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by

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Boundary Concentrated Finite Element Methods

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Abstract

A method with optimal (up to logarithmic terms) complexity for solving elliptic problems is proposed. The method relies on interior regularity but the solution may have globally low regularity due to rough boundary data or geometries. Elliptic regularity results, high order approximation results, and an efficient preconditioner are presented.

The method is utilized to realize, with linear-logarithmic complexity, an accurate and data-sparse approximations to the associated elliptic Poincaré-Steklov operators. Further applications include the treatment of exterior boundary value problems and its use in the framework of domain decomposition methods.

Keywords: *hp*-finite element methods, preconditioning, data-sparse approximation to Poincaré-Steklov operator, meshes refined toward boundary
AMS Subject Classification: 65N35,65F10, 35D10

1 Introduction

In this paper, we present the *boundary concentrated finite element method.* This method is designed to solve numerically elliptic boundary value problems with low global Sobolev regularity. The coefficients of the underlying PDE, however, are assumed to be smooth so that, owing to interior elliptic regularity, the low global Sobolev regularity is due to boundary effects such as low-regularity boundary data or geometries. The key idea of the method is to exploit this interior regularity in the framework of the *hp*-version of the finite element method (*hp*-FEM) by using low order elements on refined meshes near the boundary and high order polynomials on large elements in the interior of the domain. The combination of mesh refinement near the boundary and polynomial degree distribution proposed in this paper concentrates most degrees of freedom in a narrow neighborhood of the boundary, explaining the name boundary concentrated FEM.
Since the boundary concentrated FEM may be viewed as a generalization of the boundary element method (BEM), we illustrate its most important properties by a side-by-side comparison with the classical BEM. In the BEM (see, e.g., [17] for an introduction to the topic), an elliptic boundary value problem on a domain $\Omega \subset \mathbb{R}^d$ is reduced to a problem posed on the boundary $\partial\Omega$ thereby effecting a dimensional reduction. This dimensional reduction immediately leads to a reduction of the problem size of the discrete problems. In the present paper, we show that

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in the "error vs. degrees of freedom" perspective, the boundary concentrated FEM achieves the same rate of convergence as the classical, low order Galerkin h-BEMs that are formulated on quasi-uniform boundary triangulations. In this respect, therefore, the boundary concentrated FEM is comparable to the classical BEM. However, it represents a generalization of the BEM in that it can be formulated for equations with variable (albeit piecewise analytic) coefficients while the BEM is effectively restricted to equations with constant coefficients because explicit knowledge of a fundamental solution is required.
A second difference between the classical BEM and the boundary concentrated FEM manifests itself in the structure of the resulting linear system of equations. The boundary concentrated FEM, being a FEM, naturally leads to sparse stiffness matrices. In contrast, the stiffness matrix in BEM is in general fully populated. We mention, however, that this drawback of the classical BEM has been successfully overcome in recent years by various compression schemes, notably the panel clustering techniques [20], multi-pole expansions (see the survey [12] and the references therein), and wavelet compression methods [10]. A generalization of the clustering techniques are the recently introduced $\mathcal{H}$-matrices, [18, 19].
A further interesting point of comparison is the cost of setting up and solving the linear system. We show in this paper that the stiffness matrix of the boundary concentrated FEM can be computed with optimal complexity $O(N)$, where N is the problem size. The classical BEM, which, as we mentioned above, achieves a comparable accuracy with the same number of degrees of freedom N, requires $O(N^2)$ operations to set up the linear system due to the fact that the stiffness matrix is fully populated. Again, only recent progress in compression schemes for the BEM has led to methods with complexity $O(N \log^q N)$ for suitable $q \in \mathbb{N}_0$.
Another important observation is that our technique leads to the accurate and data-sparse approximation of complexity $O(N \log N)$ to Poincaré-Steklov operators associated with elliptic equations with variable coefficients. This generalizes previously known methods for equations with piecewise constant coefficients in polygonal domains such as [22–25].
In this paper we present a complete theory in the two-dimensional setting. Many results, however, have analogs in higher dimensions. In particular, the regularity assertion (Theorem 1.4) and hp-approximation results on shape regular meshes (Theorem 2.13) can be extended in a straightforward way to three dimensions. Preconditioning techniques for hp-FEM/hp-BEM in three dimensions have recently been proposed, [1, 15, 35], and we expect that these ideas can be employed for the successful development of preconditioners for the boundary concentrated FEM in 3D.

The paper is organized as follows: We start with a formulation of the model problem and provide analytic regularity results for the solution. In Section 2, we present convergence results for the hp-FEM applied to the model problem and show that the method yields the same optimal convergence rate as the h-BEM on quasi-uniform meshes. In Section 3, we address the question of efficiently solving the resulting linear system. For Dirichlet problems we show that the condition number of the linear system grows only polylogarithmically with the problem size. For Neumann problems, we exhibit a block-diagonal preconditioner such that the condition number of the preconditioned system grows again polylogarithmically. We show in Section 4 how the boundary concentrated FEM can be employed to realize an application of the Poincaré-Steklov operator with linear-logarithmic complexity with respect to the boundary degrees of freedom (both in operation count and memory requirement). So far, for the sake of simplicity, our discussion has been mainly restricted to interior problems with analytic coefficients. Further applications of our approach in the case of exterior problems and in the domain decomposition framework (piecewise smooth data with respect to a regular geometric decomposition) are addressed in Section 5. Numerical experiments in Section 6 illustrate the theoretical results of Sections 2 and 3.

1.1 Notation

For a Lipschitz domain $\Omega \subset \mathbb{R}^2$, the Sobolev spaces $H^k(\Omega)$, $H_0^k(\Omega)$, $k \in \mathbb{N}_0$, are defined in the standard way. Fractional order Sobolev spaces $H^s(\Omega)$ are defined by interpolation (the real method) between integer order Sobolev spaces. Negative order space such as $H^{-1}(\Omega)$ are defined by duality: $H^{-s}(\Omega) = (H_0^s(\Omega))'$. Spaces on the boundary $\partial\Omega$ are defined in the usual way:

$$H^s(\partial\Omega) = \begin{cases} \{u|_{\partial\Omega} \mid u \in H^{s+1/2}(\Omega)\} & \text{if } s > 0 \\ L^2(\partial\Omega) & \text{if } s = 0 \\ (H^{-s}(\partial\Omega))' & \text{if } s < 0. \end{cases}$$

We mention at this point the important fact that for *polygonal domains* Ω, the spaces $H^s(\partial\Omega)$, $|s| < 3/2$ are invariant under piecewise smooth changes of parametrization of $\partial\Omega$. In particular, the parametrization $\varphi : [0, L) \to \partial\Omega$ by arc length provides an isomorphism $u \mapsto u \circ \varphi$ from $H^s(\partial\Omega)$ to $H^s_{per}([0, L))$ for $|s| < 3/2$. Here, for $s > 0$ we set $H^s_{per}([0, L)) := \{u \in H^s(\mathbb{R}) \mid u \text{ is } L\text{-periodic}\}$ with the corresponding topology and $H^{-s}_{per}([0, L)) = \left(H^s_{per}([0, L))\right)'$ for $s < 0$.
Duality pairings will be denoted by $\langle \cdot, \cdot \rangle$ with subscripts indicating with respect to which spaces the pairing is taken. Since the spaces $H^{1/2}(\partial\Omega)$, $H^{-1/2}(\partial\Omega)$ arise frequently in this paper, we abbreviate

$$Y := H^{1/2}(\partial\Omega), \qquad Y' := H^{-1/2}(\partial\Omega). \tag{1.1}$$

1.2 Problem Class

For simplicity of exposition, we will restrict our attention to problems formulated on polygons, and we will not consider the case of mixed boundary conditions. That is, we consider for a *polygonal* Lipschitz domain $\Omega \subset \mathbb{R}^2$ either the Dirichlet problem

$$\mathcal{L}u = f \in L^2(\Omega), \qquad \text{in } \Omega, \tag{1.2a}$$

$$\gamma_0 u = \lambda \in H^{1/2}(\partial\Omega) \qquad \text{on } \partial\Omega, \tag{1.2b}$$

or the Neumann problem

$$\mathcal{L}u = f \in L^2(\Omega) \qquad \text{in } \Omega, \tag{1.3a}$$

$$\gamma_1 u = \psi \in H^{-1/2}(\partial\Omega) \qquad \text{on } \partial\Omega. \tag{1.3b}$$

Here, the differential operator $\mathcal{L}$ is given by

$$\mathcal{L}u := -\nabla \cdot (A\nabla u) + b \cdot \nabla u + a_0 u \tag{1.4}$$

with uniformly (in $x \in \overline{\Omega}$) symmetric positive definite matrix $A = (a_{ij})_{i,j=1}^2$; the vector-valued function b, and the scalar valued function a_0 are assumed to be analytic on $\overline{\Omega}$. The operator γ_0

is the trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and $\gamma_1 := \sum_{i,j=1}^{2} n_i a_{ij} \partial_j$ is the co-normal derivative operator. We assume that the operator $\mathcal{L}$ generates an $H^1(\Omega)$-elliptic bilinear form

$$B(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}\partial_j u \partial_i v + \sum_{i=1}^{2} b_i \partial_i u v + a_0 uv \, dx, \tag{1.5}$$

i.e.,

$$c_0 \|u\|_{1,\Omega}^2 \le B(u,u) \le c_1 \|u\|_{1,\Omega}^2 \qquad \forall u \in V, \tag{1.6}$$

where we introduced the space $V \subset H^1(\Omega)$ in the standard way as

$$V := \begin{cases} H_0^1(\Omega) & \text{if the Dirichlet problem (1.2) is considered} \\ H^1(\Omega) & \text{if the Neumann problem (1.3) is considered.} \end{cases} \tag{1.7}$$

The boundary value problems (1.2), (1.3) are understood in the usual, variational sense. Solving (1.2) is equivalent to

$$\text{Find } u \in H^1(\Omega) \text{ with } \gamma_0 u = \lambda \text{ and } B(u,v) = \int_\Omega f\, v\, dx \qquad \forall v \in H_0^1(\Omega). \tag{1.8}$$

Solving (1.3) reads:

$$\text{Find } u \in H^1(\Omega) \text{ s.t. } B(u,v) = \int_\Omega f\, v\, dx + \langle \psi, \gamma_0 v \rangle_{Y' \times Y} \qquad \forall v \in H^1(\Omega). \tag{1.9}$$

1.3 Assumptions on the data

In this paper we make the following assumptions on the data:

$$\begin{array}{c} \text{the coefficients } A,\ b,\ a_0, \text{ and the right-hand side } f \text{ are analytic on } \overline{\Omega} \text{ and} \\ \text{the solution } u \in H^{1+\delta}(\Omega) \text{ for some } \delta \in (0,1]. \end{array} \tag{1.10}$$

Such a situation arises, for example, if the boundary data λ, ψ are not smooth and/or of the domain Ω is merely a Lipschitz domain.

The problem class under consideration may be viewed as a generalization of the setting of the classical BEM in that, while the boundary input data are allowed to be rough, the coefficients of the differential equation are smooth on Ω. The particular case of constant coefficients and homogeneous right-hand side, which is the setting of the BEM, is a special case.

Remark 1.1 The method proposed in this paper could be easily adapted to the case of piecewise analytic coefficients A, b, a_0 and right-hand side f. The further discussion on this topic can be found in Section 5.3. ■

Remark 1.2 Methodologically, the analysis of the present paper is closely related to the classical *hp*-FEM, [37, 39]. In the classical *hp*-FEM, stronger regularity assumptions are made, namely, piecewise analyticity of the boundary $\partial\Omega$ and the boundary data λ, ψ is stipulated. These stronger regularity assumptions imply stronger regularity results for the solution u. In the classical *hp*-FEM, these stronger regularity assertions for u are exploited to design exponentially convergent methods by using meshes that are graded geometrically towards few singularities located at the boundary. Our weaker regularity assumptions (1.10) require geometric refinement towards the whole boundary and lead to algebraic rates of convergence only. Nevertheless, the algebraic rates obtained in this paper are optimal (in the sense of n-widths) for the class of problems characterized by the regularity assumptions (1.10). ■

Remark 1.3 Our regularity assumption (1.10) makes strong smoothness assumptions on the right-hand side f. However, the techniques presented in this paper could be employed for methods for solving

$$\mathcal{L}u = f, \qquad \gamma_1 u = \psi$$

with $f \in H^{-1+\delta}(\Omega)$ and $u \in H^{1+\delta}(\Omega)$, $\delta \in (0, 1/2)$. In that case, let $u_0 \in H_0^{1+\delta}(\Omega)$ be the particular solution of $\mathcal{L}u = f$ in Ω solving

$$B(u_0, v) = \int_\Omega f(x) v dx \qquad \forall v \in H_0^1(\Omega). \tag{1.11}$$

For the remaining $\mathcal{L}$-harmonic component of the solution $u_H = u - u_0 \in V_H$, where

$$V_H = \{v \in V : B(v, z) = 0 \quad \forall z \in H_0^1(\Omega)\}, \tag{1.12}$$

we have the equation

$$B(u_H, v) = \int_{\partial\Omega} \psi v ds + \langle \gamma_1 u_0, v \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \qquad \forall v \in H^1(\Omega) \tag{1.13}$$

and note that u_H solves an equation satisfying the regularity assumptions (1.10). For finite element discretizations, we may use different ansatz spaces to approximate the solutions to equations (1.11) and (1.13). The only constraint is that these spaces have to provide *the same trace space on* $\partial\Omega$. In particular, we obtain $u = u_0 + u_H$ with $u_0 \in H^{1+\delta}(\Omega)$, $u_H \in \widetilde{\mathcal{B}}^2_{1-\delta}$, where the countably normed space $\widetilde{\mathcal{B}}^2_{1-\delta}$ is defined in (1.17) below. ■

1.4 Regularity of the solution

The key to efficiently treating (1.2), (1.3) numerically are precise regularity assertions for their solutions. In the case analyzed in the classical *hp*-FEM (see Remark 1.2), the regularity of the solution u is best described in terms of the countably normed spaces $\mathcal{B}^2_\beta$, [3, 4]. This regularity assertion allows for a rigorous proof of exponential convergence of the *hp*-FEM on suitably chosen meshes, [37]. We are interested in the case of the weakened regularity assumptions (1.10). That is, we seek regularity assertions for the solution u to the differential equation

$$\mathcal{L}u = f \qquad \text{on } \Omega \tag{1.14}$$

for f analytic on $\overline{\Omega}$; the boundary conditions—of Dirichlet, Neumann, or mixed type—however, may be rough. By standard interior regularity, [32, Chapter 5], any solution u to (1.14) is analytic on Ω but control of higher order derivatives is lost as one approaches the boundary $\partial\Omega$. Yet, it is possible to measure the blow-up of higher order derivatives near the boundary in terms of weighted spaces. A very precise control, which is suitable for the *hp*-FEM error analysis below, is achieved with the countably normed space $\widetilde{\mathcal{B}}^2_\beta$ that we define as follows: We introduce the distance function r by

$$r(x) := \operatorname{dist}(x, \partial\Omega) \tag{1.15}$$

and define for $\beta \in [0, 1)$ the space $H^2_\beta(\Omega)$ as the completion of $C^\infty(\Omega)$ under the norm

$$\|u\|^2_{H^2_\beta(\Omega)} := \|u\|^2_{H^1(\Omega)} + \|r^\beta \nabla^2 u\|^2_{L^2(\Omega)}. \tag{1.16}$$

For analytic coefficients in the differential operator $\mathcal{L}$, the regularity of the solutions to (1.14) can be described in terms of countably normed spaces, akin to the spaces $\mathcal{B}^2_\beta(C,\gamma)$ introduced in [3,4]. Specifically, for C, $\gamma > 0$, $\beta \in [0,1)$ we define the space $\widetilde{\mathcal{B}}^2_\beta(C,\gamma)$ by

$$\widetilde{\mathcal{B}}^2_\beta(C,\gamma) = \{u \in H^2_\beta(\Omega) \mid \|u\|_{H^2_\beta(\Omega)} \le C, \quad \|r^{\beta+p}\nabla^{p+2}u\|_{L^2(\Omega)} \le C\gamma^p p! \qquad \forall p \in \mathbb{N}\}. \tag{1.17}$$

We then have the following result (see Theorem A.1 for the proof, where in fact the assumptions on the right-hand side f are slightly weaker).

Theorem 1.4 *Let Ω be a Lipschitz domain. Let A, b, a_0, f be analytic on $\overline{\Omega}$ and assume that $u \in H^{1+\delta}(\Omega)$, $\delta \in (0,1]$, solves (1.14). Then u is analytic on Ω, and there exist C, $\gamma > 0$ depending only on Ω, A, b, a_0, δ, and $\|u\|_{H^{1+\delta}(\Omega)}$ such that*

$$u \in \widetilde{\mathcal{B}}^2_{1-\delta}(C,\gamma).$$

Remark 1.5 An analogous result can be formulated if the data A, b, c, f are piecewise analytic. ■

It is of interest to state conditions under which a solution u to (1.14) satisfies $u \in H^{1+\delta}(\Omega)$. For example, for a general Lipschitz domain Ω the solution u of the Dirichlet problem (1.2) satisfies the following shift theorem, [33]:

$$\|u\|_{H^{1+\delta}(\Omega)} \le C_\delta \left[\|f\|_{H^{-1+\delta}(\Omega)} + \|\lambda\|_{H^{1/2+\delta}(\partial\Omega)}\right], \qquad \delta \in [0,1/2), \tag{1.18}$$

provided the right-hand side of (1.18) is finite. (1.18) represents a shift theorem with restriction $\delta \in [0,1/2)$. Shift theorems where one can shift further (i.e., $\delta \ge 1/2$) are known for piecewise smooth boundaries (e.g., polygons). Using the techniques of [13], [5] shows that for a polygon $\Omega \subset \mathbb{R}^2$ there exists $\delta_0 \in (1/2,1]$ (depending on Ω and A) such that the solution u of the Dirichlet problem (1.2) satisfies

$$\|u\|_{H^{1+\delta}(\Omega)} \le C_\delta \left[\|f\|_{H^{-1+\delta}(\Omega)} + \|\lambda\|_{H^{1/2+\delta}(\partial\Omega)}\right], \qquad \delta \in [0,\delta_0). \tag{1.19}$$

We recall further that for convex polygons $\delta_0 = 1$.

