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by

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TEMPORAL ASYMPTOTICS FOR THE P'TH POWER NEWTONIAN FLUID IN ONE SPACE DIMENSION

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ABSTRACT. In this paper we consider a variety of initial-boundary value problems for the p'th power Newtonian fluid in one space dimension. We extend previously known results for the ideal gas case to a more general p'th power gas law, subject to the pinned endpoints and Dirichlet or Neumann temperature boundary conditions. The exponential convergence of the temperature, velocity and density is established for generic initial data. The estimates for different boundary conditions are presented in a unified manner.

1. INTRODUCTION.

This article is concerned with initial-boundary value problems for a p'th power Newtonian fluid (see [D]) undergoing longitudinal one-dimensional motions. The governing equations, in Lagrangian form, are the following

$$\begin{aligned} \xi_t &= \nu_x, \\ \nu_t &= \left(-\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}\right)_x, \\ c_v\theta_t &= \left(-\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}\right)\nu_x + \left(\kappa\frac{\theta_x}{\xi}\right)_x, \end{aligned} \tag{G}$$

where ξ (specific volume), ν (velocity), θ (absolute temperature) are unknown functions of $(x,t) \in [0,1] \times [0,\infty)$, and $\mu, \kappa, c_v > 0$ are given constants. We impose the zero velocity endpoints condition

$$\nu(0,t) = 0 = \nu(1,t), \tag{V}$$

along with either the Dirichlet boundary temperature condition

$$\theta(0,t) = \Theta = \theta(1,t), \tag{D}$$

where $\Theta > 0$ is a prescribed constant, or the Neumann condition

$$\theta_x(0,t) = 0 = \theta_x(1,t). \tag{N}$$

The initial data are given by

$$\xi(x,0) = \xi_0(x), \qquad \nu(x,0) = \nu_0(x), \qquad \theta(x,0) = \theta_0(x), \tag{I}$$

and the specific volume ξ and the absolute temperature θ are subject to the physical constraints

$$\xi > 0 \quad \text{and} \quad \theta > 0. \tag{C}$$

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The two initial-boundary value problems given by (G)(V)(I)(C), along with the Dirichlet condition (D) or Neuman condition (N) will be referred to as $(IBVP)_D$ and $(IBVP)_N$, respectively. For convenience and without loss of generality we may assume $\int_0^1 \xi_0(x)\mathrm{d}x = 1$. Then it follows from conservation of mass and condition (V) that

$$\int_0^1 \xi(x,t)\mathrm{d}x = 1. \tag{1.1}$$

The governing equations (G) arise from the conservation of mass, momentum and energy, for the viscous, heat-conducting p'th power gas, whose pressure $\mathcal{P}$ and internal energy e are given by

$$\mathcal{P} = \frac{\theta}{\xi^p}, \qquad e = c_v\theta,$$

with the pressure exponent $p \geq 1$ and constant specific heat $c_v > 0$. The total stress S, involving both elastic and viscous contributions, and the heat flux q arising from Fourier's law of heat conduction appear as

$$S = -\frac{\theta}{\xi^p} + \mu\frac{\nu_x}{\xi}, \qquad q = -\kappa\frac{\theta_x}{\xi}.$$

Here constants $\mu, \kappa > 0$ are viscosity and heat conductivity of the fluid. We thus see that the momentum and energy equations in (G) take the form of the balance laws

$$\nu_t = S_x, \tag{M}$$

$$\left(e + \frac{1}{2}\nu^2\right)_t = (S\nu - q)_x\,. \tag{E}$$

Initial-boundary value problems for the compressible Navier-Stokes ($p = 1$) equation in one space dimension have been extensively studied following the seminal paper [KS], where the existence and uniqueness of global classical solutions to the corresponding $(IBVP)_N$ was established. The ideas of this work have been extended to a variety of other physically natural boundary conditions (e.g. stress free) and more general pressure laws, along with weak existence theorems in Sobolev and BV spaces (see [H], [S], [KN], [Ka], [J], [N1], [L], [CHT]). The temporal asymptotics of solutions have also been studied, and exponential rates of convergence established (see [N2] and [HL]); the one gap being the treatment of the Dirichlet temperature condition (D).

In this paper we prove the exponential convergence of the specific volume, velocity and temperature to their respective equilibrium values (Theorem II). The main step in the argument is the derivation of the pointwise uniform bounds on the specific volume (see Theorem I (i)). A central difficulty here, that is associated with the pinned endpoint boundary condition, is the presence of an a priori unknown impulse $\int_0^t S(1,\tau)\mathrm{d}\tau$, arising at the boundary. We obtain the requisite bound, namely

$$0 < \underline{\xi} < \xi(x,t) < \overline{\xi}, \quad \forall (x,t) \in [0,1] \times [0,\infty),$$

through an analysis of the momentum balance, in combination with estimates following from the entropy identity (**??**) and convexity arguments.

