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QUASICONVEX HULLS IN SYMMETRIC MATRICES

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ABSTRACT. We analyze the semiconvex hulls of the subset K in symmetric matrices given by $K = \{F \in \mathbf{M}^{2\times 2} : F^T = F, |F_{11}| = a, |F_{12}| = b, |F_{22}| = c\}$ that was first considered by Dacorogna&Tanteri [Commun. in PDEs 2001]. We obtain explicit formulae for the polyconvex, the quasiconvex, and the rank-one convex hull for $ac - b^2 \geq 0$ and show in particular that the quasiconvex and the polyconvex hull are different if strict inequality holds. For $ac - b^2 < 0$ we obtain a closed form for the polyconvex and the rank-one convex hull.

1. INTRODUCTION

The central notion of convexity in the vector valued calculus of variations is quasiconvexity (in the sense of Morrey [14]). Recall that a real valued function f defined on the space $\mathbf{M}^{m\times n}$ of all real $m \times n$ matrices is quasiconvex if there exists an open domain Ω in $\mathbf{R}^n$ such that

$$\frac{1}{|\Omega|}\int_\Omega W(F)\mathrm{d}x \leq \frac{1}{|\Omega|}\int_\Omega W(F + D\phi)\mathrm{d}x$$

for all $F \in \mathbf{M}^{m\times n}$ and $\phi \in C_0^\infty(\Omega; \mathbf{R}^m)$.

In particular motivated by applications to problems in materials science (see, e.g, [1, 5, 9, 16]), there has been an increasing interest in the mathematical analysis of variational integrals for which the energy density W is not quasiconvex. If we assume that $W \geq 0$ with $K = \{X : W(X) = 0\} \neq \emptyset$, then a typical question is to characterize the set of all matrices F such that

$$\inf_{\substack{u \in W^{1,\infty}(\Omega;\mathbf{R}^m) \\ u(x)=Fx \text{ on } \partial\Omega}} \frac{1}{|\Omega|}\int_\Omega W(Du)\mathrm{d}x = 0.$$

This set is called the quasiconvex hull of K and it describes in the context of nonlinear elasticity theory the set of all affine deformations of $\partial\Omega$ with arbitrarily small stored energy. In nice analogy to the definition of the convex hull K^{c} of a set, an equivalent characterization of K^{qc} is given by [20]

$$K^{\mathrm{qc}} = \{F \in \mathbf{M}^{m\times n} : f(F) \leq \sup_{X\in K} f(X) \ \forall f : \mathbf{M}^{m\times n} \to \mathbf{R} \text{ quasiconvex}\}.$$

Despite the fundamental importance of quasiconvex hulls, only very few explicit examples are available in the literature (see, e.g., [2, 3, 4, 19]). In most of these examples, the quasiconvex hull coincides with two closely related hulls, the rank-one

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convex hull K^{rc} and the polyconvex hull K^{pc} of K. The definition of these hulls is analogous to the definition of the quasiconvex hull where one replaces quasiconvexity by rank-one convexity and polyconvexity, respectively. Here we say that a function $f : \mathbf{M}^{m\times n} \to \mathbf{R}$ is rank-one convex if it is convex on all rank-one lines, that is, the functions $\phi(t) = f(F + tR)$ are convex in t for all $F \in \mathbf{M}^{m\times n}$ and for all R with $\mathrm{rank}(R) = 1$. It is polyconvex if there exists a convex function g of the vector $M(F)$ of all minors of F with $f(F) = g(M(F))$. For $m = n = 2$, the case of interest in this note, g is a convex function from $\mathbf{R}^5$ into $\mathbf{R}$ with $f(F) = g(F, \det F)$. Since rank-one convexity is a necessary condition for quasiconvexity and polyconvexity a sufficient one, it follows that

$$K^{\mathrm{rc}} \subseteq K^{\mathrm{qc}} \subseteq K^{\mathrm{pc}}.$$

As a consequence, one obtains a characterization of K^{qc} for all sets K for which the rank-one convex and the polyconvex hull coincide. While this identity has been established in certain cases with high symmetry, it does not hold in general. Indeed, a nice example of a set in 3×2 matrices for which the rank-one convex hull is different from the quasiconvex hull can be found in [13]. It is an open question whether $K^{\mathrm{rc}} = K^{\mathrm{qc}}$ for 2×2 matrices. A positive answer was recently given in [15] for the case that K is a subset of the diagonal 2×2 matrices. In the proof one crucially uses the fact that the intersection of the rank-one cone with the diagonal matrices consists of two lines. The case of symmetric 2×2 matrices is already much more challenging. In this case, the rank-one cone still has a very simple geometric structure. If one uses the coordinates

$$(\xi, \eta, \zeta) = \begin{pmatrix} \zeta + \xi & \eta \\ \eta & \zeta - \xi \end{pmatrix},$$

then it is given by the standard cone $\zeta^2 = \xi^2 + \eta^2$. The methods in [15], however, do not apply since the set of rank-one directions is not linearly independent. The geometric insight into the structure of the rank-one cone in symmetric matrices is also at the heart of the surprising example of a set of five points without rank-one connections which is the range of the gradient of a Lipschitz function that is not affine [10].