2 Discretization by hp-FEM

2.1 Abstract FEM

2.1.1 Formulation

The finite element method (FEM) is obtained from the weak formulations (1.8), (1.9) by replacing the space V with finite dimensional space. For a space $V_N \subset H^1(\Omega)$ the FEM for the Neumann problem (1.9) reads:

$$\text{Find } u_N \in V_N \text{ s.t.} \quad B(u_N, v) = \int_\Omega f\, v\, dx + \langle \psi, \gamma_0 v\rangle_{Y'\times Y} \qquad \forall v \in V_N. \tag{2.1}$$

For the Dirichlet problem (1.2), we introduce the space

$$Y_N := V_N|_{\partial\Omega} = \{\gamma_0 v \mid v \in V_N\} \subset H^{1/2}(\partial\Omega). \tag{2.2}$$

For an approximation $\lambda_N \in Y_N$ to λ we can then define the FEM for (1.8) as

$$\text{Find } u_N \in V_N \text{ s.t. } u_N = \lambda_N \qquad \text{and} \qquad B(u_N, v) = \int_\Omega f\, v\, dx \qquad \forall v \in V_N \cap H_0^1(\Omega). \tag{2.3}$$

The coercivity assumption (1.6) ensures existence of the finite element approximation u_N. Furthermore, by Céa's Lemma there is $C > 0$ independent of V_N such that

$$\|u - u_N\|_{H^1(\Omega)} \leq C \inf_{v \in V_N} \|u - v\|_{H^1(\Omega)} \tag{2.4}$$

for the solution u_N of (2.1) and

$$\|u - u_N\|_{H^1(\Omega)} \leq C \inf_{\substack{v \in V_N \\ \gamma_0 v = \lambda_N}} \left\{ \|u - v\|_{H^1(\Omega)} + \|\lambda_N - \lambda\|_{H^{1/2}(\partial\Omega)} \right\} \tag{2.5}$$

for the solution u_N of the Dirichlet problem (2.3).
In practice, the approximations λ_N are obtained with the aid of a linear operator $P_N : H^{1/2}(\partial\Omega) \to Y_N$ by setting $\lambda_N = P_N \lambda$. In most of the present paper, we will choose this operator P_N to be the L^2-projection Q_N, i.e., for $\lambda \in L^2(\partial\Omega)$ the function $Q_N \lambda$ is defined by

$$\langle Q_N \lambda, v \rangle_{0,\partial\Omega} = \langle \lambda, v \rangle_{0,\partial\Omega} \qquad \forall v \in Y_N. \tag{2.6}$$

2.1.2 Discrete harmonic extension

In the paper, we will only consider families of approximation spaces V_N that are sufficiently large in the following sense: There exists $\widetilde{C} > 0$ such that

$$\inf\{\|u - v\|_{H^1(\Omega)} \,|\, v \in V_N \text{ and } v|_{\partial\Omega} = u|_{\partial\Omega}\} \leq \widetilde{C} \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) \text{ with } u|_{\partial\Omega} \in Y_N \tag{2.7}$$

Remark 2.1 Condition (2.7) is satisfied for all approximation spaces considered in this paper. For the standard piecewise linear spaces, condition (2.7) may be verified with the aid of the Clément-type interpolation operator of [38]. For the high-order spaces employed in the present paper, the corresponding *hp*-Clément-type interpolation operator is constructed in [31]. ∎

Condition (2.7) is required for the discrete harmonic extension to have the following stability properties:

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Assume that a family of approximation spaces $V_N \subset H^1(\Omega)$ satisfies (2.7). Then there exists $C > 0$ such that the discrete harmonic extension operator $E_N : Y_N \to V_N$ given by*

$$B(E_N u, v) = 0 \qquad \forall v \in V_N \cap H_0^1(\Omega) \tag{2.8}$$

is stable

$$\|E_N u\|_{H^1(\Omega)} \leq C \|u\|_{H^{1/2}(\partial\Omega)} \qquad \forall u \in Y_N.$$

Moreover, the Galerkin orthogonality implies

$$B(z, z) = B(z - E_N(\gamma_0 z), z - E_N(\gamma_0 z)) + B(E_N(\gamma_0 z), E_N(\gamma_0 z)) \qquad \forall z \in V_N.$$

2.2 Geometric meshes and hp-FEM spaces

2.2.1 The geometric mesh

For simplicity of notation, we will restrict our attention to triangulations consisting of *affine triangles*. We emphasize, however, that an extension to quadrilateral elements is possible. The triangulation $\mathcal{T} = \{K\}$ of Ω consists of elements K. Each element K is the image $F_K(\hat{K})$ of the equilateral reference triangle

$$\hat{K} = \left\{(x,y) \,|\, 0 < x < 1,\, 0 < y < \sqrt{3}\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right)\right\}$$

under the *affine* map F_K. We furthermore assume that the triangulation $\mathcal{T}$ is γ-*shape-regular*, i.e.,

$$h_K^{-1}\|F_K'\|_{L^\infty(T)} + h_K\|(F_K')^{-1}\|_{L^\infty(T)} \leq \gamma \qquad \forall K \in \mathcal{T}. \tag{2.9}$$

Here, h_K denotes the diameter of the element K. Of particular importance to us will be the "geometric meshes", which are strongly refined meshes near the boundary $\partial\Omega$:

Definition 2.3 (geometric mesh) *A γ-shape-regular (cf. (2.9)) mesh $\mathcal{T}$ is called a* geometric mesh *with* boundary mesh size h *if there exist c_1, $c_2 > 0$ such that for all $K \in \mathcal{T}$:*

1. *if $\overline{K} \cap \partial\Omega \neq \emptyset$, then $h \leq h_K \leq c_2 h$;*
2. *if $\overline{K} \cap \partial\Omega = \emptyset$, then $c_1 \inf_{x \in K} \operatorname{dist}(x, \partial\Omega) \leq h_K \leq c_2 \sup_{x\in K} \operatorname{dist}(x, \partial\Omega)$.*

A typical example of a geometric mesh is depicted in Fig. 1. Note that the restriction to the boundary $\partial\Omega$ of a geometric mesh is a quasi-uniform mesh, which justifies speaking of a "boundary mesh size h."

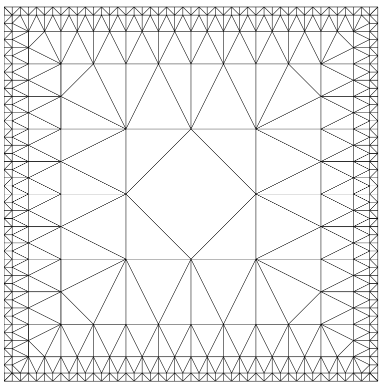

Figure 1: Example of a geometric mesh in the sense of Definition 2.3.

Remark 2.4 An important algorithmic issue is the automatic generation of geometric meshes. Such meshes can be generated with the algorithm of [36]. ∎

2.2.2 hp-FEM spaces

In order to define hp-FEM spaces on a mesh $\mathcal{T}$, we associate a polynomial degree $p_K \in \mathbb{N}$ with each element K, collect these p_K in the polynomial degree vector $\mathbf{p} := (p_K)_{K\in\mathcal{T}}$ and set

$$\begin{aligned} S^{\mathbf{p}}(\Omega,\mathcal{T}) &:= \{u \in H^1(\Omega) \,|\, u \circ F_K \in \mathcal{P}_{p_K}(\hat{K}) \quad \forall K \in \mathcal{T}\}, &(2.10)\\ S_0^{\mathbf{p}}(\Omega,\mathcal{T}) &:= S^{\mathbf{p}}(\Omega,\mathcal{T}) \cap H_0^1(\Omega), &(2.11)\end{aligned}$$

where for $p \in \mathbb{N}$ we introduced the space of all polynomials of degree p as

$$\mathcal{P}_p(\hat{K}) = \operatorname{span}\{x^i y^j \,|\, 0 \le i+j \le p\}.$$

For the approximation of solutions to (1.14) on geometric meshes (in the sense of Definition 2.3), the so-called *linear degree vector* is a particularly useful polynomial degree distribution:

Definition 2.5 (linear degree vector) *Let $\mathcal{T}$ be a geometric mesh in the sense of Definition 2.3. A polynomial degree vector $\mathbf{p} = (p_K)_{K\in\mathcal{T}}$ is said to be a* linear degree vector with slope $\alpha > 0$ *if*

$$1 + \alpha c_1 \log\frac{h_K}{h} \le p_K \le 1 + \alpha c_2 \log\frac{h_K}{h}.$$

Here, $h := \min_{K\in\mathcal{T}}$ is a measure for the mesh-size of the quasi-uniform mesh $\mathcal{T}|_{\partial\Omega}$.

Remark 2.6 Linear degree vectors $\mathbf{p}$ have the additional property that the polynomial degree varies slowly, i.e., there exists $C > 0$ such that

$$C^{-1} p_{K'} \le p_K \le C p_{K'} \qquad \forall K, K' \in \mathcal{T} \quad \text{with } \overline{K} \cap \overline{K'} \neq \emptyset. \tag{2.12}$$

■

We conclude this section by showing that for geometric meshes in the sense of Definition 2.3, the number of elements of $\mathcal{T}$ is proportional to the number of elements on the boundary. Similarly, for linear degree vectors (Definition 2.5), the dimension $\dim S^{\mathbf{p}}(\Omega,\mathcal{T})$ is proportional to the number of unknowns on the boundary:

Proposition 2.7 *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h. Let $\mathbf{p}$ be a linear degree vector with slope $\alpha > 0$ on $\mathcal{T}$. Then there exists $C > 0$ depending only on the shape-regularity constant γ and the constants c_1, c_2, α of Definitions 2.3, 2.5 such that*

$$\begin{aligned} \sum_{K\in\mathcal{T}} 1 &\le Ch^{-1}, \\ \dim S^{\mathbf{p}}(\Omega,\mathcal{T}) &\sim \sum_{K\in\mathcal{T}} p_K^2 \le Ch^{-1}, \\ \max_{K\in\mathcal{T}} p_K &\le C|\log h|. \end{aligned}$$

Proof. We will only prove the second estimate as the first one is proved similarly.

$$\sum_{K\in\mathcal{T}} p_K^2 = \sum_{K\in\mathcal{T}:\overline{K}\cap\partial\Omega\neq\emptyset} p_K^2 + \sum_{K\in\mathcal{T}:\overline{K}\cap\partial\Omega=\emptyset} p_K^2. \tag{2.13}$$

For the first sum, we note that the assumptions on a geometric mesh $\mathcal{T}$ and the linear degree vector give that $p_K \leq C$ for all $K \in \mathcal{T}$ with $\overline{K} \cap \partial\Omega \neq \emptyset$. Thus,

$$\sum_{K \in \mathcal{T}:\overline{K}\cap\partial\Omega\neq\emptyset} p_K^2 \leq C \sum_{K \in \mathcal{T}:\overline{K}\cap\partial\Omega\neq\emptyset} 1 \leq Ch^{-1}.$$

For the second sum in (2.13) we bound

$$\begin{aligned} \sum_{K \in \mathcal{T}:\overline{K}\cap\partial\Omega=\emptyset} p_K^2 &\leq C \sum_{K \in \mathcal{T}:\overline{K}\cap\partial\Omega=\emptyset} \int_K \frac{1+|\ln(r(x)/h)|^2}{r^2(x)}\,dx \\ &\leq C \int_{x\in\Omega, r(x)\geq ch} \frac{1+|\ln(r(x)/h)|^2}{r^2(x)}\,dx \leq C\int_{c'h}^{\infty} \frac{1+|\ln(z/h)|^2}{z^2}\,dz \leq Ch^{-1}, \end{aligned}$$

where in the penultimate step we locally flattened the boundary with Lipschitz maps. The integral represents the integration normal to the boundary whereas the integration in the tangential direction was absorbed in the constant C. ∎

2.3 *hp*-FEM Approximation on geometric meshes

2.3.1 Approximation on the boundary $\partial\Omega$

If $\mathcal{T}$ is a geometric mesh and $V_N = S^{\mathbf{p}}(\Omega, \mathcal{T})$ with linear degree vector $\mathbf{p}$ then the space Y_N defined in (2.2) is a space of piecewise polynomials of fixed, low degree (depending on α and the constants c_1, c_2 appearing in Definition 2.5) on a quasi-uniform mesh. It can be shown (with the aid of Proposition C.3) that for $0 \leq s < 3/2$ the L^2-projector Q_N is in fact stable on $H^s(\partial\Omega)$; in particular, therefore, $Y_N \subset H^s(\partial\Omega)$ for $0 \leq s < 3/2$. This allows us to extend the operator Q_N by duality to an operator $H^{-s}(\partial\Omega) \to Y_N$ with $0 \leq s < 3/2$ by

$$\langle Q_N u, v\rangle_{0,\partial\Omega} := \langle u, v\rangle_{H^{-s}(\partial\Omega)\times H^s(\partial\Omega)} \qquad \forall u \in H^{-s}(\partial\Omega) \quad \forall v \in Y_N. \tag{2.14}$$

By a slight abuse of notation, the extended operator is again denoted by Q_N. It has the following properties:

Lemma 2.8 *Let Ω be a polygon and let $\mathcal{T}$ be a geometric mesh with boundary mesh size h in the sense of Definition 2.3. Let $\mathbf{p}$ be a linear degree vector given by Definition 2.5, $V_N := S^{\mathbf{p}}(\Omega, \mathcal{T})$, Y_N defined by (2.2), Q_N the L^2 projection given by (2.6) and (2.14). Then*

$$\begin{aligned} \|Q_N u\|_{H^s(\partial\Omega)} &\leq C_s \|u\|_{H^s(\partial\Omega)} \quad \forall u \in H^s(\partial\Omega), \quad 0 \leq |s| < 3/2; &(2.15)\\ \|u - Q_N u\|_{H^s(\partial\Omega)} &\leq C_{s,s'} h^{s'-s} \|u\|_{H^{s'}(\partial\Omega)} \quad \forall u \in H^{s'}(\partial\Omega), \ -3/2 < s \leq s' < 3/2. &(2.16) \end{aligned}$$

The constants C_s, $C_{s,s'}$ depend only on the s, s', and the constants appearing in Definitions 2.3, 2.5.

Proof. Let $\varphi : [0, L) \to \partial\Omega$ be a parametrization by arclength. The fact that Ω is a polygon together with Lemma C.1 implies that the map $u \mapsto u \circ \phi$ is an isomorphism between the Sobolev spaces $H^s(\partial\Omega)$ and $H^s_{per}([0, L))$, $0 \leq s < 3/2$. The stability result (2.15) for $s \geq 0$ now follows from Proposition C.3 and by duality for $s \in (-3/2, 0)$. For $0 \leq s$, the approximation result (2.16) follows from (2.15) and standard approximation results in the usual way. The case $s \in (-3/2, 0)$ is again obtained by duality. ∎

2.3.2 Approximation of $\widetilde{\mathcal{B}}^2_\beta$-functions from $S^{\mathbf{p}}(\Omega,\mathcal{T})$

Our *hp*-FEM approximation results for functions of $\widetilde{\mathcal{B}}^2_\beta$ will be based on the following lemma:

Lemma 2.9 *Let $\hat{K}$ be the reference triangle with edges Γ_i, $i \in \{1,2,3\}$. Let $\hat{u}$ be analytic on $\overline{\hat{K}}$ and assume that*

$$\|\nabla^{n+2}\hat{u}\|_{L^2(\hat{K})} \leq C_u \gamma_u^n n! \qquad \forall n \in \mathbb{N}_0$$

for some C_u, $\gamma_u > 0$. Let $c \in (0,1]$. Then for each p, p_1, p_2, $p_3 \in \mathbb{N}$ with

$$cp \leq p_i \leq p \qquad i \in \{1,2,3\}$$

there exists a polynomial $\pi_p \in \mathcal{P}_p(\hat{K})$ with

1. *$\pi_p|_{\Gamma_i} = i_{p_i,\Gamma_i}(u|_{\Gamma_i})$ for $i \in \{1,2,3\}$. Here, i_{p_i,Γ_i} denote the Gauss-Lobatto interpolant of degree p_i on edge Γ_i.*
2. *$\|u - \pi_p\|_{W^{1,\infty}(\hat{K})} \leq C C_u e^{-bp}$.*

The constants C, $b > 0$ depend only on c and γ_u.

