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by

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1 Introduction

On Kähler manifolds, there exist deep relationships between holomorphic and harmonic objects, between algebraic data and analytic variational principles. For example, through the work of Narasimhan-Seshadri [NS], Donaldson [D1, D2, D3] and Uhlenbeck-Yau [UY] we know that an irreducible holomorphic bundle on a compact Kähler manifold X possesses a Hermite-Einstein (H-E) metric if and only if it satisfies the algebraic condition of stability. As such a H-E metric is flat if the first and second Chern numbers of the bundle vanish, this yields a possibility to construct flat connections, i.e. linear representations of the fundamental group $\pi_1(x)$. In the opposite direction, Hitchin [H] saw how to construct a holomorphic bundle E with a so-called Higgs field θ on a compact Riemann surface X from such a representation $\rho : \pi_1(x) \to \mathrm{GL}(n, \mathbb{C})$ via the associated equivariant harmonic map $h : \tilde{X} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$ on the universal cover $\tilde{X}$. Simpson [S1, S3] extended Hitchin's theory to compact Kähler manifolds and established an important connection with complex variations of Hodge structures.

For many reasons, however, the theory becomes fully satisfactory only when one can also include quasi-compact Kähler manifolds, the analogues of quasi-projective manifolds, the smooth complements of the singularities of algebraic varieties. Thus, let $X = \overline{X} \setminus D$ where $\overline{X}$ is a compact Kähler manifold with Kähler form ω_0 and D a divisor in $\overline{X}$ with normal crossings as its only possible singularities. In that case, for the bundles on X under consideration, one needs some condition as one approaches the compactifying divisor D, the so-called parabolic structures as introduced by Mehta-Seshadri [MS], Maruyama-Yokogawa [MY], Biquard [B, B1, B2], Li-Narasimhan [LN,

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LN1]. The general notion is in [L], and Li shows that the existence of the H-E metrics on the bundle over a quasi-compact Kähler manifold is essentially equivalent to the stability of the parabolic structure.

As the present paper will also deal with the quasi-compact case, let us beriefly describe the previous results in order to put our work into its proper perspective.

Simpson [S2] (see also [LW, LW1]) studied Higgs bundles over punctured Riemann surfaces and established the relations between flat connections, representations of the fundamental group and complex variations of Hodge structures analogously to the compact case.

We now turn to quasi-compact Kähler manifolds of arbitrary dimension. Suppose that $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ is an irreducible representation. Jost-Zuo [JZ,JZ1] construct a ρ-equivariant harmonic map from $\tilde{X}$ to $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$, where $\tilde{X}$ is the universal covering of X. This result is the starting point of the present paper. We show that the harmonic map introduces a parabolic Higgs bundle (E, θ) with logarithmic poles. Let ω be a complete Poincaré like Kähler metric (cf. [CG], Section3). We prove that the curvature of the metric is bounded with respect to ω. We consider the $\mathbb{C}^*$ action on the Higgs bundle $(E, \lambda\theta)$, and by solving the Hermitian-Yang-Mills equation, we get a flat connection D_λ on $(E, \lambda\theta)$. We prove that, as $\lambda \to 0$, the connection D_λ converges to a new flat connection D_0. We apply the existence and the compactness to non-abelian Hodge theory. In particular, we prove a non-abelian Hodge (p, q)-type theorem, which generalises the results due to Jost-Zuo [JZ2] and Katzarkov-Pantev [KP] in the compact case. We also prove that any discrete subgroup of a Lie group of rank ≥ 2 that is not of Hodge type can't be the fundamental group of a quasi-compact Kähler manifold. For example, $\mathrm{SL}(n, \mathbb{Z})$ $(n \geq 3)$ can't be the fundamental group of a quasi-compact Kähler manifold. This result generalises Simpson's theorem in the compact case.

We would like to point out that the analytic ideas in proving the existence of a H-E metric on the Higgs bundle $(E, \lambda\theta)$ are applicable in greater generality, as soon as one knows how to construct an initial metric with bounded pseudo curvature with respect to the complete metric ω.

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2 Construction of pluriharmonic maps

We recall the ideas in [JZ, JZ1] to construct a pluriharmonic map and derive an estimation for its energy density. We recall some notations in [JZ, JZ1]. Let $\overline{X}$ be a compact Kähler manifold, D a divisor in $\overline{X}$ with normal crossings. We write $D = \sum_{i=1}^m D_i$ where the irreducible components D_i of D are smooth and meet transversely.

For each $i = 1, \cdots, m$, we choose a metric on the line bundle $[D_i]$ defined by the divisor D_i. Let σ_i be the canonical section of D_i which vanishes on D_i. We may assume that its length $|\sigma_i| < 1$. We put $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_m$, which is a section of $[D]$, then $|\sigma| = \Pi_i |\sigma_i| < 1$.