In this paper, we show how the geometry of the rank-one matrices in the space of all symmetric matrices can be used to characterize the semiconvex hulls in an interesting test case. Following Dacorogna&Tanteri [7], we define the set K for constants a, b, $c > 0$ by

$$K = \{F \in \mathbf{M}^{2\times 2} : F^T = F,\ |F_{11}| = a, |F_{12}| = b, |F_{22}| = c\}.$$

Before we state our main result, we define the lamination convex hull K^{lc} of a set K which is well-adopted to constructions and of importance in the proof of Theorem 1.1 below. Motivated by the observation that F_1, $F_2 \in K$ with $\mathrm{rank}(F_1 - F_2) = 1$ implies that the line segment $\lambda F_1 + (1-\lambda)F_2$, $\lambda \in [0, 1]$, belongs to K^{rc}, we set

$$K^{\mathrm{lc}} = \bigcup_{i=0}^{\infty} K^{(\mathrm{i})},$$

where $K^{(0)} = K$ and

$$K^{(\mathrm{i}+1)} = K^{(\mathrm{i})} \cup \big\{ F = \lambda F_1 + (1-\lambda) F_2 : F_1,\, F_2 \in K^{(\mathrm{i})}, \\ \mathrm{rank}(F_1 - F_2) = 1,\, \lambda \in (0,1) \big\}.$$

By definition, $K^{\mathrm{lc}} \subseteq K^{\mathrm{rc}}$. We are now in a position to state the main result of this paper.

Theorem 1.1. *Let*

$$K = \Big\{ F = \begin{pmatrix} x & y \\ y & z \end{pmatrix} : |x| = a,\, |y| = b,\, |z| = c \Big\}$$

with constants $a,\, b,\, c > 0$. Then

$$K^{\mathrm{pc}} = \Big\{ F \in K^{\mathrm{c}} : (x-a)(z+c) \le y^2 - b^2,\, (x+a)(z-c) \le y^2 - b^2 \Big\}.$$

Moreover, the following assertions hold:
i) If $ac - b^2 < 0$ then

$$K^{(2)} = K^{\mathrm{lc}} = K^{\mathrm{rc}} = \big\{ F \in K^{\mathrm{c}} : |y| = b \big\}.$$

ii) If $ac - b^2 \ge 0$ then $K^{(4)} = K^{\mathrm{lc}} = K^{\mathrm{rc}} = K^{\mathrm{qc}}$ and

$$K^{\mathrm{qc}} = \big\{ F \in K^{\mathrm{pc}} : (x-a)(z-c) \ge (|y| - b)^2, \\ (x+a)(z+c) \ge (|y| - b)^2 \big\}.$$

Remark 1.2. *It is an open problem to find a formula for the quasiconvex hull of K in the case $ac - b^2 < 0$.*

Remark 1.3. *A short calculation shows that the additional inequalities in the definition of K^{lc} are true for all $F \in K^{\mathrm{pc}}$ if $ac - b^2 = 0$ and that consequently $K^{\mathrm{lc}} = K^{\mathrm{pc}}$. This was already shown in Dacorogna&Tanteri* [7]*. The authors also obtained the formula for K^{lc} in the case $ac - b^2 < 0$ and observed that K^{lc} is always contained in the intersection of the convex hull of K with the exterior of the two hyperboloids $(x-a)(z+c) = y^2 - b^2$ and $(x+a)(z-c) = y^2 - b^2$. However, they did not identify the latter set as K^{pc}.*

The rest of the paper is organized as follows: We derive the formula for the polyconvex hull of K in Section 2. The formulae for the lamination convex hulls in statements i) and ii) in the theorem are obtained in Sections 3 and 4, respectively. Section 5 finally contains the proof for the representation of the quasiconvex hull for $ac - b^2 \ge 0$.