The next step in the proof of convergence involves establishing a global L^2 estimate on the temperature gradient:

$$\int_0^\infty \int_0^1 \theta_x^2(x,t)\mathrm{d}x\mathrm{d}t < \infty$$

(see Theorem I (ii)). Here, the main difficluty arises from the Dirichlet temperature condition, due to the a priori unknown energy flux $\int_0^t [q(0,\tau) - q(1,\tau)]\mathrm{d}\tau$, through the boundary. This is circumvented by identifying a thermodynamic potential, the availablity [1], which is adapted to either temperature boundary condition (D) or (N), and then serves as a Lyapunov function for solutions.

The layout of this paper is as follows. In Section 2 we state the main theorems, while in Section 3 we gather some preliminary global estimates relevant for their proofs. The key pointwise estimate on the specific volume is obtained in Section 4, followed by the proof of the global L^2 temperature gradient estimate in Section 5. The final two Sections are devoted to establishing the exponential convergence of solutions to equilibrium states.

2. Main results.

The existence theory for the initial-boundary value problems may be conveniently formulated in terms of the spaces of Hölder continuous functions $C^{2+\alpha}[0,1]$, and $C^{\alpha,\alpha/2}([0,1]\times[0,\infty))$ which arise naturally in the theory of parabolic partial differential equations, see [K]. Throughout the paper we adopt the convention that any constant that appears will depend at most on the $C^{2+\alpha}$ norms of the initial data, $\min_{x\in[0,1]}\xi_0(x)$ and $\min_{x\in[0,1]}\theta_0(x)$. Also, we denote such generic "small" constants by λ, and "large" constants by Λ.

We first state an existence and uniqueness result which follows directly from [W].

Theorem . *Consider the initial-boundary value problems given by* $(IBVP)_D$ *or* $(IBVP)_N$. *Set* $\alpha \in (0,1)$ *and let the initial data* $\xi_0, \nu_0, \theta_0 \in C^{2+\alpha}[0,1]$ *satisfy the physical constraints* $\xi_0, \theta_0 > 0$ *and be compatible with the relevant boundary conditions. Then there exists a unique classical solution* (ξ,ν,θ) *on* $[0,1]\times[0,\infty)$ *with* $\xi \in C^{1+\alpha,1+\alpha/2}([0,1]\times[0,\infty))$, $\nu,\theta \in C^{2+\alpha,1+\alpha/2}([0,1]\times[0,\infty))$.

Our first main result concerns uniform pointwise bounds on the specific volume and global L^2 bounds on the temperature gradient.

Theorem I. *Let* (ξ,ν,θ) *be as in the previous Theorem. There exist constants* $\underline{\xi}, \bar{\xi} > 0$, *such that*

$$\text{(i)} \quad \underline{\xi} \le \xi(x,t) \le \overline{\xi},$$

$$\text{(ii)} \quad \int_0^\infty \int_0^1 \theta_x^2 \mathrm{d}x\mathrm{d}t < \infty.$$

Based on the above estimates we are able to establish the exponential convergence of solutions to equilibrium states. Here, a distinction arises between the

[1] Ericksen refers to this as balistic free energy [E].

$(IBVP)_D$ and $(IBVP)_N$ with respect to the limiting temperature. More precisely, seting

$$\bar{\Theta} = \begin{cases} \Theta & \text{for } (IBVP)_D, \\ \dfrac{1}{c_v}\displaystyle\int_0^1 \left(c_v\theta_0 + \frac{1}{2}\nu_0^2\right) \mathrm{d}x & \text{for } (IBVP)_N, \end{cases}$$

we have the following convergence result:

Theorem II. *Let* (ξ, ν, θ) *be as before. Then*

$$\text{(i)} \quad \int_0^1 \left(\xi_x^2 + \nu_x^2 + \theta_x^2\right)(x,t)\mathrm{d}x \le \Lambda e^{-\lambda t},$$

$$\text{(ii)} \quad \max_{x\in[0,1]} \left(|\xi(x,t) - 1| + |\nu(x,t)| + |\theta(x,t) - \bar{\Theta}|\right) \le \Lambda e^{-\lambda t}.$$

The established convergence rate is a reflection of the underlying parabolic structure of the governing equations (G) induced by the presence of viscosity and heat conduction.

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