Proof. The case $p_1 = p_2 = p_3 = p$ is considered in [28, Thm. 6.2.6]. The extension to the present case is straightforward. ■

Proposition 2.10 *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h as defined in Definition 2.3. Let $\mathbf{p}$ be a linear degree vector on $\mathcal{T}$ with slope $\alpha > 0$. Let $u \in \widetilde{\mathcal{B}}^2_\beta(C_u,\gamma_u)$ for some $\beta \in [0,1)$, C_u, $\gamma_u > 0$. Then there exist C, $b > 0$ depending only on shape-regularity constant γ and the constants c_1, c_2 of Definition 2.3 as well as C_u, γ_u, β such that*

$$\inf\left\{\|u - v\|_{H^1(\Omega)} \,|\, v \in S^{\mathbf{p}}(\Omega,\mathcal{T})\right\} \leq C h^{1-\beta} + C h^{b\alpha}. \tag{2.17}$$

In terms of degrees of freedom, we have by setting $N = \dim S^{\mathbf{p}}(\Omega,\mathcal{T})$ that

$$h \sim N^{-1}.$$

Proof. For $u \in \widetilde{\mathcal{B}}^2_\beta(C_u,\gamma_u)$ we define

$$C_K^2 := \sum_{n=0}^{\infty} \frac{1}{(2\gamma_u)^{2n}(n!)^2}\|r^{n+\beta}\nabla^{n+2}u\|^2_{L^2(K)}.$$

The assumption $u \in \widetilde{\mathcal{B}}^2_\beta(C_u,\gamma_u)$ guarantees

$$\sum_{K\in\mathcal{T}} C_K^2 \leq \sum_{n=0}^{\infty} \frac{1}{(2\gamma_u)^{2n}(n!)^2}\|r^{n+\beta}\nabla^{n+2}u\|^2_{L^2(\Omega)} \leq C_u^2 \sum_{n=0}^{\infty}\frac{1}{2^{2n}} = \frac{4}{3}C_u^2.$$

Hence, we conclude that $u \in \widetilde{\mathcal{B}}^2_\beta(C_u,\gamma_u)$ implies

$$\|r^{n+\beta}\nabla^{n+2}u\|_{L^2(K)} \leq C_K (2\gamma_u)^n n! \qquad \forall n \in \mathbb{N}_0, \quad \forall K \in \mathcal{T}, \tag{2.18a}$$

$$\sum_{K\in\mathcal{T}} C_K^2 \leq \frac{4}{3}C_u^2. \tag{2.18b}$$

We explicitly construct an element of $S^{\mathbf{p}}(\Omega,\mathcal{T})$ with the desired approximation properties. To that end, we first assume, as we may, that $p_K = 1$ for all elements abutting $\partial\Omega$. Next, we associate with each edge e of $\mathcal{T}$ a polynomial degree $p_e := \min\{p_K \,|\, e \text{ is an edge of element } K\}$. After these preparations, we construct the approximant element by element. We distinguish the cases $\overline{K}\cap\partial\Omega \neq \emptyset$ and $\overline{K}\cap\partial\Omega = \emptyset$.
Using Theorem B.4 and a scaling argument, we obtain for all elements K abutting on $\partial\Omega$ that the linear interpolant Iu satisfies

$$\|u - Iu\|_{H^1(K)} \le C h_K^{1-\beta}\|r^\beta \nabla^2 u\|_{L^2(K)} \le C h^{1-\beta} C_K.$$

For the elements not abutting on $\partial\Omega$, we employ Lemma 2.9. Using (2.18a) we see that the pull-back $\hat{u} := u\circ F_K$ satisfies

$$\begin{aligned}\|\nabla^{n+2}\hat{u}\|^2_{L^2(\hat{K})} &\le C h_K^{2(n+1)}\|\nabla^{n+2}u\|^2_{L^2(K)} \le C h_K^{2-2\beta}\|r^{n+\beta}\nabla^{n+2}u\|^2_{L^2(K)}\\ &\le C C_K\, h_K^{2(1-\beta)}(2\gamma_u)^{2n}(n!)^2,\end{aligned}$$

with $C>0$ independent of n and K. The approximant Iu of Lemma 2.9 then satisfies

$$\|u - Iu\|_{H^1(K)} \le C C_K h_K^{1-\beta} e^{-bp_K}$$

for some C, $b>0$ independent of the element K. We note that the interpolant constructed element-wise in this fashion is indeed an element of $S^{\mathbf{p}}(\Omega,\mathcal{T})$ (the edge polynomial degrees p_i in Lemma 2.9 are taken to be the polynomial degrees p_e of the corresponding edges e).
Using $p_K \ge c\alpha\ln(h_K/h)$, we arrive at

$$\|u - Iu\|_{H^1(K)} \le C C_K h_K^{1-\beta-\alpha b'} h^{\alpha b'}$$

for some $b'>0$. Exploiting that $h_K \ge ch$, a simple calculation reveals that

$$h_K^{1-\beta-\alpha b'} h^{\alpha b'} \le C h^{\min\{1-\beta,\alpha b'\}}.$$

We thus conclude in view of (2.18b) that

$$\sum_{K\in\mathcal{T}}\|u - Iu\|^2_{H^1(K)} \le C\sum_{K\in\mathcal{T}} C_K^2 h^{\min\{1-\beta,\alpha b'\}} \le C h^{\min\{1-\beta,\alpha b'\}},$$

which is the desired estimate. The bound for the dimension of $S^{\mathbf{p}}(\Omega,\mathcal{T})$ follows from Proposition 2.7. ■

Remark 2.11 The meshes $\mathcal{T}$ considered here consist of triangles only. Likewise, the approximation result in Proposition 2.10 is formulated for triangles only. This restriction is not essential and was done for simplicity of exposition only. The approximation results can be formulated for meshes consisting of non-affine elements (quadrilaterals, curved elements) as well. To handle this case, it is required that the element maps F_K for elements K not abutting on the boundary the element be analytic (with a controlled domain of analyticity; see, e.g., [28]) and that the error on elements abutting $\partial\Omega$ be $O(h^{1-\beta})$. ■

Proposition 2.10 is a result for unconstrained approximation in $H^1(\Omega)$. For treating Dirichlet problems, constrained approximation as in (2.5) is required. This is accomplished in the following corollary.

Corollary 2.12 *Assume the hypotheses of Proposition 2.10 and additionally* $u \in H^{2-\beta}(\Omega) \cap \widetilde{B}^2_\beta(C_u, \gamma_u)$ *for some* C_u, $\gamma_u > 0$, $\beta \in (0,1)$. *Let* Y_N *be the restriction of* $S^{\mathbf{p}}(\Omega, \mathcal{T})$ *to* $\partial\Omega$ *as given by (2.2). Let* Q_N *be the* L^2*-projection into* Y_N *(cf. (2.6)). Then*

$$\inf\{\|u - v\|_{H^1(\Omega)} \,|\, v \in S^{\mathbf{p}}(\Omega, \mathcal{T}) \text{ such that } \gamma_0 v = Q_N(\gamma_0 u)\} \leq C\left[h^{1-\beta} + h^{b\alpha}\right]. \qquad (2.19)$$

The constants C, $b > 0$ *depend only on the shape-regularity constant* γ, *the constants* c_1, c_2 *appearing in Definition 2.3, and* C_u, γ_u, β, $\|u\|_{H^{2-\beta}(\Omega)}$.

Proof. We first observe that the trace theorem gives $\|\gamma_0 u\|_{H^{3/2-\beta}(\partial\Omega)} \leq C\|u\|_{H^{2-\beta}(\Omega)}$. Lemma 2.8 and Proposition 2.10 therefore imply the existence of $v_N \in S^{\mathbf{p}}(\Omega, \mathcal{T})$ such that

$$\begin{aligned} \|u - v_N\|_{H^1(\Omega)} &\leq C\left[h^{1-\beta} + h^{b\alpha}\right], \\ \|\gamma_0 u - Q_N(\gamma_0 u)\|_{H^{1/2}(\partial\Omega)} &\leq Ch^{1-\beta}\|u\|_{H^{2-\beta}(\Omega)}. \end{aligned}$$

The desired result now follows with the aid of the discrete harmonic extension operator E_N of Lemma 2.2: Since $\gamma_0 v_N - Q_N(\gamma_0 u) \in Y_N$, we get that the function $\widetilde{v}_N := v_N - E_N(\gamma_0 v_N - Q_N(\gamma_0 u)) \in V_N$ satisfies $\gamma_0 \widetilde{v}_N = Q_N(\gamma_0 u)$ and

$$\begin{aligned} \|\widetilde{v}_N - u\|_{H^1(\Omega)} &\leq \|u - v_N\|_{H^1(\Omega)} + \|E_N(\gamma_0 v_N - Q_N(\gamma_0 u))\|_{H^1(\Omega)} \\ &\leq \|u - v_N\|_{H^1(\Omega)} + C\|\gamma_0 v_N - Q_N(\gamma_0 u)\|_{H^{1/2}(\partial\Omega)} \\ &\leq \|u - v_N\|_{H^1(\Omega)} + C\left[\|\gamma_0(v_N - u)\|_{H^{1/2}(\partial\Omega)} + \|\gamma_0 u - Q_N(\gamma_0 u)\|_{H^{1/2}(\partial\Omega)}\right]. \end{aligned}$$

The result now follows. ■

Corollary 2.12 allows us to finally formulate an approximation result for the *hp*-FEM on geometric meshes applied to (1.2) and (1.3):

Theorem 2.13 *Let* $\mathcal{T}$ *be a geometric mesh with boundary mesh size* h *as defined in Definition 2.3. Let* $\mathbf{p}$ *be a linear degree vector on* $\mathcal{T}$ *with slope* $\alpha > 0$ *(cf. Definition 2.5). Let* Q_N *be the* L^2*-projection onto* $Y_N = S^{\mathbf{p}}(\Omega, \mathcal{T})|_{\partial\Omega}$.
Let $u \in H^{1+\delta}(\Omega)$, $\delta \in (0,1)$, *be the solution to (1.2) (resp. the solution to (1.3)) with coefficients* A, b, a_0, *and right-hand side* f *analytic on* $\overline{\Omega}$. *Then the FE-solution* u_N *given by (2.3) (resp. (2.1)) satisfies*

$$\|u - u_N\|_{H^1(\Omega)} \leq C\left[h^\delta + h^{b\alpha}\right]. \qquad (2.20)$$

The constants C, $b > 0$ *depend only on the shape-regularity constant* γ, *the constants* c_1, c_2 *appearing in Definition 2.3, and the data* A, b, c, f, Ω.

Proof. Theorem 1.4 implies that the solution $u \in H^{1+\delta}(\Omega)$ is in $\widetilde{B}^2_{1-\delta}(C_u, \gamma_u)$ for some C_u, $\gamma_u > 0$. In view of the best approximation properties (2.5), (2.4), the assertion (2.20) now follows from Corollary 2.12. ■

For α sufficiently large the boundary concentrated FEM achieves the optimal rate of convergence

$$\|u - u_N\|_{H^1(\Omega)} \leq Ch^\delta = O(N^{-\delta}).$$

bdy. cond.	$p = p(N)$	DOF	cond (A)	cond (A^c)	cond $(C^{-1}A)$
Dirichlet	$O(\log N)$	N	$O(p^4(1+\log p))$	$O(p(1+\log p))$	$O(1+\log^2 p)$
Neumann	$O(\log N)$	N	$O(Np^4(1+\log p))$	$O(Np(1+\log p))$	$O(1+\log^2 p)$

Table 1: Conditioning of the hp-FEM stiffness matrices: $N = \#$ elements, $p = \max_{K\in\mathcal{T}} p_K$.

3 hp-FEM solution procedure

Choosing the slope α of the polynomial degree vector sufficiently large, we obtain for the FEM approximation the optimal rate $\|u - u_N\|_{H^1(\Omega)} \le CN^{-\delta}$. In the present section we discuss how the FE solution u_N can be computed with complexity $O(N \log^2 N)$.

We consider iterative methods for solving the Dirichlet and Neumann problems on geometric meshes in the sense of Definition 2.3. We restrict our attention to the symmetric positive definite case, i.e., we take in (1.4)

$$b_0 = 0, \quad a_0 > 0.$$

The main results of this section are collected in Table 1: A is the stiffness matrix, A^c stands for the statically condensed stiffness matrix (with the shape functions discussed in Example 3.1), and C stands for the preconditioners proposed here.

We focus in this section on the design of preconditioners for Neumann problems. The reason for our concentrating on this case can be seen in Table 1: While the Dirichlet problem is fairly well-conditioned (the condition number without preconditioning grows only polylogarithmically in N since $p = O(\log N)$) the Neumann problem leads to at least linearly (in N) growing conditioning numbers, thus requiring preconditioning.

Our approach for the design of a preconditioner for the Neumann problem is based on the results of [2] (see also [37, Sec. 4.7]).

3.1 Shape functions and assembling

3.1.1 Element shape functions

In order to set up the stiffness matrix, bases of the polynomial spaces $\mathcal{P}_p$ have to be chosen. It is customary in p- and hp-FEM to split a basis of $\mathcal{P}_{p_K}$ into *vertex*, *side*, and *internal* shape functions. These three types of shape functions are characterized as follows:

1. *Vertex shape functions $\mathcal{V}$:* These are the usual linear nodal shape functions, which are equal to one in one node and vanish on the edge opposite that node. We write $\widetilde{\mathcal{V}} = \operatorname{span}\mathcal{V}$.

2. *Side shape functions $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$:* The side shape functions from $\mathcal{S}_i$ are associated with the edge Γ_i of $\partial\hat{K}$ and vanish on the edges Γ_j for $j \ne i$. We write $\widetilde{\mathcal{S}}_i = \operatorname{span}\mathcal{S}_i$ and $\widetilde{\mathcal{S}} := \operatorname{span}\{\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3\}$.

3. *Internal shape functions $\mathcal{I}$:* The functions from $\mathcal{I}$ vanish on $\partial\hat{K}$. We write $\widetilde{\mathcal{I}} = \operatorname{span}\mathcal{I}$.

The side and internal shape functions are not uniquely defined with the above separation. An important consideration for an actual choice is the conditioning of the resulting stiffness matrix. We will discuss this point further below. One possible choice of a basis of $\mathcal{P}_p$ is based on Lagrange interpolation polynomials with respect to the Gauss-Lobatto points on the sides, which we elaborate in the following example.

Example 3.1 Denote by v_i, $i = 1, \ldots, 3$, the three vertices of $\hat{K}$ and by Γ_i, $i = 1, \ldots, 3$ the three edges (we assume $\Gamma_1 = \{(x, 0) \in \mathbb{R}^2 \,|\, 0 < x < 1\}$). Let $p_i \in \mathbb{N}$ be polynomial degrees associated with the edges Γ_i and let $p \in \mathbb{N}$ be the polynomial degree of the internal shape functions. We then define vertex shape functions $\mathcal{V}$, side shape functions $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, and internal shape functions $\mathcal{I}$ as follows:

$$\begin{aligned}
\mathcal{V} &:= \text{ the usual linear nodal shape functions } n_i \text{ with } n_i(v_j) = \delta_{ij},\\
\mathcal{S}_1 &:= \left\{ l_{j,p_1}(x) \frac{y - \sqrt{3}x}{x} \frac{y + \sqrt{3}(x-1)}{1-x} \,|\, j = 1, \ldots, p_1 - 1 \right\},\\
\mathcal{I} &:= \{ y(y - \sqrt{3}x)(y + \sqrt{3}(x-1)) L_i(x) L_j(y) \,|\, 0 \le i + j \le \frac{(p-3)(p-2)}{2} \}.
\end{aligned}$$

Here, the polynomials L_i are the Legendre polynomials scaled to the interval $[0,1]$. The polynomials l_{j,p_1} are the Lagrange interpolation polynomials with respect to the $p_1 + 1$ Gauss-Lobatto points on $[0,1]$: Letting $0 = x_0 < x_1 < \cdots < x_{p_1} = 1$ be the zeros of the polynomials $x \mapsto x(1-x)L'_{p_1}(x)$, the functions $l_{j,p_1}(x)$ are defined by

$$l_{j,p_1}(x) := \prod_{\substack{i=0\\ i \neq j}}^{p_1} \frac{x - x_i}{x_j - x_i}, \qquad j = 0, \ldots, p_1. \tag{3.1}$$

The side shape functions $\mathcal{S}_2$, $\mathcal{S}_3$ are obtained similarly with p_1 replaced with p_2 (resp. p_3) and a suitable coordinate transformation. ■

3.1.2 Global bases and assembling

The decomposition of a basis $\mathcal{B}_K$ of $\mathcal{P}_{p_K}$ facilitates the assembly process in hp-FEM with variable polynomial degree. For a detailed discussion of this procedure we refer for example to [11, 34]. The basis of Example 3.1, however, may serve to illustrate the main point. The topological entities "edge" and "element" carry a polynomial degree: the polynomial degree associated with an element K is p_K, whereas the polynomial degree p_e associated with an edge $e = \overline{K} \cap \overline{K'}$ is $p_e := \min\{p_K, p_{K'}\}$. This edge polynomial degrees p_e then correspond to the polynomial degree p_i, $i = 1, \ldots, 3$, in Example 3.1; since the side shape functions are Lagrange interpolation polynomials on the edges, the assembly process is straightforward.
We introduce the *assembly operator* $\mathcal{A}_{K \in \mathcal{T}}$ of [21] to combine the bases $\mathcal{B}_K$ of the spaces $\mathcal{P}_{p_K}$ into a global basis $\mathcal{B}$ of the FE-space V_N:

$$\mathcal{B} = \mathop{\mathcal{A}}_{K \in \mathcal{T}} \mathcal{B}_K.$$

One can also assemble only the functions from $\mathcal{V}$, $\mathcal{S}$, or $\mathcal{I}$:

$$V_{\mathcal{V}} = \mathop{\mathcal{A}}_{K \in \mathcal{T}} \mathcal{V}_K, \qquad V_{\mathcal{S}} = \mathop{\mathcal{A}}_{K \in \mathcal{T}} \mathcal{S}_K, \qquad V_{\mathcal{I}} = \mathop{\mathcal{A}}_{K \in \mathcal{T}} \mathcal{I}_K. \tag{3.2a}$$

Of course, the functions of $V_{\mathcal{V}}$ are just the standard piecewise linear hat functions spanning the space $S^1(\Omega, \mathcal{T})$. The shape functions of $V_{\mathcal{I}}$ are supported by a single element and the shape function of $V_{\mathcal{S}}$ are supported by at most two adjacent elements. We can further split $V_{\mathcal{S}}$

$$V_{\mathcal{S}} = \oplus_{\text{edges } e} V_e, \qquad V_e = \{ v \in V_{\mathcal{S}} \,|\, v_{|_{e'}} = 0 \quad \text{for all edges } e' \neq e \}. \tag{3.2b}$$

