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by

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1 Introduction

On Kähler manifolds, there exist deep relationships between holomorphic and harmonic objects, between algebraic data and analytic variational principles. For example, through the work of Narasimhan-Seshadri [NS], Donaldson [D1, D2, D3] and Uhlenbeck-Yau [UY] we know that an irreducible holomorphic bundle on a compact Kähler manifold $X$ possesses a Hermite-Einstein (H-E) metric if and only if it satisfies the algebraic condition of stability. As such a H-E metric is flat if the first and second Chern numbers of the bundle vanish, this yields a possibility to construct flat connections, i.e. linear representations of the fundamental group $\pi_1(x)$. In the opposite direction, Hitchin [H] saw how to construct a holomorphic bundle $E$ with a so-called Higgs field $\theta$ on a compact Riemann surface $X$ from such a representation $\rho : \pi_1(x) \to$ GL$(n, \mathbb{C})$ via the associated equivariant harmonic map $h : \tilde{X} \to$ GL$(n, \mathbb{C})/U(n)$ on the universal cover $\tilde{X}$. Simpson [S1, S3] extended Hitchin’s theory to compact Kähler manifolds and established an important connection with complex variations of Hodge structures.

For many reasons, however, the theory becomes fully satisfactory only when one can also include quasi-compact Kähler manifolds, the analogues of quasi-projective manifolds, the smooth complements of the singularities of algebraic varieties. Thus, let $X = \tilde{X} \setminus D$ where $\tilde{X}$ is a compact Kähler manifold with Kähler form $\omega_0$ and $D$ a divisor in $\tilde{X}$ with normal crossings as its only possible singularities. In that case, for the bundles on $X$ under consideration, one needs some condition as one approaches the compactifying divisor $D$, the so-called parabolic structures as introduced by Mehta-Seshadri [MS], Maruyama-Yokogawa [MY], Biquard [B, B1, B2], Li-Narasimhan [LN,

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LN1]. The general notion is in [L], and Li shows that the existence of the H-E metrics on the bundle over a quasi-compact Kähler manifold is essentially equivalent to the stability of the parabolic structure.

As the present paper will also deal with the quasi-compact case, let us briefly describe the previous results in order to put our work into its proper perspective.

Simpson [S2] (see also [LW, LW1]) studied Higgs bundles over punctured Riemann surfaces and established the relations between flat connections, representations of the fundamental group and complex variations of Hodge structures analogously to the compact case.

We now turn to quasi-compact Kähler manifolds of arbitrary dimension. Suppose that \( \rho : \pi_1(X) \to \text{GL}(n, \mathbb{C}) \) is an irreducible representation. Jost-Zuo [JZ, JZ1] construct a \( \rho \)-equivariant harmonic map from \( \tilde{X} \) to \( \text{GL}(n, \mathbb{C})/U(n) \), where \( \tilde{X} \) is the universal covering of \( X \). This result is the starting point of the present paper. We show that the harmonic map introduces a parabolic Higgs bundle \( (E, \theta) \) with logarithmic poles. Let \( \omega \) be a complete Poincaré like Kähler metric (cf. [CG], Section 3). We prove that the curvature of the metric is bounded with respect to \( \omega \). We consider the \( \mathbb{C}^* \) action on the Higgs bundle \( (E, \lambda \theta) \), and by solving the Hermitian-Yang-Mills equation, we get a flat connection \( D_\lambda \) on \( (E, \lambda \theta) \). We prove that, as \( \lambda \to 0 \), the connection \( D_\lambda \) converges to a new flat connection \( D_0 \). We apply the existence and the compactness to non-abelian Hodge theory. In particular, we prove a non-abelian Hodge \((p,q)\)-type theorem, which generalises the results due to Jost-Zuo [JZ2] and Katzarkov-Pantev [KP] in the compact case. We also prove that any discrete subgroup of a Lie group of rank \( \geq 2 \) that is not of Hodge type can’t be the fundamental group of a quasi-compact Kähler manifold. For example, \( \text{SL}(n, \mathbb{Z}) \) \( (n \geq 3) \) can’t be the fundamental group of a quasi-compact Kähler manifold. This result generalises Simpson’s theorem in the compact case.

We would like to point out that the analytic ideas in proving the existence of a H-E metric on the Higgs bundle \( (E, \lambda \theta) \) are applicable in greater generality, as soon as one knows how to construct an initial metric with bounded pseudo curvature with respect to the complete metric \( \omega \).

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2 Construction of pluriharmonic maps

We recall the ideas in [JZ, JZ1] to construct a pluriharmonic map and derive an estimation for its energy density. We recall some notations in [JZ, JZ1]. Let \( \mathcal{X} \) be a compact Kähler manifold, \( D \) a divisor in \( \mathcal{X} \) with normal crossings. We write \( D = \sum_{i=1}^m D_i \) where the irreducible components \( D_i \) of \( D \) are smooth and meet transversely.

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For each $i = 1, \ldots, m$, we choose a metric on the line bundle $[D_i]$ defined by the divisor $D_i$. Let $\sigma_i$ be the canonical section of $D_i$ which vanishes on $D_i$. We may assume that its length $|\sigma_i| < 1$. We put $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_m$, which is a section of $[D]$, then $|\sigma| = \Pi_i |\sigma_i| < 1$.

One can introduce (cf. [CG], Section 3) a complete Poincaré like Kähler metric

$$\omega = \sqrt{-1} \sum_{i=1}^m \partial \bar{\partial} \log |\sigma_i|^{-2} + C \omega_0$$

on $X = \overline{X} \setminus D$, where $\omega_0$ is a Kähler metric on $\overline{X}$. Set

$$\omega = \frac{\sqrt{-1}}{2} \gamma_{\alpha \beta} d z^\alpha \wedge d \bar{z}^\beta.$$

We define $\Delta_\delta = \{ x \in X \mid |\sigma| = \delta \}$ and $X_\delta = \{ x \in X \mid |\sigma| > \delta \}$. So, we have $\partial X_\delta = \Delta_\delta$. Set $\delta_i = \{ x \in X \mid |\sigma_i| = \delta \}$ for $i = 1, \ldots, m$. It is clear that $\Delta_\delta$ can be considered as the boundary of a family of disks, and at the singular points $D_i \cdot D_j$, it can be considered as a torus.

Let $N$ be a simply connected complete Riemannian manifold with non-positive sectional curvature, with isometry group $G := I(N)$. Let $\rho : \pi_1(X) \to G$ be a homomorphism from the fundamental group of $X$ into the isometry group of $N$. We say that $\rho$ is reductive ([JZ, Definition 1.1]) if there exists a complete totally geodesic subspace $Z$ of $N$ stabilized by $\rho(\pi_1(X))$ with the property that, for every totally geodesic subspace $Z'$ of $Z$ which has no Euclidean factor, $\rho(\pi_1(X))$ does not fix any point in the sphere at infinity of $Z'$. Let $d(\cdot, \cdot)$ be the distance function on $N$.

We divide the smooth irreducible components $D_i$ of $D$ into two sets,

$$D^1 = \{ D_i \mid \text{for every small loop } \gamma \text{ around } D_i, \inf_{y \in Y} d(y, \rho(\gamma)y) = 0 \}$$

$$D^2 = \{ D_i \mid \text{for every small loop } \gamma \text{ around } D_i, \inf_{y \in Y} d(y, \rho(\gamma)y) > 0 \}.$$

It is clear that $D = D^1 + D^2$ since the local monodromy groups have only one generator.

We say that $\rho$ is stabilizing near $D^1$ (Definition 1 in [JZ1]). We say that $g \in G$ is a hyperbolic element if $\inf_{y \in Y} d(y, gy) = \lambda_g > 0$ (Definition 1.2 in [JZ]). So, for every loop $\gamma$ around $D_i \in D^2$, $\rho(\gamma)$ is a hyperbolic element.

Near the divisors in $D^1$, we can construct a $\rho$-equivariant map with finite energy (cf. [JZ1]). For completeness, we give the details here.

Let $N(\infty)$ be the sphere at infinity of $N$. For any $x \in X$, we set

$$G_x := \{ g \in G \mid gx = x \}.$$

We say that a subgroup $P$ of $G$ is parabolic if there exists $x \in N(\infty)$ with $P \subset G_x$.

For every $x \in N(\infty)$, we can define a horospherical flow $\phi_t = \phi_{tx} : N \to N$ as follows (see [JZ, Section 2]):

$$3$$
For $p \in N$, we let $\gamma_{px}$ be the unique geodesic from $p$ to $x$ and put
$$\phi_t(p) := \gamma_{px}(t).$$

It is proved in [JZ1] (Lemma 1) that, for any $x \in N(\infty)$, $\phi_{t,x}$ commutes with $G_x$ and
$$d(\phi_{t_1}(p), \phi_{t_2}(q)) \leq d(\phi_{t_1}(p), \phi_{t_1}(q)),$$
for any $t_2 < t_1$, $p, q \in N$.

Let $k : \tilde{X} \to N$ be a smooth $\rho$-equivariant map, where $\tilde{X}$ is the universal covering
of $X$. Let
$$\Sigma_\delta := \{ z \in X \mid \min |\sigma_i| = \delta, \ D_i \in D^1 \}$$
and
$$\Sigma^1_\delta := \{ z \in \Sigma_\delta \mid \text{number } \{ j \in \{1, \cdots, m\} \text{ with } |\sigma_j| = \delta, \ D_j \in D^1 \} = l \}.$$

For sufficiently small $\delta > 0$, each component of
$$U_\delta := \bigcup \{ \Sigma_\eta \mid 0 < \eta < \delta \}$$
is a neighbourhood of a component of $D$ in $\overline{X}$; $\partial U_\delta = \Sigma_\delta$. We will deform the map
$k$ into a finite energy map on $U_\delta$. It suffices to treat one end. In the sequel we will
consider $k$ as a map from $X$ into the quotient of $N$ by $\rho(\pi_1(X))$. We first construct
a map on $\Sigma^1_\delta$. We assume that $\Sigma^1_\delta$ is locally given by $|z^1|^2 = \delta$ and $\Sigma^1 = \Omega^1 \times S^1$
for a certain region $\Omega^1$ in $z^1 = 0$. We let $\pi : \Omega^1 \times S^1 \to \Omega^1$ be the projection,
and $s : \Omega^1 \to \Omega^1 \times S^1$ be a smooth section. Each $S^1$-fiber then is parametrized by
$\theta \in [0, 2\pi)$ proportional to arclength with its intersection with $s(\Omega^1)$ as initial point
and the orientation for which $U_\delta$ is to the left. By assumption $\rho(g)$ is contained in
some parabolic group $G_x, x \in N(\infty)$. We let $\phi_t = \phi_{t,x}$ be the associated horocyclic
flow. For each $w_0 \in \Omega^1$, we choose a lift $z_0$ in $X$ of $s(w_0)$ and map the lift of the
$S^1$-fiber with initial point $z_0$ onto the geodesic arc from $k(z_0)$ to $\rho(g)k(z_0)$ with a
parametrization proportion to arclength. We thus obtain a map $h$ from the lift of $\Sigma^1_\delta$
to $\tilde{X}$ into $N$. On the punctured disk $D^*$ bounded by an $S^1$-fiber, we introduce polar
coordinates $0 < r \leq 1, 0 \leq \theta < 2\pi$. The lift $\tilde{D}^*$ of this punctured disk in $\tilde{X}$ either
again is a punctured disk, or it is a strip with coordinates $0 < r \leq 1, \theta \in C$; in the
latter case, the intersection with a fundamental domain of $X$ again may be assumed
to have coordinates $0 < r \leq 1, 0 \leq \theta < 2\pi$. We extend the above boundary map $h$ to
this lift by defining
$$h^*(r, \theta) := \phi_{\alpha^1,\beta^1}(h(1, \theta)).$$

Near the divisors in $D^2$, we can do as in [JZ]. We recall the details.

