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by

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Few Remarks on Differential Inclusions *

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Abstract In this paper we analyse the methodology of the theory of differential inclusions. First we emphasize that any sequence of piece-wise affine functions with successive elements obtained by perturbations of preceding ones in the sets of their affinity converges strongly. This gives a simple algorithm to construct sequences of approximate solutions which converge to exact ones (neither specific choice suggested by the method of convex integration nor Baire category methodology is required). Then we suggest a functional which is defined in the set of admissible functions and which measures maximal oscillations produced by sequences of admissible functions weakly convergent to given ones. This functional can be used to prove that the set of stable solutions is dense (residual) in the closure of the set of admissible functions both via the Baire category lemma or via specific choice of strictly convergent sequences.

Key words Differential inclusions, Baire categories, convex integration, integral functionals, weak-strong convergence

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1 Introduction

In this paper we address the issues of existence and stability of solutions of the problem

$$Du \in K,\ u = g \text{ on } \partial\Omega,\ u \in W^{1,\infty}(\Omega;\mathbf{R}^m). \tag{1.1}$$

Everywhere in this paper we assume that U, K are *bounded and compact* subsets of the set of $m\times n$ matrices $\mathbf{R}^{m\times n}$, respectively. The set Ω is an open bounded subset of $\mathbf{R}^n$ with Lipschitz boundary. We reserve the notation l_A for affine functions with the gradient equal to A.

We say that a function $g \in W^{1,1}(\Omega;\mathbf{R}^m)$ is piece-wise affine if there exists an at most countable family of disjoint open subsets Ω_j of Ω with $\operatorname{meas}(\partial\Omega_j) = 0$ such that $\operatorname{meas}(\Omega \setminus \cup_j\Omega_j) = 0$ and the restriction of g to each of these sets is an affine function.

We call a function $u \in W^{1,\infty}$ *admissible* for the problem (1.1) if it is piece-wise affine, $u = g$ on $\partial\Omega$, and $Du \in (U \cup K)$ a.e. in Ω. In case the value of u at the boundary is not specified the function u is called *admissible.*

To state the first result we need

Definition 1.1 *We say that a sequence of piece-wise affine functions* $u_i : \Omega \to \mathbf{R}^m$ is obtained by perturbation *if for each element u_i of the sequence there exists an at most countable family of disjoint open subsets Ω^i_j of Ω, $j \in \mathbf{N}$, such that* $\operatorname{meas}(\Omega \setminus \cup_j\Omega^i_j) = 0$ *and for each $j \in \mathbf{N}$ we have:* $\operatorname{meas}(\partial\Omega^i_j) = 0$*, u_i is affine in Ω^i_j, $u_i = u_{i+k}$ on $\partial\Omega^i_j$ for all $k \in \mathbf{N}$.*

Theorem 1.2 *We assume that the sequence of piece-wise affine functions $u_i : \Omega \to \mathbf{R}^m$ is obtained by perturbation. We assume also that $\{u_i, i \in \mathbf{N}\}$ is a weakly compact set in $W^{1,1}(\Omega;\mathbf{R}^m)$. Then the sequence u_i converges strongly in $W^{1,1}(\Omega;\mathbf{R}^m)$, i.e. $u_i \to u_\infty$ in $W^{1,1}(\Omega;\mathbf{R}^m)$ as $i \to \infty$.*

Corollary 1.3 *Assume that a piece-wise affine function $u_1 : \Omega \to \mathbf{R}^m$ is admissible for the problem (1.1). Then each sequence u_i, which is obtained by perturbation and which has the property*

$$\operatorname{dist}(Du_i, K) \to 0 \text{ in } L^1(\Omega),\ i \to \infty, \tag{1.2}$$

converges to a solution u_∞ of the problem (1.1).

In particular, if for each $A \in U$ there exists a sequence of admissible functions u_i with $u_i = l_A$ on $\partial\Omega$, $i \in \mathbf{N}$, and such that (1.2) holds, then each problem (1.1) with admissible boundary data g has a solution.

The second part of the assertion is an existence result first established in [**S1**]. Another form of the same result is due to B.Kirchheim [**Ki**].

Corollary 1.4 *Assume that for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $A \in (U \setminus \cup_{x \in K} B(x, \epsilon))$ *then we can find a piece-wise affine function* $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *with the properties*

$$D(\phi + l_A) \in (U \cup K), \ ||D\phi||_{L^1(\Omega; \mathbf{R}^{m \times n})} \geq \delta \,\mathrm{meas}\, \Omega.$$

Then the problem (1.1) possesses a solution for each admissible function g.

Proof of Corollary 1.3 is straightforward. In the case of Corollary 1.4 one has to select a sequence u_i by perturbation in such a way that each perturbation is "almost maximal". Then the sequence u_i converge strongly and $\mathrm{dist}(Du_i, K) \to 0$ in L^1, otherwise the selection would not been "almost maximal". The detailed explanations can be found in §3, where we prove the result. This argument presents an implicit method to construct a sequence of admissible functions u_i with $\mathrm{dist}(Du_i, K) \to 0$.