2. The polyconvex hull of K.

Among the different notions of convexity, polyconvexity has the most similarities with classical convexity. One instance is the following representation for the polyconvex hull K^{pc} (see [19]),

$$K^{\mathrm{pc}} = \big\{ F \in \mathbf{M}^{2\times 2} : (F, \det F) \in \widetilde{K}^{\mathrm{c}} \big\}, \tag{2.1}$$

where

$$\widetilde{K} = \big\{ (F, \det F) : F \in K \big\} \subset \mathbf{R}^5.$$

By definition, K consists of symmetric matrices, and therefore $\widetilde{K}$ and $\widetilde{K}^{\mathrm{c}}$ are contained in a four-dimensional subspace of $\mathbf{R}^5$. We restrict our calculations to this subspace by the identifications

$$K = \left\{ \begin{pmatrix} a \\ c \\ b \end{pmatrix}, \begin{pmatrix} -a \\ c \\ b \end{pmatrix}, \begin{pmatrix} a \\ -c \\ b \end{pmatrix}, \begin{pmatrix} -a \\ -c \\ b \end{pmatrix}, \begin{pmatrix} a \\ c \\ -b \end{pmatrix}, \begin{pmatrix} -a \\ c \\ -b \end{pmatrix}, \begin{pmatrix} a \\ -c \\ -b \end{pmatrix}, \begin{pmatrix} -a \\ -c \\ -b \end{pmatrix} \right\}$$

and

$$\widetilde{K} = \big\{(x,z,y,xz-y^2) : (x,z,y) \in K\big\}.$$

We denote the eight points in $\widetilde{K}$ by $\tilde{f}_1, \dots, \tilde{f}_8$.

Since K is a finite set, $\widetilde{K}^{\mathrm{c}}$ is a polyhedron in $\mathbf{R}^4$, which is the intersection of a finite number of half spaces. Moreover, on each face of $\widetilde{K}^{\mathrm{c}}$ we must have at least four points in $\widetilde{K}$ that span a three-dimensional hyperplane in $\mathbf{R}^4$. A short calculation shows that the following list of six normals completely describes the convex hull of $\widetilde{K}$:

$$\begin{aligned} n_1 &= (c,a,0,-1), & n_2 &= (-c,a,0,1,), \\ n_3 &= (c,-a,0,1), & n_4 &= (-c,-a,0,-1), \\ n_5 &= (0,0,1,0), & n_6 &= (0,0,-1,0). \end{aligned}$$

It turns out that the hyperplanes defined by $n_1, \dots, n_4$ contain six points in K,

$$\begin{aligned} \langle \tilde{f}_4, n_1\rangle = \langle \tilde{f}_8, n_1\rangle &= -3ac + b^2 < ac + b^2 = \langle \tilde{f}_i, n_1\rangle, \quad i \notin \{4,8\}, \\ \langle \tilde{f}_3, n_2\rangle = \langle \tilde{f}_7, n_2\rangle &= -3ac - b^2 < ac - b^2 = \langle \tilde{f}_i, n_2\rangle, \quad i \notin \{3,7\}, \\ \langle \tilde{f}_2, n_3\rangle = \langle \tilde{f}_6, n_3\rangle &= -3ac - b^2 < ac - b^2 = \langle \tilde{f}_i, n_3\rangle, \quad i \notin \{2,6\}, \\ \langle \tilde{f}_1, n_4\rangle = \langle \tilde{f}_5, n_4\rangle &= -3ac + b^2 < ac + b^2 = \langle \tilde{f}_i, n_4\rangle, \quad i \notin \{1,5\}, \end{aligned}$$

and that the faces of the polyhedron defined by n_5 and n_6 contain four points,

$$\begin{aligned} \langle \tilde{f}_j, n_5\rangle &= -b < b = \langle \tilde{f}_i, n_5\rangle, \qquad i = 1,2,3,4,\ j = 5,6,7,8, \\ \langle \tilde{f}_j, n_6\rangle &= -b < b = \langle \tilde{f}_i, n_6\rangle, \qquad i = 5,6,7,8,\ j = 1,2,3,4. \end{aligned}$$

In view of the representation (2.1) for the polyconvex hull and the formulae for the normals, this implies that all points in K^{pc} must satisfy the convex inequality

$$|y| \le b \tag{2.2}$$

as well as the additional inequalities

$$\begin{aligned} cx + az - (xz - y^2) &\le ac + b^2, & -cx + az + (xz - y^2) &\le ac - b^2, \\ cx - az + (xz - y^2) &\le ac - b^2, & -cx - az - (xz - y^2) &\le ac + b^2, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} -(x-a)(z-c) &\le -y^2 + b^2, & (x+a)(z-c) &\le y^2 - b^2, \\ (x-a)(z+c) &\le y^2 - b^2, & -(x+a)(z+c) &\le -y^2 + b^2. \end{aligned} \tag{2.3}$$