One can introduce (cf. [CG], Section3) a complete Poincaré like Kähler metric

$$\omega = \sqrt{-1} \sum_{i=1}^{m} \partial\overline{\partial} \log \log |\sigma_i|^{-2} + C\omega_0$$

on $X = \overline{X} \setminus D$, where ω_0 is a Kähler metric on $\overline{X}$. Set

$$\omega = \frac{\sqrt{-1}}{2} \gamma_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta.$$

We define $\Delta_\delta = \{ \ x \in X \mid |\sigma| = \delta \ \}$ and $X_\delta = \{ \ x \in X \mid |\sigma| > \delta \ \}$. So, we have $\partial X_\delta = \Delta_\delta$. Set $\delta_\delta^i = \{ \ x \in X \mid |\sigma_i| = \delta \ \}$ for $i = 1, \cdots, m$. It is clear that Δ_δ can be considered as the boundary of a family of disks, and at the singular points $D_i \cdot D_j$, it can be considered as a torus.

Let N be a simply connected complete Riemannian manifold with non-positive sectional curvature, with isometry group $G := I(N)$. Let $\rho : \pi_1(X) \to G$ be a homomorphism from the fundamental group of X into the isometry group of N. We say that ρ is reductive ([JZ, Definition 1.1]) if there exists a complete totally geodesic subspace Z of N stabilized by $\rho(\pi_1(X))$ with the property that, for every totally geodesic subspace Z' of Z which has no Euclidean factor, $\rho(\pi_1(X))$ does not fix any point in the sphere at infinity of Z'. Let $d(\cdot, \cdot)$ be the distance function on N.

We divide the smooth irreducible components D_i of D into two sets,

$$D^1 = \{ \ D_i \mid \text{for every small loop } \gamma \text{ around } D_i \ , \inf_{y \in Y} d(y, \rho(\gamma)y) = 0 \ \}$$

$$D^2 = \{ \ D_i \mid \text{for every small loop } \gamma \text{ around } D_i \ , \inf_{y \in Y} d(y, \rho(\gamma)y) > 0 \ \}.$$

It is clear that $D = D^1 + D^2$ since the local momodromy groups have only one generator.

We say that ρ is stabilizing near D^1 (Definition 1 in [JZ1]). We say that $g \in G$ is a hyperbolic element if $\inf_{y \in Y} d(y, gy) =: \lambda_g > 0$ (Definition 1.2 in [JZ]). So, for every loop γ around $D_i \in D^2$, $\rho(\gamma)$ is a hyperbolic element.

Near the divisors in D^1, we can construct a ρ-equivariant map with finite energy (cf. [JZ1]). For completeness, we give the details here.

Let $N(\infty)$ be the sphere at infinity of N. For any $x \in$, we set

$$G_x := \{ \ g \in G \mid gx = x \ \}.$$

We say that a subgroup P of G is parabolic if there exists $x \in N(\infty)$ with $P \subset G_x$. For every $x \in N(\infty)$, we can define a horospherical flow $\phi_t = \phi_{tx} : N \to N$ as follows (see [JZ, Section 2]):

For $p \in N$, we let γ_{px} be the unique geodesic from p to x and put

$$\phi_t(p) := \gamma_{px}(t).$$

It is proved in [JZ1] (Lemma 1) that, for any $x \in N(\infty)$, $\phi_{t,x}$ commutes with G_x and $d(\phi_{t_1}(p), \phi_{t_1}(q)) \leq d(\phi_{t_2}(p), \phi_{t_2}(q))$, for any $t_2 < t_1$, $p, q \in N$.

Let $k : \tilde{X} \to N$ be a smooth ρ-equivariant map, where $\tilde{X}$ is the universal covering of X. Let

$$\Sigma_\delta := \{ \ z \in X \ | \ \min |\sigma_i| = \delta, \ D_i \in D^1 \ \}$$

and

$$\Sigma_\delta^l := \{ \ z \in \Sigma_d \mid \text{number } \{j \in \{1, \cdots, m\} \text{ with } |\sigma_j| = \delta, \ D_j \in D^1\} = l \ \}.$$