3.2 Cost of setting up the stiffness matrix and local condensation

We first show that setting up the stiffness matrix and the optional local static condensation can be performed with optimal complexity on geometric meshes with linear degree vectors. To see that, we introduce the elementwise bilinear form B_K by restricting B to the element K:

$$B_K(u,v) := \int_K (A\nabla u\cdot\nabla v + a_0 uv)\,dx.$$

Furthermore, for functions u, $v \in V_N$, we write $u_K := u|_K$, $v_K := v|_K$. With this understanding we can write

$$B(u,v) = \sum_{K\in\mathcal{T}} B_K(u_K, v_K). \tag{3.3}$$

The actual evaluation of $B_K(u_K, v_K)$ is performed by integrating on the reference element $\hat{K}$ instead of K. That is, writing $\hat{u} = u_K \circ F_K$, $\hat{v} = v_K \circ F_K$, we set $B_K(u,v) := \hat{B}_K(\hat{u},\hat{v})$, where

$$\begin{aligned}\hat{B}_K(\hat{u},\hat{v}) &:= \int_{\hat{K}} \hat{A}\nabla\hat{u}\cdot\hat{v} + \hat{a}_0\hat{u}\hat{v}\,dx,\\ \hat{A} &:= \left(F_K'\right)^{-\top}(A\circ F_K)\left(F_K'\right)^{-1}\det F_K', \qquad \hat{a}_0 = a_0\circ F_K\cdot\det F_K'.\end{aligned}$$

In practice, the integration over $\hat{K}$ that is required for the evaluation of B_K is performed with a quadrature rule with $O(p_K^2)$ points. In a standard *hp*-FEM code, the bilinear form B_K defines the element stiffness matrix

$$A_K = \left(B_K(u,v)\right)_{u,v\in\mathcal{V}\cup\mathcal{S}\cup\mathcal{I}},$$

which is an $O(p_K)\times O(p_K)$-matrix. Finally, in the assembly the local stiffness matrices A_K are combined into the global stiffness matrix A. As in [21], we write this assembly process as

$$A = \mathop{\mathcal{A}}_{K\in\mathcal{T}} A_K.$$

In *hp*-FEM it is also customary to perform local static condensation. The partitioning of the basis of $\mathcal{P}_K$ in vertex, side, and internal shape functions implies a corresponding block structure of A_K:

$$A_K = \begin{pmatrix} A_K^{\mathcal{V}\mathcal{V}} & A_K^{\mathcal{V}\mathcal{S}} & A_K^{\mathcal{V}\mathcal{I}}\\ & A_K^{\mathcal{S}\mathcal{S}} & A_K^{\mathcal{S}\mathcal{I}}\\ sym. & & A_K^{\mathcal{I}\mathcal{I}}\end{pmatrix}.$$

Since $A_K^{\mathcal{I}\mathcal{I}}$ is invertible, one can form the following Schur complement

$$A_K^c := A_K^{\mathcal{E}\mathcal{E}} - A_K^{\mathcal{E}\mathcal{I}}\left(A_K^{\mathcal{I}\mathcal{I}}\right)^{-1}\left(A_K^{\mathcal{E}\mathcal{I}}\right)^\top,$$

where we introduce the notion of *external shape functions*

$$\mathcal{E} := \mathcal{V}\cup\mathcal{S}. \tag{3.4}$$

The condensed global stiffness matrix A^c is obtained by assembling the condensed local matrices A_K^c:

$$A^c := \mathop{\mathcal{A}}_{K\in\mathcal{T}} A_K^c. \tag{3.5}$$

An important observation is that the condensed stiffness matrix A^c is the stiffness matrix corresponding to elementwise discrete harmonic external shape functions.

Proposition 3.2 *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h and let $\mathbf{p}_K$ be a linear degree vector with slope α in the sense of Definition 2.5. Assume that for all elements quadrature rule with $O(p_K^2)$ points are used. Then the stiffness matrix A is generated with work*

$$W(A) = O(N),$$

where $N \sim h^{-1}$. Additionally, the local static condensation, i.e., forming the Schur complement with respect to the internal shape functions is performed with work $W(A^c) = O(N)$.

Proof. The cost of setting up the local stiffness matrix with $O(p_K^2 \times p_K^2)$ entries using numerical quadrature with $O(p_K^2)$ points is $O(p_K^6)$. Thus, the total cost to set up the global stiffness matrix is

$$W \sim \sum_{K \in \mathcal{T}} (p_K^2)^3 = O(N)$$

where the last bound is obtained by arguments similar to those in the proof of Proposition 2.7. Also computing the condensed stiffness matrix A_K^c is done with work $O((p_K^2)^3) = O(p_K^6)$. Again, summing this work estimate over all elements K of the geometric mesh we arrive at $W(A^c) = O(N)$. ■

Remark 3.3 The presence of numerical quadrature introduces variational crimes. It can be shown, [30], that the bilinear form $\tilde{B}$ obtained by numerical quadrature induces an inner product on V_N that is equivalent with the inner product generated by B, and the approximation result Theorem 2.13 still holds. ■

3.3 The Dirichlet problem

The aim of the present section is to show that the stiffness matrix resulting from the discretization of a Dirichlet problem is fairly well-conditioned. We have

Proposition 3.4 *Given a geometric mesh $\mathcal{T}$ with boundary mesh size h, let $\mathbf{p}$ be a polynomial degree distribution satisfying (2.12) and let the element shape functions be taken as described in Example 3.1. Then there exists $C > 0$ independent of h and $\mathbf{p}$ such that the condensed stiffness matrix A^c corresponding to the Dirichlet problem has l^2-condition number*

$$\kappa(A^c) \leq C|\mathbf{p}|(1 + \log|\mathbf{p}|), \tag{3.6}$$

where $|\mathbf{p}| = \max_{K\in\mathcal{T}} p_K$. In the case of Neumann boundary conditions, there holds

$$\kappa(A^c) \leq Ch^{-1}\,|\mathbf{p}|(1 + \log|\mathbf{p}|). \tag{3.7}$$

Proof. The $O(1)$-condition number estimate for the case $p = 1$ has been proved in [41]. For the reader's convenience, however, we include a simple proof; we refer to the examples in Section 6 for numerical verifications of this result. The bound is based on the embedding results in weighted Sobolev spaces (i.e., the well-known Hardy inequality, see, e.g., [13]): there exists the constant $C = C(\Omega)$, such that

$$\|r^{-1}(x)u\|_{0,\Omega} \leq C\,\|u\|_{1,\Omega}, \qquad \forall\, u \in H_0^1(\Omega). \tag{3.8}$$

Recall that for any $u \in H_0^1(\Omega)$ there holds $B(u,u) \asymp \|\nabla u\|_{0,\Omega}^2$. Now, for the usual linear/bilinear nodal shape functions n_i, any $u \in S_0^{\mathbf{1}}(\Omega, \mathcal{T})$ can be written in the form $u = \sum_i u_i n_i$; we then obtain

$$\sum_{i=1}^N u_i^2 \leq c \sum_{K \in \mathcal{T}} h_K^{-2} \|u\|_{0,K}^2 \leq c \sum_{K \in \mathcal{T}} \|r^{-1} u\|_{0,K}^2 = c\|r^{-1}u\|_{0,\Omega}^2 \leq c\|\nabla u\|_{0,\Omega}^2. \qquad (3.9)$$

Combining (3.9) with the trivial estimate $\|\nabla u\|_{0,\Omega}^2 \leq c \sum_{i=1}^N u_i^2$, which holds due to the shape regularity of mesh, we arrive at the desired bound $B(u,u) \asymp \sum_{i=1}^N u_i^2$ for all $u \in S_0^{\mathbf{1}}(\Omega, \mathcal{T})$. For the case $p \geq 1$ and Dirichlet boundary conditions, we refer to [29, Theorem 2.2]. The case of the Neumann boundary conditions is obtained combining the results of [29] for the p-dependence with those of [6, Theorem 4.1] for the h-dependence. ■

Applied to the case of linear degree vector $\mathbf{p}$, Proposition 3.4 yields that in the case of a Dirichlet problem the condensed stiffness matrix satisfies

$$\kappa(A^c) \leq C \log N (1 + \log\log N)$$

because $|\mathbf{p}| \leq C \log N = C|\log h|$. Thus, solving Dirichlet problems on geometric meshes can be accomplished efficiently by simple CG-iterations. Note that the condition number $\kappa(A)$, of the full hp-FEM stiffness matrix is shown to be $O(|\mathbf{p}|^4 \log|\mathbf{p}|)$. In this situation the CG-iterations again lead to the linear-logarithmic complexity.

Applied to the Neumann problem on geometric meshes with linear degree vectors, Proposition 3.4 yields a bound of the form $\kappa(A^c) \leq CN \log N(1 + \log\log N)$. In this case, preconditioning seems to be desirable for the efficient solution of the resulting linear system. We propose a preconditioner in the following two subsections.

3.4 Neumann problem: Reduction to preconditioning on piecewise linear spaces

The bilinear form B is expressed in (3.3) as a sum of element contributions. The preconditioner C is also constructed elementwise:

$$C(u,v) := \sum_{K \in \mathcal{T}} C_K(u_K, v_K). \qquad (3.10)$$

For the construction of the preconditioner, we use the fact that by our discussion in Section 3.1.1 the space $\mathcal{P}_{p_K}$ can be written as $\mathcal{P}_{p_K} = \widetilde{\mathcal{V}} \oplus_{i=1}^3 \widetilde{\mathcal{S}}_i \oplus \widetilde{\mathcal{I}}$, where the polynomial degrees associated with sets of side shape functions $\mathcal{S}_i$ and the internal shape functions $\mathcal{I}$ implicitly depend on K. Correspondingly, a function $u_K \in \mathcal{P}_{p_K}$ can be written as

$$u_K = u_K^{\mathcal{V}} + \sum_{i=1}^3 u_K^{\mathcal{S}_i} + u_K^{\mathcal{I}}. \qquad (3.11)$$

We then define the element contributions C_K of the preconditioner C as

$$C_K(u,v) = B_K(u_K^{\mathcal{V}}, v_K^{\mathcal{V}}) + \sum_{i=1}^3 B_K(u_K^{\mathcal{S}_i}, v_K^{\mathcal{S}_i}) + B_K(u_K^{\mathcal{I}}, v_K^{\mathcal{I}}). \qquad (3.12)$$

We mentioned in Section 3.1.1 that the splitting of a basis of $\mathcal{P}_{p_K}$ in vertex, side, and internal shape functions is not unique. The following result, which is due to [2], asserts that for discrete harmonic side shape functions the preconditioner C given by (3.10) is only weakly dependent on $\mathbf{p}$ and independent of the mesh:

Proposition 3.5 ([2]) *Let $\mathcal{T}$ be a shape regular mesh consisting of triangles. Assume that the side shape functions are discrete harmonic, i.e.,*

$$B_K(u,v) = 0 \qquad \forall u \in \widetilde{\mathcal{S}} \quad \forall v \in \widetilde{\mathcal{I}}. \tag{3.13}$$

Then there exist c_1, $c_2 > 0$ depending only on the coefficients A, a_0 and the shape-regularity constant γ such that

$$c_1 B(u,u) \le C(u,u) \le c_2(1+\ln|\mathbf{p}|)^2 B(u,u) \qquad \forall u \in S^{\mathbf{p}}(\Omega,\mathcal{T}),$$

where $|\mathbf{p}| = \max_{K\in\mathcal{T}} p_K$.

Remark 3.6 The condition (3.13) can be achieved by a process akin to the local static condensation described in Section 3.2. By Proposition 3.2, the condition (3.13) can be enforced with work $O(N)$. ■

Remark 3.7 When solving the system by local static condensation as described in Section 3.2, the condition (3.13) should be substituted by

$$B_K(u,v) = 0 \qquad \forall u \in \widetilde{\mathcal{E}} \quad \forall v \in \widetilde{\mathcal{I}}. \tag{3.14}$$

It is easy to see that the energy minimization property of the discrete B-harmonic functions now implies the same condition number for the preconditioner (3.12), (3.14) as in Proposition 3.5. On the other hand, the former preconditioner (3.12), (3.13) can be shown to have the same condition number as the modified one. ■

We now analyze the cost of applying the preconditioner C, i.e., solving the variational problem

$$C(u,v) = l(v) \qquad \forall v \in V_N.$$

By the decomposition (3.2) of the basis $\mathcal{B}$ of V_N, the sought function $u \in V_N$ can be written in the form

$$u = u^{\mathcal{V}} + \sum_{\text{edges } e} u^e + \sum_{K\in\mathcal{T}} u_K^{\mathcal{I}}, \qquad u^{\mathcal{V}} \in S^1(\Omega,\mathcal{T}), \quad u_e \in \operatorname{span} V_e, \quad u_K^{\mathcal{I}} \in \operatorname{span} V_{\mathcal{I}}. \tag{3.15}$$

Computing the components $u^{\mathcal{V}}$, u^e, $u_K^{\mathcal{I}}$ amounts to solving a global problem corresponding to a discretization with piecewise linear functions and two sets of local problems:

$$\begin{aligned} B(u^{\mathcal{V}},v) &= l(v) \qquad \forall v \in V_{\mathcal{V}} && (3.16)\\ B(u^e,v) &= l(v) \qquad \forall v \in V_e \qquad \text{for all edges } e && (3.17)\\ B(u_K^{\mathcal{I}},v) &= l(v) \qquad \forall v \in V_{\mathcal{I}} \qquad \forall K \in \mathcal{T} && (3.18) \end{aligned}$$

We now show that solving (3.17) and (3.18) can be accomplished with work $O(N)$ on geometric meshes with linear degree vectors:

Proposition 3.8 *Let $\mathcal{T}$ be a geometric mesh with boundary mesh size h and* $\mathbf{p}$ *be a linear degree vector. Then the problems (3.17) and (3.18) can be solved with work $W = O(N)$, where $N \sim h^{-1}$.*

Proof. First, we note that the stiffness matrices corresponding to (3.17), (3.18) are submatrices of the global stiffness matrix; this guarantees that the problems can be set up with optimal complexity $O(N)$. For the solution step, we observe that the problems decouple into problems associated with single elements or two adjacent elements. Specifically, using Gaussian elimination for each element with cost $O(p_K^6)$, we arrive at the total cost for the solution of (3.18)

$$W \leq C \sum_{K \in \mathcal{T}} p_K^6 = O(N),$$

by a reasoning as in the proof of Proposition 2.7. The solution of (3.17) decouples into problems associated with each edge of the mesh and a calculation shows again that the required work is $O(N)$. ■

Since $\dim S^1(\Omega, \mathcal{T}) = O(N)$, the cost for solving (3.16) is at least $O(N)$; hence, by Proposition 3.8, the total cost of applying the preconditioner C is controlled by the cost of solving (3.16).

3.5 Efficient solution of piecewise linear discretization

The analysis of the preceding section allowed us to restrict our attention to the case of piecewise linear approximation on geometric meshes $\mathcal{T}$. By the general theory of preconditioning, it suffices to find a spectrally equivalent bilinear form $\widetilde{B}$ on $S^1(\Omega, \mathcal{T}) \times S^1(\Omega, \mathcal{T})$. To that end, the space $S^1(\Omega, \mathcal{T})$ is decomposed further

$$S^1(\Omega, \mathcal{T}) = V_{\mathcal{H}} \oplus \left(S^1(\Omega, \mathcal{T}) \cap H_0^1(\Omega) \right), \tag{3.19}$$

where, for the $S^1(\Omega, \mathcal{T})$-discrete harmonic extension operator

$$\begin{aligned} E_{\mathcal{V}} : \gamma_0(S^1(\Omega, \mathcal{T})) &\to S^1(\Omega, \mathcal{T}) \\ u &\mapsto E_{\mathcal{V}} u \quad \text{with} \quad B(E_{\mathcal{V}} u, v) = 0 \qquad \forall v \in S^1(\Omega, \mathcal{T}) \cap H_0^1(\Omega), \end{aligned}$$

the space $V_{\mathcal{H}}$ is given by

$$V_{\mathcal{H}} := \text{Range}\, E_{\mathcal{V}}.$$

By definition, decomposition (3.19) provides also the B-orthogonal splitting, see Lemma 2.2,

$$B(u, v) = B(E_{\mathcal{V}} \gamma_0 u, E_{\mathcal{V}} \gamma_0 v) + B(u - E_{\mathcal{V}} \gamma_0 u, v - E_{\mathcal{V}} \gamma_0 v) \qquad \forall u, v \in S^1(\Omega, \mathcal{T}). \tag{3.20}$$

Due to Proposition 3.4, any spectrally equivalent approximation to the first term on the right hand side of (3.20) yields the desired bilinear form $\widetilde{B}$. For simplicity of exposition, we now assume that Ω is simply connected. Following the standard construction in the domain decomposition methods we apply a circulant preconditioning matrix. Let $F_\Omega : \widehat{C} \to \partial\Omega$ be the bi-Lipschitz mapping providing the global parameterization of $\partial\Omega$ by 2π-periodic function. Assume that our quasi-uniform partitioning of $\mathcal{T}_{|\partial\Omega}$ is the image of the uniform grid $\mathcal{T}_{\widehat{C}}$ on $\widehat{C} := [0, 2\pi]$. Let $\Delta_{\widehat{C},h}$ be the discrete Laplacian defined on the set $\mathcal{T}_{\widehat{C}}$ and associated with the corresponding FE space of periodic piecewise linear functions $V_h(\widehat{C})$,

$$\langle -\Delta_{\widehat{C},h} u, v \rangle_{0,\widehat{C}} = \int_{\widehat{C}} \nabla u \cdot \nabla v \, ds + \int_{\widehat{C}} uv \, ds \qquad \forall\, u, v \in V_h(\widehat{C}). \tag{3.21}$$

The bilinear form $\widetilde{B}$ is then defined on $S^1(\Omega,\mathcal{T}) \times S^1(\Omega,\mathcal{T})$ by

$$\widetilde{B}(u,v) := \langle(-\Delta_{\widehat{C},h})^{1/2}(\gamma_0 u \circ F_\Omega), (\gamma_0 v \circ F_\Omega)\rangle_{0,\widehat{C}} + B(u - E_\mathcal{V}\gamma_0 u, v - E_\mathcal{V}\gamma_0 v). \tag{3.22}$$

The symmetric positive definite bilinear form $\widetilde{B}$ is spectrally equivalent with B on $S^1(\Omega,\mathcal{T})$ since

$$\langle(-\Delta)^{1/2}(\gamma_0 u \circ F_\Omega), \gamma_0 u \circ F_\Omega\rangle \sim \|\gamma_0 u\|^2_{1/2,\partial\Omega} \sim \|E_\mathcal{V}(\gamma_0 u)\|^2_{1,\partial\Omega}.$$

We are thus left with the efficient solution of the problem

$$\text{Find } u \in S^1(\Omega,\mathcal{T}) \text{ s.t. } \widetilde{B}(u,v) = l(v) \qquad \forall v \in S^1(\Omega,\mathcal{T}). \tag{3.23}$$

Problem (3.23) can be solved in four steps:

Algorithm 3.9 (Preconditioner for p.w. linear discretization)

1. *Determine* $\tilde{u} \in S^1(\Omega,\mathcal{T}) \cap H^1_0(\Omega)$ *as the solution of*

$$B(\tilde{u},v) = l(v) \qquad \forall v \in S^1(\Omega,\mathcal{T}) \cap H^1_0(\Omega). \tag{3.24}$$

   This can be done efficiently by CG-iteration, because the stiffness matrix has uniformly bounded condition number.