We now assert that this system of inequalities is equivalent to the conditions

$$|x| \le a, \quad |z| \le c, \quad |y| \le b \tag{2.4}$$

describing the convex hull of K and two additional inequalities

$$(x+a)(z-c) \le y^2 - b^2, \quad (x-a)(z+c) \le y^2 - b^2. \tag{2.5}$$

This proves the formula for the polyconvex hull of K. In fact, the sum of the two upper and the two lower inequalities in (2.3) implies

$$az \le ac \quad \text{and} \quad -az \le ac,$$

and the sum of the two left and the two right inequalities, respectively, gives

$$cx \le ac \quad \text{and} \quad -cx \le ac.$$

Therefore $|z| \le c$ and $|x| \le a$ and this proves that (2.2) and (2.3) imply (2.4) and (2.5). Conversely, if the convex inequalities $|x| \le a$, $|z| \le c$, and $|y| \le b$ in (2.4) hold, then $x-a \le 0$, $z-c \le 0$ and $-y^2+b^2 \ge 0$. Consequently $-(x-a)(z-c) \le -y^2+b^2$. Similarly, we have $x+a \ge 0$, $z+c \ge 0$ and thus $-(x+a)(z+c) \le -y^2+b^2$, as asserted. This concludes the proof of the formula for K^{pc} for all parameters a, b, $c > 0$.

3. The lamination convex hull of K for $ac - b^2 < 0$.

We now turn towards proving the formula for K^{lc} and we assume first that $ac - b^2 < 0$. We let

$$\mathcal{A} = \{F \in K^{\mathrm{c}} : |y| = b\}.$$

In this case, none of the matrices in $\mathcal{A}$ with $y = b$ is rank-one connected to any of the matrices in $\mathcal{A}$ with $y = -b$, and the assertion follows essentially from the following locality property of the rank-one convex hull.

Proposition 3.1 ([10, 11, 12, 17])**.** *Assume that K is compact and that K^{rc} consists of two compact components C_1 and C_2 with $C_1 \cap C_2 = \emptyset$. Then*

$$K^{\mathrm{rc}} = (K \cap C_1)^{rc} \cup (K \cap C_2)^{rc}.$$

Clearly, all elements in $\mathcal{A}$ can be constructed using the rank-one connections between the four matrices in K with $y = b$ and $y = -b$, respectively. The observation is now that the polyconvex hull is not connected, since $K^{\mathrm{pc}} \cap \{F : |y| \le \epsilon\} = \emptyset$ for $\epsilon > 0$ so small that $\epsilon^2 < b^2 - ac$. Indeed, summation of the two inequalities in the definition of K^{pc} implies $ac - xz \ge b^2 - y^2$ or, equivalently, $0 > ac - b^2 + y^2 \ge xz$. Thus necessarily either $x > 0$ and $z < 0$ or $x < 0$ and $z > 0$. In the former case the first inequality cannot hold since

$$(z-a)(z+c) \le y^2 - b^2 \quad \Leftrightarrow \quad 0 \le x(z+c) - az \le ac - b^2 + y^2 < 0.$$

In the latter case the second inequality is violated. We may now apply Proposition 3.1 and we conclude that $K^{\mathrm{lc}} = K^{\mathrm{rc}} = \mathcal{A}$.

4. The lamination convex hull of K for $ac - b^2 \ge 0$.

Assume now that $ac - b^2 \ge 0$, and let $\mathcal{A}$ be given by

$$\mathcal{A} = \{F \in K^{\mathrm{pc}} : (x-a)(z-c) \ge (|y|-b)^2,\ (x+a)(z+c) \ge (|y|-b)^2\}.$$

By symmetry, we may suppose in the following arguments that $y \ge 0$. Then this set is described by three types of inequalities, namely the *stripes*

$$|x| \le a, \quad |z| \le c, \quad |y| \le b \tag{4.1}$$

defining the convex hull of K, the *hyperboloids*

$$(x-a)(z+c) \le y^2 - b^2, \quad (x+a)(z-c) \le y^2 - b^2, \tag{4.2}$$

in the definition of K^{pc}, and the *cones*

$$(x-a)(z-c) \geq (y-b)^2, \quad (x+a)(z+c) \geq (y-b)^2. \tag{4.3}$$

To simplify notation, we write

$$X = \begin{pmatrix} \xi & \eta \\ \eta & \zeta \end{pmatrix}$$

Since $\mathcal{A}$ is compact, it suffices to prove that all points $X \in \mathcal{A}$ that satisfy equality in at least one of the inequalities in the definition of $\mathcal{A}$ can be constructed as laminates. To see this, assume that X lies in the interior of $\mathcal{A}$. The idea is to split X along a rank-one line in two rank-one connected matrices $X^{\pm}$ that satisfy equality in at least one of the inequalities in the definition of $\mathcal{A}$. We set

$$t^- = \sup\{t < 0 : X + tw \otimes w \text{ satisfies one equality in } \mathcal{A}\},$$
$$t^+ = \inf\{t > 0 : X + tw \otimes w \text{ satisfies one equality in } \mathcal{A}\}.$$

By assumption, $t^- < 0 < t^+$ and we define $X^{\pm} = X + t^{\pm} w \otimes w$. Then $X = (t^- X^+ - t^- F^+)/(t^+ - t^-)$ and it suffices to show that $X^{\pm}$ are contained in K^{lc}.