For sufficiently small $\delta > 0$, each component of

$$U_\delta \ : \ = \cup \{ \Sigma_\eta \mid 0 < \eta < \delta \}$$

is a neighbourhood of a component of D in $\overline{X}$; $\partial U_\delta = \Sigma_\delta$. We will deform the map k into a finite energy map on U_δ. It suffices to treat one end. In the sequel we will consider k as a map from X into the quotient of N by $\rho(\pi_1(X))$. We first construct a map on Σ_δ^1. We assume that Σ_δ^1 is locally given by $|z^1|^2 = \delta$ and $\Sigma_\delta^1 = \Omega^1 \times S^1$ for a certain region Ω^1 in $z^1 = 0$. We let $\pi \ : \ \Omega^1 \times S^1 \to \Omega^1$ be the projection, and $s \ : \ \Omega^1 \to \Omega^1 \times S^1$ be a smooth section. Each S^1-fiber then is parametrized by $\theta \in [0, 2\pi)$ proportional to arclength with its intersection with $s(\Omega^1)$ as initial point and the orientation for which U_δ is to the left. By assumption $\rho(g)$ is contained in some parabolic group G_x, $x \in N(\infty)$. We let $\phi_t = \phi_{t,x}$ be the associated horospherical flow. For each $w_0 \in \Omega^1$, we choose a lift z_0 in X of $s(w_0)$ and map the lift of the S^1-fiber with initial point z_0 onto the geodesic arc from $k(z_0)$ to $\rho(g)k(z_0)$ with a parametrization proportion to arclength. We thus obtain a map h from the lift of Σ_δ^1 to $\tilde{X}$ into N. On the punctured disk D^* bounded by an S^1-fiber, we introduce polar coordinates $0 < r \leq 1$, $0 \leq \theta < 2\pi$. The lift $\tilde{D^*}$ of this punctured disk in $\tilde{X}$ either again is a punctured disk, or it is a strip with coordinates $0 < r \leq 1$, $\theta \in C$; in the latter case, the intersection with a fundamental domain of X again may be assumed to have coordinates $0 < r \leq 1$, $0 \leq \theta < 2\pi$. We extend the above boundary map h to this lift by defining

$$h^*(r, \theta) \ : \ = \phi_{\alpha \log \log \frac{2}{r}}(h(1, \theta)),$$

Near the divisors in D^2, we can do as in [JZ]. We recall the details.

Each hyperbolic element in N can be represented by translation along some geodesic, in the quotient $N_0 = N/\rho(\pi_1(X))$, this yields a closed geodesic c. We then map each circle in Σ_δ^i onto the corresponding closed geodesic proportionally to arclength. This can be done continuously, because the two elements of $\rho(\pi_1(X))$ corresponding to the circles around D_i and D_j near $D_i \cdot D_j$ commute. For each

$i = 1, \cdots, m$ we set $S_\delta^i = \cup_{w \in D_i} B_{w,\delta}$, where every $B_{w,\delta}$ can be holomorphically identified with $\{ z \in C \mid |z| \leq \delta \}$, and $z = 0$ corresponding to $w \in D_i$. We also identify $X \cap B_{w,\delta} =: B^*_{w,\delta}$ with $B^*_{w,\delta} = \{ z \in C \mid 0 < |z| \leq \delta \}$ or with the semi-infinite cylinder $A_\delta = [\log \delta, \infty) \times S^1$. The above map from Σ_δ^i to N induces a map from $\partial A_1 = \{0\} \times S^1$ to N, mapping ∂A_1 proportionally to arclength onto a closed geodesic γ in N. We denote the map by $\overline{g}_\gamma(\theta)$ $(\theta \in S^1)$. We extend $\overline{g}_\gamma(\theta)$ to all of A_1 by putting

$$g_\gamma(s, \theta) = \overline{g}_\gamma(\theta).$$

Performing this construction on all punctured disks $B^*_{w,\delta}$, we obtain a continuous map v_1 from Σ_1 to N which is harmonic on each such punctured disk. We extend v_1 continuously to $v \ : \ X \to N$ in the required homotopy class.

We use the notations in [JZ]:

$$S_\delta'^i = S_1^i \setminus S_\delta^i, \quad S_\delta'' = X \setminus \cup_{i=1}^m S_\delta^i, \quad S_\delta' = \cup_{i=1}^m S_\delta'^i.$$

We set

$$v_\delta = v_1|_{\Sigma_\delta},$$

and consider the harmonic map

$$u_\delta \ : \ X_\delta \to N$$

with boundary values

$$u_\delta|_{\Sigma_\delta} = v_\delta|_{\Sigma_\delta}$$

and inducing the homomorphism $\rho \ : \ \pi_1(X) \to G$.

It is proved in [JZ] that $u_\delta \to u$ as $\delta \to 0$ and u is a harmonic map. We shall recall some details and derive an estimate for the energy density of u, which will be important in our subsequent applications.

For any point $p \in D$, we choose a neighbourhood U_p of p. Assume that $(z_1, \cdots, z_n)$ is a coordinate system in U_p such that $U_p \cap D = \{z_1 \cdots z_j = 0\}$. Then we can see that, in $U_p \setminus D$, ω is quasi isometric to

$$\sum_{i=1}^{j} |\sigma_i|^{-2} \log^{-2} |\sigma_i| dz_i \wedge d\overline{z}_i + \sum_{i=j+1}^{n} dz_i \wedge d\overline{z}_i.$$

For simplicity of notation, we assume that $\dim_C \overline{X} = 2$.