Recently an interesting issue of stability was raised by B.Kirchheim, who proved that the set of stable solutions is dense in the closure of admissible functions [**Ki**]. Here we call $u \in W^{1,\infty}$ a *stable solution* of (1.1) if each sequence of functions admissible for the problem (1.1) converges to u in $W^{1,1}$ provided it converges to u weakly in $W^{1,1}$.

Note that one can use the above algorithm of almost maximal oscillations to construct stable solutions provided the oscillations produced by admissible perturbations of linear functions l_A, with $A \in (U \cup K)$, coincide with those produced by the gradients of admissible functions converging weakly to linear ones. The later requirement is difficult to verify in such a general situation, which we consider in this paper, but one can do this in particular cases, e.g. in the case of convex sets U. However one can easily address the issues of existence and stability in the abstract setting, like in [**Ki**], measuring maximal oscillations produced by the gradients of admissible functions on given ones.

To state the results we will deal with the gradients of admissible functions. We assume that S is a weakly compact subset of $L^1(\Omega; \mathbf{R}^l)$. Then elements of S form an equiintegrable set, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that $||\xi||_{L^1(\tilde{\Omega}; \mathbf{R}^m)} \leq \epsilon$ provided $\tilde{\Omega} \subset \Omega$ and meas $\tilde{\Omega} \leq \delta$. Moreover there exists a strictly convex function $\theta : \mathbf{R}^l \to \mathbf{R}$ with the superlinear growth at infinity and such that the family $\theta(\xi)$, $\xi \in S$, is equi-integrable. In fact any of these two properties is equivalent to the relative weak compactness of the set S, cf. e.g. [**McS**].

Since weakly compact subsets are metrizable there is a metric ρ, which generates the weak topology in S. We will reserve notations $\rightharpoonup$ and $\to$ for the weak and strong convergences, respectively.

Definition 1.5 *Let S be a weakly compact subset of $L^1(\Omega;\mathbf{R}^l)$ and let θ be a strictly convex nonnegative C^1-regular function such that the family $\{\theta(\xi) : \xi \in S\}$ is equiintegrable. Given $\xi \in S$ we define*

$$\mathrm{ind}(\xi) = \sup_{\xi_i \rightharpoonup \xi, \xi_i \in S} \limsup_{i\to\infty} \int_\Omega \{\theta(\xi_i(x)) - \theta(\xi(x))\} dx.$$

Any sequence $\xi_i \in S$, $i \in \mathbf{N}$, such that $\xi_i \rightharpoonup \xi$ in L^1 and

$$\int_\Omega \theta(\xi_i(x))dx \to \int_\Omega \theta(\xi(x))dx + \mathrm{ind}(\xi), \ i \to \infty,$$

will be called a sequence generating maximal oscillations *(associated with ξ).*

We use this terminology since the sequence ξ_i, $i \in \mathbf{N}$, generates maximal oscillations in the sense that the action of the corresponding Young measure (on θ) is maximal in the class of all Young measures produced by those sequences of elements in S, which converge weakly to ξ. In fact it would be appropriate to involve Young measures in analysis. However we avoid doing that to make this paper accessible for those who are unfamiliar with Young measure theory.

We will see in §2 that $\xi_i \to \xi$ in L^1 for all sequences $\xi_i \in S$, $i \in \mathbf{N}$, with $\xi_i \rightharpoonup \xi$ in L^1 if and only if $\mathrm{ind}(\xi) = 0$. The following results explain how to make use of the functional $\xi \to \mathrm{ind}(\xi)$.

Theorem 1.6 *Let S be a weakly compact subset of L^1. The function $\xi \in (S,\rho) \to \mathrm{ind}(\xi)$ is upper semicontinuous. If ξ_i, $i \in \mathbf{N}$, is a sequence, which generates maximal oscillations associated with ξ, then $\mathrm{ind}(\xi_i) \to 0$, $i \to \infty$.*

Given $\epsilon > 0$ the set S_ϵ of the elements ξ of S with $\mathrm{ind}(\xi) < \epsilon$ is open and dense in (S,ρ). In particular the set

$$S_0 := \{\xi \in S : \mathrm{ind}(\xi) = 0\}$$

is dense (and residual) in the space (S,ρ) and for each $\xi_0 \in S_0$ we have $\xi_i \to \xi_0$ in L^1 provided $\xi_i \in S$, $i \in \mathbf{N}$, and $\xi_i \rightharpoonup \xi_0$ in L^1.

Theorem 1.7 *Let S be a weakly compact subset of L^1. Given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\limsup_{i\to\infty} ||\xi_i - \xi||_{L^1} < \epsilon$$

provided $\xi, \xi_i \in S$, $\mathrm{ind}(\xi) \leq \delta$ and $\xi_i \rightharpoonup \xi$ in L^1.

Given $\xi_0 \in S$ and $\epsilon_0 > 0$ we can find a sequence $\xi_i \in S$ such that $\rho(\xi_0, \xi_i) \leq \epsilon_0$, $\forall i \in \mathbf{N}$, $\xi_i \to \xi_\infty$ in L^1 and $\mathrm{ind}(\xi_i) \to \mathrm{ind}(\xi_\infty) = 0$ as $i \to \infty$.