Assume thus that $X \in \mathcal{A}$ satisfies equality in at least one inequality in the definition of $\mathcal{A}$. We have to prove that this implies $X \in K^{\mathrm{lc}}$. This is immediate for the convex inequalities $|x| \leq a$, $|y| \leq b$, and $|z| \leq c$. For example, if $\xi = a$, then by (4.2) $|\eta| = b$ and by symmetry we may assume that $\eta = b$. Then (4.1) implies that $\zeta = \lambda c + (1-\lambda)(-c)$ for some $\lambda \in [0,1]$ and thus

$$X = \lambda \begin{pmatrix} a & b \\ b & c \end{pmatrix} + (1-\lambda) \begin{pmatrix} a & b \\ b & -c \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} - \begin{pmatrix} a & b \\ b & -c \end{pmatrix} = 2ce_2 \otimes e_2.$$

The argument is similar for $|\zeta| = c$. Finally, if $|\eta| = b$ and $\eta \geq 0$, then

$$(\xi, \eta) \in \operatorname{conv}\big\{(a,c), (-a,c), (a,-c), (-a,-c)\big\},$$

and therefore $X \in K^{(2)}$.

Assume next that X lies on the surface of one of the cones

$$(x-a)(z-c) \geq (y-b)^2, \quad (x+a)(z+c) \geq (y-b)^2.$$

These cones are the rank-one cones centered at points in K, and we may suppose that X is contained in the rank-one cone C_1 given by

$$C_1 = \Big\{F : \det\Big[F - \begin{pmatrix} a & b \\ b & c \end{pmatrix}\Big] = (x-a)(z-c) - (y-b)^2 = 0\Big\};$$

the argument is similar in the other case. The cone C_1 intersects the part of the boundary of the convex hull of K that is contained in the plane $\{y = -b\}$, which by the foregoing arguments is contained in $K^{(2)}$. We will show that X belongs to a rank-one segment between a point G in this intersection and the point $F_1 \in K$, where F_1 and G are given by

$$F_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \overline{x} & -b \\ -b & \overline{z} \end{pmatrix}, \qquad |\overline{x}| \leq a,\ |\overline{z}| \leq c.$$

This implies $X \in K^{(3)} \subset K^{\mathrm{lc}}$. In order to prove this fact, let

$$R = F_1 - F = \begin{pmatrix} a-\xi & b-\eta \\ b-\eta & c-\zeta \end{pmatrix}.$$

By assumption, $\det R = 0$, and we seek a $t \in \mathbf{R}$ such that

$$F_1 + tR = \begin{pmatrix} a + t(a-\xi) & b + t(b-\eta) \\ b + t(b-\eta) & c + t(c-\xi) \end{pmatrix} = G.$$

This implies

$$t = -\frac{2b}{b-\eta}$$

and thus

$$\overline{x} = a - \frac{2b(a-\xi)}{b-\eta}, \quad \overline{z} = c - \frac{2b(c-\xi)}{b-\eta}.$$

Clearly $\overline{x} \le a$ and we only have to check that $\overline{x} \ge -a$, or equivalently

$$\frac{a}{b} \ge \frac{a-\xi}{b-\eta}.$$

To establish this inequality, we subtract the equality $(x-a)(z-c) = (y-b)^2$ in the definition of C_1 from the inequality $(x+a)(z-c) \le y^2 - b^2$ in the definition of K^{pc}, and we obtain that X satisfies

$$2a(\zeta - c) \le (-2b)(b-\eta).$$

Therefore, again in view of the definition of C_1,

$$\frac{a}{b} \ge \frac{b-\eta}{c-\zeta} = \frac{a-\xi}{b-\eta},$$

and this proves the bounds for $\overline{x}$; the arguments for $\overline{z}$ are similar. Since $G \in K^{(2)}$ we conclude

$$X = \frac{1+t}{t} F_1 - \frac{1}{t} G = \frac{b+\eta}{2b} F_1 + \frac{b-\eta}{2b} G \in K^{(3)}.$$