Let $\gamma = \det(\gamma_{\alpha\overline{\beta}})$, we have

$$\begin{aligned} E(u_\delta) &= \int_{S_\delta''} | \bigtriangledown u_\delta|^2 dV \\ &= \int_{S_\delta''} \gamma^{1\overline{1}}(z_1, z_2) \frac{\partial u_\delta}{\partial z_1} \frac{\partial u_\delta}{\partial \overline{z}_1} \gamma(z_1, z_2) dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \end{aligned}$$

$$
\begin{aligned}
& +\int_{S''_\delta} \gamma^{2\overline{2}}(z_1,z_2)\frac{\partial u_\delta}{\partial z_2}\frac{\partial u_\delta}{\partial \overline{z}_2}\gamma(z_1,z_2)dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +\int_{S''_\delta} \gamma^{1\overline{2}}(z_1,z_2)\frac{\partial u_\delta}{\partial z_1}\frac{\partial u_\delta}{\partial \overline{z}_2}\gamma(z_1,z_2)dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +\int_{S''_\delta} \gamma^{2\overline{1}}(z_1,z_2)\frac{\partial u_\delta}{\partial z_2}\frac{\partial u_\delta}{\partial \overline{z}_1}\gamma(z_1,z_2)dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
\geq\ & C\int_{S'_\delta} \gamma_{2\overline{2}}(0,z_2)\frac{1}{|z_2|^2(\log|z_2|^2)^2}\frac{\partial u_\delta}{\partial z_1}\frac{\partial u_\delta}{\partial \overline{z}_1}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +C\int_{S'_\delta} \gamma_{1\overline{1}}(z_1,0)\frac{1}{|z_1|^2(\log|z_1|^2)^2}\frac{\partial u_\delta}{\partial z_2}\frac{\partial u_\delta}{\partial \overline{z}_2}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +\int_{S'_\delta} O(|z_1|)\frac{\partial u_\delta}{\partial z_1}\frac{\partial u_\delta}{\partial \overline{z}_1}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +\int_{S'_\delta} O(|z_2|)\frac{\partial u_\delta}{\partial z_2}\frac{\partial u_\delta}{\partial \overline{z}_2}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
\geq\ & (-\log\delta)\int_{D_2} l^2\frac{\gamma_{2\overline{2}}(0,z_2)}{|z_2|^2(\log|z_2|^2)^2}dz_2\wedge d\overline{z}_2 \\
& +(-\log\delta)\int_{D_1} l^2\frac{\gamma_{1\overline{1}}(z_1,0)}{|z_1|^2(\log|z_1|^2)^2}dz_1\wedge d\overline{z}_1 \\
& +\int_{S'_\delta} O(|z_1|)\frac{\partial u_\delta}{\partial z_1}\frac{\partial u_\delta}{\partial \overline{z}_1}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2 \\
& +\int_{S'_\delta} O(|z_2|)\frac{\partial u_\delta}{\partial z_2}\frac{\partial u_\delta}{\partial \overline{z}_2}dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2
\end{aligned}
$$

where l is the length of the image geodesic c.

On the other hand, we have

$$
\begin{aligned}
E(v|_{S''_\delta}) &= E(v|_{S'_\delta}) + E(v|_{X\setminus S'_\delta}) \\
&\leq E(v|_{S'_\delta}) + C \\
&= (-\log\delta)\int_{D_2} l^2\frac{\gamma_{2\overline{2}}(0,z_2)}{|z_2|^2(\log|z_2|^2)^2}dz_2\wedge d\overline{z}_2 \\
&\quad +(-\log\delta)\int_{D_1} l^2\frac{\gamma_{1\overline{1}}(z_1,0)}{|z_1|^2(\log|z_1|^2)^2}dz_1\wedge d\overline{z}_1 + C. \qquad (1)
\end{aligned}
$$

Since

$$E(u_\delta) \leq E(v|_{S''_\delta}),$$

we get

$$E(v|_{S''_\delta}) - C \leq E(u_\delta) \leq E(v|_{S''_\delta}). \qquad (2)$$

If $0 < \delta < \eta < 1$, we have

$$E(v|_{S''_\delta}) = E(v|_{S''_\eta}) + E(v|_{S'_\delta \setminus S'_\eta}) \tag{3}$$

and

$$E(u_\delta|_{S''_\delta}) = E(u_\eta|_{S''_\eta}) + E(u_\delta|_{S'_\delta \setminus S'_\eta}). \tag{4}$$

Using an argument similar to the one used in obtaining (1), one gets

$$E(v|_{S'_\delta \setminus S'_\eta}) \leq E(u_\delta|_{S'_\delta \setminus S'_\eta}) + C \tag{5}$$

By (3), (4) and (5), we have

$$E(u_\eta|_{S''_\eta}) \leq E(v|_{S''_\eta}) + C$$

where C is a positive constant independent of δ (may depend on η).

By a standard argument (cf. [JZ, 1.4]), one can show that, if ρ is reductive, u_δ sub-converges to a harmonic map u in $C^2(\Omega, N)$ for any $\Omega \subset\subset X$. Since the sectional curvature of N is non-positive, we have

$$\triangle |du|^2_\omega \geq -k|du|^2_\omega$$

where $\triangle$ is the Laplacian with respect to the metric ω, and $|du|^2_\omega$ is the energy density of u with respect to the metric ω. By the sub-mean value inequality and (1), we obtain

$$|du|^2_\omega \leq C(-\log|\sigma|)^2.$$

Let $d_\rho(x) := d_\rho(u, v)$, where $d_\rho(u, v)$ is the ρ-equivariant distance between u and v. Using the estimate for the energy density, we obtain an estimate for d_ρ near the divisor, namely

$$d_\rho \leq C(-\log|\sigma|).$$

One can show that u is in fact pluriharmonic by an argument in the proof of Lemma 1.1 in [JZ]. So we proved the following theorem.