Each hyperbolic element in $N$ can be represented by translation along some
geodesic, in the quotient $N_0 = N/\rho(\pi_1(X))$, this yields a closed geodesic $c$. We
then map each circle in $\Sigma^1_\delta$ onto the corresponding closed geodesic proportionally
to arclength. This can be done continuously, because the two elements of $\rho(\pi_1(X))$
the circles around $D_i$ and $D_j$ near $D_i \cdot D_j$ commute. For each
\( i = 1, \ldots, m \) we set \( S_\delta^i = \bigcup_{w \in D_i} B_{w, \delta}, \) where every \( B_{w, \delta} \) can be holomorphically identified with \( \{ \, z \in C \mid |z| \leq \delta \, \}, \) and \( z = 0 \) corresponding to \( w \in D_i. \) We also identify \( X \cap B_{w, \delta} =: B_{w, \delta}^* \) with \( B_{w, \delta} = \{ \, z \in C \mid 0 < |z| \leq \delta \, \} \) or with the semi-infinite cylinder \( A_{\delta} = [\log \delta, \infty) \times S^1. \) The above map from \( \Sigma_\delta^i \) to \( N \) induces a map from \( \partial A_1 = \{0\} \times S^1 \) to \( N, \) mapping \( \partial A_1 \) proportionally to arclength onto a closed geodesic \( \gamma \) in \( N. \) We denote the map by \( \overline{g}_\gamma(\theta) \) (\( \theta \in S^1 \)). We extend \( \overline{g}_\gamma(\theta) \) to all of \( A_1 \) by putting

\[
g_\gamma(s, \theta) = \overline{g}_\gamma(\theta).
\]

Performing this construction on all punctured disks \( B_{w, \delta}^* \), we obtain a continuous map \( v_1 \) from \( \Sigma_1 \) to \( N \) which is harmonic on each such punctured disk. We extend \( v_1 \) continuously to \( v : X \to N \) in the required homotopy class.

We use the notations in [JZ]:

\[
S_\delta^i = S_1^i \setminus S_\delta^i, \quad S_\delta^u = X \setminus \bigcup_{i=1}^m S_\delta^i, \quad S_\delta^i = \bigcup_{i=1}^m S_\delta^i.
\]

We set

\[
v_\delta = v_1|_{\Sigma_\delta},
\]

and consider the harmonic map

\[
u_\delta : X_\delta \to N
\]

with boundary values

\[
u_\delta|_{\Sigma_\delta} = v_\delta|_{\Sigma_\delta}
\]

and inducing the homomorphism \( \rho : \pi_1(X) \to G. \)

It is proved in [JZ] that \( u_\delta \to u \) as \( \delta \to 0 \) and \( u \) is a harmonic map. We shall recall some details and derive an estimate for the energy density of \( u \), which will be important in our subsequent applications.

For any point \( p \in D \), we choose a neighbourhood \( U_p \) of \( p \). Assume that \((z_1, \ldots, z_n)\) is a coordinate system in \( U_p \) such that \( U_p \cap D = \{z_1 \cdots z_j = 0\} \). Then we can see that, in \( U_p \setminus D, \omega \) is quasi isometric to

\[
\sum_{i=1}^j |\sigma_i|^{-2} \log^{-2} |\sigma_i| dz_i \wedge d\overline{z}_i + \sum_{i=j+1}^n dz_i \wedge d\overline{z}_i.
\]

For simplicity of notation, we assume that \( \dim_C \overline{X} = 2. \)

Let \( \gamma = \det(\gamma_{\alpha\beta}) \), we have

\[
E(u_\delta) = \int_{S_\delta^u} |\nabla u_\delta|^2 dV
\]

\[
= \int_{S_\delta^u} \gamma^{-1}(z_1, z_2) \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} \gamma(z_1, z_2) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2
\]

\[5\]
\[ \begin{align*}
&\quad\quad + \int_{s^2_\delta} \gamma_2^2(z_1, z_2) \frac{\partial u_\delta}{\partial z_2} \frac{\partial u_\delta}{\partial \overline{z}_2} \gamma(z_1, z_2) d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \\
&\quad\quad + \int_{s^2_\delta} \gamma_1^2(z_1, z_2) \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} \gamma(z_1, z_2) d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \\
&\quad\quad + \int_{s^2_\delta} \gamma_2^2(z_1, z_2) \frac{\partial u_\delta}{\partial z_2} \frac{\partial u_\delta}{\partial \overline{z}_2} \gamma(z_1, z_2) d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \\
&\quad\quad \geq C \int_{s^2_\delta} \gamma_2^2(0, z_2) \frac{1}{|z_2|^2 (\log |z_2|^2)^2} \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \\
&\quad\quad + C \int_{s^2_\delta} \gamma_1^2(z_1, 0) \frac{1}{|z_1|^2 (\log |z_1|^2)^2} \frac{\partial u_\delta}{\partial z_2} \frac{\partial u_\delta}{\partial \overline{z}_2} d z_2 \wedge d \overline{z}_2 \wedge d z_1 \wedge d \overline{z}_1 \\
&\quad\quad + \int_{s^2_\delta} O(|z_1|) \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \\
&\quad\quad + \int_{s^2_\delta} O(|z_2|) \frac{\partial u_\delta}{\partial z_2} \frac{\partial u_\delta}{\partial \overline{z}_2} d z_2 \wedge d \overline{z}_2 \wedge d z_1 \wedge d \overline{z}_1
\end{align*} \]

where \( l \) is the length of the image geodesic \( c \).

On the other hand, we have

\[ E(v|s^\delta) = E(v|s^\delta) + E(v|\chi \setminus s^\delta) \leq E(v|s^\delta) + C \]

\[ = (- \log \delta) \int_{D_2} \rho^2 \frac{\gamma_2^2(0, z_2)}{|z_2|^2 (\log |z_2|^2)^2} d z_2 \wedge d \overline{z}_2 \]

\[ + (- \log \delta) \int_{D_1} \rho^2 \frac{\gamma_1^2(z_1, 0)}{|z_1|^2 (\log |z_1|^2)^2} d z_1 \wedge d \overline{z}_1 \]

\[ + \int_{s^2_\delta} O(|z_1|) \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} d z_1 \wedge d \overline{z}_1 \wedge d z_2 \wedge d \overline{z}_2 \]

\[ + \int_{s^2_\delta} O(|z_2|) \frac{\partial u_\delta}{\partial z_2} \frac{\partial u_\delta}{\partial \overline{z}_2} d z_2 \wedge d \overline{z}_2 \wedge d z_1 \wedge d \overline{z}_1 + C. \tag{1} \]

Since

\[ E(u_\delta) \leq E(v|s^\delta), \]

we get

\[ E(v|s^\delta) - C \leq E(u_\delta) \leq E(v|s^\delta). \tag{2} \]
If $0 < \delta < \eta < 1$, we have
\[ E(v|s^\eta_\delta) = E(v|s^\eta_\delta) + E(v|s^\eta_\delta \setminus s^\eta_\delta) \]  
and
\[ E(u_\delta|s^\eta_\delta) = E(u_\eta|s^\eta_\delta) + E(u_\delta|s^\eta_\delta \setminus s^\eta_\delta). \]  
Using an argument similar to the one used in obtaining (1), one gets
\[ E(v|s^\eta_\delta \setminus s^\eta_\delta) \leq E(u_\delta|s^\eta_\delta \setminus s^\eta_\delta) + C \]  
By (3), (4) and (5), we have
\[ E(u_\eta|s^\eta_\delta) \leq E(v|s^\eta_\delta) + C \]
where $C$ is a positive constant independent of $\delta$ (may depend on $\eta$).

By a standard argument (cf. [JZ, 1.4]), one can show that, if $\rho$ is reductive, $u_\delta$ sub-converges to a harmonic map $u$ in $C^2(\Omega, N)$ for any $\Omega \subset \subset X$. Since the sectional curvature of $N$ is non-positive, we have
\[ \Delta |du|^2 \geq -k|du|^2 \]
where $\Delta$ is the Laplacian with respect to the metric $\omega$, and $|du|^2$ is the energy density of $u$ with respect to the metric $\omega$. By the sub-mean value inequality and (1), we obtain
\[ |du|^2 \leq C(- \log |\sigma|)^2. \]
Let $d_\rho(x) := d_\rho(u, v)$, where $d_\rho(u, v)$ is the $\rho$-equivariant distance between $u$ and $v$. Using the estimate for the energy density, we obtain an estimate for $d_\rho$ near the divisor, namely
\[ d_\rho \leq C(- \log |\sigma|). \]

One can show that $u$ is in fact pluriharmonic by an argument in the proof of Lemma 1.1 in [JZ]. So we proved the following theorem.

**Theorem 2.1** Let $N = GL(n, \mathbb{C})/U(n)$. Let $\tilde{X}$ be the universal covering of $X$. Assume that $\rho$ is reductive. Then there is a $\rho$-equivariant pluriharmonic map
\[ u : \tilde{X} \to Y \]
with
\[ |du|_\omega \leq C(- \log |\sigma|). \]  
The matrix $u$ in $GL(n, \mathbb{C})/U(n)$ is $O((- \log |\sigma|)^\beta)$, for some $\beta \in \mathbb{R}$.  