It remains to consider the case that $X \in \mathcal{A}$ satisfies equality in one of the inequalities defining the one-sheeted hyperboloids. Assume thus that

$$(\xi + a)(\zeta - c) = \eta^2 - b^2.$$

The idea is to use the geometric property of one-sheeted hyperboloids H already observed by Šverák [19], namely that for each point F on H there exist two straight lines intersecting at F that are contained in H, and that correspond to rank-one lines in the space of symmetric matrices. More precisely, we seek solutions $w = (u, v) \in \mathbf{S}^1$ of

$$X + tw \otimes w \in H \quad \text{or} \quad (\xi + tu^2 + a)(\zeta + tv^2 - c) = (\eta + tuv)^2 - b^2.$$

This is equivalent to the quadratic equation

$$u^2(\zeta - c) + v^2(\xi + a) = 2uv\eta.$$

Since $u = 0$ and $v = 0$ are only solutions for $\xi = -a$ and $\zeta = c$, respectively, we may assume that u, $v \neq 0$. In this case there are two solutions for the ratio $\tau = u/v$, given by

$$\tau_{1,2} = \frac{\eta \pm b}{\zeta - c}.$$

The strategy is now to split X into two points $X^\pm$ along one of these rank-one lines that satisfy equality in at least two of the inequalities in the definition of $\mathcal{A}$. Let

$$t^- = \sup\{t < 0 : X + tw \otimes w \text{ realizes two equalities in } \mathcal{A}\},$$
$$t^+ = \inf\,\{t > 0 : X + tw \otimes w \text{ realizes two equalities in } \mathcal{A}\}.$$

By assumption, $t^- < 0 < t^+$ and we define $X^\pm = X + t^\pm w \otimes w$. In view of the foregoing arguments, the matrices $X^\pm$ belong either to $K^{(3)}$ or to the intersection $\widetilde{H}$ of the two hyperboloids,

$$\widetilde{H} = \big\{F : (x+a)(z-c) = y^2 - b^2,\, (x-a)(z+c) = y^2 - b^2\big\}.$$

The formula for the lamination convex hull is therefore established if we show that $\widetilde{H} \subset K^{\mathrm{lc}}$. By symmetry it suffices again to prove this for all $F \in \widetilde{H}$ with $y \geq 0$. Now, if $F \in \widetilde{H}$, then

$$az = cx, \qquad \text{and} \qquad xz - ac = y^2 - b^2.$$

Thus the intersection of the two hyperboloids can be parameterized for $y \geq 0$ by

$$t \mapsto \Big(\sigma\sqrt{\frac{a}{c}}\sqrt{t^2 + ac - b^2}, t, \sigma\sqrt{\frac{c}{a}}\sqrt{t^2 + ac - b^2}\Big), \quad \sigma \in \{\pm 1\}, \quad t \geq 0.$$

We may assume that $\sigma = 1$. In this case the inequality $(x-a)(z-c) \geq (y-b)^2$ in the definition of $\mathcal{A}$ is equivalent to $(ac-b^2)(b-t)^2 \leq 0$ and this implies $t = b$, and thus $F \in K$, if $ac - b^2 > 0$. If $ac - b^2 = 0$, then the intersection of the hyperboloids coincides with the rank-one line between

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ and } \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -a & b \\ b & -c \end{pmatrix} \text{ and } \begin{pmatrix} a & -b \\ -b & c \end{pmatrix},$$

and consequently $F \in K^{(1)}$. This proves the formula for the lamination convex hull.

5. The quasiconvex hull of K for $ac - b^2 \geq 0$.

It remains to prove that for $ac - b^2 \geq 0$ all points in $K^{\mathrm{pc}} \setminus K^{\mathrm{lc}}$ can be separated from K (or equivalently from K^{lc}) with quasiconvex functions. Recall that by Remark 1.3 the quasiconvex and the polyconvex hull coincide for $ac - b^2 = 0$. We may therefore assume in the following that $ac - b^2 > 0$. We divide the proof of this assertion into three steps. First we show that the additional inequalities in the definition of K^{lc} are only active for $x,\, z \geq 0$ or $x,\, z \leq 0$. Then we construct a sufficiently rich family of quasiconvex functions that separates points from K, and finally we prove the theorem.