One can apply any of the theorems 1.6 or 1.7 to obtain

Corollary 1.8 *Assume that U and K satisfy the assumptions of the second part of Corollary 1.3, i.e. for each $A \in U$ there exists a sequence u_i of admissible functions with $u_i = l_A$ on $\partial\Omega$ and with $\mathrm{dist}(Du_i, K) \to 0$ in L^1 as $i \to \infty$.*

Let also g be a piece-wise affine function with $Dg \in (U \cup K)$ a.e. and let D be the closure of the set of admissible (for the problem) functions in L^∞ topology. Then the set of stable solutions of the problem (1.1) is dense (residual) in the space (D, L^∞).

In the last ten years interest to the problems (1.1) was partially motivated by successful models proposed for studying behavior of crystal lattices undergoing solid-solid phase transformations, see [**BJ1**], [**BJ2**]. The idea was to study the behavior of crystal lattices in general assuming that their elements have certain preferable affine deformations - so-called unstressed microlocal deformations; K is the set of matrices corresponding to the gradients of these deformations. The paper [**BJFK**] is a survey of results in this direction, for more recent progress see also [**Ki**], [**CKi**], [**KiP**]. For other applications see e.g. books [**DM7**], [**DM8**].

There have been developed two competitive methods to construct solutions of the problems (1.1). One goes back to ideas of Gromov, Nash and Kuiper, see [**N**], [**G**] and [**K**], to construct sequences of approximate solutions selecting successive elements as sufficiently small perturbations of preceding ones that allows to control convergence of the gradients in L^1-norm. This approach was pushed in the papers by S.Müller & V.Šverák [**MuSv1-4**] who suggested the first explicit proof of the result in [**G**, p.218] and generalized it. The ultimate existence result in this direction is the second part of Corollary 1.3, which has been first proved in [**S1**]; for the nonhomogeneous version see [**MuS**]. However the authors were interested in applications and the methodological analysis remained incomplete. Theorem 1.2 and its corollaries say that a way to choose approximate solutions converging to an exact one is, in fact, rather simple, one has to take any sequence u_i obtained by perturbation with $\mathrm{dist}(Du_i, K) \to 0$ in L^1, $i \to \infty$.

Another approach was developed by Italian school. It is based on construction of integral functionals, which are upper semicontinuous with respect to the weak convergence and which have nonnegative integrands vanishing in the set K. The Baire category lemma allows to prove that the set of solutions of the problem (1.1) is dense in the closure of the set of admissible functions in the weak topology, provided the subsets of these functions with values of the functional minorizing given sufficiently small $\epsilon > 0$ are open and dense in the space.

In the scalar case this approach was developed in the papers [**C**], [**Br**], [**BrFl**], [**DeBP**] with the optimal construction and results obtained in the paper [**BrFl**], with some extraremarks related to regularity of g added in [**DeBP**]. The vector-valued case was treated by Dacorogna & Marcellini in the papers and books [**DM1-9**], [**DMT**], [**DT**]. In the first works [**DM1-6**] the authors pushed the case when U is convex and K is the null set of a nonnegative quasiconcave integrand defined in U (recall that quasiconcavity of integrands characterizes upper semicontinuity of associated integral functionals with respect to the weak convergence in the Sobolev space). Later, see [**DM7-9**], [**DMT**], [**DT**], they suggested an abstract functional, which allowed them to prove the existence result of the paper [**S1**], i.e. the result of the second part of Corollary 1.3, under special extra assumptions on U and K, see e.g. [**S1**, §3] or [**MuS**, §5] for details. The result was sufficient to consider some applied problems of interest. However it was not clear whether one can recover the existence result in full generality via the Baire category lemma since it was difficult to guess which functional can do the job. Theorem 1.6 says that a right choice is $u \to \mathrm{ind}(Du)$.

In fact this functional gives a match between the convex integration approach to select a strongly convergent sequence of approximate solutions via control of L^∞-norm of the perturbations and the attempts to use various versons of Baire lemma to select the set of solutions as the intersection of the sets of approximate ones, i.e. those where the value of the functional does not exceed a small given $\epsilon > 0$. Arguments of the proof of Theorem 1.6 show how to use the functional $u \to \mathrm{ind}(Du)$ to prove the existence and stability results via Baire lemma. Another way to use this functional is to recover the results constructing strongly converging sequences g_i, $i \in \mathbf{N}$, such that the difference $||Dg_{i+1} - Dg_i||_{L^1}$ is controlled by maximal oscillations associated with Dg_i, cf. Theorem 1.7 and its proof, that is reminiscent of the idea of the convex integration approach.

Our methodological analysis was pushed by the work of B.Kirchheim [**Ki**] who considered the mapping $u \to Du$ ($L^\infty \to L^1$), which is defined in the set of admissible functions, and used the fact that such mappings are continuous in a dense set since they are pointwise limits of continuous ones, i.e. they are Baire-1 functions. To see this one can take the mappings $u \to Du_\epsilon$, where u_ϵ are ϵ-mollifications, as approximating mappings.

As we see now the facts behind the results are that the admissible functions with the gradients producing maximal oscillations are almost stable in the sense of the functional $u \to \text{ind}Du$ and then one can construct the functions with zero oscillations applying any of the theorems 1.6 or 1.7. Of course theorems 1.6 and 1.7 give only an abstract way to find solutions. Theorem 1.2, which is based on a new fact, presents a very simple algorithm to construct sequences of approximate solutions with the gradients converging strongly. The limit functions are always solutions of the problem (1.1).