2. *Determine* $\gamma_0 u$ *as the solution of*

$$\langle(-\Delta_{\widehat{C},h})^{1/2}(\gamma_0 u \circ F_\Omega), (\gamma_0 v \circ F_\Omega)\rangle_{0,\widehat{C}} = l(v) - B(\tilde{u},v) \qquad \forall v \in S^1(\Omega,\mathcal{T}). \tag{3.25}$$

   Note that the right hand side in (3.25) depends only on $\gamma_0 v$*, which makes it possible to compute it with the test functions supported within one near-boundary grid layer. Due to the special structure of the operator* $\Delta_{\widehat{C},h}$ *on uniform meshes, solving (3.25) is efficiently accomplished by a forward and a backward Fast Fourier Transform.*

3. *Compute (approximately) the discrete* B*-harmonic extension* $E_\mathcal{V}\gamma_0 u$*. This can be achieved with the aid of explicit extension operator [16] or by CG-iteration of the problem*

$$B(E_\mathcal{V}\gamma_0 u, v) = 0 \qquad \forall v \in S^1(\Omega,\mathcal{T}) \cap H^1_0(\Omega),$$

   since the stiffness matrix corresponding to this problem has condition number bounded uniformly in N *by Proposition 3.4.*

4. *Set* $u := \tilde{u} + E_\mathcal{V}\gamma_0 u$.

Algorithm 3.9 has the following complexity:

Proposition 3.10 *If Steps 1, 3 in Algorithm 3.9 are solved iteratively with a tolerance* $\varepsilon = O(N^{-q})$*, then the solution of (3.23) with Algorithm 3.9 requires* $O(N \log N)$ *floating point operations.*

Proof. The solution of the problem in Step 1 requires $O(N|\log\varepsilon|)$ work. The cost of the FFT is $O(N \log N)$. Finally, the calculation of the harmonic extension by the CG-iteration requires $O(N|\log\varepsilon|)$ arithmetic operations, where $\varepsilon > 0$ is the desired accuracy. Setting $\varepsilon = O(N^{-q})$ completes the proof. ∎

3.6 Remarks on implementations with static condensation

In computational practice one would likely base an iterative solution fully on local static condensation. One of the advantages of such a procedure is the reduction of the size of the problem that is solved iteratively. We recall the classical static condensation based scheme:

Algorithm 3.11 (Solution based on local static condensation)

1. *Compute the local stiffness matrices* A_K *and the local load vectors* l_K.
2. *Compute the condensed local stiffness matrices* A_K^c *(note that this enforces (3.14)) and compute the condensed load vectors* l_K^c.
3. *Assemble* $A^c = \mathcal{A}_{K\in\mathcal{T}} A_K^c$ *and assemble the condensed load vectors* l^c *(see, e.g., [39]).*
4. *Solve the linear system* $A^c x = l^c$ *by a preconditioned CG-iteration with the modified preconditioner of Algorithm 3.12 below.*
5. *Sweep through the mesh and solve for the internal degrees of freedom.*

Note that with the exception of solving the condensed system $A^c x = l^c$, all steps of this algorithm can be accomplished with work $O(N)$. For Dirichlet problems, Proposition 3.4 shows that the condition number of the matrix A^c grows polylogarithmically with the problem size and can thus be treated efficiently by CG-iterations. For the Neumann problem, a preconditioner C^c similar to the one introduced above is required.
Static condensation on the element level can be interpreted as choosing new nodal shape functions $V_{\mathcal{V}}^c$ and side shape functions $V_{\mathcal{S}}^c$. Specifically, the mapping Z that accomplishes this transformation of shape functions is given by

$$Z : u \quad\mapsto\quad Zu, \qquad Zu \text{ satisfies} \begin{cases} Zu|_e = u|_e & \forall \text{ edges } e \\ B(Zu, v) = 0 & \forall v \in V_{\mathcal{I}}. \end{cases} \tag{3.26}$$

The mapping Z maps the set of piecewise linears $V_{\mathcal{V}}$ and the the set of side shape functions $V_{\mathcal{S}}$ onto discrete harmonic nodal shape functions $V_{\mathcal{V}}^c := ZV_{\mathcal{V}}$ and discrete harmonic side shape functions $V_{\mathcal{S}}^c := ZV_{\mathcal{S}}$, respectively. The preconditioner C^c employed for the solution of the condensed linear system can then be realized with the following algorithm:

Algorithm 3.12 (Preconditioner C^c for statically condensed stiffness matrix)
Output: solution $u = u_{\mathcal{V}^c} + u_{\mathcal{S}^c} \in \operatorname{span} V_{\mathcal{V}}^c \cup V_{\mathcal{S}}^c$ *such that*

$$C^c(u, v) = l(v) \qquad \forall v \in \operatorname{span} V_{\mathcal{V}}^c \cup V_{\mathcal{S}}^c.$$

1. (a) *Determine* $\widetilde{u} \in \operatorname{span} V_{\mathcal{V}}^c \cap H_0^1(\Omega)$ *as the solution to*

$$B(\tilde{u}, v) = l(v) \qquad \forall v \in \operatorname{span} V_{\mathcal{V}}^c \cap H_0^1(\Omega). \tag{3.27}$$

This can be done efficiently by a CG-iteration as the stiffness matrix has uniformly bounded condition number (cf. [29]). Note that the stiffness matrix corresponding to (3.27) is a submatrix of A^c.

(b) *Determine* $\gamma_0 u$ *as the solution of*

$$\langle (-\Delta_{\widehat{C},h})^{1/2}(\gamma_0 u \circ F_\Omega), (\gamma_0 v \circ F_\Omega)\rangle_{0,\widehat{C}} = l(v) - B(\tilde{u}, v) \qquad \forall v \in \operatorname{span} V^c_{\mathcal{V}}.$$

(c) *Find* $E^c\gamma_0 u \in \operatorname{span} V^c_{\mathcal{V}}$ *such that*

$$B(E^c\gamma_0 u, v) = 0 \qquad \forall v \in \operatorname{span} V^c_{\mathcal{V}}. \tag{3.28}$$

This could be accomplished by a CG-iteration as the stiffness matrix corresponding to (3.28) has uniformly bounded condition number by Proposition 3.4. Note again that the stiffness matrix is a submatrix of A^c.

(d) *Set* $u_{\mathcal{V}^c} := \tilde{u} + E^c\gamma_0 u$.

2. *Introduce* $V^{c,e} := \{v \in V^c_{\mathcal{S}} \,|\, v_{|_{e'}} = 0 \quad$ *for all edges* $e' \neq e\}$. *Then* $u_{\mathcal{S}^c} \in \operatorname{span} V^c_{\mathcal{S}}$ *is given by* $u_{\mathcal{S}^c} = \sum_{\text{edges } e} u^e$ *with* $u^e \in V^{c,e}$ *solving*

$$B(u^e, v) = l(v) \qquad \forall v \in V^{c,e}. \tag{3.29}$$

Note that the stiffness matrices for these edge-based problems are submatrices of A^c. *Solving all edge problems (3.29) can be achieved with work* $O(N)$ *by Gaussian elimination.*

We will not analyze the scheme consisting of Algorithms 3.11, 3.12. In view of the following lemma, however, we expect complexity estimates for this scheme as in Proposition 3.10.

Lemma 3.13 *The mapping* Z *of (3.26) is an isomorphism between* $S^1(\Omega, \mathcal{T})$ *and* $ZS^1(\Omega, \mathcal{T})$, *i.e., for some* $C > 0$ *there holds*

$$C^{-1}\|Zu\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq C\|Zu\|_{H^1(\Omega)} \qquad \forall u \in S^1(\Omega, \mathcal{T}). \tag{3.30}$$

Proof. The lower bound $\|Zu\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega)}$ follows from the fact that the energy norm is equivalent to the H^1-norm and the energy minimization properties of the mapping Z. For the upper bound, we write Ω as the union of elements. For a fixed element K, we write $\widehat{u}$, $\widehat{Zu}$ for the pull-backs to the reference element $\hat{K}$ of the functions $u|_K$, $Zu|_K$. We then calculate using the trace theorem on $\hat{K}$ and exploiting that $\widehat{u}$ is linear:

$$\|\nabla\widehat{u}\|_{L^2(\hat{K})} \sim |\widehat{u}|_{H^{1/2}(\partial\hat{K})} = |\widehat{Z}u|_{H^{1/2}(\partial\hat{K})} \leq C\|\nabla\widehat{Z}u\|_{L^2(\hat{K})}$$

where the implied constant in the $\sim$ notation and the constant C are independent of the polynomial degree p_K and the element K. Scaling to the physical element K and summing over all elements yields:

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla Zu\|_{L^2(\Omega)}.$$

Since $u = Zu$ on $\partial\Omega$, we arrive at the upper bound $\|u\|_{H^1(\Omega)} \leq C\|Zu\|_{H^1(\Omega)}$ in (3.30). ■

4 Approximation to Poincaré-Steklov operators

4.1 Poincaré-Steklov Operators in Elliptic Problems

In some applications, the solution u to (1.2) or (1.3) is not the principal quantity of interest but rather the missing data for a complete set of Cauchy data. The Poincaré-Steklov operator T (also known as the Dirichlet-to-Neumann map) is defined as

$$\begin{aligned} T : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ \lambda &\mapsto \gamma_1 u, \quad u \text{ solves (1.2) with } f = 0. \end{aligned} \tag{4.1}$$

Likewise, we define the Steklov-Poincaré operator S (also called the Neumann-to-Dirichlet map) by

$$\begin{array}{rcl} S : H^{-1/2}(\partial\Omega) & \to & H^{1/2}(\partial\Omega) \\ \psi & \mapsto & \gamma_0 u, \quad u \text{ solves (1.3) with } f = 0. \end{array} \tag{4.2}$$

We note that the operators S, T are in fact inverses to each other, i.e., $S^{-1} = T$. Akin to the situations in (1.18), (1.19), the operator T admits a shift theorem. For general Lipschitz domains Ω, [8, Lemma 3.7] asserts for each $s \in [0, 1/2]$ the existence of $C_s > 0$ such that

$$\|T\lambda\|_{H^{-1/2+s}(\partial\Omega)} \le C_s \|\lambda\|_{H^{1/2+s}(\partial\Omega)} \qquad \forall \lambda \in H^{1/2+s}(\partial\Omega). \tag{4.3}$$

For polygonal domains Ω, this shift theorem holds in a larger range. While this is closely related to (1.19), a precise reference seems to be missing, and we therefore formulate this as an assumption:

Assumption 4.1 *There exists $\delta_0 > 1/2$ such that for each $\delta \in [0, \delta_0)$ there exists $C_\delta > 0$ with*

$$\|Tu\|_{H^{-1/2+\delta}(\partial\Omega)} \le C_\delta \|u\|_{H^{1/2+\delta}(\partial\Omega)} \qquad \forall u \in H^{1/2+\delta}(\partial\Omega). \tag{4.4}$$

Remark 4.2 For the case of Laplace's equation (and thus the case of constant coefficients), Assumption 4.1 can be verified as follows. Let $\delta_0 \in (1/2, 1]$ be defined by (1.19). For general Lipschitz domains Ω, the estimate (4.3) covers the case $\delta \in [0, 1/2]$. For $\delta \in (1/2, 1)$, combining [9, Lemma 2.11], [9, Lemma 2.7] gives

$$\|Tu\|_{H^{-1/2+\delta}(\partial\Omega)} \le C\|u\|_{H^{1+\delta}(\Omega)} \le C_\delta \|\lambda\|_{H^{1/2+\delta}(\partial\Omega)},$$

where the second estimate follows from the regularity result (1.19). ∎

In the case of convex polygons, we have $\delta_0 = 1$.

Remark 4.3 The case $b = 0$, $a_0 = 0$ does not fall directly in our framework because assumption (1.6) is not satisfied. The modification of considering the energy space $V = H^1(\Omega)/\mathbb{R}$, as is standard for Laplace's equation, can be carried out in the case of non-constant matrix A as well.

4.2 *hp*-FEM Approximation of Poincaré-Steklov operators T and S

We recall that the projector Q_N of (2.6) can be extended to an operator on $H^{-1/2}(\partial\Omega)$ by (2.14).

4.2.1 Approximation of the Steklov–Poincaré operator

Viewing the dual space Y_N' as a subspace of $H^{-1/2}(\partial\Omega)$, we define the approximation S_N to the Steklov–Poincaré operator S as

$$\begin{array}{rcl} S_N : Y_N' & \to & Y_N \\ \Psi & \mapsto & S_N\Psi := \gamma_0 u_N, \end{array}$$

where $u_N \in V_N$ solves the following discrete Neumann problem:

$$B(u_N, v) = \langle \Psi, v \rangle_{0,\partial\Omega} \qquad \forall v \in V_N.$$

The error analysis for S_N is rather straightforward.

Theorem 4.4 *Under the assumptions of Theorem 2.13 (with $f = 0$) there holds*

$$\|S\Psi - S_N\Psi\|_{1/2,\partial\Omega} \leq C\left[h^{\delta} + h^{b\alpha}\right]. \tag{4.5}$$

Proof. Applying the trace theorem, we obtain

$$\|S\Psi - S_N\Psi\|_{1/2,\partial\Omega} = \|\gamma_0 u - \gamma_0 u_N\|_{1/2,\partial\Omega} \leq C\|u - u_N\|_{1,\Omega}.$$

Now (2.20) yields (4.5). ∎

4.2.2 Approximation of the Poincaré–Steklov operator

The approximation

$$\begin{array}{rcl} T_N : Y_N & \to & Y_N' \\ \lambda & \mapsto & T_N\lambda \end{array} \tag{4.6}$$

to the Poincaré–Steklov operator T defines an element of the dual space Y_N' via

$$\langle T_N\lambda, v\rangle_{0,\partial\Omega} = B(u_N, \widetilde{v}) \qquad \forall v \in Y_N,$$

where $\widetilde{v} \in V_N$ is an arbitrary extension of v, $\gamma_0\widetilde{v} = v$, and $u_N \in V_N$ satisfies

$$\gamma_0 u_N = \lambda \quad \text{ and } \quad B(u_N, v) = 0 \quad \forall\, v \in V_N \cap H_0^1(\Omega).$$

Theorem 4.5 *Let δ_0 be given by Assumption 4.1. Under the hypotheses of Theorem 2.13 (with $f = 0$) there holds for arbitrary $\overline{\delta} \in [0, \delta] \cap [0, \delta_0)$*

$$\|T\lambda - T_N Q_N\lambda\|_{-1/2,\partial\Omega} \leq C_{\overline{\delta}}\left[h^{\overline{\delta}} + h^{b\alpha}\right]. \tag{4.7}$$

Proof. Using $Q_N T_N Q_N = T_N Q_N$, we write

$$T - T_N Q_N = (\mathrm{Id} - Q_N)T + Q_N T(\mathrm{Id} - Q_N) + Q_N(T - T_N)Q_N. \tag{4.8}$$

The first two terms in (4.8) lead to estimates of the form

$$\|(\mathrm{Id} - Q_N)T\lambda\|_{-1/2,\partial\Omega} \leq C_{\overline{\delta}} h^{\overline{\delta}}\|T\lambda\|_{-1/2+\delta,\partial\Omega} \leq C_{\overline{\delta}} h^{\overline{\delta}}\|\lambda\|_{1/2+\overline{\delta},\partial\Omega}, \tag{4.9}$$

$$\|Q_N T(\mathrm{Id} - Q_N)\lambda\|_{-1/2,\partial\Omega} \leq C\|(\mathrm{Id} - Q_N)\lambda\|_{1/2,\partial\Omega} \leq C h^{\overline{\delta}}\|\lambda\|_{1/2+\overline{\delta},\partial\Omega}, \tag{4.10}$$

where we exploited the stability and approximation properties of the projector Q_N given in Lemma 2.8 and used the shift theorem for T as detailed in Assumption 4.1. For the third term in (4.8), we first introduce for the elliptic extension $E_N : Y_N \to V_N$. By a reasoning similar to that in the proof of Corollary 2.12, we get for $v \in H^{1/2}(\partial\Omega)$

$$\|E_N(Q_N v)\|_{H^1(\Omega)} \leq C\|Q_N v\|_{H^{1/2}(\partial\Omega)} \leq C\|v\|_{H^{1/2}(\partial\Omega)}, \tag{4.11}$$

where we used the stability result (2.15) in the last step. For the treatment of the third term in (4.8), we next let $u \in H^1(\Omega)$ be the solution to (1.2) with $f = 0$ and Dirichlet boundary conditions λ; $\tilde{u} \in H^1(\Omega)$ solves (1.2) with $f = 0$ and boundary conditions $Q_N\lambda$; and $u_N \in V_N$ is the hp-FEM approximation to u given by (2.3). Reasoning as in [23, Lemma 6.1] the definitions of the operators T, T_N imply

$$\begin{aligned}\|Q_N(T-T_N)Q_N\lambda\|_{-1/2,\partial\Omega} &= \sup_{v\in H^{1/2}(\partial\Omega)} \frac{\langle (T-T_N)Q_N\lambda, Q_Nv\rangle_{0,\partial\Omega}}{\|v\|_{1/2,\partial\Omega}} \\ &= \sup_{v\in H^{1/2}(\partial\Omega)} \frac{B(\tilde u - u_N, E_N(Q_Nv))}{\|v\|_{H^{1/2}(\partial\Omega)}} \\ &\le C\|\tilde u - u_N\|_{H^1(\Omega)} \le C\left[\|u-u_N\|_{H^1(\Omega)} + \|u-\tilde u\|_{H^1(\partial\Omega)}\right] \\ &\le C\left[\|u-u_N\|_{H^1(\Omega)} + \|\lambda - Q_N\lambda\|_{H^{1/2}(\partial\Omega)}\right].\end{aligned}$$

$\|\lambda - Q_N\lambda\|_{H^{1/2}(\partial\Omega)}$ can be bounded as required by (4.10) and Theorem 2.13 allows us to bound $\|u-u_N\|_{H^1(\Omega)}$ in the desired fashion. ■

Remark 4.6 We employed the $L^2(\partial\Omega)$-projection Q_N in the definition of T_N. Other projections could, in principle, be used as well. Essential is that the projector P_N has for $\delta \in [0, \delta_0)$ the following stability and approximation properties:

$$\begin{aligned}\|P_Nu\|_{H^{-1/2}(\partial\Omega)} &\le C\|u\|_{H^{-1/2}(\partial\Omega)}, \\ \|P_Nu\|_{H^{1/2+\delta}(\partial\Omega)} &\le C_\delta\|u\|_{H^{1/2+\delta}(\partial\Omega)}, \\ \|u - P_Nu\|_{H^{1/2}(\partial\Omega)} &\le C_\delta h^\delta\|u\|_{H^{1/2+\delta}(\partial\Omega)}.\end{aligned}$$

■

Due to the results of Section 3 (see also Table 1), the implementation of the finite-dimensional operators S_N and T_N has the linear-logarithmic complexity $O(N\log^q N)$ with respect to N, except in the Case (R ii) with Neumann boundary conditions, where we arrive at the cost $O(N^{3/2}\log^q N)$.