7
3 Estimation of the curvature

Let $V$ be a $\text{GL}(n, \mathbb{C})$ bundle with flat connection $D$. Introducing a metric $g$ leads to the decomposition

$$D = D_g + \theta$$

where $D_g$ preserves the metric. Let $\rho : \pi(X) \to \text{GL}(n, \mathbb{C})$ be the representation defined by the flat bundle $V$. A metric on $V$ can be considered as a $\rho$-equivariant map

$$g : X \to \text{GL}(n, \mathbb{C})/\text{U}(n).$$

We have

$$dg = \theta,$$

and $h$ is harmonic if and only if

$$D_g^*\theta = 0,$$

where $D_g^*$ is the adjoint operator of $D_g$ with respect to the metric $g$. In this case, we say that the metric $g$ is harmonic. We then decompose into types:

$$D = D' + D'',$$

$$\theta = \theta^{1,0} + \theta^{0,1}.$$

We have $(D'')^2 = 0$, and hence $E := (V, D'')$ is a holomorphic bundle. Thus $(E, \theta^{1,0})$ is a Higgs bundle. A Higgs bundle $(E, \theta^{1,0})$ on $X$ consists of a holomorphic bundle $E$ together with

$$\theta^{1,0} : E \to \Omega^{1,0}(X) \otimes E$$

satisfying the integrability condition

$$\theta^{1,0} \wedge \theta^{1,0} = 0.$$

Let $(E, \theta^{1,0})$ be a Higgs bundle on $X$. The analogue of the $\overline{\partial}$-operator on $E$ now is

$$D^2 = \overline{\partial} + \theta^{1,0}.$$

A metric on $E$ then defines

$$D^1 = \partial + \theta^{0,1}.$$

**Theorem 3.1** Let $u$ be the pluriharmonic map obtained in Theorem 2.1. Let $\theta = du$. Let $F_g$ be the curvature of the metric connection $D_g = \partial + \overline{\partial}$. We have $|F_g|_\omega \leq C$, where $| \cdot |_\omega$ is the norm with respect to the metric $\omega$.

The purpose of this section is to prove the theorem. The theorem can be seen as a generalization of Theorem 1 in [S2] to the higher dimensional case.

**Proof:** For simplicity of notation, we prove the result in the complex surface case, as the proof for the higher dimensional case is exactly the same. It suffices [JZ, 1.6]
to prove the estimate near the point $D_1 \cdot D_2$. We choose local coordinates so that $X = B^*_1 \times B^*_1$ where $B^*_1 = \{ z \in C \mid 0 < |z| < 1 \}$. Assume that

$$\theta^{1,0} = \theta_1 dz_1 + \theta_2 dz_2.$$ 

Since $\theta^{1,0} \wedge \theta^{1,0} = 0$, we have

$$\theta_1 \theta_2 = \theta_2 \theta_1.$$ 

Let $\lambda_j$ be the eigenvalues of $\theta_1$. It is clear that $\lambda_j$ is a (multivalued) holomorphic function. By Theorem 2.1, we have

$$|\lambda_j| \leq \frac{C}{|z_i|}.$$ 

As in [S2] (Section 2), we may divide up the eigenvalues into groups $\lambda^1_1, \ldots, \lambda^k_i$ with the property that

$$|\lambda_j^i - a^i \frac{dz_1}{z_1}| \leq \frac{C}{|z_1|^{1-\epsilon}}, \quad (7)$$

for any $0 < \epsilon < \frac{1}{\text{rank} E}$, where $a^i$ ($i = 1, \ldots$) are complex numbers. We decompose $E = \oplus E^1_i$ holomorphically so that $E^1_i$ is the sum of the generalized eigenspaces for the eigenvalues $\lambda^1_j$. We define a new endomorphism valued one form $\phi_1$, semi-simple with eigenvalues $a^i \frac{dz_1}{z_1}$, and eigenspaces $E^1_i$. The subspaces $E^1_i$ are preserved by $\phi_1$ and $\theta_1$. Similarly, we can also divide up the eigenvalues of $\theta_2$ into groups $\nu^1_1, \ldots, \nu^k_i$, with the property that

$$|\nu_j^i - b^i \frac{dz_2}{z_2}| \leq \frac{C}{|z_2|^{1-\epsilon}}, \quad (8)$$

where $b^i$ ($i = 1, \ldots$) are also complex numbers. We can also decompose $E = \oplus E^2_i$ holomorphically and define an endomorphism-valued one form $\phi_2$, semi-simple with eigenvalues $b^i \frac{dz_2}{z_2}$, and eigenspaces $E^2_i$. The spaces $E^2_i$ are preserved by $\phi_2$ and $\theta_2$.

Because $\theta_1 \theta_2 = \theta_2 \theta_1$, we can have a holomorphic decomposition $E = \oplus E_i$ so that $E_i$ are preserved by $\phi_1$, $\phi_2$, $\theta_1$, and $\theta_2$, which is a refined decomposition of both $E = \oplus E^1_i$ and $E = \oplus E^2_i$. Choose a basis $\{e_k\}$ of $E$, orthonormal with respect to the harmonic metric, compatible with the decomposition. We write everything in terms of this basis. We have

$$\theta_i = \beta_i + \tau_i,$$

$$\phi_i = \alpha_i + q_i$$

for $i = 1, 2$, where $\beta_i$ and $\alpha_i$ are diagonal matrices of one forms, and $\tau_i, q_i$ are upper triangular matrices of one forms.

It is obvious that

$$\beta_i = \alpha_i + \gamma_i$$

with

$$|\alpha_i| \leq \frac{C}{|z_i|}, \quad |\gamma_i| \leq \frac{C}{|z_i|^{1-\epsilon}}.$$
Let
\[ M^+ = \oplus_{i>j} \Hom(E_i, E_j) \subset \End(E), \]
\[ M^0 = \oplus_i \End(E_i) \subset \End(E), \]
\[ M^- = \oplus_{i<j} \Hom(E_i, E_j) \subset \End(E). \]

One can show (cf. [S2, Section 2]) that \( q_i \in M^+ \). We write \( \tau_i = \tau_i^0 + \tau_i^+ \) with \( \tau_i^+ \in M^+, \tau_i^0 \) upper triangular in \( M_i^0 \). Note that the spaces \( M^0 \) and \( M^+ \) are single valued, so \( \tau_i^+ \) and \( \gamma_i^0 + \tau_i^0 \) are single valued.

Theorem 3.1 follows from the following lemma.

**Lemma 3.2** We have
\[ |\tau_i^0| \leq \frac{C}{|z_i|(- \log |z_i|)}, \]
\[ |\tau_i^+| \leq C|z_i|^{-1+\epsilon} \quad \text{and} \quad |q_i| \leq C|z_i|^{-1+\epsilon}, \]
for \( i = 1, 2 \), where \(| \cdot |\) is with respect to the harmonic metric on the bundle and the restriction metric \( \omega_0 \) on \( X \).

**Proof:** It suffices to derive the estimates in local coordinates around \( D_1 \cdot D_2 \). We use \( B_1^0 \times B_1^0 \) as the coordinates as before. Because \( \omega_0 \) is equivalent to the Euclidean metric in \( B_1^0 \times B_1^0 \), we assume that it is the Euclidean metric. We prove the estimates for \( i = 1 \), the proof for \( i = 2 \) is the same. In the sequel, we will omit the subscript for simplification.

It is proved in [S2, P.729] that
\[ \nabla \log(|\beta|^2 + |\tau|^2) \geq C \frac{|\tau|^4}{|\beta|^2 + |\tau|^2}, \]
where \( \nabla \) is the Laplace operator with respect to the Euclidean metric \( \omega_0 \).

We shall modify Simpson’s proof [S2] to our case.

We first show that \( |\tau| \leq \frac{C}{|z_i|} \). Set \( h = |\beta|^2 + |\tau|^2 \), we shall show that \( h \leq \frac{C}{|z_i|} \).

Set \( m_\epsilon = \frac{C_2 |z_i|}{|z_i| - \epsilon} \) for \( \epsilon > 0 \), we have
\[ \nabla \log m_\epsilon \leq \frac{2}{C_2} m_\epsilon. \]

If at an interior point \( h > m_\epsilon \), we have \( h \geq \frac{C_0}{|z_i|} \) and
\[ \nabla \log h \geq C_1 h. \]

Choosing \( C_2 \) large enough so that \( C_2 \geq C_0 \) and \( \frac{2}{C_2} \leq C_1 \), we obtain
\[ \nabla \log(h/m_\epsilon) \geq C_1 (h - m_\epsilon) > 0. \]
Therefore $h/m_\epsilon$ can’t achieve a maximum value greater than 1 in the interior. Since $h/m_\epsilon = 0$ on the boundary $|z_1| = \epsilon$ and $h$ is bounded on the boundary $|z_1| = 1$, we have

$$h \leq C_3 m_\epsilon.$$ 

Letting $\epsilon \to 0$ yields the claim.

We then show that

$$|\tau| \leq \frac{C}{|z_1|(-\log |z_1|)}.$$ 

Let $a = \sqrt{\sum_i |a_i|^2}$. We may assume that $a \neq 0$, otherwise, the claim clearly holds. Set

$$k = \log\left(\frac{|z_1|^2}{a}(|\tau|^2 + |\tau|^2)\right).$$

Since $|\beta|^2 = \frac{a}{|z_1|^2} + b$ with $b \leq \frac{c}{|z_1|^2 - \epsilon^2}$, we have

$$k = \log\left(1 + \frac{|z_1|^2 b}{a} + |z_1|^2 |\tau|^2 a^{-1}\right),$$

and thus

$$c(|z_1|^2 b + |z_1|^2 |\tau|^2) \leq k \leq C(|z_1|^2 b + |z_1|^2 |\tau|^2).$$

It suffices to show that

$$k \leq \frac{C}{(-\log |z_1|)^2}.$$ 

Set

$$p_\epsilon(z_1, z_2) = \frac{C_2}{(-\log |z_1|)^2} + \epsilon(-\log |z_1|).$$

Calculating directly, one gets

$$\Delta p_\epsilon \leq -\frac{6 p_\epsilon^2}{C_2 |z_1|^2}.$$ 

It is clear that, at the point where $k \geq \frac{C_1}{(-\log |z_1|)^2}$, we have

$$\Delta k \geq c_1 \frac{k^2}{|z_1|^4}.$$ 

So, applying the maximum principle again, we get

$$k \leq p_\epsilon.$$ 

This proves the claim.