5.1. **Reduction to the case $x,\, y,\, z \geq 0$.** By symmetry we may always assume that $y \geq 0$. In this case the formula for K^{lc} contains the additional inequalities

$$(x+a)(z+c) \geq (y-b)^2, \quad (x-a)(z-c) \geq (y-b)^2. \tag{5.1}$$

Assume for example that $F \in K^{\mathrm{pc}}$ with $x \leq 0$ and $z \geq 0$. The inequalities in (5.1) can be rewritten as

$$(x \pm a)(z \pm c) \geq b^2 - y^2 + 2y^2 - 2by.$$

It follows from $F \in K^{\mathrm{pc}}$ that $-(x+a)(z-c) \geq b^2 - y^2$. The foregoing inequalities are thus true if

$$(x \pm a)(z \pm c) \geq -(x+a)(z-c) + 2y^2 - 2by$$

is satisfied. The equation with the minus and the plus sign are equivalent to

$$2x(z-c)+2y(b-y)\geq 0 \quad \text{and} \quad 2z(x+a)+2y(b-y)\geq 0, \tag{5.2}$$

respectively. Since by assumption $x \leq 0$, $z \leq c$, and $y \in [0,b]$, the first inequality in (5.2) holds and this implies the first inequality (5.1). Similarly, the second inequality in (5.2) is true in view of $z \geq 0$ and $x \geq -a$, and consequently the second inequality in (5.1) follows.

5.2. **Construction of quasiconvex functions.** From now on we assume that x, y, $z \geq 0$ and that $x \neq a$, $z \neq c$ and $y \neq b$ (see Section 4). We need to show that all points in K^{pc} with $(x-a)(z-c) < (y-b)^2$ can be separated from K by quasiconvex functions. This will be done using the Šverák's remarkable result that the functions

$$g_\ell(F) = \begin{cases} |\det F| & \text{if the index of } F \text{ is } \ell, \\ 0 & \text{otherwise,} \end{cases}$$

are quasiconvex on symmetric matrices, see [18]. Here the index of the symmetric matrix F is the number of its negative eigenvalues.

We begin by calculating the intersection of the boundary of the cone $(x-a)(z-c) \geq (y-b)^2$ with K^{pc} for fixed $y \in [0,b)$. This intersection can be parameterized by

$$t \mapsto \begin{pmatrix} t & y \\ y & c+(y-b)^2/(t-a) \end{pmatrix}, \quad t \in I_y = \Big[\,\frac{ay}{b}, a+\frac{b(y-b)}{c}\,\Big],$$

and we write $t \mapsto F(y,t)$ or $t \mapsto F_{y,t}$ for simplicity. A short calculation shows that $|I_y| = (ac-b^2)(b-y)/(bc) > 0$. We define quasiconvex functions $f_{y,t}$ on the space of all symmetric matrices by

$$f_{y,t}(F) = g_0(F-F_{y,t}), \quad y \in [0,b),\, t \in I_y,$$

and show first that $f_{y,t} = 0$ on K. In order to do this, it suffices to prove that all the matrices of the form $F - F(y,t)$ with $F \in K$ are not positive definite. In fact,

$$\det\Big[\begin{pmatrix} a & \pm b \\ \pm b & \pm c \end{pmatrix} - F_{y,t}\Big] = (a-t)(\pm c - c) + (y-b)^2 - (\pm b - y)^2 \leq 0,$$

and thus all matrices of the form $F - F_{y,t}$, with $F \in K$ and $F_{11} = a$ are not positive definite. Moreover,

$$\Big[\begin{pmatrix} -a & \pm b \\ \pm b & \pm c \end{pmatrix} - F_{y,t}\Big] = \begin{pmatrix} -a-t & \pm b - y \\ \pm b - y & \pm c - c - \frac{(y-b)^2}{t-a} \end{pmatrix},$$

and consequently all the matrices $X = F - F_{y,t}$ with $F \in K$ and $F_{11} = -a$ satisfy $X_{11} \leq 0$ and are therefore not positive definite.

5.3. **Separation of points from K^{lc} with quasiconvex functions.** Recall that we assume that

$$X = \begin{pmatrix} \xi & \eta \\ \eta & \zeta \end{pmatrix} \quad \text{with } \xi,\, \eta,\, \zeta \geq 0 \text{ and } \xi \neq a,\, \zeta \neq c,\, \eta \neq b.$$

We have to show that all matrices $X \in K^{\mathrm{pc}}$ with

$$(\xi - a)(\zeta - c) < (\eta - b)^2 \tag{5.3}$$

can be separated from K by a quasiconvex function. We will achieve this by analyzing different regions for ξ which are related to the intersection of K^{qc} with