Remark 4.7 In the case of piecewise constant coefficients in a polygonal domain, an efficient method for matrix-vector product with the discrete Poincaré-Steklov operators was developed in [22–25]. It is based on a sparse h-FEM approximation to the Schur-complement matrix

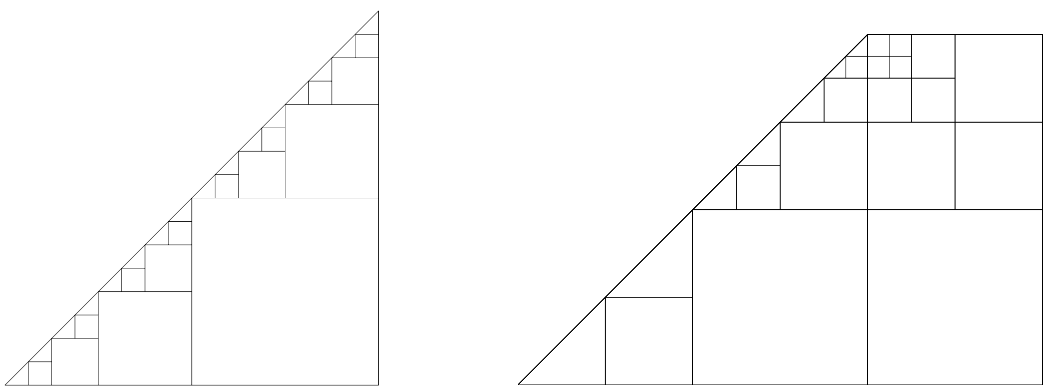

Figure 2: Uniformly (left) and nested (right) refined interface.

on a rectangle combined with the reduction of the PDE to the *refined interface*. The idea of

the refined interface is illustrated in Fig. 2, where the left and right pictures correspond to the quasi-uniform and locally refined meshes, respectively. In this way the Schur-complement matrix in each $n \times n$ rectangular subdomain is treated with $O(n \log^2 n)$ arithmetic operations using a truncated Fourier representation.
Another example of the refined interface is given in Figure 3, where non-conforming decompositions in Figure 3, (left) corresponds to the geometric meshes in Figure 1. Figure 3, (right) corresponds to the case of composite meshes refined towards the corner points related to the situation in Remark 1.2. In the case of symmetric and positive definite operators with piecewise constant coefficients the spectrally equivalent multilevel interface preconditioners, see [24, 25], lead to the complexity $O(N \log^3 N)$ for solving the Schur-complement equation on the refined interfaces depicted in Figures 2 and 3. The method of refined interface was shown to have the memory requirements $O(N \log N)$ with respect to the number of boundary degrees of freedom N. The approach was also applied to the biharmonic, Stokes and Lamé equations, see [23] and references therein.

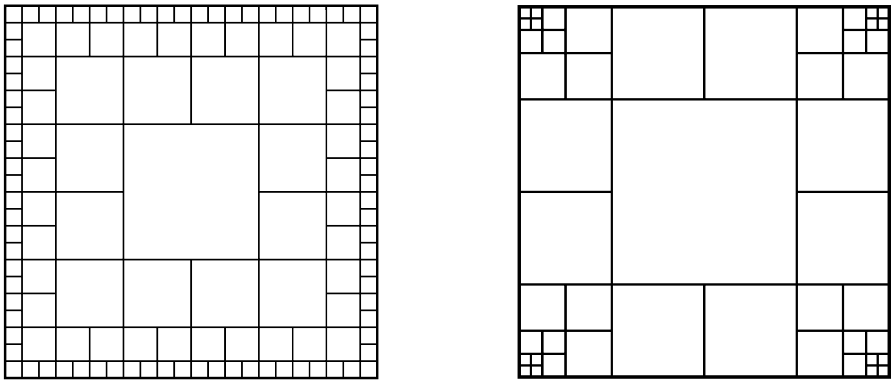

Figure 3: Refined interface corresponding to Fig. 1 (left) and to Remark 1.2 (right).

In this way, the boundary concentrated *hp*-FEM presented in this paper extends the above mentioned methods to the case of variable (piecewise analytic) coefficients. ■

5 Further Applications

5.1 Relation to boundary integral equations

Consider the case of constant coefficients. The results on the sparse approximation to the Poincaré-Steklov operators directly apply to the construction of asymptotically optimal solvers for the classical boundary integral equations involving weakly singular, hypersingular and double layer harmonic potential operators V, D and K, respectively, defined by

$$\begin{aligned} Vu(x) &= \int_\Gamma g(x,y)u(y)dy, & Ku(x) &= \int_\Gamma \frac{\partial}{\partial n_y} g(x,y)u(y)dy, \\ K'u(x) &= \int_\Gamma \frac{\partial}{\partial n_x} g(x,y)u(y)dy, & Du(x) &= -\int_\Gamma \frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y} g(x,y)u(y)dy, \end{aligned} \tag{5.1}$$

where $g(x, y)$ denotes the fundamental solution for the corresponding elliptic operator $\mathcal{L}$, and $\Gamma = \partial\Omega$ a Lipschitz domain $\Omega \in \mathbb{R}^2$. The approach is based on the representation of the inverse to the above mentioned boundary integral operators in terms of interior T_1, S_1, and exterior T_2, S_2, Poincaré–Steklov mappings proposed in [22]. Given a Hilbert space H and an element $g \in H'$, we denote $H_g := \{v \in H : \langle v, g\rangle = 0\}$. The following theorem shows the relationship between these Poincaré-Steklov operators and classical boundary integral operators.

Theorem 5.1 [22]. *The operator $V^{-1} : H^{1/2}_{g_0}(\Gamma) \to H^{-1/2}_1(\Gamma)$ has the representation*

$$V^{-1} = T_1 + T_2 \,, \tag{5.2}$$

where g_0 is the Robin potential on Γ satisfying $K' g_0 = -\frac{1}{2} g_0$. The following formulae hold

$$(\frac{1}{2}I - K)^{-1} z = (I + S_2 \cdot T_1) z, \qquad \forall\, z \in H^{1/2}(\Gamma) \tag{5.3}$$

$$(\frac{1}{2}I + K)^{-1} z = (I + S_1 \cdot T_2) z \,, \qquad \forall\, z \in H^{1/2}_{g_0}(\Gamma) \,. \tag{5.4}$$

The operator $D^{-1} : H^{-1/2}_1(\Gamma) \to H^{1/2}_{g_0}(\Gamma)$ has the representation

$$D^{-1} = S_1 + S_2. \tag{5.5}$$

A remarkable consequence of the above statement is that whenever some linear complexity scheme is constructed for the operators T_i and S_i, $i = 1, 2$, we immediately obtain a linear complexity approximation to the inverse of the classical boundary integral operators in question. We refer to [18,19] for an alternative approach to data-sparse approximations of T and S based on $\mathcal{H}$-matrix arithmetic.

5.2 Application to Exterior Boundary Value Problems

BEM are very often applied to exterior domain problems. In this subsection, we want to briefly show how the boundary concentrated FEM can be adapted to this setting.
In the exterior domain $\Omega_e := \mathbb{R}^d \setminus \Omega$, we consider the Dirichlet problem

$$\mathcal{L}_e u := -\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + c(x) u \;=\; f \qquad \text{in } \Omega_e, \tag{5.6a}$$

$$\gamma_0 u \;=\; \lambda \qquad \text{on } \partial\Omega, \tag{5.6b}$$

with similar assumptions on the data as in Section 1.3. In addition, we assume that b, a_0 and f have bounded support Ω_0, such that $\mathcal{L}_e = -\Delta$ in $\mathbb{R}^2 \setminus \Omega_0$, and that u satisfies the "radiation condition" of the form

$$|u(x)| = O(|x|^{-1}), \qquad |\nabla u| = O(|x|^{-2}), \quad |x| \to \infty \tag{5.7}$$

(this implies a compatibility condition, which suppresses the logarithmic growth at infinity that solutions to Laplace's equation on exterior domains typically exhibit). We approximate (5.7) by imposing homogeneous Neumann conditions on the auxiliary boundary Γ_∞ with dist $(\Gamma_\infty, \partial\Omega) = R = O(N^{1/2})$. As above, N denotes the number of degrees of freedom on $\partial\Omega_e$. Following [25] (see also the references therein), we use a mesh on Int $(\Gamma_\infty) \setminus \Omega$ that is a geometric mesh in the sense of Definition 2.3 (see Fig. 4). The number of levels is again estimated by $\log R = O(\log N)$. We stress that the approximation and solution schemes remain verbatim as for the interior problem.
Our arguments indicate that in the framework of boundary concentrated hp-FEM, there is no essential difference between solving exterior and interior problems in the case of smooth coefficients, and we expect again complexity $O(N \log^q N)$ for the approximation of the exterior Poincaré-Steklov operators.

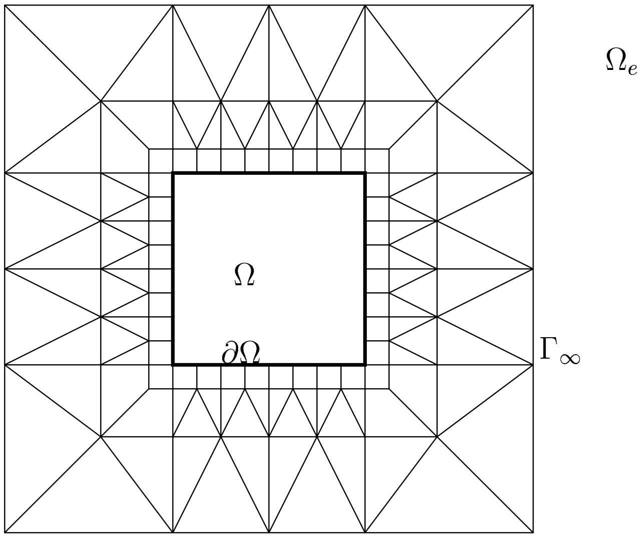

Figure 4: Geometric mesh on an exterior domain.

5.3 Application in Domain Decomposition

The efficient realization of the Poincaré-Steklov operators can also be employed in the context of domain decomposition methods. Assume the domain Ω to be composed of $M_0 \geq 1$ non-overlapping polygons Ω_i, $\overline{\Omega} = \cup_{i=1}^{M_0} \overline{\Omega}_i$ with $\Gamma := \cup_{i=1}^{M_0} \Gamma_i \setminus \partial\Omega$. Assume a bilinear form $B(\cdot, \cdot)$ is written as a sum of subdomain contributions and consider the problem of finding $u \in H_0^1(\Omega)$

$$\sum_{i=1}^{M_0} B_i(u_{|\Omega_i}, v_{|\Omega_i}) = B(u, v) = F(v) = \int_\Omega f(x) v dx + \sum_{i=1}^{M} \langle \psi_i, v \rangle_{L^2(\Gamma_i)} \qquad \forall v \in H_0^1(\Omega).$$

where the local continuous forms $B_i : V_i \times V_i \to \mathbb{R}$ are supposed to be $H_0^1(\Omega_i)$-elliptic with $V_i := H^1(\Omega_i)$ and given $\psi_i \in H^{-1/2}(\Gamma_i)$. Letting $u_{0,i} \in H_0^1(\Omega_i)$ be the particular solutions in Ω_i,

$$B_i(u_{0,i}, v) = \int_{\Omega_i} f(x) v dx \qquad \forall v \in H_0^1(\Omega_i) \tag{5.8}$$

and introducing the trace space on Γ,

$$Y_\Gamma = \{u = z_{|\Gamma} : \ z \in H_0^1(\Omega)\}, \qquad \|u\|_{Y_\Gamma} = \inf_{z \in H_0^1(\Omega): z_{|\Gamma} = u} \|z\|_{H^1(\Omega)},$$

we transform the above problem to the interface equation

$$u_{|\Gamma} \in Y_\Gamma : \quad B_\Gamma(u_{|\Gamma}, v) := \sum_{i=1}^{M} \langle T_i u_i, v_i \rangle_{0,\Gamma_i} = \sum_{i=1}^{M} \langle g_i, v \rangle_{\Gamma_i} \quad \forall v \in Y_\Gamma, \tag{5.9}$$

where $g_i = \psi_i - \gamma_{1,i} u_{0,i}$ and $u_i = u_{|\Gamma_i}$, $v_i = v_{|\Gamma_i}$. The local Poincaré-Steklov operators $T_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ are defined by the B_i-harmonic extensions, see (4.6). Since the bilinear form $B_\Gamma(\cdot, \cdot) : Y_\Gamma \times Y_\Gamma \to \mathbb{R}$ is continuous and coercive, the equation (5.9) is uniquely solvable in Y_Γ for any $g_i \in H^{-1/2}(\Gamma_i)$ providing the trace $u_{|\Gamma}$. Thus, our boundary concentrated FEM can be directly applied for solving the interface equation (5.9) by using the *hp*-FEM approximation to each individual operator T_i. In this way, we apply the geometric mesh refined towards the interface Γ.

6 Numerics

The main goal of this section is to confirm the principal features of our method (approximation power and conditioning for both full and linear-subspace stiffness matrices) for a simple model problem. Specifically, we consider

$$-\Delta u = 0 \quad \text{on } \Omega = (0,1)^2, \qquad u|_{\partial\Omega} = \begin{cases} \sin \pi x & \text{if } y = 0 \\ 0 & \text{else} \end{cases} \tag{6.1}$$

with the exact solution $u(x,y) = \sin \pi x \, \frac{\sinh(\pi(1-y))}{\sinh \pi}$. Our calculations are performed with the code CONCEPTS, [26]. For quadrilateral elements, this general hp-FEM code employs the so-called "Babuška-Szabó" shape functions, which are the tensor products of the following 1D-shape functions defined on $(-1,1)$ (we refer to [37, 39] for the details):

$$\varphi_1(x) = \frac{1}{2}(1-x), \qquad \varphi_2(x) = \frac{1}{2}(1+x), \qquad \varphi_i(x) = \frac{1}{\|L_{i-3}\|_{L^2(-1,1)}} \int_{-1}^{x} L_{i-3}(t)\, dt, \quad i \geq 3.$$

Here, the polynomials L_i are the usual Legendre polynomials. First, we check the convergence result of Theorem 2.13, which asserts that for suitable linear degree vector, the hp-FEM yields $\|u - u_N\|_{H^1(\Omega)} \leq Ch$. We check this assertion for the meshes and linear degree vectors depicted in Fig. 5. The mesh size on the portion $y = 0$ of $\partial\Omega$ takes the role of the boundary mesh size h in the statement of Theorem 2.13. The convergence behavior (H^1-error vs. boundary mesh

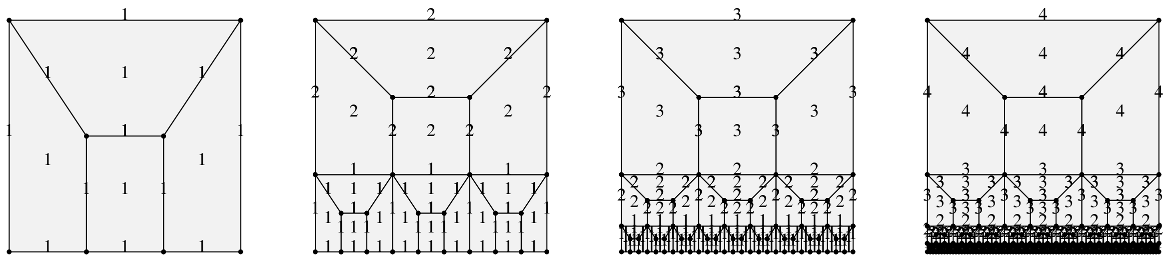

Figure 5: Geometric mesh for levels 1–4 and linear degree vector.