It is proved in [S2, p. 731] that

$$\tau^+ = (1 + f)q.$$
where \( f \) is an operator with norm bounded by \( \frac{C}{(-\log |z_1|)} \), and \(|q| \leq \frac{C}{|z_1|(-\log |z_1|)} \). We have [S2, p. 731-732]

\[
\Delta \log |\phi|^2 \geq c \frac{||\phi, \bar{\gamma}||^2}{|\phi|^2},
\]

and

\[
||\phi, \bar{\gamma}||^2 \geq \frac{|q|^2}{|z_1|^2},
\]

so

\[
\Delta \log |\phi|^2 \geq c|q|^2.
\]

Since

\[
|\phi|^2 = |\alpha|^2 + |q|^2 = \frac{a}{|z_1|^2} \left( 1 + \frac{|z_1|^2|q|^2}{a} \right),
\]

we have

\[
\Delta \log \left( 1 + \frac{|z_1|^2|q|^2}{a} \right) \geq c|q|^2.
\]

Let \( k = \log(1 + \frac{|z_1|^2|q|^2}{a}) \), we have

\[
c|z_1|^2|q|^2 \leq k \leq C|z_1|^2|q|^2,
\]

and

\[
\Delta k \geq -c\frac{k}{|z_1|^2}.
\]

Set

\[
p_{\nu} = |z_1|^{-\nu} + \epsilon(-\log |z_1|), \quad \text{if } \nu^2 < c.
\]

Then the maximum principle yields that \( k \leq C|z_1|^{-\nu} \). So,

\[
|q|^2 \leq \frac{C}{|z_1|^{2-\nu}}
\]

and consequently,

\[
|\tau^+| \leq C(1 + \frac{1}{(-\log |z_1|)}) \frac{1}{|z_1|^{2-\nu}} \leq C \frac{1}{|z_1|^{2-\nu}},
\]

for any \( 0 < \epsilon < \nu \). This proves the lemma.
4 Filtrations and Chern numbers

In this section, for simplicity we again assume that $X$ is of complex dimension 2. In the higher dimensional case, the results and the proofs are the same. Recall that $E$ is a holomorphic vector bundle over $X$, $\theta := \theta_{1,0}$ is a Higgs 1-form on $X$, $(E, \theta)$ is a Higgs bundle. Suppose that

$$D = D_g + \theta + \overline{\theta}$$

where $\overline{\theta}$ is defined by $(\theta u, v)_g = (u, \overline{\theta} v)_g$. By Theorem 3.1 we know that $F_{D_g}$, the curvature of $D_g$ satisfies $|F_{D_g}|_{\omega} \leq C$. Applying the extension theorem [CG, Theorem 1], we know that $E$ can be extended to $\overline{E}$ across the divisor $D$ as a coherent sheaf.

**Theorem 4.1** Suppose that $s$ is a flat section of $E$ with respect to the connection $D$. There exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ ($i = 1, 2$) such that

$$c|\sigma_1|^\alpha|\sigma_2|^\beta \leq |s|_g \leq C|\sigma_1|^\alpha|\sigma_2|^\beta.$$

**Proof:** We have

$$d\langle s, s \rangle_g = 2\langle D_g s, s \rangle_g = -4\langle \theta s, s \rangle_g.$$

So,

$$|d\langle s, s \rangle_g| \leq \frac{C_1}{\sigma_1} |\langle s, s \rangle_g| |dz_1| + \frac{C_2}{\sigma_2} |\langle s, s \rangle_g| |dz_2|.$$

The theorem obviously follows from the above inequality.

Because of Theorem 3.1, we can define

$$E^{\alpha, \beta} = \{ s \text{ is a flat section of } E \text{ near } D | |s|_g \leq |\sigma_1|^{\alpha-\epsilon}|\sigma_2|^{\beta-\epsilon} \text{ for any } \epsilon > 0 \}.$$

It is clear that $E^{\alpha, \beta} \subset E^{\alpha', \beta'}$ if $\alpha \geq \alpha'$, $\beta \geq \beta'$, $E^{\alpha, \beta} = \cap_{\gamma < \alpha, \delta < \beta} E^{\gamma, \delta}$, $E^{\alpha+1, \beta} = \sigma_1 E^{\alpha, \beta}$, $E^{\alpha, \beta+1} = \sigma_2 E^{\alpha, \beta}$, and

$$j_! E = \cup_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} E^{\alpha, \beta} \quad (9)$$

where $\sigma_i$ ($i=1,2$) is the canonical section of the divisor $D_i$ and $j : X \to \overline{X} = X \cup D$ is the inclusion.

Finally, we have

$$\theta : E^{\alpha, \beta} \to E^{\alpha, \beta} \otimes \Omega^1_X(\log D)$$

where $\Omega^1_X(\log D)$ is the sheaf of logarithmic differentials at $D$. The identity (9) gives a filtered regular Higgs bundle.

For the uniqueness of the harmonic metrics, we have

**Theorem 4.2** Suppose that $u_1$ and $u_2$ are two $\rho$-equivariant harmonic maps, that $\theta_1 = du_1$ and $\theta_2 = du_2$ induce the same filtration $j_! E = \cup_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} E^{\alpha, \beta}$, and that $\text{Res}_D(\theta_1, E^{\alpha, \beta}) = \text{Res}_D(\theta_2, E^{\alpha, \beta})$ for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then $\exists g \in \mathcal{U}(n)$ such that $u_1 = gu_2$. 

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Proof: Let $f := d_p(u_1, u_2)$, where $d_p(u_1, u_2)$ is the $p$-equivariant distance between $u_1$ and $u_2$. It suffices to show that $d_p(u_1, u_2) \equiv \text{Constant}$. Since $u_1$ and $u_2$ are $p$-equivariant harmonic maps, we have

$$\triangle f \geq 0. \quad (10)$$

Since $\theta_1 = du_1$ and $\theta_2 = du_2$ induce the same filtration and

$$\text{Res}_D(\theta_1, E^{\alpha, \beta}) = \text{Res}_D(\theta_2, E^{\alpha, \beta}),$$

we have

$$A := \theta_1 - \theta_2 : E^{\alpha, \beta} \to E^{\alpha + \epsilon, \beta + \epsilon} \otimes \Omega^1_X(\log D)$$

for some $\epsilon > 0$ and consequently,

$$|df|_0 \leq C(|\sigma|^{-1 + \epsilon}). \quad (11)$$

By (10) and (11), one can get that $f \equiv \text{Constant}$. This proves the theorem.

More generally, we have

**Theorem 4.3** Suppose that for the flat connection $D = D_{g_1} + \theta_1 = D_{g_2} + \theta_2$ (decomposed with respect to two different harmonic metrics $g_1$ and $g_2$), $\theta_1$ and $\theta_2$ induce the same filtration $j_i E = \cup_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} E^{\alpha, \beta}$ and $\text{Res}_D(\theta_1, E^{\alpha, \beta}) = \text{Res}_D(\theta_2, E^{\alpha, \beta})$ for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then $g_1$ and $g_2$ are mutually bounded.

**Proof:** We consider the identity map $e$ from $(E, g_1)$ to $(E, g_2)$ as a section of $E^* \otimes E$. Since $\text{Res}_D(\theta_1, E^{\alpha, \beta}) = \text{Res}_D(\theta_2, E^{\alpha, \beta})$, we have

$$A := \theta_1 - \theta_2 : E^{\alpha, \beta} \to E^{\alpha + \epsilon, \beta + \epsilon} \otimes \Omega^1_X(\log D)$$

for some $\epsilon > 0$.

We introduce a Kähler metric

$$\omega_\gamma := \sqrt{-1} \sum_{j=1}^{2} \partial \bar{\partial} |\sigma_j|^{2-\gamma} + C_\gamma \omega_0$$

for $0 < \gamma < 2$. Then by Lemma 4.1 in [S2], we get

$$\triangle_\gamma \log |e|^2 \geq -C \quad (12)$$

where $\triangle_\gamma$ is the Laplace operator with respect to $\omega_\gamma$.

Since $g_1$ and $g_2$ induce the same filtration, we have

$$|e|^2 \leq C|\sigma|^{-\delta},$$

for any $\delta > 0$. So,

$$\log |e|^2 \leq -\delta \log |\sigma| + C.$$
Therefore (12) holds weakly on $X$. Since the Sobolev inequality holds on $(X, \omega)$ (cf. [L, Section 4]), Moser’s [Mo] iterative argument shows that

$$\log |\epsilon|^2 \leq C < \infty.$$ 

This proves the theorem.

We define the first and second parabolic Chern numbers as follows: Let $\alpha_l = \alpha$ if $r - \text{rank} E^{0,0}|_{D_1} < l \leq r - \text{rank} E^{0,0}|_{D_1}$, and $\beta_l = \beta$ if $r - \text{rank} E^{0,\beta}|_{D_2} < l \leq r - \text{rank} E^{0,\beta}|_{D_2}$ ($l = 1, \cdots, r$). Set

$$\alpha^1 = \begin{pmatrix} \alpha_1 & \cdots & \\ & \ddots & \alpha_r \end{pmatrix}$$

and

$$\alpha^2 = \begin{pmatrix} \beta_1 & \cdots & \\ & \ddots & \beta_r \end{pmatrix}$$

$$\text{Par } C_1 = C_1(E) + \sum_{i=1}^{2} \text{tr} \alpha^i \deg[D_i]$$

where $C_1(E)$ is the first Chern number of $E$.

$$\text{Par } C_2 = C_2(E) + \frac{1}{2} \left( 2 \sum_{\alpha} \alpha \deg(E^{0,\beta}|_{D_1})/E^{0,0}|_{D_1}) + 2 \sum_{\beta} \beta \deg(E^{0,\beta}|_{D_2})/E^{0,\beta}|_{D_2}) + \sum_{i=1}^{2} \text{tr} (\alpha^i)^2 D_i^2 - 2 \sum_{i=1}^{2} \text{tr} \alpha^i \deg(E[D_i]) - \sum_{i,j=1}^{2} \text{tr} \alpha^i \alpha^j D_i \cdot D_j \right).$$

where $C_2(E)$ is the second Chern number of $E$, $D_i \cdot D_j = \int_x C_1([D_i]) \wedge C_1([D_j]$ is the intersection number of $D_i$ and $D_j$ ($i, j = 1, 2$).