We place some auxiliary propositions in §2. In §3 we prove propositions 1.2-1.4. Propositions 1.6-1.8 will be proved in §4.

2 Some auxiliary results

In the proofs of the results we will be using some standard facts on integral functionals with strictly convex integrands. In this section we can assume that Ω is a bounded measurable subset of $\mathbf{R}^n$.

Given *a nonnegative strictly convex* C^1-regular function $\theta : \mathbf{R}^l \to \mathbf{R}$ we define the integral functional $\xi \in L^1(\Omega; \mathbf{R}^l) \to J(\xi)$ as follows

$$J(\xi) = \int_\Omega \theta(\xi(x))dx, \text{ if } \theta(\xi) \in L^1(\Omega),$$

$$J(\xi) = \infty, \text{ otherwise.}$$

Theorem 2.1 *For any $\xi \in L^1(\Omega; \mathbf{R}^l)$ we have*
1) $\liminf_{i\to\infty} J(\xi_i) \geq J(\xi)$ if $\xi_i \rightharpoonup \xi$ in L^1,
2) in case $J(\xi) < \infty$ the convergence $J(\xi_i) \to J(\xi)$ implies the convergence $||\xi_i - \xi||_{L^1} \to 0$.

We include a proof for convenience of the readers. See also [**S2**] and [**S3**] for more general results in this direction.

Proof

We can find an increasing sequence of compact subsets Ω_k of Ω with meas$(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$ and such that the restrictions of ξ to these sets are continuous. Let $f(\cdot) := D\theta(\xi(\cdot))$. Then $f\Big|_{\Omega_k} : \Omega_k \to \mathbf{R}^l$, $k \in \mathbf{N}$, are continuous functions. The convergence $\xi_i \rightharpoonup \xi$ in $L^1(\Omega; \mathbf{R}^l)$ implies the convergence

$$\int_{\Omega_k} \langle \xi_i(x), f(x) \rangle dx \to \int_{\Omega_k} \langle \xi(x), f(x) \rangle dx, \ \forall k \in \mathbf{N}.$$

Therefore given $k \in \mathbf{N}$ we have

$$\liminf_{i \to \infty} \int_{\Omega_k} \{\theta(\xi_i(x)) - \theta(\xi(x))\} dx =$$

$$\liminf_{i \to \infty} \int_{\Omega_k} \{\theta(\xi_i(x)) - \theta(\xi(x)) - \langle f(x), \xi_i(x) - \xi(x) \rangle\} dx \geq 0 \tag{2.1}$$

since the expression under the integral is nonnegative for a.e. $x \in \Omega$.

This proves the first part of the theorem. To prove the second one notice that if $\xi_i \rightharpoonup \xi$ in L^1 and $||\xi_i - \xi||_{L^1(\Omega;\mathbf{R}^l)} > \epsilon$, $\forall i \in \mathbf{N}$, then for some k we have $||\xi_i - \xi||_{L^1(\Omega_k;\mathbf{R}^l)} > \epsilon/2$, $\forall i \in \mathbf{N}$. Moreover we can find $\delta > 0$ such that for every $i \in \mathbf{N}$ we have

$$\text{meas}\,\{x \in \Omega_k : 1/\delta > |\xi(x)|,\ 1/\delta > |\xi_i(x) - \xi(x)| > \delta\} > \delta.$$

Then the integrals in (2.1) are bounded from below by a positive constant. Therefore

$$\text{liminf}_{i \to \infty}\{J(\xi_i) - J(\xi)\} > 0,$$

which is a contradiction with the assumption $J(\xi_i) \to J(\xi)$. The contradiction shows that $||\xi_i - \xi||_{L^1(\Omega;\mathbf{R}^l)} \to 0$ as $i \to \infty$.

QED

We will be using one more fact.

Lemma 2.2 *Let S be a weakly compact subset of $L^1(\Omega; \mathbf{R}^l)$ and let $\theta : \mathbf{R}^l \to \mathbf{R}$ be a nonnegative strictly convex C^1-regular function. Given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$liminf_{i \to \infty} J(\xi_i) \geq J(\xi) + \delta$$

provided $\xi, \xi_i \in S$, $J(\xi) < \infty$, $\xi_i \rightharpoonup \xi$ in L^1, and $liminf_{i \to \infty}||\xi_i - \xi||_{L^1} \geq \epsilon$.

Proof will use a sharper version of the arguments involved in the proof of Theorem 2.1.

Given $M > 0$ consider the function

$$\lambda_M(\eta) := \inf_{v \in \mathbf{R}^l} \{\theta(v) - \theta(A) - \langle D\theta(A), v - A\rangle : |v - A| \geq \eta, |A| \leq M\}. \quad (2.2)$$

The function λ_M is nondecreasing with $\lambda_M(0) = 0$ and $\lambda_M(\eta) > 0$ for $\eta > 0$.