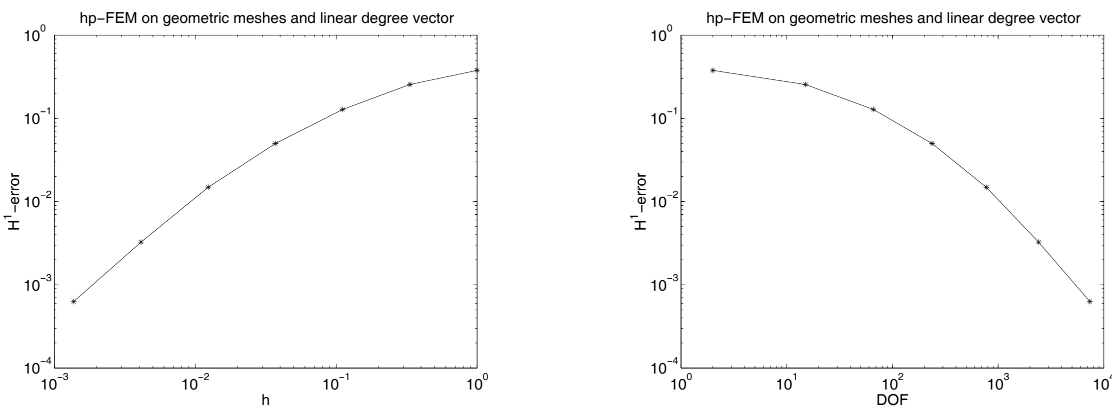

Figure 6: H^1-error for hp-BEM vs. h and N

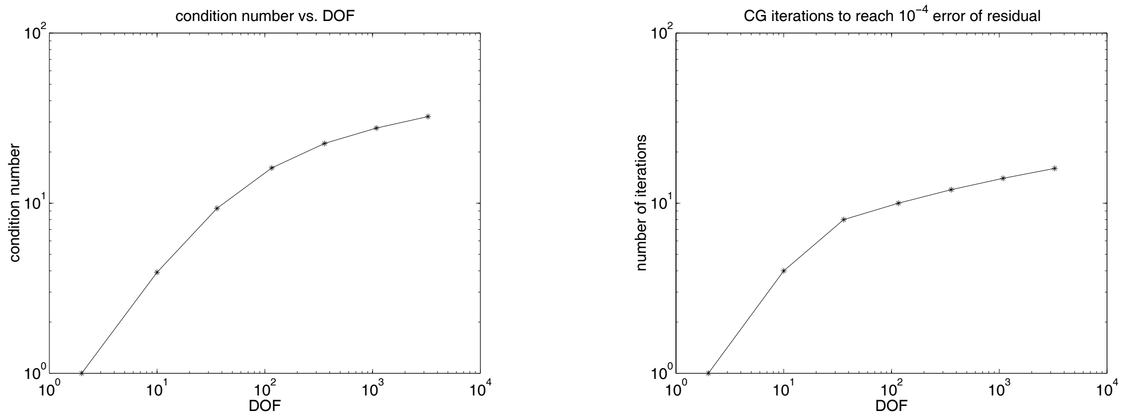

Figure 7: Condition number of stiffness matrix number of CG iterations required for a residual of 10^{-4} using p.w. linear elements.

level	6	7	8	9	10	11	12
N_{it}	27	30	32	35	37	38	41
N_Γ	376	760	1528	3064	6136	12280	24568
N_Ω	665	1417	2937	5993	12121	24300	48953

Table 2: Iteration count in the case $p = 1$ with the mesh in Fig. 1–left.

size h and the number of degrees of freedom DOF $= \dim S^{\mathbf{p}}(\mathcal{T}, \Omega))$ is given in Fig. 6.
We now turn to our results in Proposition 3.4 concerning the conditioning of the stiffness matrix. For the present Dirichlet problem and $p = 1$, Proposition 3.4 asserts that the condition number is bounded uniformly in h for the meshes of Fig. 5. The numerical results can be found in Fig. 7. A second numerical example for the condition number estimates for the stiffness matrix is shown in Table 2. For meshes as depicted in Fig. 1 (left) and polynomial degree $p = 1$, we present in Table 2 the number of boundary nodes N_Γ, the number N_Γ of nodes in Ω, as well as the number CG iterations N_{it} (with diagonal preconditioning) to reach a residual with l^2-norm below 10^{-6}.
We finally consider the condition number of the full stiffness matrix on the meshes and polynomial degree distributions as depicted in Fig. 5. From [27] and a reasoning as in the proof of Proposition 3.4 (see also [29]), the condition number of the full stiffness matrix is bounded by

$$\kappa(A) \leq C p^4, \tag{6.2}$$

where C is independent of h. Fig. 8 presents the number of CG iterations (without preconditioning) to reach a residual of 10^{-6}. The numerical results are slightly better than the growth of $O(p^2)$ expected from (6.2).

A Appendix: Analytic Regularity Results

In the present section, we are interested in analytic regularity results for solutions to the equation

$$\mathcal{L}u := -\nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u + a_0(x)u = f(x) \quad \text{on } \Omega. \tag{A.1}$$

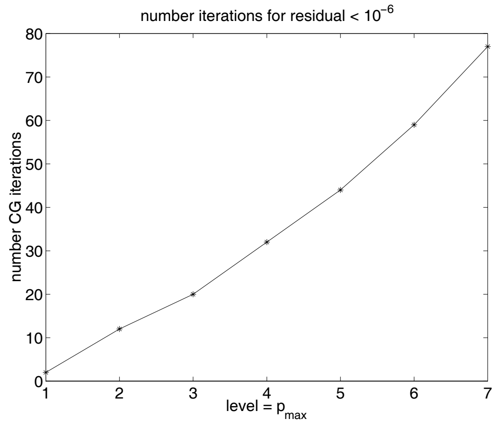

Figure 8: Iterations count for the full system vs. the number of levels.

Here, the symmetric matrix A is uniformly positive definite and b is a vector. We furthermore assume the coefficients A, b, c to be analytic; i.e., we stipulate the existence of C_d, $\gamma_d > 0$ such that

$$\|\nabla^p A\|_{L^\infty(\Omega)} + \|\nabla^p b\|_{L^\infty(\Omega)} + \|\nabla^p a_0\|_{L^\infty(\Omega)} \le C_d \gamma_d^p p! \qquad \forall p \in \mathbb{N}_0 . \tag{A.2}$$

In aim of the present section is the proof of the following analytic regularity result for the solution u of (A.1):

Theorem A.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega$. Let the distance function r be given by (1.15). Let f be analytic on Ω and satisfy for some $\delta \in (0,1]$*

$$\|r^{p+1-\delta}\nabla^p f\|_{L^2(\Omega)} \le C_f \gamma_f^p p! \qquad \forall p \in \mathbb{N}_0 \tag{A.3}$$

for some C_f, $\gamma_f > 0$. Finally, let $u \in H^{1+\delta}(\Omega)$ solve (A.1) with data A, b, a_0 satisfying (A.2). Then there exist C, $\gamma > 0$ depending only on C_d, C_f, γ_d, γ_f such that

$$\|r^{p+1-\delta}\nabla^{p+2} u\|_{L^2(\Omega)} \le C\gamma^p p! \|u\|_{H^{1+\delta}(\Omega)} \qquad \forall p \in \mathbb{N}_0 .$$

Remark A.2 The restrictions on the data A, b, c are not minimal: "blow-up" akin to that in (A.3) would be possible.
The theorem can also extended to the case of piecewise analytic data A, b, c.

In order to prove Theorem A.1, we start with the following lemma.

Lemma A.3 *Let B_R be a ball of radius $R \le 1$. Assume that A, b, c satisfy (A.2) with Ω replaced with B_R. Assume that f satisfies on B_R*

$$\|\nabla^p f\|_{L^2(B_R)} \le C_f \gamma_f^p p! R^{-p-1+\delta} \qquad \forall p \in \mathbb{N}_0$$

for some C_f, $\gamma_f > 0$, $\delta \in (0,1]$. Let $u \in H^{1+\delta}(B_R)$ solve

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f \qquad \text{on } B_R.$$

Then for every $c \in (0,1)$ there exist constants C, $\gamma > 0$ depending only on C_d, γ_d, C_f, γ_f, δ, and c such that

$$\|\nabla^{p+2} u\|_{L^2(B_{cR})} \le C R^{-p-1+\delta} \gamma^p p! \left[\|u\|_{H^1(B_R)} + |\nabla u|_{H^\delta(B_R)} \right],$$

where $|\nabla u|_{H^\delta(B_R)}$ stands for the Aronszajn-Slobodeckij norm.

Proof. Using the techniques of [32], we have (see [28] for the details)

$$R^p \|\nabla^{p+2} u\|_{L^2(B_{cR})} \leq C\gamma^p p! \left[R^{-1}\|\nabla u\|_{L^2(B_R)} + R^{-2}\|u\|_{L^2(B_R)} + C_f R^{-1+\delta}\right], \tag{A.4}$$

where C, $\gamma > 0$ depend only on γ_f, C_d, and γ_d. Let l be an arbitrary linear function. Then $u - l$ satisfies

$$L(u-l) = \tilde{f} := f - Ll.$$

From the assumptions on the data, we then conclude that for all $p \in \mathbb{N}_0$

$$\|\nabla^p \tilde{f}\|_{L^2(B_R)} \leq C_f \gamma_f^p p! R^{-p-1+\delta} + C\gamma^p p! \|l\|_{H^1(B_R)} \leq \tilde{C}\tilde{\gamma}^p R^{-p-1+\delta}\left[C_f + R^{1-\delta}\|l\|_{H^1(B_R)}\right],$$

where the constants $\tilde{C}$, $\tilde{\gamma} > 0$ depend on C_d, γ_d, and γ_f. Applying (A.4) with u replaced with $u - l$, we obtain

$$R^p\|\nabla^{p+2}u\|_{L^2(B_{cR})} \leq C\gamma^p p! \left[R^{-1}\|\nabla(u-l)\|_{L^2(B_R)} + R^{-2}\|u-l\|_{L^2(B_R)} + R^{-1+\delta}C_f + \|l\|_{H^1(B_R)}\right]. \tag{A.5}$$

The assumption $u \in H^{1+\delta}(B_R)$ implies the existence of a linear function l such that

$$\|u-l\|_{L^2(B_R)} + R\|\nabla(u-l)\|_{L^2(B_R)} \leq CR^{1+\delta}|\nabla u|_{H^\delta(B_R)},$$

where $|\cdot|_{H^\delta(B_R)}$ denotes the Aronszajn-Slobodeckij semi norm. We conclude using $R \leq 1$

$$R^{1-\delta+p}\|\nabla^{p+2}u\|_{L^2(B_{cR})} \leq C\gamma^p p! \left[|\nabla u|_{H^\delta(B_R)} + \|u\|_{H^1(B_R)} + C_f\right].$$

■

Proof of Theorem A.1. Using the Besicovitch covering theorem (see, e.g., [42]), we can construct a covering of Ω by a countable collection $\mathcal{B} = \{B_i \,|\, i \in \mathbb{N}\}$ of closed balls B_i with the following properties:

1. $B_i = B_{r_i}(x_i)$ where $r_i = c' \operatorname{dist}(x_i, \partial\Omega)$ for some fixed $c' \in (0,1)$;
2. there exists $N \in \mathbb{N}$ such that for all $x \in \Omega$: $|\{i \in \mathbb{N} \,|\, x \in B_i\}| \leq N$;
3. there exists $c \in (0,1)$ such that $\Omega \subset \cup_{i\in\mathbb{N}} B_{cr_i}(x_i)$.

Set

$$C_i^2 := \sum_{p\in\mathbb{N}_0} \frac{1}{(2\gamma_f)^{2p}(p!)^2}\|r^{p+1-\delta}\nabla^p f\|^2_{L^2(B_i)}.$$

The properties of the covering $\mathcal{B}$ and the assumption (A.3) imply

$$\begin{aligned} r_i^{-p-1+\delta}\|\nabla^p f\|_{L^2(B_i)} &\leq C_i\left(\frac{c'}{1-c'}\right)^{p+1-\delta}(2\gamma_f)^p p! \qquad \forall p \in \mathbb{N}_0, \\ \sum_{i\in\mathbb{N}} C_i^2 &\leq NC_f^2 \sum_{p\in\mathbb{N}_0}\frac{1}{(2\gamma_f)^{2p}(p!)^2}\gamma_f^{2p}(p!)^2 = \frac{4}{3}NC_f^2. \end{aligned}$$

From Lemma A.3 we get

$$r_i^{p+1-\delta}\|\nabla^{p+2}u\|_{L^2(B_{cr_i}(x_i))} \leq \gamma^p p! \left[C_i + \|u\|_{H^1(B_i)} + |\nabla u|_{H^\delta(B_i)}\right].$$

Using the fact that $\Omega \subset \cup_i B_{cr_i}(x_i)$ and the finite overlap property of the covering $\mathcal{B}$, we get by summation over all balls B_i

$$\begin{aligned}
\|r^{p+1-\delta}\nabla^{p+2}u\|^2_{L^2(\Omega)} &\leq \left(\frac{1+c'}{c'}\right)^{p+1-\delta} \sum_{i\in\mathbb{N}} r_i^{2(p+1-\delta)} \|\nabla^{p+2}u\|^2_{L^2(B_{cr_i}(x_i))} \\
&\leq C(\gamma^p p!)^2 \sum_{i\in\mathbb{N}} C_i^2 + \|u\|^2_{H^1(B_i)} + |\nabla u|^2_{H^\delta(B_i)} \\
&\leq C(\gamma^p p!)^2 \left[C_f^2 + \|u\|^2_{H^{1+\delta}(\Omega)}\right]
\end{aligned}$$

for suitable constants C, γ depending on γ_f and the covering $\mathcal{B}$. ■

The restriction $\delta \in [0,1]$ in Theorem A.1 can be removed:

Corollary A.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and r be defined by (1.15). Let $k = \kappa + \delta$ with $\kappa \in \mathbb{N}_0$, $\delta \in [0,1)$ be given. Assume that $u \in H^{1+k}(\Omega)$ satisfies (A.1) with coefficients A, b, c satisfying (A.2) and f satisfying*

$$\|f\|_{H^{\kappa-1}(\Omega)} + \|r^{p+1-\delta}\nabla^{p+\kappa} f\|_{L^2(\Omega)} \leq C_f \gamma_f^p p! \qquad \forall p \in \mathbb{N}_0. \tag{A.6}$$

Then there exist C, $\gamma > 0$ such that

$$\|r^{p+1-\delta}\nabla^{p+2+\kappa}u\|_{L^2(\Omega)} \leq C\gamma^p p! \left[C_f + \|u\|_{H^{1+k}(\Omega)}\right].$$

Proof. The corollary is proved by induction on $\kappa \in \mathbb{N}_0$. For $\kappa = 0$, the result holds true by Theorem A.1. Assuming that it holds for all $0 \leq \kappa' < \kappa$ for some $\kappa \in \mathbb{N}$, we show that it holds for $\kappa + 1$. The induction hypothesis implies that

$$\|r^{p+1-\delta}\nabla^{p+2+\kappa'}u\|_{L^2(\Omega)} \leq C\left[C_f + \|u\|_{H^{1+\delta+\kappa'}(\Omega)}\right]\gamma^p p! \qquad \forall p \in \mathbb{N}_0, \quad 0 \leq \kappa' < \kappa.$$

Differentiating (A.1) κ times, it is easy to see that $D^\alpha u$ with $|\alpha| = \kappa$ satisfies a differential equation of the form

$$LD^\alpha u = f_\alpha := D^\alpha f + \tilde{u}_\alpha$$

where $\tilde{u}_\alpha = \sum_{|\beta|\leq\kappa+1} \lambda_{\alpha,\beta} D^\beta u$ for some analytic functions $\lambda_{\alpha,\beta}$. The induction hypothesis and the assumptions on f imply that

$$\|r^{p+1-\delta}\nabla^p f_\alpha\|_{L^2(\Omega)} \leq C\left[C_f + \|u\|_{H^{1+\delta+\kappa}(\Omega)}\right]\gamma^p p! \qquad \forall p \in \mathbb{N}_0.$$

(See Lemma 2.3.3 of [28] for a rigorous proof that products $\lambda_{\alpha,\beta}D^\beta u$ satisfy the desired bounds.) Theorem A.1 therefore allows us to conclude the induction argument. ■

B Appendix: Compact Embeddings

For $\delta \in (0,1)$ and a domain $\Omega \subset \mathbb{R}^2$ we define as usual

$$\|u\|^2_{\widetilde{H}^\delta(\Omega)} := \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy + \int_\Omega \frac{|u(x)|^2}{|\mathrm{dist}\,(x,\partial\Omega)|^{2\delta}}\,dx \tag{B.1}$$

For completeness' sake, we include the proof of a lemma that is due to von Petersdorff [40]:

Lemma B.1 *Let $\mathcal{B} = \{B_i \,|\, i \in \mathbb{N}\}$ be a covering of a domain $\Omega \subset \mathbb{R}^2$ that satisfies a* finite overlap property, *i.e., there exists $N \in \mathbb{N}$ such that*

$$\sup_{x\in\Omega} \operatorname{card} \{i \in \mathbb{N} \,|\, x \in B_i\} \leq N. \tag{B.2}$$

Then

$$\sum_{i,j\in\mathbb{N}} \int_{B_i}\int_{B_j} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy \leq N\left(3+\frac{\pi}{\delta}\right)\sum_{i\in\mathbb{N}} \|u\|^2_{\widetilde{H}^\delta(B_i)}.$$