Set

$$C_1(E, g) = \int_X \frac{\sqrt{-1}}{2\pi} \text{tr} F_g \wedge * \omega_0,$$

$$C_2(E, g) = - \int_X \frac{1}{8\pi^2} (\text{tr} F_g \wedge \text{tr} F_g - \text{tr} F_g \wedge F_g).$$

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Theorem 4.4 Let $u$ be the pluriharmonic map obtained in Theorem 2.1. Let $\theta = du$. Let
\[ j_* E = \bigcup_{n \in \mathbb{R}, \beta \in \mathbb{R}} E^{\alpha, \beta} \]
be the filtration introduced by $(g, \theta)$. Then $\text{Par } C_1 = C_1(E, g)$ and $\text{Par } C_2 = C_2(E, g)$.

Proof: We choose a Hermitian metric $g_0$ on the bundle $E$. Set $g_0^{-1} g = g_1$, then $\text{tr } F_g = \text{tr } F_{g_0} + \bar{\partial} \partial \log \det g_1$.

\[ C_1(E, g) = \deg E + \frac{\sqrt{-1}}{2\pi} \int_X \bar{\partial} \partial \log \det g_1 \wedge * \omega \]

Set
\[ S^1 = \left( \begin{array}{c} |\sigma_1|^\alpha_1 \\ \vdots \\ |\sigma_1|^\alpha_r \end{array} \right) \]
and
\[ S^2 = \left( \begin{array}{c} |\sigma_2|^\beta_1 \\ \vdots \\ |\sigma_2|^\beta_r \end{array} \right) \]

We have
\[ C_1(E, h) = \deg E + \frac{\sqrt{-1}}{2\pi} \int_X \bar{\partial} \partial \log \det \left( \prod_i (S^i)^2 \right) \wedge * \omega \]
\[ + \frac{\sqrt{-1}}{2\pi} \int_X \bar{\partial} \partial \log \det \left( \prod_i (S^i)^{-2} g_1 \right) \wedge * \omega \]

By the Poincaré-Lelong formula ([SABK], Ch.II, Section 1, Theorem 2), we have
\[ C_1(E, g) = \text{Par } C_1(E) + \frac{\sqrt{-1}}{2\pi} \int_X \bar{\partial} \partial \log \det \left( \prod_i (S^i)^{-2} g_1 \right) \wedge * \omega \]

It is not difficult to see that $\log \det \left( \prod_i (S^i)^{-2} g_1 \right)$ can be extended smoothly to $\overline{X}$, so the last term on the right hand side of the identity vanishes. This proves the first part of the theorem.

We have
\[ C_1(E, g) \wedge C_1(E, g) \]
\[ = \left( \frac{\sqrt{-1}}{2\pi} \right)^2 (\text{tr } F_{g_0} + \bar{\partial} \partial \log \det g_1) \wedge (\text{tr } F_{g_0} + \bar{\partial} \partial \log \det g_1) \]
\[ = \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \left( (\text{tr } F_{g_0})^2 + 2 \text{tr } F_{g_0} \wedge \bar{\partial} \partial \log \det g_1 \right. \]
\[ \left. + \bar{\partial} \partial \log \det g_1 \wedge \bar{\partial} \partial \log \det g_1 \right) \]
So, 
\[
\int_X C_1(E, g) \wedge C_1(E, g)
= \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X (\text{tr} F_{\varphi_0})^2 \\
+ 2 \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr} F_{\varphi_0} \wedge \overline{\partial} \log \det(\Pi_i (S^i)^2) \\
+ 2 \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr} F_{\varphi_0} \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \\
+ \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \overline{\partial} \log \det(\Pi_i (S^i)^2) \wedge \overline{\partial} \log \det(\Pi_i (S^i)^2) \\
+ \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \\
+ 2 \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \overline{\partial} \log \det(\Pi_i (S^i)^2) \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i).
\]

Since \( \log \det(\Pi_i (S^i)^{-2} g) \) can be extended smoothly to \( \overline{\mathcal{X}} \), we have
\[
\int_X \text{tr} F_{\varphi_0} \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \\
= \int_X \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \\
= \int_X \overline{\partial} \log \det(\Pi_i (S^i)^2) \wedge \overline{\partial} \log \det(\Pi_i (S^i)^{-2} g_i) \\
= 0.
\]

We therefore have
\[
\int_X C_1(E, g) \wedge C_1(E, g) = \int_X C_1(E) \wedge C_1(E) \\
+ 2 \sum_{i=1}^{2} \text{tr} \alpha^i \deg(E|D_i) + \sum_{i,j=1}^{2} (\text{tr} \alpha^i \text{tr} \alpha^j D_i \cdot D_j).
\]

It is clear that
\[
F_g = F_{\varphi_0} + \overline{\partial}(g_1^{-1} \partial_{\varphi_0} g_1).
\]

So,
\[
F_g \wedge F_g = F_{\varphi_0} \wedge F_{\varphi_0} + 2 F_{\varphi_0} \wedge \overline{\partial}(g_1^{-1} \partial_{\varphi_0} g_1) + \overline{\partial}(g_1^{-1} \partial_{\varphi_0} g_1) \wedge \overline{\partial}(g_1^{-1} \partial_{\varphi_0} g_1)
\]

Note that
\[
\overline{\partial}(g_1^{-1} \partial_{\varphi_0} g_1) = \overline{\partial}((S g_1)^{-1} \partial_{\varphi_0} (S g_1)) - \overline{\partial}(g_1^{-1} (S^{-1} \partial_{\varphi_0} S) g_1)
\]

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where \( S = \Pi_i(S^i)^{-2} \). Since \((Sg)\) can be seen as an endomorphism of \( E \), we have
\[
\left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(F_g \wedge F_g) \\
= \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(F_{g_0} \wedge F_{g_0}) \\
- 2 \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(F_{g_0} \wedge \overline{\partial}(g_0^{-1}(S^{-1} \partial_{g_0} S)g_1)) \\
+ \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(\overline{\partial}(g_1^{-1}(S^{-1} \partial_{g_0} S)g_1) \wedge \overline{\partial}(g_1^{-1}(S^{-1} \partial_{g_0} S)g_1))
\]
A simple calculation shows that
\[
\left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(F_{g_0} \wedge \overline{\partial}(g_1^{-1}(S^{-1} \partial_{g_0} S)g_1)) \\
= \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(F_{g_0} \wedge \overline{\partial}(S^{-1} \partial_{g_0} S)) \\
= 2 \sum_{\alpha} \alpha \deg(E^0_{\alpha} |_{D_1} / E^{\alpha+0}_{\alpha} |_{D_1}) + 2 \sum_{\beta} \beta \deg(E^{0,\beta} |_{D_2} / E^{0,\beta+} |_{D_2})
\]
Similarly, we have
\[
\left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(\overline{\partial}(g_1^{-1}(S^{-1} \partial_{g_0} S)g_1) \wedge \overline{\partial}(g_1^{-1}(S^{-1} \partial_{g_0} S)g_1)) \\
= \left( \frac{\sqrt{-1}}{2\pi} \right)^2 \int_X \text{tr}(\overline{\partial}(S^{-1} \partial_{g_0} S) \wedge \overline{\partial}(S^{-1} \partial_{g_0} S)) \\
= \sum_{i=1}^2 \text{tr}(\alpha_i^2 D_i^2)
\]
This completes the proof of the theorem.

The Theorem clearly implies the following corollary.

**Corollary 4.5** Let \((E, \theta)\) be the Higgs bundle introduced by the harmonic map \( u \) obtained in Theorem 2.1. We have \( \text{Par } C_1 = 0 \) and \( \text{Par } C_2 = 0 \).

**Proof:** Calculating directly, one gets
\[
C_2(E, g) = -\frac{1}{8\pi^2} \int_X (\text{tr} F_g \wedge \text{tr} F_g - \text{tr} F_g \wedge F_g) = -\frac{1}{8\pi^2} \int_X (\text{tr} F_D \wedge \text{tr} F_D - \text{tr} F_D \wedge F_D).
\]
Since \( D = D_g + \theta \) is flat, we can see that \( \text{Par } C_2 = C_2(E, g) = 0 \). Similarly, one can also show that \( \text{Par } C_1 = 0 \). This proves the corollary.
5 The existence of flat connections

Let $\Lambda$ be the contraction with respect to the complete Kähler metric $\omega$. A Hermitian-Einstein (H-E) metric $H$ is a Hermitian metric on $E$ with the property that

$$\Lambda(F_H + [\theta, \theta_H^*]) = \mu I,$$

where $F_H$ is the curvature of the Hermitian connection of $H$, and $\theta^*$ is defined by $(\theta u, v)_H = (u, \theta_H^* v)_H$, $\mu$ is a constant, $I$ is the identity endomorphism of $E$.

We show, in Section 3 (Theorem 3.1), that there exist a Hermitian metric $g$ and a Higgs field $\theta$ on $E$ satisfying

$$|F_g|_\omega \leq C, \text{ and } |\theta|_\omega \leq C,$$

and the connection $D = D_g + \theta$ is flat. We consider the connection

$$D^\lambda = D_g + \lambda \theta \text{ for } \lambda \neq 0.$$

It is clear that $D^\lambda$ is a flat connection if and only if $|\lambda| = 1$. Using the heat flow method, we will deform a H-E metric $H_0^\lambda$ on the Higgs bundle $(E, \lambda \theta)$ and show that $D_{H_0^\lambda} + \lambda \theta$ is a flat connection.

**Theorem 5.1** For any $\lambda \neq 0$, there is a metric $H_0^\lambda$ on the Higgs bundle $(E, \lambda \theta)$ so that $D^\lambda_0 = D_{H_0^\lambda} + \lambda \theta$ is flat. The parabolic structure introduced by $H_0^\lambda$ is the same as that introduced by $g$. As $\lambda \rightarrow 0$, the connection $D^\lambda_0 \rightarrow D_0$ in $C^1(X)$ and $D_0$ is a flat connection on $E$.

**Proof:** We use the same notation as that in Section 3, set

$$D^1 = D = D_g + \theta \text{ and } D^\lambda = D_g + \lambda \theta.$$

By Theorem 3.1, we can see that

$$|F_{D^\lambda}|_\omega \leq C,$$

where $C > 0$ is independent of $\lambda$.

For any Hermitian metric $H$ on the bundle $E$, we set

$$D^\lambda_H = D_H + \lambda \theta \text{ and } F^\lambda_H = D^\lambda_H \cdot D^\lambda_H.$$

Let $(F^\lambda_H)^\perp = F^\lambda_H - \frac{1}{\text{tr}F^\lambda_H}F^\lambda_H$ be the trace free part of $F^\lambda_H$.