Since the elements of S are equiintegrable there exists $M > 0$ such that for each $\xi \in S$ there exists a set $\Omega_{M,\xi}$ with $\xi\Big|_{\Omega_{M,\xi}}$ continuous, with $|\xi| \leq M$ in $\Omega_{M,\xi}$ and with

$$\sup_{\chi \in S} ||\chi||_{L^1(\Omega \setminus \Omega_{M,\xi})} \leq \epsilon/4.$$

Therefore if $\xi, \xi_i \in S$, $i \in \mathbf{N}$, and if $\xi_i \rightharpoonup \xi$ in L^1 with

$$\liminf_{i \to \infty} ||\xi_i - \xi||_{L^1(\Omega;\mathbf{R}^l)} \geq \epsilon$$

then

$$\liminf_{i \to \infty} ||\xi_i - \xi||_{L^1(\Omega_{M,\xi};\mathbf{R}^l)} \geq \epsilon/2.$$

The equiintegrability also implies that we have even more: for some $\eta > 0$

$$\text{meas}\,\{x \in \Omega_{M,\xi} : |\xi_i(x) - \xi(x)| > \eta\} > \eta,$$

if $i \in \mathbf{N}$ is sufficiently large. Then (2.1), (2.2) imply that

$$\liminf_{i \to \infty} \{J(\xi_i) - J(\xi)\} \geq \lambda(\eta)\eta$$

and the result of Lemma 2.2 follows with $\delta = \lambda(\eta)\eta$.

QED

Recall also a standard version of the Vitaly covering theorem, cf. e.g. [**Sa**, p.109].

Let Ω be an open bounded set with meas $(\partial\Omega) = 0$. Given an open set $\tilde{\Omega}$ and $\epsilon > 0$ there exists a decomposition of $\tilde{\Omega}$ into disjoint sets $x_i + \epsilon_i\bar{\Omega}$ with $\epsilon_i < \epsilon$, $i \in \mathbf{N}$, and a set of zero measure.

Let $u \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$. We define the function $\tilde{u}$ as follows: $\tilde{u}(x) = \epsilon_i u((x - x_i)/\epsilon_i)$ for $x \in x_i + \epsilon_i\bar{\Omega}$, $i \in \mathbf{N}$, $\tilde{u}(x) = 0$ otherwise. Then $\tilde{u} \in W_0^{1,\infty}(\tilde{\Omega}; \mathbf{R}^m)$.

It is easy to see that

Proposition 2.3 *For any continuous function* $\theta : \mathbf{R}^{m\times n} \to \mathbf{R}$ *we have*

$$\frac{1}{\text{meas }\Omega}\int_\Omega \theta(Du(x))dx = \frac{1}{\text{meas }\tilde{\Omega}}\int_{\tilde{\Omega}} \theta(D\tilde{u}(x))dx.$$

In particular for each subset K *of* $\mathbf{R}^{m\times n}$ *we have*

$$\frac{1}{\text{meas }\tilde{\Omega}}\int_{\tilde{\Omega}} \text{dist}(D\tilde{u}(x),K)dx = \frac{1}{\text{meas }\Omega}\int_\Omega \text{dist}(Du(x),K)dx.$$

Note also that we can make L^∞*-norm of* $\tilde{u}$ *arbitrary small by taking* $\epsilon := \sup_{i\in\mathbf{N}}\epsilon_i$ *sufficiently small.*

3 Proof of propositions 1.2-1.4

We start with the proof of Corollary 1.4. Theorem 1.2 will be proved later. Corollary 1.3 is a straightforward consequence of Theorem 1.2.

Proof of Corollary 1.4

Without loss of generality we can assume $g = l_A$. We construct a sequence of admissible functions g_i by induction. Let $g_1 = l_A$. Given $i \in \mathbf{N}$ we define g_{i+1} as an admissible perturbation of g_i in each of the open disjoint sets Ω_j^i, where g_i is affine; here $\text{meas}\,(\partial\Omega_j^i) = 0$, $\forall j$, $\text{meas}\,(\Omega \setminus \cup_j \Omega_j^i) = 0$. It means that $g_{i+1} = g_i + \phi_i$ is an admissible function and $\phi_i\Big|_{\partial\Omega_j^i} = 0$, $\forall j$. Moreover we assume that

$$||D\phi_i||_{L^1(\Omega;\mathbf{R}^{m\times n})} \geq \sup_\phi ||D\phi||_{L^1(\Omega;\mathbf{R}^{m\times n})} - 1/2^i, \tag{3.1}$$

where ϕ stay in the set of all admissible perturbations.

Theorem 1.2 implies $g_i \to g_\infty$ in $W^{1,1}(\Omega;\mathbf{R}^m)$ as $i \to \infty$. The function g_∞ is a solution of (1.1) provided

$$\text{dist}(Dg_i, K) \to 0 \text{ in } L^1(\Omega),\ i \to \infty. \tag{3.2}$$

The latter will be proved by contradiction.

Assume that (3.2) fails, i.e. there exists a subsequence g_k, $k \in \mathbf{N}$, (not relabeled) and $\epsilon > 0$ such that

$$\text{meas}\,\{x \in \Omega : \text{dist}(Dg_k(x), K) > \epsilon\} > \epsilon,\ \forall k.$$

The set

$$\Omega_k := \{x \in \Omega : \text{dist}(Dg_k(x), K) > \epsilon\}$$

can be represented as follows

$$\Omega_k = \cup_{j\in I_k}\Omega_j^k$$

with $I_k \subset \mathbf{N}$. Note that g_k is affine in each set Ω_j^k, $j \in I_k$, with the gradient lying in $U \setminus \cup_{x\in K} B(x,\epsilon)$.