Proof. We write the double integrals of the double sum as

$$\int_{B_i}\int_{B_j} = \int_{B_i\setminus B_j}\int_{B_j\setminus B_i} + \int_{B_i\setminus B_j}\int_{B_j\cap B_i} + \int_{B_i\cap B_j}\int_{B_j\setminus B_i} + \int_{B_i\cap B_j}\int_{B_j\cap B_i} \tag{B.3}$$

Using the finite overlap property (B.2), we bound

$$\begin{aligned}
\sum_{i,j} \int_{B_i\cap B_j}\int_{B_j\cap B_i} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy &\leq N\sum_i \int_{B_i}\int_{B_i} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy \\
&\leq N\sum_{i\in\mathbb{N}} \|u\|^2_{\widetilde{H}^\delta(B_i)}.
\end{aligned}$$

Completely analogously, we conclude

$$\begin{aligned}
\sum_{i,j} \int_{B_i\setminus B_j}\int_{B_j\cap B_i} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy &\leq N\sum_{i\in\mathbb{N}} \|u\|^2_{\widetilde{H}^\delta(B_i)} \\
\sum_{i,j} \int_{B_i\cap B_j}\int_{B_j\setminus B_i} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy &\leq N\sum_{j\in\mathbb{N}} \|u\|^2_{\widetilde{H}^\delta(B_j)}.
\end{aligned}$$

For the first term in (B.3), we bound

$$\sum_{i,j} \int_{B_i\setminus B_j}\int_{B_j\setminus B_i} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy \leq 2\sum_{i,j} \int_{B_i\setminus B_j}\int_{B_j\setminus B_i} \frac{|u(x)|^2+|u(y)|^2}{|x-y|^{2+2\delta}}\,dxdy.$$

We bound

$$\begin{aligned}
\sum_{i,j} \int_{B_i\setminus B_j}\int_{B_j\setminus B_i} \frac{|u(x)|^2}{|x-y|^{2+2\delta}}\,dxdy &\leq \sum_i \int_{B_i} |u(x)|^2 \sum_j \int_{B_j\setminus B_i} \frac{1}{|x-y|^{2+2\delta}}\,dy\,dx \\
&\leq N\sum_i \int_{B_i} |u(x)|^2 \int_{\Omega\setminus B_i} \frac{1}{|x-y|^{2+2\delta}}\,dy\,dx \\
&\leq \frac{\pi N}{\delta}\sum_i \int_{B_i} |u(x)|^2 |\operatorname{dist}(x,\partial B_i)|^{-2\delta}\,dx \leq \frac{\pi N}{\delta}\sum_i \|u\|^2_{\widetilde{H}^\delta(B_i)}
\end{aligned}$$

Combining these estimates completes the proof. ∎

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let r be defined by (1.15). For $k \in \mathbb{N}$ and $\beta \in (0,1)$ introduce the norm

$$\|u\|^2_{H^k_\beta(\Omega)} := \|u\|^2_{H^{k-1}(\Omega)} + \|r^\beta \nabla^k u\|^2_{L^2(\Omega)}. \tag{B.4}$$

The spaces $H^k_\beta(\Omega)$ are then defined as the completion of $C^\infty(\overline{\Omega})$ under this norm. We have

Lemma B.2 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, $\beta \in (0,1)$. Then for each $0 \le \delta < \min\{1/2, 1-\beta\}$ there exists $C > 0$ such that*

$$\|u\|_{H^\delta(\Omega)} \le C\|u\|_{H^1_\beta(\Omega)} \qquad \forall u \in H^1_\beta(\Omega).$$

Proof. Let $\mathcal{B} = \{B_i \,|\, i \in \mathbb{N}\}$ be the cover of Ω by balls given in the proof of Theorem A.1. For each i, let $\hat{u}_i = u \circ F_i$, where the affine map $F_i : \hat{B} \to B_i$ maps the unit ball $\hat{B}$ onto B_i. By Sobolev's embedding theorem there exists, for each $\varepsilon > 0$, a constant $C_\varepsilon > 0$ such that

$$\|\hat{u}_i\|^2_{\widetilde{H}^\delta(\hat{B})} \le C_\varepsilon \|\nabla \hat{u}_i\|^2_{L^2(\hat{B})} + \varepsilon \|\hat{u}_i\|^2_{L^2(\hat{B})}.$$

Here, we used the assumption $\delta < 1/2$. We will choose $\varepsilon > 0$ sufficiently small at the end of the proof. Scaling back to the balls B_i, we conclude

$$r_i^{-2+2\delta}\|u\|^2_{\widetilde{H}^\delta(B_i)} \le C_\varepsilon \|\nabla u\|^2_{L^2(B_i)} + C\varepsilon r_i^{-2}\|u\|^2_{L^2(B_i)}.$$

Thus, using the properties of the covering

$$\|u\|^2_{\widetilde{H}^\delta(B_i)} \le C_\varepsilon \|r^{1-\delta}\nabla u\|^2_{L^2(B_i)} + C\varepsilon \|r^{-\delta}u\|^2_{L^2(B_i)}.$$

Summing over all balls B_i of the covering and using its overlap property together with Lemma B.1, we arrive at

$$|u|^2_{H^\delta(\Omega)} \le C\sum_i \|u\|^2_{\widetilde{H}^\delta(B_i)} \le C_\varepsilon \|r^{1-\delta}\nabla u\|^2_{L^2(\Omega)} + C\varepsilon \|r^{-\delta}u\|^2_{L^2(\Omega)}.$$

As $\delta < 1/2$, we get from Theorem 1.4.4.3 of [13] $\|r^{-\delta}u\|_{L^2(\Omega)} \le C\|u\|_{H^\delta(\Omega)}$. Thus, choosing ε sufficiently small, we arrive at the desired bound. ■

A consequence of Lemma B.2 is the following

Theorem B.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\beta \in (0,1)$. Then for each $0 \le \delta < \min\{1/2, 1-\beta\}$ the embedding $H^2_\beta(\Omega) \subset H^{1+\delta}(\Omega)$ is compact. In particular, $H^2_\beta(\Omega) \subset C^0(\overline{\Omega})$.*

Proof. The embedding $H^2_\beta(\Omega) \subset C^0(\overline{\Omega})$ follows from $H^{1+\delta}(\Omega) \subset C^0(\overline{\Omega})$, valid for all $\delta > 0$ by Sobolev's embedding theorem. Because $H^{1+\delta}(\Omega) \subset H^{1+\delta'}(\Omega)$ is compactly embedded for $0 \le \delta < \delta'$, it suffices to show that for each δ there exists $C > 0$ such that $\|u\|_{H^{1+\delta}(\Omega)} \le C\|u\|_{H^2_\beta(\Omega)}$. This estimate follows from Lemma B.2 applied to ∇u. ■

Theorem B.4 *Let $\hat{K}$ be the reference square or the reference triangle. Let $r(x) = \operatorname{dist}(x, \hat{K})$, $\beta \in (0,1)$. For $u \in H^2_\beta(\hat{K})$ let Iu be the linear (if $\hat{K} = T$) or the bilinear (if $\hat{K} = S$) interpolant. Then*

$$\|u - Iu\|_{H^2_\beta(\hat{K})} \le C\|r^\beta \nabla^2 u\|_{L^2(\hat{K})}.$$

Proof. Let A_1, A_2, A_3 be three vertices of $\hat{K}$. Exploiting the compactness result of Theorem B.3 in the same way as in the proof of Lemma 4.16 of [37], we obtain the existence of $C > 0$ such that

$$\|u\|^2_{H^2_\beta(\hat{K})} \le C\left[\|r^\beta \nabla^2 u\|^2_{L^2(\hat{K})} + \sum_{i=1}^3 |u(A_i)|^2\right] \qquad \forall u \in H^2_\beta(\hat{K}).$$

As $u(A_i) = Iu(A_i)$ by construction, the result follows. ■

C Sobolev spaces on the boundary of polygons

Lemma C.1 *Let $I = [a,b]$, $I' = [a',b']$ be intervals. Let $\varphi : I \to I'$ be a piecewise smooth bijection. Then for $|s| < 3/2$ the map $u \mapsto u \circ \phi$ is an isomorphism between $H^s(I)$ and $H^s(I')$. The same result holds for the spaces $H^s_{per}(I)$, $H^s_{per}(I')$ of periodic functions, if the piecewise smooth function ϕ satisfies additionally $\varphi(a) = \varphi(b)$.*
In particular, for a polygon Ω, let $\varphi : I \to \partial\Omega$ be the parametrization by arc length. Then the map $u \mapsto u \circ \phi$ provides an isomorphism between the spaces $H^s(\partial\Omega)$ and $H^s_{per}(I)$ for $|s| < 3/2$.

Proof. This result is due to P. Grisvard; see, e.g., [9, Cor. 2.8]. ∎

For a mesh $\mathcal{T} = \{K\}$, we define the piecewise polynomial spaces $S^{p,k}(I,\mathcal{T})$, p, $k \in \mathbb{N}_0$ as follows:

$$S^{p,k}(I,\mathcal{T}) = \{u \in H^k(I) \,|\, u|_K \text{ is a polynomial of degree } p\}.$$

Lemma C.2 *Let $I \subset \mathbb{R}$ be an interval and $\mathcal{T}$ a quasi-uniform mesh on I with mesh size h, i.e., the nodes $x_0 < x_1 < \cdots x_N$ of the mesh satisfy $\gamma^{-1}h \le x_{i+1} - x_i \le \gamma h$, $i = 0,\ldots,N-1$, for some $\gamma > 0$. Then for every $\varepsilon \in [0,1/2)$ and every $p \in \mathbb{N}_0$ there exists $C_{\varepsilon,p} > 0$ such that*

$$\|u\|_{H^\varepsilon(I)} \le C_{\varepsilon,p} h^{-\varepsilon} \|u\|_{L^2(I)} \qquad \forall u \in S^{p,0}(I,\mathcal{T}), \tag{C.1}$$

$$\|u\|_{H^{1+\varepsilon}(I)} \le C_{\varepsilon,p} h^{-(1+\varepsilon)} \|u\|_{L^2(I)} \qquad \forall u \in S^{p,1}(I,\mathcal{T}). \tag{C.2}$$

Proof. We first show (C.1). (C.1) is trivially valid for $\varepsilon = 0$. Let therefore $\varepsilon \in (0,1/2)$. We characterize the norm $\|\cdot\|_{H^\varepsilon(I)}$ using the K-functional; that is, we have for all $u \in H^\varepsilon(I)$:

$$\|u\|^2_{H^\varepsilon(I)} \sim \int_0^\infty t^{-2\varepsilon-1} K^2(u,t)\,dt, \qquad K^2(u,t) := \inf_{v \in H^1(I)} \|u - v\|^2_{L^2(I)} + |v'|^2_{L^2(I)}.$$

We choose the function v in the infimum appropriately. For $t \ge h$ we take $v \equiv 0$ and get

$$\int_h^\infty t^{-2\varepsilon-1} K^2(u,t)\,dt \le \int_h^\infty t^{-2\varepsilon-1} \|u\|^2_{L^2(I)}\,dt = \frac{1}{2\varepsilon} h^{-2\varepsilon} \|u\|^2_{L^2(I)}. \tag{C.3}$$

In the range $t \in (0,h)$ we proceed as follows. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a smooth function with $0 \le \varphi(x) \le 1$ that is supported by $[-1,1]$ and that satisfies $\varphi \equiv 1$ on $[-1/2,1/2]$. For $\delta > 0$ we set $\varphi_\delta(x) := \varphi(x/\delta)$. For $t \in (0,h)$ we then set

$$\psi_t(x) := 1 - \sum_{i=0}^N \varphi_{t/\gamma}(x - x_i)$$

(γ is the quasi-uniformity constant of the mesh $\mathcal{T}$) and choose the function v in the infimum defining $K(t,u)$ as

$$v(x) := \psi_t(x) u(x).$$

Noting the support properties of ψ_t and standard polynomial inverse estimates, we arrive at

$$\begin{aligned}
\|u - v\|^2_{L^2(I)} &= \|(1 - \psi_t)u\|^2_{L^2(I)} \le \gamma^{-1} t \|u\|^2_{L^\infty(I)} \le C_p t h^{-1} \|u\|^2_{L^2(I)}, \\
\|v'\|^2_{L^2(I)} &\le \sum_{i=0}^{N-1} \|(u\psi_t)'\|^2_{L^2(x_i,x_{i+1})} \le 2 \sum_{i=0}^{N-1} \|u'\|^2_{L^2(x_i,x_{i+1})} + \|u\|^2_{L^\infty(x_i,x_{i+1})} \|\psi_t'\|^2_{L^2(x_i,x_{i+1})} \\
&\le C \sum_{i=0}^{N-1} h^{-2} \|u\|^2_{L^2(x_i,x_{i+1})} + h^{-1} t^{-1} \|u\|^2_{L^2(x_i,x_{i+1})} \le C h^{-2} \left(1 + \frac{h}{t}\right) \|u\|^2_{L^2(I)}.
\end{aligned}$$

A straightforward calculation then shows

$$\int_0^h t^{-1-2\varepsilon} K^2(t,u)\,dt \le C h^{-2\varepsilon} \|u\|^2_{L^2(I)},$$

where the constant C depends on p, ε, I, and γ, but is independent of u and h. This proves (C.1). For (C.2), we note again that the case $\varepsilon = 0$ is a standard polynomial inverse estimate. For $\varepsilon \in (0, 1/2)$, we bound

$$\|u\|_{H^{1+\varepsilon}(I)} \le C\left[\|u\|_{L^2(I)} + \|u'\|_{L^2(I)} + \|u'\|_{H^\varepsilon(I)}\right] \le Ch^{-1}\|u\|_{L^2(I)} + C\|u'\|_{H^\varepsilon(I)}.$$

Since $u' \in S^{p-1,0}(I,\mathcal{T})$, we may apply (C.2) to the second term to get

$$\|u\|_{H^{1+\varepsilon}(I)} \le Ch^{-1}\|u\|_{L^2(I)} + C_{\varepsilon,p}h^{-\varepsilon}\|u'\|_{L^2(I)} \le C_{\varepsilon,p}h^{-1-\varepsilon}\|u\|_{L^2(I)}.$$

■

Proposition C.3 *Let $I \subset \mathbb{R}$ be an interval, $\mathcal{T}$ be a quasi-uniform mesh on I with quasi-uniformity constant γ. Let $P : L^2(I) \to S^{p,1}(I,\mathcal{T})$ be a linear operator with*

i. $\|Pu\|_{L^2(I)} \le C_{stable}\|u\|_{L^2(I)}$ for all $u \in L^2(I)$;

ii. $Pu = u$ for all $u \in S^{1,1}(I,\mathcal{T})$.

Then for every $\varepsilon \in [0, 3/2)$ there exists a constant $C_\varepsilon > 0$ depending only on p, C_{stable}, γ, and ε such that

$$\|Pu\|_{H^\varepsilon(I)} \le C_\varepsilon \|u\|_{H^\varepsilon(I)}.$$

Proof. Let $u \in H^\varepsilon(I)$ be arbitrary. For simultaneous approximation in Sobolev spaces (see, e.g., [7]) there holds for some $C_\varepsilon > 0$ independent of u and h: Let $q_u \in S^{1,1}(I,\mathcal{T})$ be a piecewise linear function that attains the infimum ($S^{1,1}(I,\mathcal{T})$ is finite-dimensional), i.e.,

$$h^\varepsilon \|u - q_u\|_{H^\varepsilon(I)} + \|u - q_u\|_{L^2(I)} \le C_\varepsilon h^\varepsilon \|u\|_{H^\varepsilon(I)}. \tag{C.4}$$

Next, exploiting the reproduction assumption (ii) and the stability assumption (i), we obtain

$$\|u - Pu\|_{L^2(I)} \le \|u - q_u\|_{L^2(I)} + \|P(u - q_u)\|_{L^2(I)} \le (1 + C_{stable})\|u - q_u\|_{L^2(I)}. \tag{C.5}$$

We can therefore estimate with Lemma C.2

$$\begin{aligned}
\|Pu\|_{H^\varepsilon(I)} &\le \|u\|_{H^\varepsilon(I)} + \|u - Pu\|_{H^\varepsilon(I)} \le \|u\|_{H^\varepsilon(I)} + \|u - q_u\|_{H^\varepsilon(I)} + \|q_u - Pu\|_{H^\varepsilon(I)} \\
&\le C_\varepsilon\|u\|_{H^\varepsilon(I)} + C_\varepsilon h^{-\varepsilon}\|q_u - Pu\|_{L^2(I)} \\
&\le C_\varepsilon\|u\|_{H^\varepsilon(I)} + C_\varepsilon h^{-\varepsilon}\left\{\|u - q_u\|_{L^2(I)} + \|u - Pu\|_{L^2(I)}\right\} \\
&\le C_\varepsilon\|u\|_{H^\varepsilon(I)} + C_\varepsilon(2 + C_{stable})h^{-\varepsilon}\|u - q_u\|_{L^2(I)} \le C\|u\|_{H^\varepsilon(I)},
\end{aligned}$$

where we used (C.5) and (C.4). This concludes the proof of the proposition. ■

Remark C.4

1. It can be checked that Proposition C.3 also holds if the linear operator P is replaced with an operator $P : H^1(I) \to S^{p,1}(I,\mathcal{T})$ that is stable in $H^1(I)$, i.e., $\|Pu\|_{H^1(I)} \le C_{stable}\|u\|_{H^1(I)}$ for all $u \in H^1(I)$ and that satisfies assumption ii of Proposition C.3.

2. It can also be checked that Proposition C.3 remains valid for periodic functions, i.e., if $P : L^2(I) \to S^{p,1}(I,\mathcal{T}) \cap H^1_{per}(I)$ satisfies $Pu = u$ for all $u \in H^s_{per}(I)$.

■

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