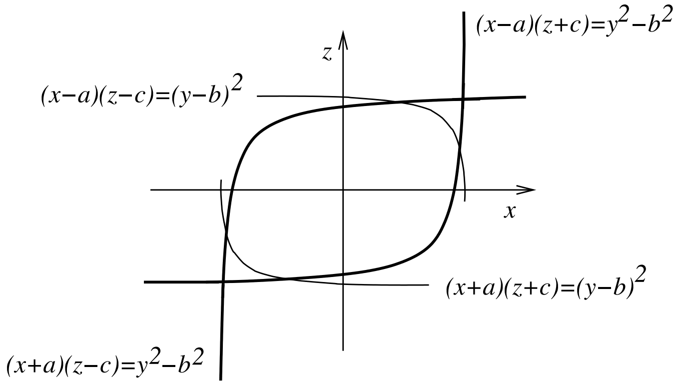

FIGURE 1. *The polyconvex hull (bounded by the thick solid lines) and the quasiconvex hull (the intersection of the four hyperbolic arcs) of K in the plane $\{y = \eta > 0\}$.*

the plane $y = \eta$. In this plane, the intersection of K^{qc} with the quadrant $x \geq 0$ and $z \geq 0$ is bounded by the three hyperbolic arcs $(x-a)(z-c) = (\eta - b)^2$ and $(x \pm a)(z \mp c) = \eta^2 - b^2$. In the following we consider four different regions for $\xi \geq 0$ which are defined by the points where two of these hyperbolic arcs intersect (see Figure 1). More precisely, the hyperbola $(x-a)(z-c) = (\eta-b)^2$ intersects the hyperbola $(x+a)(z-c) = \eta^2 - b^2$ for $x_1 = a\eta/b$ and the hyperbola $(x-a)(z+c) = \eta^2 - b^2$ for $x_2 = a + b(\eta - b)/c$. The four cases now correspond to $\xi \in [0, x_1]$, $\xi \in (x_1, x_2)$, $\xi = x_2$, and $\xi \in (x_2, a)$, respectively. We begin with the last case first.

Case a) Assume that $\xi > a + b(\eta - b)/c$. If $(\xi - a)(\zeta + c) \leq \eta^2 - b^2$, then

$$\zeta \geq -c + \frac{b^2 - \eta^2}{a - \xi} > -c - \frac{c(b^2 - \eta^2)}{b(\eta - b)} = \frac{c\eta}{b}.$$

We define

$$G_\eta = F\big(\eta, a + \frac{b(\eta - b)}{c}\big), \; Z = X - G_\eta = \begin{pmatrix} \xi - a - b(\eta - b)/c & 0 \\ 0 & \zeta - c\eta/b \end{pmatrix},$$

then Z is positive definite and in view of Section 5.2 the function $g_0(F - G_\eta)$ separates X from K^{lc}. On the other hand, if $(\xi - a)(\zeta + c) > \eta^2 - b^2$, then X does not belong to K^{pc}.

Case b) Assume that $\xi = a + b(\eta - b)/c$. We assert that in view of (5.3) we may find an $\widetilde{x} \in I_\eta = (a\eta/b, \xi)$ such that

$$Z = X - F(\eta, \widetilde{x}) = \begin{pmatrix} \xi - \widetilde{x} & 0 \\ 0 & \zeta - c - (\eta - b)^2/(\widetilde{x} - a) \end{pmatrix}$$

is positive definite. This follows easily since X is positive definite if and only if $\xi > \widetilde{x}$ and

$$\zeta - c - \frac{(\eta - b)^2}{\widetilde{x} - a} > 0 \quad \text{or} \quad (\widetilde{x} - a)(\zeta - c) - (\eta - b)^2 < 0.$$

In view of (5.3) we can choose $\widetilde{x} < \xi$ close enough to x such that the latter inequality holds. Therefore we can separate X from K^{lc} with the function $g_0(F - F(\eta, \widetilde{x}))$.

Case c) Assume that $\xi \in \big(a\eta/b, a + b(\eta - b)/c\big)$. The conclusion follows as in case b), since we can choose by continuity $\widetilde{x} \in \big(a\eta/b, \xi\big)$ such that $X - F(\eta, \widetilde{x})$ is positive definite.

Case d) Assume that $\xi \in [0, a\eta/b]$. We assert that no point in $K^{\rm pc}$ satisfies (5.3). If (5.3) holds, then

$$\zeta > c + \frac{(\eta - b)^2}{\xi - a}.$$

However, for

$$x = \widetilde{x} = \frac{a\eta}{b} \quad \text{and} \quad z = \widetilde{z} = c + \frac{(\eta - b)^2}{\widetilde{x} - a}$$

the inequality $(x + a)(z - c) \le \eta^2 - b^2$ is satisfied with equality. If

$$\xi \le \frac{a\eta}{b} \quad \text{and} \quad \zeta > c + \frac{(\eta - b)^2}{\widetilde{x} - a},$$

then $(\xi + a)(\zeta - c) > \eta^2 - b^2$, a contradiction. This concludes the proof of the theorem.

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