By Proposition 2.3 and by the assumptions of Corollary 1.4 given $A \in (U\setminus\cup_{x\in K}B(x,\epsilon))$ and given an open bounded set $\tilde{\Omega}$ there exists a piece-wise affine function $\phi_A \in W_0^{1,\infty}(\tilde{\Omega};\mathbf{R}^m)$ such that

$$A + D\phi_A \in (U\cup K) \text{ a.e. in } \tilde{\Omega},\ ||D\phi_A||_{L^1(\tilde{\Omega};\mathbf{R}^{m\times n})} \geq \delta \operatorname{meas} \tilde{\Omega}.$$

The construction of g_i, see (3.1), then implies

$$||Dg_{k+1} - Dg_k||_{L^1} \geq \delta \operatorname{meas}\, \Omega_k - \frac{1}{2^k} \geq \delta\epsilon - \frac{1}{2^k},\ \forall k \in \mathbf{N},$$

that gives a contradiction with the convergence $||g_k - g_\infty||_{W^{1,1}} \to 0$.

QED

Proof of Theorem 1.2

Given $i \in \mathbf{N}$ we assume that Ω_j^i, $j \in \mathbf{N}$, are disjoint and open subsets of Ω with $\operatorname{meas}(\Omega\setminus\cup_j\Omega_j^i) = 0$; we also assume that for each $j \in \mathbf{N}$ the function u_i is affine in Ω_j^i, $u_i = u_{i+k}$ on $\partial\Omega_j^i$, $\forall k \in \mathbf{N}$, and $\operatorname{meas}(\partial\Omega_j^i) = 0$. Since $\{Du_i, i \in \mathbf{N}\}$ is a weakly compact subset in L^1 we can find a nonnegative strictly convex C^1-regular integrand $\theta : \mathbf{R}^{m\times n} \to \mathbf{R}$ such that the sequence $\theta(Du_i)$ is equiintegrable. Note that

$$i \to \int_\Omega \theta(Du_i(x))dx$$

is an increasing sequence. In fact, for each set Ω_j^i we have $Du_i = A_j^i$ in Ω_j^i and

$$\int_{\Omega_j^i} Du_{i+k}(x)dx = A_j^i \operatorname{meas}\, \Omega_j^i,\ k \geq 1;$$

therefore

$$\int_{\Omega_j^i} \theta(Du_{i+k}(x))dx \geq \theta(A_j^i) \operatorname{meas}\, \Omega_j^i = \int_{\Omega_j^i} \theta(Du_i(x))dx,\ k \geq 1. \tag{3.3}$$

Assume that the sequence u_i does not converge in $W^{1,1}(\Omega;\mathbf{R}^m)$. Then we can find $\epsilon > 0$ and a subsequence u_i (not relabeled) such that

$$||Du_i - Du_{i+1}||_{L^1(\Omega;\mathbf{R}^{m\times n})} \geq \epsilon,\ \forall i \in \mathbf{N}.$$

Note that each subsequence of a sequence obtained by perturbation (see Definition 1.1) is also a sequence obtained by perturbation.

For each $M > 0$ we define

$$\Omega_{M,i} := \cup_{\{j:|A_j^i|\leq M\}}\Omega_j^i.$$

Equiintegrability of Du_i implies that for sufficiently large M we have

$$||Du_i - Du_{i+1}||_{L^1(\Omega_{M,i};\mathbf{R}^{m\times n})} \geq \epsilon/2.$$

Moreover there exists $\delta > 0$ such that

$$\text{meas}\,\{x \in \Omega_{M,i} : |Du_i(x) - Du_{i+1}(x)| > \delta\} > \delta.$$

Let

$$\lambda_M(\eta) := \inf\{\theta(v) - \theta(A) - \langle D\theta(A), v - A\rangle : |A| \leq M, |v - A| \geq \eta\}.$$

Then λ_M is a nondecreasing function. Strict convexity of θ implies $\lambda_M(0) = 0$, $\lambda_M(\eta) > 0$ if $\eta > 0$. The inequalities (3.3) imply

$$\int_\Omega \{\theta(Du_{i+1}(x)) - \theta(Du_i(x))\}dx \geq \int_{\Omega_{M,i}} \{\theta(Du_{i+1}(x)) - \theta(Du_i(x))\}dx =$$

$$\sum_{\{j:|A_j^i|\leq M\}} \int_{\Omega_j^i} \{\theta(Du_{i+1}(x)) - \theta(A_j^i) - \langle D\theta(A_j^i), Du_{i+1}(x) - A_j^i\rangle\}dx \geq$$

$$\sum_{\{j:|A_j^i|\leq M\}} \int_{\Omega_j^i} \lambda(|Du_{i+1}(x) - A_j^i|)dx \geq \lambda(\delta)\delta > 0.$$

This gives a contradiction with the fact that the sequence

$$i \to \int_\Omega \theta(Du_i(x))dx$$

is bounded.