Set $X_\beta = \{ x \in X \mid \log |\sigma|^2 > -\beta \}$. We solve the heat equation

$$\begin{cases}
  H^{-1} \frac{dH}{dt} = -\sqrt{-1} \Lambda(F^\lambda_H)^\perp \\
  H|_{t=0} = g \\
  \det H = \det g
\end{cases} \quad (13)$$

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on $X_\beta$ with Dirichlet boundary condition $H|_{\partial X_\beta} = g$. By Lemma 6.1 in [S1] we have
\[(\Delta - \frac{\partial^2}{\partial t^2})|\Lambda(F^\lambda_{H_\beta}) |^2_{H_\beta} = 2|\overline{\Lambda}(F^\lambda_{H_\beta}) |^2_{H_\beta} \geq 0.\] So, $\Lambda(F^\lambda_{H_\beta}) |^2_{H_\beta} \geq 0$. The maximum principle implies
\[|\Lambda(F^\lambda_{H_\beta}) |^2_{H_\beta} \leq C\] (14)
for all $(x, t) \in X_\beta \times [0, \infty)$.

It is clear (cf. [D4] and [S1]) that the above equation has a solution $H_\beta$ for all time and that a subsequence of $H_\beta(t)$ converges in $C^2(\overline{X_\beta})$ to $H_\beta(t)$ satisfying $\Lambda(F^\lambda_{H_\beta}) |^2 = 0$, with $H_\beta|_{\partial X_\beta} = g$ and $\det H_\beta = \det g$. We set $H_\beta = g$ outside $X_\beta$.

Let $S(E)$ denote the real vector bundle of selfadjoint endomorphisms of $E$ and $S(\text{End}E)$ consists of the elements of $\text{End}(\text{End}E)$ which are selfadjoint with respect to the metric $tr(AB^*)$. For two metrics $K$ and $H = Ke^s$, we (c.f. [S1], Section 4) define the following functional
\[M(K, H) = \sqrt{-1} \int_X tr(S\Lambda F_K) + \int_X (\Psi(S)(D^* S), D^* S)_K,\]
where $\Psi : S(E) \rightarrow S(\text{End}E)$ is constructed as in [S1] from the function
\[\Psi(\lambda_1, \lambda_2) = \frac{e^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1) - 1}{(\lambda_2 - \lambda_1)^2},\]
i.e., if we write $A = \sum_{i, j} A_{ij} e_i^* \otimes e_j$, then
\[\Psi(S)(A) = \sum_{i, j} \Psi(\lambda_i, \lambda_j) A_{ij} e_i^* \otimes e_j,\]
here $\{e_i\}$ is an orthonormal basis of eigenvectors of $S$ with eigenvalues $\lambda_i$ and $\{e_i^*\}$ is the dual basis of $\{e_i\}$ in $E^*$. For the details we refer the readers to [S1]. It is proved in [LW] (see also [S1]) that
\[M(g, H_\beta) \leq C.\]

We set
\[\omega_\alpha = \sqrt{-1} \sum_{i=1}^m \partial \overline{\partial} \log^{-\alpha} |\sigma_i|^{-2} + C_\alpha \omega_0\]
where $\alpha > 0$ $(i = 1, \cdots, m)$. It is proved in [CL] (Theorem 1.1) that the Sobolev inequality holds on the manifold $(X, \omega_\alpha)$. Let $A_\alpha$ be the contraction with respect to the Kähler metric $\omega_\alpha$, let $\nabla_\alpha$ be the gradient operator of the metric $\omega_\alpha$, and let $\Delta_\alpha$ be the Laplace operator of the metric $\omega_\alpha$.

Let $h_\beta = g^{-1} H_\beta$. In $X_\beta$, we have ([S1], Lemma 3.1 (c))
\[\Delta_\alpha \log \frac{1}{\text{rank}E} tr h_\beta \geq - (|\Lambda_\alpha(F^\lambda_g)|_g + |\Lambda_\alpha(F^\lambda_{H_\beta})|_{H_\beta}),\]
\[\log \frac{1}{\text{rank}E} tr h_\beta|_{\partial X_\beta} = 0,\]
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so, we obtain,
\[
\triangle_{\alpha} \frac{1}{\text{rank}E} \log^\alpha tr h_\beta \geq -\frac{(|\Lambda_\alpha(F^\lambda_g)|_{h_\beta} + |\Lambda_\alpha(F^\lambda_{H_{g_\beta}})|_{H_{g_\beta}})}{(-\log |\sigma|)^\alpha} + 2 \nabla_{\alpha} \log \frac{1}{\text{rank}E \cdot \nabla_{\alpha}} tr h_\beta \cdot \nabla_{\alpha} \frac{1}{(-\log |\sigma|)^\alpha} - C_\alpha.
\]

By Theorem 3.1 and (14), we have
\[
\frac{|\Lambda_\alpha(F^\lambda_g)|_{h_\beta} + |\Lambda_\alpha(F^\lambda_{H_{g_\beta}})|_{H_{g_\beta}}}{(-\log |\sigma|)^\alpha} \leq C.
\]

It is clear that
\[
|\nabla_{\alpha} \frac{1}{(-\log |\sigma|)^\alpha}| \leq C_\alpha.
\]

Hence by Moser’s iterative argument [Mo] (see also [G-T] Ch.8), we have
\[
\sup_{X_\beta} \log \frac{1}{\text{rank}E} tr h_\beta \leq C_\alpha (1 + \int_{X_\beta} \log \frac{1}{\text{rank}E} tr h_\beta dV).
\]

Set \(e^{s_\beta} = h_\beta = g^{-1} H_\beta\) in \(X_\beta\) and \(s_\beta = 0\) outside \(X_\beta\) we have
\[
\sup_X \frac{|s_\beta|_{g}}{(-\log |\sigma|)^\alpha} \leq C_1 + C_2 \|S_\beta\|_{L^2(X, \omega)}.
\]

By an argument similar to the one used in the proof of Proposition 5.3 in [S1] (see also [LW]), we can see that
\[
\sup_X \frac{|s_\beta|_{g}}{(-\log |\sigma|)^\alpha} \leq C,
\]
becaus the parabolic Higgs bundle is stable. Therefore
\[
|s_\beta|_{g} \leq C(-\log |\sigma|)^\alpha \quad \text{and} \quad |H_\beta|_{g} \leq Ce^{(-\log |\sigma|)^\alpha}.
\]

Because ([S1], Lemma 3.1 (c))
\[
\triangle tr h_\beta \geq 2\sqrt{-1} tr (h_\beta \Lambda(F^\lambda_g)) + 2 (D^\infty h_\beta)_{h_\beta}^{-\frac{1}{2}}_{g},
\]
for any compact set \(X_0 \subset X\), we have
\[
\int_{X_0} |D^\infty h_\beta|_g^2 dV \leq C(X_0).
\]

By the diagonal argument, we obtain a sequence \(\beta_i \to \infty\), such that \(h_i = h_{\beta_i} \to h_0\) in \(L^2(X_0)\), for any \(X_0 \subset X\). Now set \(h_{ij} = h_i^{-1} h_j\), then in \(X_0\), there holds
\[
\Delta \log(tr h_{ij}) \geq 0
\]

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for sufficiently large $i, j$. By Moser’s iterative argument again, we have for any $X_0 \subset \subset X$

$$\sup_{X_0} \log(\text{tr}(h_{ij})) \leq C \left( \int_X (\log(\text{tr}(h_{ij})))^2 dV \right)^{\frac{1}{2}}.$$ 

So $h_i \to h$ in $C^0(X_0)$ for any $X_0 \subset \subset X$.

Clearly $H_i = gh_i \to H_0 = gh_0$ in $C^0(X_0)$ and $H_i$ are bounded locally in $L^p_2$ for any $1 < p < \infty$ (see the remark following Lemma 6.4 in [S1]). So by going to a subsequence, we have $H_i \to H'_0$ weakly in $L^p_{2, \text{loc}}$. Thus $F_{H'_0}$ is defined and $(F_{H'_0})^\perp = 0$. Elliptic regularity results imply that $H'_0$ is smooth.

We set $H_0 = e^{-\phi} H'_0$, where $\phi$ is a smooth real valued function on $X$, we have

$$\sqrt{-\Delta} F_{H_0}^\lambda = \sqrt{-\Delta} F_{H'_0}^\lambda + \triangle \phi I = (\triangle \phi + \frac{\sqrt{-\Delta}}{\text{rank}E} \text{tr}\Lambda F_{H'_0}^\lambda) I.$$ 

It is clear that $f = \frac{\sqrt{-\Delta}}{\text{rank}E \sqrt{d(X)}} \int_X \text{tr}\Lambda F_{H'_0}^\lambda dV - \frac{\sqrt{-\Delta}}{\text{rank}E} \text{tr}\Lambda F_{H'_0}^\lambda \in L^\infty(X, \omega)$, and we have $\int_X f dV = 0$. We consider the closed subspace

$$L^2_0(X, \omega) = \{ \varphi \in L^2(X, \omega) \mid \int_X \varphi dV = 0 \}$$

of $L^2(X, \omega)$. $\triangle$ is a closed operator from $\mathcal{D}(\triangle) \cap L^2_0(X)$ to $L^2_0(X)$. Since the Poincaré inequality holds on $(X, \omega)$ ([Li] Theorem 2.4, see also [CL], Theorem 1.1), we know that $0$ does not belong to the spectrum of $\triangle$ in $L^2_0(M)$. So we can get a smooth function $\phi$ such that $\triangle \phi = f$ in $X$. By an argument similar to the one used in the proof of (15) we have

$$|\phi| \leq C(-\log|\sigma|). \quad (16)$$

Hence

$$\Lambda F_{H_0}^\lambda = \mu I.$$ 

That is $H_0$ is a H-E metric. By (15) and (16), we have

$$|H_0|_g \leq e^{-(-\log|\sigma|)^\nu}. \quad (17)$$

By the estimate (17), note that

$$\int_{\partial X, \beta} \frac{1}{|\sigma|(-\log|\sigma|)^{1-\alpha}} ds = O\left(\frac{1}{\beta^{1-\alpha}}\right) \to 0 \text{ as } \beta \to \infty,$$

we can see that the parabolic structure introduced by $H_0$ is the same as that introduced by $g$ and that the metric $H_0$ gives the same parabolic first and second Chern numbers.
We choose \( f_\beta = \max \{0, 1 + \frac{\log|\beta|^2}{\beta}\} \). By Theorem 7.3 in [S1], we have

\[
\int_{X_\beta} f_\beta tr(F_{H_\beta}^\lambda \wedge F_{H_\beta}^\lambda) \wedge \omega_n^{-2} \leq \int_{X_\beta} f_\beta tr(F_g^\lambda \wedge F_g^\lambda) \wedge \omega_n^{-2} + C^{\frac{1}{\beta}} \int_{X_\beta} |s_\beta|_g |F_g^\lambda|_g dV.
\]

