$, we define mixed polar surface area functions

$$S_l(D_1, ..., D_l, x) = S_l(W_1, ..., W_l). \tag{4.7}$$

We call $S_l(D, x) = S_l(W, ..., W)$ the lth polar surface are function of D. We also define a mixed polar volume,

$$V_{k+1}^*(D_1, ..., D_{k+1}) = \frac{1}{V_{k+1}(u_1, ..., u_{k+1})} \tag{4.8}$$

where $V_{k+1}(u_1, ..., u_{k+1})$ defined as in (4.1). We also write, $\forall 0 \leq l \leq k+1$, $V_l^*(D) = V_{k+1}^*(D, ..., D, B, ..., B)$, where B is the unit ball centered at the origin in $\mathbb{R}^{n+1}$, D appears l times, and B appears $k + 1 - l$ times in the formula.

We note that if D is convex, D is polar centrized if and only if the Steiner point of the polar of D is the origin. If D is convex, $V_l^*(D)$ in Definition 4.2 is the is the reciprocal of the lth quermassintegral of the polar of D. The geometric quantities of D and its polar D^* in this case are related by some important inequalities, like Blascke-Santalo inequality, Mahler's conjecture. When D is a centrally symmetric convex body and $l = n+1$, by the work of [4], $V(D)V(D^*) \geq c_n$ for some positive constant c_n depending only on the dimensionality.

As an application, we have the following consequences of Theorem 3.2 and Theorem 4.1.

Theorem 4.3. *Suppose D_1, D_2 are two k^*-convex domains in $\mathbb{R}^{n+1}$. If kth polar surface area functions of D_1 and D_2 are the same, i.e.,*

$$S_l(D_1, x) = S_l(D_2, x), \quad \forall x \in S^n,$$

then, the distance functions of D_1, D_2 are equal upto a linear function. In particular, if both D_1 and D_2 are polar centrized, then $D_1 = D_2$.

Theorem 4.4. *Suppose $D_1, ..., D_{k+1}$ are k^*-convex domains in $\mathbb{R}^{n+1}$,then we have the following Alexandrov-Fenchel inequality for the mixed polar volumes:*

$$(V^*_{k+1}(D_1, ..., D_{k+1}))^2 \leq V^*_{k+1}(D_1, D_1, D_3 ..., D_{k+1}) V^*_{k+1}(D_2, D_2, D_3 ..., D_{k+1}),$$

with the equality if and only if the distance functions of D_1 and D_2 are equal upto a linear function. In particular, if both D_1, D_2 are polar centerized, then $D_1 = \lambda D_2$ for some $\lambda > 0$.

The above theorem indicates that the reciprocal of the mixed polar volume is log-concave. Therefore, one may deduce a sequence of inequalities for k^*-convex domains from Theorem 4.1 as in the convex case (see section 20 in [5], section 6.4 in [16] and appendix in [14]). In particular, one can obtain the corresponding Brunn-Minkowski inequality and quermassintegral inequalities for V^*.

Corollary 4.5. *Suppose D_1, D_2 are k^*-convex, then for $0 \leq t \leq 1$,*

$$V^*_{k+1}((1-t)D_1 + tD_2)^{\frac{-1}{k+1}} \geq (1-t) V^*_{k+1}(D_1)^{\frac{-1}{k+1}} + t V^*_{k+1}(D_2)^{\frac{-1}{k+1}},$$

if D_1, D_2 are polar centralized, the equality for some $0 < t < 1$ holds if and only if $D_1 = \lambda D_2$ for some $\lambda > 0$. If D is k^*-convex, then for $0 \leq i < j < l \leq k+1$,

$$(V^*_j(D))^{l-i} \leq (V^*_i(D))^{l-j} (V^*_l(D))^{j-i}.$$

if D is polar centralized, the equality holds if and only if D is a ball centered at the origin. In particular, if we let σ_n be the volume of the unit ball B in $\mathbb{R}^{n+1}$,

$$\sigma_n^{i-j} (V^*_j(D))^{k-i} \leq (V^*_i(D))^{k-j},$$

if D is polar centralized, the equality holds if and only if D is a ball centered at the origin.

We discuss some problems arising from the subject we discussed.

(1). It is important to study the Steiner and Schwarz symmetrization processes for k^*-convex bodies ([18]), that would provide us better understanding of the Hessian equation (3.1).

(2). The next is prescribing k^* surface area function problem. The equivalent analytical problem is to find a positive solution of the equation (3.1).Of course, φ has to satisfy $\int_{S^n} x\varphi(x) = 0$. The existence and regularity of admissible solutions have been established in [13]. The questions is when an admissible solution is positive? This means that, to find a necessary and sufficient conditions for φ such that the above equation has a positive solution. We note that if u is a solution of (3.1), $u(x) + l(x)$ is also a solution for any linear function $l(x)$. Therefore, it is required to put some restriction on the solutions. One of that is to require u orthogonal to the span of $x_1, ..., x_{n+1}$, that corresponds to find a polar centralized k^*-convex body. The uniqueness of the positive solutions for the problem is a consequence of Theorem 3.2.

(3). Equation (3.1) has another important geometric connection, it is related to the Christoffel-Minkowski problem. In that case, one looks for *convex* solutions of (3.1). The intermediate Christoffel-Minkowski problem is still unsolved. A sufficient condition was obtained in a recent paper [12], further references of this problem can be found in [12].

(4). Equation (3.1) is a model case for general equation of the form:

$$F(\{u_{ij} + \delta_{ij}u) = \varphi.$$

F is assumed to be elliptic and concave. One would like to understand the uniqueness and existence questions for the equation. We also want to know when a solution is convex. When F is a quotient of Hessians, the equation related to prescribing Weingarten curvatures on outer normals, we refer [3], [8] and [11] for further references.

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