Theorem 2.1 *Let $N = GL(n, \mathbb{C})/U(n)$. Let $\widetilde{X}$ be the universal covering of X. Assume that ρ is reductive. Then there is a ρ-equivariant pluriharmonic map*

$$u \; : \; \widetilde{X} \to Y$$

with

$$|du|_\omega \leq C(-\log|\sigma|). \tag{6}$$

The matrix u in $GL(n, \mathbb{C})/U(n)$ is $O((-\log|\sigma|)^\beta)$, for some $\beta \in \mathbb{R}$.

3 Estimation of the curvature

Let V be a $\mathrm{GL}(n,\mathbb{C})$ bundle with flat connection D. Introducing a metric g leads to the decomposition

$$D = D_g + \theta$$

where D_g preserves the metric. Let $\rho \ : \ \pi_1(X) \to \mathrm{GL}(n,\mathbb{C})$ be the representation defined by the flat bundle V. A metric on V can be considered as a ρ-equivariant map

$$g \ : \ X \to \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n).$$

We have

$$dg = \theta,$$

and h is harmonic if and only if

$$D_g^* \theta = 0,$$

where D_g^* is the adjoint operator of D_g with respect to the metric g. In this case, we say that the metric g is harmonic. We then decompose into types:

$$D = D' + D''$$

$$\theta = \theta^{1,0} + \theta^{0,1}.$$

We have $(D'')^2 = 0$, and hence $E := (V, D'')$ is a holomorphic bundle. Thus $(E, \theta^{1,0})$ is a Higgs bundle. A Higgs bundle $(E, \theta^{1,0})$ on X consists of a holomorphic bundle E together with

$$\theta^{1,0} \ : \ E \to \Omega^{1,0}(X) \otimes E$$

satisfying the integrability condition

$$\theta^{1,0} \wedge \theta^{1,0} = 0.$$

Let $(E, \theta^{1,0})$ be a Higgs bundle on X. The analogue of the $\overline{\partial}$-operator on E now is

$$D^2 = \overline{\partial} + \theta^{1,0}.$$

A metric on E then defines

$$D^1 = \partial + \theta^{0,1}.$$

Theorem 3.1 *Let u be the pluriharmonic map obtained in Theorem 2.1. Let $\theta = du$. Let F_g be the curvature of the metric connection $D_g = \partial + \overline{\partial}$. We have $|F_g|_\omega \leq C$, where $|\cdot|_\omega$ is the norm with respect to the metric ω.*

The purpose of this section is to prove the theorem. The theorem can be seen as a generalization of Theorem 1 in [S2] to the higher dimensional case.

Proof: For simplicity of notation, we prove the result in the complex surface case, as the proof for the higher dimensional case is exactly the same. It suffices [JZ, 1.6] to prove the estimate near the point $D_1 \cdot D_2$. We choose local coordinates so that $X = B_1^* \times B_1^*$ where $B_1^* = \{\ z \in C \mid 0 < |z| < 1\ \}$. Assume that

$$\theta^{1,0} = \theta_1 dz_1 + \theta_2 dz_2.$$

Since $\theta^{1,0} \wedge \theta^{1,0} = 0$, we have

$$\theta_1\theta_2 = \theta_2\theta_1.$$

Let λ_j be the eigenvalues of θ_1. It is clear that λ_j is a (multivalued) holomorphic function. By Theorem 2.1, we have

$$|\lambda_j| \leq \frac{C}{|z_1|}.$$

As in [S2] (Section 2), we may divide up the eigenvalues into groups $\lambda_1^i, \cdots, \lambda_{l_i}^i$ with the property that

$$|\lambda_j^i - a^i \frac{dz_1}{z_1}| \leq \frac{C}{|z_1|^{1-\epsilon}}, \tag{7}$$

for any $0 < \epsilon < \frac{1}{rankE}$, where a^i $(i = 1, \cdots)$ are complex numbers. We decompose $E = \oplus E_i^1$ holomorphically so that E_i^1 is the sum of the generalized eigenspaces for the eigenvalues λ_j^i. We define a new endomorphism valued one form ϕ_1, semi-simple with eigenvalues $a^i \frac{dz_1}{z_1}$, and eigenspaces E_i^1. The subspaces E_i^1 are preserved by ϕ_1 and θ_1. Similarly, we can also divide up the eigenvalues of θ_2 into groups $\nu_1^i, \cdots, \nu_{k_i}^i$ with the property that

$$|\nu_j^i - b^i \frac{dz_2}{z_2}| \leq \frac{C}{|z_2|^{1-\epsilon}}, \tag{8}$$

where b^i $(i = 1, \cdots)$ are also complex numbers. We can also decompose $E = \oplus E_i^2$ holomorphically and define an endomorphism-valued one form ϕ_2, semi-simple with eigenvalues $b^i \frac{dz_2}{z_2}$, and eigenspaces E_i^2. The spaces E_i^2 are preserved by ϕ_2 and θ_2.