Using (17), we get

\[
\int_{X_\beta} f_\beta tr(F_{H_\beta}^\lambda \wedge F_{H_\beta}^\lambda) \wedge \omega_n^{-2} \leq \int_{X_\beta} f_\beta tr(F_g^\lambda \wedge F_g^\lambda) \wedge \omega_n^{-2} + \frac{C}{\beta^{1-n}} \int_{X_\beta} |F_g^\lambda|_g dV.
\]

By the Riemann bilinear relations, one gets

\[
tr(F_{H_\beta}^\lambda \wedge F_{H_\beta}^\lambda) \wedge \omega_n^{-2} \geq -C|\Lambda F_{H_\beta}^\lambda|^2\omega^n.
\]

Since

\[
\sup_{X_\beta} |\Lambda(F_{H_\beta}^\lambda)^{\frac{1}{\beta}}| \leq \sup_{X_\beta} |\Lambda(F_g^\lambda)^{\frac{1}{\beta}}| \leq C
\]

and \( tr F_{H_\beta}^\lambda = tr F_g^\lambda \), we have

\[
tr(F_{H_\beta}^\lambda \wedge F_{H_\beta}^\lambda) \wedge \omega_n^{-2} \geq -C\omega^n.
\]

Letting \( \beta \to \infty \), using Fatou’s lemma we obtain

\[
\int_X tr(F_0^\lambda \wedge F_0^\lambda) \wedge \omega_n^{-2} \leq \int_X tr(F_g^\lambda \wedge F_g^\lambda) \wedge \omega_n^{-2} = 0.
\]

Note that \( C_1(E, H_0) = C_1(E, g) = 0 \), applying the Bogomolov-Gieseker inequality (c.f. [S1], Proposition 3.4), we get

\[
\int_X tr(F_0^\lambda \wedge F_0^\lambda) \wedge \omega_n^{-2} \geq 0,
\]

therefore,

\[
\int_X tr(F_0^\lambda \wedge F_0^\lambda) \wedge \omega_n^{-2} = 0,
\]

which implies that \( D_{H_0}^\lambda \) is a flat connection. Using the estimate (17), we can see that \( D_{H_0}^\lambda \) converges to a flat connection as \( \lambda \to 0 \), in \( C^2(U) \) where \( U \) is a small neighbourhood of \( D \). We can see that the convergence also holds in \( C^{2}_\text{loc} (X) \), otherwise, the second Chern number of the flat connection will decrease which contradicts the stability of the Higgs bundle. This proves the theorem.

Now we prove the uniqueness of the flat connection.

**Theorem 5.2** Suppose that \( H_1 \) and \( H_2 \) be two Hermitian-Einstein metrics on the bundle \( E \) with \( |H_1|_g = O(e^{-c|x|^2}) \) and \( |H_1|_g = O(e^{-c|x|^2}) \), then \( H_1 = UH_2 \), where \( U \) is a unitary matrix.
Proof: We set $h = H_1^{-1}H_2$, then we have

$$\log \text{tr} h = O((-\log |\sigma|)^\alpha), \quad \triangle \log \text{tr} h \geq 0.$$  
Multiplying the last identity by $f_\beta^2 \log \text{tr} h$ and integrating, we obtain:

$$\int_X f_\beta^2 |\nabla \log \text{tr} h|^2 dV \leq \frac{C}{\beta^2} \int_X |\nabla \log |\sigma||^2 \log^2 \text{tr} h dV$$

so,

$$\int_X f_\beta^2 |\nabla \log \text{tr} h|^2 dV \leq \frac{C}{\beta^{1-2\nu}}.$$  
Letting $\beta \to \infty$ yields the theorem.

6 Applications to algebraic geometry

In this section, we consider applications of the previous results.

**Theorem 6.1** Let $f : X \to Y$ be a smooth morphism between quasi-projective manifolds $X$ and $Y$, and let $\rho : \pi_1(X) \to G$ be a Zariski dense representation into an almost simple group $G$, which does not factor through $f$. If the restriction $\rho : \pi_1(f^{-1}(y)) \to G$ for some $y \in Y$ comes from a variation of Hodge structure, then $\rho$ comes from a variation of Hodge structure.

In other words, Theorem 6.1 just says that if the equivariant pluri-harmonic map $u : f^{-1}(y) \to G/K$ restricted to $f^{-1}(y)$ is holomorphic, then $u$ itself is holomorphic. Before starting to prove Theorem 6.1 we need some preparations. Let $G$ be an almost simple algebraic group and let $M(X, G)$ denote the moduli space of semi-simple representations of $\pi_1(X)$ into $G$.

Let $f : X \to Y$ be a smooth morphism from $X$ to $Y$. Fix $y \in Y$ and let $X_y$ denote the fibre of $f$ over $y$. The inclusion $i : X_y \to X$ induces a morphism $i^* : M(X, G) \to M(X_y, G)$.

**Lemma 6.2** Let $[\rho] \in M(X_y, G)$, with a Zariski-dense image in $G$. Then $i^{*\rho^{-1}}([\rho])$ consists of finitely many points.

**Proof:** Consider the exact sequence of homotopy groups

$$\pi_1(X_y, x_0) \to \pi_1(X, x_0) \to \pi_1(Y, y) \to 1,$$

and take finitely many elements $\{\gamma_1, ..., \gamma_m\}$ from $\pi_1(X)$ such that together with $\pi_1(X_y)$ they generate $\pi_1(X)$. So, it is sufficient to show that there are only finitely many possibilities of values of $\rho(\gamma_i) \in G$.

For any element $\alpha \in \pi_1(X_y, x_0)$, take the product $\gamma_1 \alpha \gamma_1^{-1}$ which lies in $\pi_1(X_y, x_0)$. Fixing a $\rho'_0 : \pi_1(X, x_0) \to G$ such that $\rho'_0|_{\pi_1(X_y, x_0)} = \rho$, one has
\[
\rho(\gamma_i\alpha\gamma_i^{-1}) = \rho'_0(\gamma_i\alpha\gamma_i^{-1}) = \rho'_0(\gamma_i)|\rho(\alpha)\rho'_0(\gamma_i)^{-1}.
\]

For any \( f : \pi_1(X, x_0) \to G \) with \( f|_{\pi_1(X_y, x_0)} = \rho \) we then have
\[
(\rho(\gamma_i)^{-1}\rho'_0(\gamma_i))\rho(\alpha)(\rho(\gamma_i)^{-1}\rho'_0(\gamma_i))^{-1} = \rho(\alpha).
\]

Note that \( \rho(\pi_1(X_y, x_0)) \) is Zariski dense in \( G \), we see that the elements \( \rho(\gamma_i)^{-1}\rho'_0(\gamma_i) \) must lie in the center \( Z(G) \). Since \( G \) is almost simple, \( Z(G) \) consists of finitely many elements. This proves the lemma.

Proof of Theorem 6.1: Let \( f : X \to Y \) be a smooth morphism between quasi-projective manifolds \( X \) and \( Y \), and let \( \overline{f} : \overline{X} \to \overline{Y} \) be a smooth compactification. Since \( \rho \) does not factor through \( f \), the restriction \( \rho : \pi_1(X_y) \to G \) is again Zariski dense. Since \( \rho \) is a semi-simple representation, by Theorem 2.1 there exists a harmonic metric \( h \) on the flat bundle \( V \) (with trivial filtrations of the local system), which makes \( V \) into a filtered Higgs bundle \( \{(E, \theta)\}_\alpha \). Restricting everything to the fibre \( X_y \), we obtain a harmonic metric \( h|_{X_y} \) on the flat bundle \( V|_{X_y} \) with trivial filtrations of the local system. The assumption in the theorem is that the corresponding Higgs bundle \( (E, \theta)|_{X_y} \) is a system of Hodge bundles. Hence, its isomorphic class is fixed by the \( \C^* \) action by Simpson’s Theorem ([S2], Page 768, Theorem 8).

For each \( t \in \C^* \) by constructing the Hermitian-Yang-Mills metric in Theorem 5.1 for the Higgs bundle \( \{(E, \theta)\}_\alpha \) we obtain a harmonic bundles \( (V_t, h_t) \) on \( X \). By Theorem 5.1 \( \{V_t, h_t\} \) converges to a harmonic bundle \( (V_0, h_0) \) as \( t \to 0 \).

Lemma 6.3 The underlying representation of \( V_0 \) is isomorphic to the original \( \rho \).

Proof: Since the isomorphic class of \( (E, \theta)\alpha|_{X_y} \) is fixed by the \( \C^* \) action, the class of the underlying representation of \( (E, \theta) \) lies in the fibre \( i^*-1(\rho|_{X_y}) \). By Lemma 6.2 \( i^*-1(\rho|_{X_y}) \) consists of finitely many points. Note that the \( \C^* \) action is continuous and contains the identity, \( p_t \simeq p_1 = \rho, \forall t \in \C^* \). Let \( t \to 0 \), we obtain \( p_0 \simeq \rho \). This proves the lemma.

Since the limiting point arising from this way is unique, for any \( t' \in \C^* \) one has \( (E_0, t'\theta_0) \simeq \lim_{t \to 0} (E, t'\theta_0) \simeq (E_0, \theta_0) \alpha \). By Simpson’s Theorem \( (E_0, \theta_0) \alpha \) is a system of Hodge bundles. By Lemma 6.3, the underlying representation of \( (E_0, \theta_0) \alpha \) is \( \rho \). Thus, \( \rho \) comes from VHS. We complete the proof of Theorem 6.1.

Theorem 6.1 has the following consequence for the non-abelian Hodge \( (p, q) \)–type theorem. Let \( f : X \to Y \) denote a smooth morphism between smooth quasi-projective manifolds \( X \) and \( Y \). Let \( G \) be an almost simple algebraic group and let \( M(X/Y, G) \) denote the the relative moduli space of the representations of the relative fundamental groups into \( G \).