QED

4 Proofs of the propositions 1.5-1.8

Proof of Theorem 1.6

First we prove upper semicontinuity of the functional

$$\xi \in (S,\rho) \to \operatorname{ind}(\xi),$$

i.e. given $\xi, \xi_i \in (S,\rho)$, $i \in \mathbf{N}$, with $\rho(\xi_i, \xi) \to 0$ we have to prove

$$\limsup_{i\to\infty} \operatorname{ind}(\xi_i) \le \operatorname{ind}(\xi).$$

By definition

$$\operatorname{ind}(\xi) := \sup_{\xi_k \in S, \rho(\xi_k,\xi)\to 0} \limsup_{k\to\infty} \{J(\xi_k) - J(\xi)\}. \tag{4.1}$$

Recall also that the functional

$$\xi \in (S,\rho) \to J(\xi) := \int_\Omega \theta(\xi(x))dx$$

is lower semicontinuous, see Theorem 2.1.

There exists a sequence $\tilde{\xi}_i \in S$ such that

$$\rho(\xi_i, \tilde{\xi}_i) \le 1/i, \ |(J(\tilde{\xi}_i) - J(\xi_i)) - \operatorname{ind}(\xi_i)| \le 1/i.$$

Since $\rho(\tilde{\xi}_i, \xi) \to 0$ and $\liminf_{i\to\infty}\{J(\xi_i) - J(\xi)\} \ge 0$ we have

$$\operatorname{ind}(\xi) \ge \limsup_{i\to\infty}\{J(\tilde{\xi}_i) - J(\xi)\} \ge$$

$$\limsup_{i\to\infty}\{J(\tilde{\xi}_i) - J(\xi_i)\} + \liminf_{i\to\infty}\{J(\xi_i) - J(\xi)\} \ge \limsup_{i\to\infty} \operatorname{ind}(\xi_i).$$

This proves upper semicontinuity of the functional

$$\xi \in (S,\rho) \to \operatorname{ind}(\xi).$$

Assume now that $\rho(\xi_i, \xi) \to 0$ and ξ_i generates maximal oscillations, i.e. $\lim_{i\to\infty}\{J(\xi_i) - J(\xi)\} = \operatorname{ind}(\xi)$. We want to show that $\operatorname{ind}(\xi_i) \to 0$. Otherwise there exist a subsequence ξ_i (not relabeled) and a sequence $\tilde{\xi}_i \in S$ with $\rho(\tilde{\xi}_i, \xi_i) \to 0$, $i \to \infty$, such that

$$\liminf_{i\to\infty}\{J(\tilde{\xi}_i) - J(\xi_i)\} = \epsilon > 0.$$

Then $\rho(\tilde{\xi_i}, \xi) \to 0$ as $i \to \infty$ and

$$\limsup_{i\to\infty}\{J(\tilde{\xi_i}) - J(\xi)\} =$$

$$\limsup_{i\to\infty}\{J(\tilde{\xi_i}) - J(\xi_i)\} + \lim_{i\to\infty}\{J(\xi_i) - J(\xi)\} \geq \epsilon + \mathrm{ind}(\xi),$$

that is a contradiction.

To prove the second part of the theorem notice that upper semicontinuity of the function $\xi \in S \to \mathrm{ind}(\xi)$ implies openness of the sets

$$S_\epsilon := \{\xi \in S : \mathrm{ind}(\xi) < \epsilon\}.$$

The set S_ϵ is dense in the space (S, ρ) since for each $\xi \in S$ we can find a sequence $\xi_i \in S$ such that $\rho(\xi_i, \xi) \to 0$, $i \to \infty$, and $\mathrm{ind}(\xi_i) \to 0$, $i \to \infty$.

By the Baire category lemma the set

$$S_0 = \cap_{i=1}^{\infty} S_{1/i}$$

is dense (residual) in the space (S, ρ). For each $\xi_0 \in S_0$ we have $\mathrm{ind}(\xi_0) = 0$. Therefore if $\xi_i \in S$, $i \in \mathbf{N}$, and $\xi_i \rightharpoonup \xi_0$ in L^1 then $J(\xi_i) \to J(\xi_0)$ and, by Theorem 2.1, $\xi_i \to \xi_0$ in L^1.

QED

Proof of Theorem 1.7

The first assertion of the theorem is an immediate consequence of Lemma 2.2. To prove the second one we fix $\xi_0 \in S$ and $\epsilon_0 > 0$. We will construct the sequence ξ_i by induction.

The definition of the functional $S : \chi \to \mathrm{ind}(\chi)$ implies $\mathrm{ind}(\chi_i) \to 0$ for each sequence $\chi_i \in S$ generating maximal oscillations associated with $\chi \in S$, see the proof of Theorem 1.6 for details.

By the first part of the theorem given $\epsilon_i \in]0, \epsilon_{i-1}/2^{i+1}[$, $i \in \mathbf{N}$, there exists $\delta_i \in]0, \epsilon_i[$ such that

$$\limsup_{k\to\infty} ||\chi_k - \chi||_{L^1} < \epsilon_i/2$$

provided $\mathrm{ind}(\chi) < \delta_i$, $\chi, \chi_k \in S$ and $\rho(\chi_k, \chi) \to 0$ as $k \to \infty$.