Because $\theta_1\theta_2 = \theta_2\theta_1$, we can have a holomorphic decomposition $E = \oplus E_i$ so that E_i are preserved by ϕ_1, ϕ_2, θ_1, and θ_2, which is a refined decomposition of both $E = \oplus E_i^1$ and $E = \oplus E_i^2$. Choose a basis $\{e_k\}$ of E, orthonormal with respect to the harmonic metric, compatible with the decomposition. We write everything in terms of this basis. We have

$$\theta_i = \beta_i + \tau_i,$$

$$\phi_i = \alpha_i + q_i$$

for $i = 1, 2$, where β_i and α_i are diagonal matrices of one forms, and τ_i, q_i are upper triangular matrices of one forms.

It is obvious that

$$\beta_i = \alpha_i + \gamma_i$$

with

$$|\alpha_i| \leq \frac{C}{|z_i|}, \quad |\gamma_i| \leq \frac{C}{|z_i|^{1-\epsilon}}.$$

Let

$$M^+ = \oplus_{i>j} Hom(E_i, E_j) \subset End(E),$$

$$M^0 = \oplus_i End(E_i) \subset End(E),$$

$$M^- = \oplus_{i<j} Hom(E_i, E_j) \subset End(E).$$

One can show (cf. [S2, Section 2]) that $q_i \in M^+$. We write $\tau_i = \tau_i^0 + \tau_i^+$ with $\tau_i^+ \in M^+$, τ_i^0 upper triangular in M_i^0. Note that the spaces M^0 and M^+ are single valued, so τ_i^+ and $\gamma_i + \tau^0$ are single valued.

Theorem 3.1 follows from the following lemma.

Lemma 3.2 *We have*

$$|\tau_i^0| \leq \frac{C}{|z_i|(-\log|z_i|)},$$

$$|\tau_i^+| \leq C|z_i|^{-1+\epsilon} \quad \text{and} \quad |q_i| \leq C|z_i|^{-1+\epsilon},$$

for $i = 1, 2$, where $|\cdot|$ is with respect to the harmonic metric on the bundle and the restriction metric ω_0 on X.

Proof: It suffices to derive the estimates in local coordinates around $D_1 \cdot D_2$. We use $B_1^* \times B_1^*$ as the coordinates as before. Because ω_0 is equivalent to the Euclidean metric in $B_1^* \times B_1^*$, we assume that it is the Euclidean metric. We prove the estimates for $i = 1$, the proof for $i = 2$ is the same. In the sequel, we will omit the subscript for simplification.

It is proved in [S2, P.729] that

$$\triangle \log(|\beta|^2 + |\tau|^2) \geq C \frac{|\tau|^4}{|\beta|^2 + |\tau|^2},$$

where $\triangle$ is the Laplace operator with respect to the Euclidean metric ω_0.

We shall modify Simpson's proof [S2] to our case.

We first show that $|\tau| \leq \frac{C}{|z_1|}$. Set $h = |\beta|^2 + |\tau|^2$, we shall show that $h \leq \frac{C}{|z_1|}$.

Set $m_\epsilon = \frac{C_2}{(|z_1|-\epsilon)^2}$ for $\epsilon > 0$, we have

$$\triangle \log m_\epsilon \leq \frac{2}{C_2} m_\epsilon.$$

If at an interior point $h > m_\epsilon$, we have $h \geq \frac{C_0}{|z_1|^2}$ and

$$\triangle \log h \geq c_1 h.$$

Choosing C_2 large enough so that $C_2 \geq C_0$ and $\frac{2}{C_2} \leq c_1$, we obtain

$$\triangle \log(h/m_\epsilon) \geq c_1(h - m_\epsilon) > 0.$$

Therefore h/m_ϵ can't achieve a maximum value greater than 1 in the interior. Since $h/m_\epsilon = 0$ on the boundary $|z_1| = \epsilon$ and h is bounded on the boundary $|z_1| = 1$, we have

$$h \leq C_3 m_\epsilon.$$

Letting $\epsilon \to 0$ yields the claim.