Theorem 6.4 (Non-abelian Hodge \( (p, q) \)–type theorem) Let \( s \) be a flat section in the space \( M(X/Y, G) \). Suppose \( s(y_0) \) is Zariski-dense and comes from a VHS from some \( y_0 \in Y \). Then \( s(y) \) comes from a VHS for all \( y \in Y \).
Proof: Let $Z$ denote the center of $G$, which is a finite group. We show first that $s$ lifts to a global representation $\rho: \pi_1(X, x_0) \to G/\mathcal{Z}$.

Since $s$ is a flat section in $M(X/Y, G)$, the class $s(y_0) = [\rho_{y_0}] \in M(X_{y_0}, G)$ is fixed by the $\pi_1(Y, y_0)$-action, i.e., for each $\beta \in \pi_1(Y, y_0)$ there exists some $g(\beta) \in G$ such that $\rho_{y_0}(\beta \alpha \beta^{-1}) = g(\beta)\rho_{y_0}(\alpha)g(\beta)^{-1}$ (in general, such $g(\beta)$ is not unique). By the same argument as in the proof of Lemma 6.2, for any two $g(\beta)$ and $g(\beta)'$, one has $g(\beta)g(\beta)'^{-1} \in \mathcal{Z}$. Thus, $g(\beta)$ is well defined in $G/\mathcal{Z}$. By taking $\rho(\beta) = g(\beta) \in G/\mathcal{Z}$ we lift $\rho_{y_0}$ to a representation $\rho: \pi_1(X) \to G/\mathcal{Z}$.

Now we are in the position to apply Theorem 6.1. Since $\rho|_{X_{y_0}}$ is Zariski dense and comes from a VHS, $\rho$ comes from a VHS. In particular, $\rho|_{X_y}$ comes from a VHS for all $y \in Y$. Since $\mathcal{Z}$ is finite, $s(y)$ comes from a VHS. This completes the proof of Theorem 6.4.

The following Theorem 6.5 is an arithmetic analogue of Theorem 6.4.

Theorem 6.5 Let $s$ be a flat section in $M(X/Y, G)$. Suppose that $s(y_0)$ takes values in the ring of algebraic integers $\mathcal{O}_K$. Then $s$ lifts to a global representation $\rho: \pi_1(X) \to G/\mathcal{Z}$ that takes values in the ring of algebraic integers $\mathcal{O}_K$.

Remark. If $\pi_1(Y)$ is replaced by the Galois group $\text{Gal}(\overline{L}/L)$ of a number field $L$ then the above statement is just Simpson’s theorem on representations from $\pi^\text{dg}_1(X/L)$ into $G(\mathcal{O}_K)$.

Proof: By Theorem 6.4, we may lift $s(y_0)$ to a global representation $\rho: \pi_1(X) \to G/\mathcal{Z}(K)$. Theorem 6.5 will follow if we can show that, for any prime ideal $p$ of $\mathcal{O}_K$, the induced representation $\rho: \pi_1(X) \to G/\mathcal{Z}(K_p)$ is $p$-bounded.

Let $\Delta$ denote the Bruhat-Tits building of $G(K_p)$. By Theorem 2.1, there exists a $\rho$-equivariant pluriharmonic map

$$u_\rho: \hat{X} \to \Delta.$$ 

Since $\rho|_{X_{y_0}}$ is $p$-bounded, $u_\rho|_{X_{y_0}}$ is constant. Since any $X_y$ is homotopic to $X_{y_0}$, their fundamental groups have the same image in $X$. Hence, $u_\rho|_{X_y}$ is constant for any $y \in Y$. That implies that $u_\rho$ factors to pluriharmonic map $\nu_\rho: \hat{Y} \to \Delta$ by the map

$$f: \hat{X} \to \hat{Y}.$$ 

We claim that $u_\rho$ is constant: otherwise $\nu_\rho: \hat{Y} \to \Delta$ is unbounded. Let $\Delta(\nu_\rho)$ denote the convex subcomplex generated by the image of $\nu_\rho$. Then $\Delta(\nu_\rho)$ is fixed by $\rho(\pi_1(X_{y_0}))$, because a point $z \in \nu_\rho(Y)$ is just the image $u_\rho(X_y)$, which is fixed by $\rho(\pi_1(X_y)) = \rho(\pi_1(X_{y_0}))$. Since $\Delta(\nu_\rho)$ contains at least one geodesic line, $\rho(\pi_1(X_{y_0}))$ fixes this line. Thus $\rho(\pi_1(X_{y_0}))$ is contained in a parabolic subgroup $P \subset G/\mathcal{Z}$. By the exact sequence of the homotopy groups

$$\pi_1(X_{y_0}) \to \pi_1(X) \to \pi_1(Y) \to 1$$

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one shows that $\bar{\rho}(\pi_1(X_{g0})) \subset \bar{\rho}(\pi_1(X)) = G/Z$ where $\bar{\rho}(\pi_1(X_{g0}))$ is normal and contained in $P$. Since $G/Z$ is a simple algebraic group, $\rho(\pi_1(X_{g0})) = \{1\}$. Thus, $s(y_b)$ is contained in $Z$. Hence, it is finite. This contradicts the Zariski density of $s(y_b)$.

Since $u_p$ is constant, $\rho$ is $p$-bounded. This proves Theorem 6.5.

We have the following application on deformations of families of abelian varieties.

Theorem 6.6 Let $\pi : Z \to B$ be a family of quasi-projective manifolds. Suppose that over one fiber $Z_b$ there is a family of polarized abelian varieties $f_b : A_b \to Z_b$ of dimension $g$, and such that the monodromy representation $\rho_b : \pi_1(Z_b) \to Sp(g, \mathbb{Z})$ lies in a finite orbit of the $\pi_1(B, b)$-action on $M(Z_b, Sp(g, \mathbb{Z})$. Then the family $f_b : A_b \to Z_b$ can be extended to a family of polarized abelian varieties $f : A \to Z$ after a base change.

Remark 1) Let $\mathcal{M}_g$ denote the moduli space of abelian varieties of dimension $g$ and with fixed polarization. If the induced map $\pi_1(Z_b) \to \pi_1(\mathcal{M}_g)$ is an isomorphism, by Weil’s local rigidity theorem for lattices in Lie groups of rank $\geq 2$ the monodromy representation $\rho_b$ satisfies the condition stated above. 2) Theorem 6.6 should be considered as a statement on variation of Hodge structure. Namely, if a representation restricted to one fibre comes from a geometric situation then the whole representation comes from a geometric situation.

Proof: Since $\rho_b \in M(Z_b, Sp(g, \mathbb{R}))$ lies in a finite orbit of the $\pi_1(B, b)$-action, after taking a finite etale cover of $B$ we may assume $\rho_b$ is fixed by the $\pi_1(B, b)$-action. Since $\rho_b$ comes from a $Z$-VHS of weight $2$, by Theorem 6.5 and Theorem 6.6 $\rho_b$ lifts to a representation $\rho : \pi_1(Z) \to Sp(g, \mathbb{R})$ and comes from a $Z$-VHS of weight $2$. (Here we may assume $\rho_b$ is Zariski dense. In general, we shall work on the Zariski closure $\rho_b(\pi_1(Z_b)) \subset Sp(g, \mathbb{R})$.) The period map $u_\rho$ corresponding to $\rho$ descends to a morphism $\phi : Z \to \mathcal{M}_g$, which extends $\phi_b : Z_b \to \mathcal{M}_g$ induced by $f_b : A_b \to Z_b$. After a base changing $Z' \to Z$ $\phi$ induces a family $f : A \to Z'$, which extends $f'_b : A'_b \to Z'_b$. This proves Theorem 6.6.

Theorem 6.7 Let $G$ be an algebraic group of rank $\geq 2$ that is not of Hodge type. Then any lattice $\Gamma \subset G$ can’t be the fundamental group of a quasi-compact Kähler manifold.

This theorem generalizes Simpson’s Theorem in the compact case. For example, $SL(n, \mathbb{Z}) (n \geq 3)$ can’t be the fundamental group of a quasi-compact Kähler manifold.

Proof of Theorem 6.7: Suppose there is a quasi-compact Kähler manifold $X$ such that $\pi_1(X) = \Gamma \subset G$. Let $\rho : \pi_1(X) \to G$ denote the induced representation. Since $\rho$ is Zariski-dense in $G$ where $G$ is almost simple, $\rho$ is, in particular, semi-simple. By Theorem 2.1 there exists a harmonic metric $h$ on the flat bundle $V$ with trivial filtrations on the local system. Let $\{(E, \theta)_{\alpha}\}_{-\infty < \alpha < \infty}$ denote the filtered Higgs sheaf corresponding to the harmonic bundle $(V, h)$. Now note that for each $t \in \mathbb{C}^*$ the filtered Higgs sheaf $\{(E, t\theta)_{\alpha}\}_{-\infty < \alpha < \infty}$, $t \in \mathbb{C}^*$ is again parabolic poly-stable,
by constructing the Hermitian-Yang-Mills metric for \( \{(E, \theta)\}_\alpha \) in Theorem 5.1 we obtain harmonic bundles \((V_t, h_t)\) on \( X \). As \( t \to 0 \), by Theorem 5.1, we can see that the harmonic bundles \((V_t, h_t)\) converge to a harmonic bundle \((V_0, h_0)\). Let \( \{(E_0, \theta_0)\}_\alpha \) denote the corresponding filtered Higgs sheaf.

Note that the continuous \( \mathbb{C}^* \) action on the representation space \( M(X, G) \) via Higgs sheaves contains the identity and all the deformations of \( \rho : \Gamma \to G \) are obtained by conjugations by Weil’s local rigidity theorem, \( \rho_t \simeq \rho_1 = \rho, \forall t \in \mathbb{C}^* \). This implies that \( \rho_0 \simeq \rho \). Since the limiting point \( \lim_{t \to 0} (V_t, h_t) \) is unique, \( \{(E_0, \theta_0)\}_\alpha \) is fixed by the \( \mathbb{C}^* \) action. Again by Simpson’s Theorem \((E_{0, \alpha}, \theta_{0, \alpha})\) is decomposed into a system of Higgs bundles \( E_{0, \alpha} = \oplus E_{0, \alpha}^{pq} \) with \( \theta_{0, \alpha} : E_{0, \alpha}^{pq} \to E_{0, \alpha}^{pq-1,q+1} \Omega^1(X) \), i.e. \( \rho \) comes from a variation of Hodge structures. Thus the Zariski closure \( \overline{\rho} = G \) is of Hodge type. But, this is a contradiction to the assumption.

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