We will also use some additional assumptions on ϵ_i to construct a sequence ξ_i with the desired properties.

We take $\xi_1 \in S$ such that $\rho(\xi_0, \xi_1) < \epsilon_1/2$ ($\epsilon_1 < \epsilon_0/2$) and $\text{ind}(\xi_1) < \delta_1$. Then we can find ϵ_2 such that

$$||\xi - \xi_1||_{L^1} < \epsilon_1/2 \text{ if } \rho(\xi, \xi_1) < \epsilon_2, \ \xi \in S.$$

We can choose then $\xi_2 \in S$ such that $\text{ind}(\xi_2) < \delta_2$ and $\rho(\xi_1, \xi_2) < \epsilon_2/2$. We can also find ϵ_3 such that

$$||\xi - \xi_2||_{L^1} < \epsilon_2/2 \text{ if } \rho(\xi, \xi_2) < \epsilon_3, \ \xi \in S.$$

We can continue this process by induction defining ξ_{i+1} and ϵ_{i+2} in such a way that $\rho(\xi_i, \xi_{i+1}) < \epsilon_{i+1}/2$, $\text{ind}(\xi_{i+1}) < \delta_{i+1}$ and

$$||\xi - \xi_{i+1}||_{L^1} < \epsilon_{i+1}/2 \text{ if } \rho(\xi, \xi_{i+1}) < \epsilon_{i+2}, \ \xi \in S.$$

Then $\rho(\xi_i, \xi_{i+1}) < \epsilon_{i+1}/2 < \epsilon_0/2^{i+1}$ and, consequently, there exists ξ_∞ such that $\rho(\xi_i, \xi_\infty) \to 0$, $i \to \infty$, and $\rho(\xi_0, \xi_\infty) < \epsilon_0$. The choice of ϵ_i implies that $\rho(\xi_i, \xi_\infty) < \epsilon_{i+1}$ and then $||\xi_i - \xi_\infty||_{L^1} < \epsilon_i/2$, i.e. $\xi_i \to \xi_\infty$ in L^1. Moreover the inequality $||\xi - \xi_\infty||_{L^1} < \epsilon_i$ holds for all ξ enough close to ξ_∞ in ρ-metric provided $\rho(\xi, \xi_i) < \epsilon_{i+1}$. This way we prove that for any sequence $\xi_k \in S$, $k \in \mathbf{N}$, the convergence $\xi_k \to \xi_\infty$ in L^1 holds provided $\rho(\xi_k, \xi_\infty) \to 0$ as $k \to \infty$. In particular we established that $\text{ind}(\xi_\infty) = 0$.

QED

Proof of Corollary 1.8

We will show how to prove the assertion via Theorem 1.6. One can also use Theorem 1.7 instead.

Let $\tilde{S}$ be the set of the gradients of admissible (for the problem) functions and let S be its closure in the weak topology of L^1. Then $S = \{Du : u \in D\}$. There exists a dense (residual) subset $S_0 \subset S$ such that for each $Du \in S_0$ we have $\text{ind}(Du) = 0$. The corresponding set $D_0 = \{u \in D : Du \in S_0\}$ is dense in the metric space (D, L^∞).

The result will be established if we show that each function $u \in D_0$ is a solution of the problem (1.1). Let u_i be a sequence of admissible (for the problem) functions such that $u_i \rightharpoonup^* u$ in $W^{1,\infty}(\Omega; \mathbf{R}^m)$. In case $\text{dist}(Du_i, K) \not\to 0$ in L^1 we can find a subsequence u_i (not relabeled) such that

$$\text{meas}\,\{x \in \Omega : \text{dist}(Du_i(x), K) > \delta\} > \delta.$$

Given $A \in (U \setminus \cup_{x \in K} B(x, \delta))$ we can find a piecewise affine function $u_A \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that $Du_A \in (U \cup K)$ a.e. in Ω and $||Du_A - A||_{L^1} \geq$

δ meas Ω. Applying Proposition 2.3 in the sets of affinity of u_i with $\epsilon = 1/j$, $j \in \mathbf{N}$, we can construct a sequence of admissible (for the problem) functions $u_j^i \in u + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that $u_j^i \rightharpoonup^* u_i$ in $W^{1,\infty}$ as $j \to \infty$ and

$$\liminf_{j\to\infty} ||Du_j^i - Du_i||_{L^1(\Omega;\mathbf{R}^m)} \geq \delta^2.$$

Lemma 2.2 then implies that $\mathrm{ind}(Du_i) \geq \delta^2 > 0$, $i \in \mathbf{N}$. This contradicts upper semicontinuity of the functional $\xi \to \mathrm{ind}(\xi)$ since $\mathrm{ind}(Du) = 0$ and $Du_i \rightharpoonup Du$ in L^1 (i.e. $\rho(Du_i, Du) \to 0$ as $i \to \infty$). Then we have both $\mathrm{dist}(Du_i, K) \to 0$ in L^1 and $\mathrm{ind}(Du_i) \to 0$. Therefore $||Du_i - Du||_{L^1} \to 0$ and, consequently, u solves the problem (1.1).

QED

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