We then show that

$$|\tau| \leq \frac{C}{|z_1|(-\log|z_1|)}.$$

Let $a = \sqrt{\sum_i |a^i|^2}$. We may assume that $a \neq 0$, otherwise, the claim clearly holds. Set

$$k = \log(\frac{|z_1|^2}{a}(|\sigma|^2 + |\tau|^2)).$$

Since $|\beta|^2 = \frac{a}{|z_1|^2} + b$ with $b \leq \frac{C}{|z_1|^{2-2\epsilon}}$, we have

$$k = \log(1 + \frac{|z_1|^2 b}{a} + |z_1|^2|\tau|^2\frac{1}{a}),$$

and thus

$$c(|z_1|^2 b + |z_1|^2|\tau|^2) \leq k \leq C(|z_1|^2 b + |z_1|^2|\tau|^2).$$

It suffices to show that

$$k \leq \frac{C}{(-\log|z_1|)^2}.$$

Set

$$p_\epsilon(z_1, z_2) = \frac{C_2}{(-\log|z_1|)^2} + \epsilon(-\log|z_1|).$$

Calculating directly, one gets

$$\triangle p_\epsilon \leq -\frac{6p_\epsilon^2}{C_2|z_1|^2}.$$

It is clear that, at the point where $k \geq \frac{C_0}{(-\log|z_1|)^2}$, we have

$$\triangle k \geq c_1\frac{k^2}{|z_1|^2}.$$

So, applying the maximum principle again, we get

$$k \leq p_\epsilon.$$

This proves the claim.

It is proved in [S2, p. 731] that

$$\tau^+ = (1 + f)q$$

where f is an operator with norm bounded by $\frac{C}{(-\log|z_1|)}$, and $|q| \leq \frac{C}{|z_1|(-\log|z_1|)}$. We have [S2, p. 731-732]

$$\triangle \log|\phi|^2 \geq c\frac{|[\phi,\overline{\theta}]|^2}{|\phi|^2},$$

and

$$|[\phi,\overline{\theta}]|^2 \geq \frac{|q|^2}{|z_1|^2},$$

so

$$\triangle \log|\phi|^2 \geq c|q|^2.$$

Since

$$|\phi|^2 = |\alpha|^2 + |q|^2 = \frac{a}{|z_1|^2}(1+\frac{|z_1|^2|q|^2}{a}),$$

we have

$$\triangle \log(1+\frac{|z_1|^2|q|^2}{a}) \geq c|q|^2.$$

Let $k = \log(1+\frac{|z_1|^2|q|^2}{a})$, we have

$$c|z_1|^2|q|^2 \leq k \leq C|z_1|^2|q|^2,$$

and

$$\triangle k \geq -c\frac{k}{|z_1|^2}.$$

Set

$$p_\epsilon = |z_1|^\nu + \epsilon(-\log|z_1|),$$

we have

$$\triangle p_\epsilon \leq c\frac{p_\epsilon}{|z_1|^2}, \quad \text{if} \quad \nu^2 < c.$$

Then the maximum principle yields that $k \leq C|z_1|^\nu$. So,

$$|q|^2 \leq \frac{C}{|z_1|^{2-\nu}}$$

and consequently,

$$|\tau^+| \leq C(1+\frac{1}{(-\log|z_1|)})\frac{1}{|z_1|^{2-\nu}} \leq C\frac{1}{|z_1|^{2-\epsilon}},$$

for any $0 < \epsilon < \nu$. This proves the lemma.

4 Filtrations and Chern numbers

In this section, for simplicity we again assume that X is of complex dimension 2. In the higher dimensional case, the results and the proofs are the same. Recall that E is a holomorphic vector bundle over X, $\theta := \theta^{1,0}$ is a Higgs 1-form on X, (E, θ) is a Higgs bundle. Suppose that

$$D = D_g + \theta + \overline{\theta}$$

where $\overline{\theta}$ is defined by $(\theta u, v)_g = (u, \overline{\theta} v)_g$. By Theorem 3.1 we know that F_{D_g}, the curvature of D_g satisfies $|F_{D_g}|_\omega \leq C$. Applying the extension theorem [CG, Theorem 1], we know that E can be extended to $\overline{E}$ across the divisor D as a coherent sheaf.

Theorem 4.1 *Suppose that s is a flat section of E with respect to the connection D. There exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ $(i = 1, 2)$ such that*

$$c|\sigma_1|^{\alpha_1}|\sigma_2|^{\beta_1} \leq |s|_g \leq C|\sigma_1|^{\alpha_2}|\sigma_2|^{\beta_2}.$$

Proof: We have

$$d\langle s, s\rangle_g = 2\langle D_g s, s\rangle_g = -4\langle \theta s, s\rangle_g.$$

So,

$$|d\langle s, s\rangle_g| \leq \frac{C_1}{|\sigma_1|}\langle s, s\rangle_g |dz_1| + \frac{C_2}{|\sigma_2|}\langle s, s\rangle_g |dz_2|.$$

The theorem obviously follows from the above inequality.

Because of Theorem 3.1, we can define

$$E^{\alpha,\beta} = \{\ s \text{ is a flat section of } E \text{ near } D \mid |s|_g \leq |\sigma_1|^{\alpha-\epsilon}|\sigma_2|^{\beta-\epsilon} \text{ for any } \epsilon > 0\ \}.$$

It is clear that $E^{\alpha,\beta} \subset E^{\alpha',\beta'}$ if $\alpha \geq \alpha'$, $\beta \geq \beta'$, $E^{\alpha,\beta} = \cap_{\gamma<\alpha,\delta<\beta} E^{\gamma,\delta}$, $E^{\alpha+1,\beta} = \sigma_1 E^{\alpha,\beta}$, $E^{\alpha,\beta+1} = \sigma_2 E^{\alpha,\beta}$, and

$$j_* E = \cup_{\alpha\in R,\beta\in\mathbb{R}} E^{\alpha,\beta} \tag{9}$$

where σ_i (i=1,2) is the canonical section of the divisor D_i and $j\ :\ X \to \overline{X} = X \cup D$ is the inclusion.

Finally, we have

$$\theta\ :\ E^{\alpha,\beta} \to E^{\alpha,\beta} \otimes \Omega^1_{\overline{X}}(\log D)$$

where $\Omega^1_{\overline{X}}(\log D)$ is the sheaf of logarithmic differentials at D. The identity (9) gives a filtered regular Higgs bundle.

For the uniqueness of the harmonic metrics, we have

Theorem 4.2 *Suppose that u_1 and u_2 are two ρ-equivariant harmonic maps, that $\theta_1 = du_1$ and $\theta_2 = du_2$ induce the same filtration $j_* E = \cup_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} E^{\alpha,\beta}$, and that $Res_D(\theta_1, E^{\alpha,\beta}) = Res_D(\theta_2, E^{\alpha,\beta})$ for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then $\exists g \in U(n)$ such that $u_1 = g u_2$.*

Proof: Let $f := d_\rho(u_1, u_2)$, where $d_\rho(u_1, u_2)$ is the ρ-equivariant distance between u_1 and u_2. It suffices to show that $d_\rho(u_1, u_2) \equiv Constant$. Since u_1 and u_2 are ρ-equivariant harmonic maps, we have

$$\triangle f \geq 0. \tag{10}$$

Since $\theta_1 = du_1$ and $\theta_2 = du_2$ induce the same filtration and

$$Res_D(\theta_1, E^{\alpha,\beta}) = Res_D(\theta_2, E^{\alpha,\beta}),$$

we have

$$A := \theta_1 - \theta_2 \;:\; E^{\alpha,\beta} \to E^{\alpha+\epsilon,\beta+\epsilon} \otimes \Omega^1_{\overline{X}}(\log D)$$

for some $\epsilon > 0$ and consequently,

$$|df|_{\omega_0} \leq C(|\sigma|^{-1+\epsilon}). \tag{11}$$

By (10) and (11), one can get that $f \equiv Constant$. This proves the theorem.

More generally, we have

Theorem 4.3 *Suppose that for the flat connection $D = D_{g_1} + \theta_1 = D_{g_2} + \theta_2$ (decomposed with respect to two different harmonic metrics g_1 and g_2), θ_1 and θ_2 induce the same filtration $j_*E = \cup_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} E^{\alpha,\beta}$ and $Res_D(\theta_1, E^{\alpha,\beta}) = Res_D(\theta_2, E^{\alpha,\beta})$ for all $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then g_1 and g_2 are mutually bounded.*

Proof: We consider the identity map e from (E, g_1) to (E, g_2) as a section of $E^* \otimes E$. Since $Res_D(\theta_1, E^{\alpha,\beta}) = Res_D(\theta_2, E^{\alpha,\beta})$, we have

$$A := \theta_1 - \theta_2 \;:\; E^{\alpha,\beta} \to E^{\alpha+\epsilon,\beta+\epsilon} \otimes \Omega^1_{\overline{X}}(\log D)$$

for some $\epsilon > 0$.

We introduce a Kähler metric

$$\omega_\gamma := \sqrt{-1}\sum_{j=1}^{2} \partial\overline{\partial}|\sigma_i|^{2-\gamma} + C_\gamma\omega_0$$

for $0 < \gamma < 2$. Then by Lemma 4.1 in [S2], we get

$$\triangle_\gamma \log|e|^2 \geq -C \tag{12}$$

where $\triangle_\gamma$ is the Laplace operator with respect to ω_γ.

Since g_1 and g_2 induce the same filtration, we have

$$|e|^2 \leq C|\sigma|^{-\delta},$$

for any $\delta > 0$. So,

$$\log|e|^2 \leq -\delta\log|\sigma| + C.$$

Therefore (12) holds weakly on $\overline{X}$. Since the Sobolev inequality holds on (X, ω_γ) (cf. [L, Section4]), Moser's [Mo] iterative argument shows that

$$\log |e|^2 \leq C < \infty.$$

This proves the theorem.

We define the first and second parabolic Chern numbers as follows: Let $\alpha_l = \alpha$ if $r - rankE^{\alpha,0}|_{D_1} < l \leq r - rankE^{\alpha+,0}|_{D_1}$, and $\beta_l = \beta$ if $r - rankE^{0,\beta}|_{D_2} < l \leq r - rankE^{0,\beta+}|_{D_2}$ $(l = 1, \cdots, r)$. Set

$$\alpha^1 = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_r \end{pmatrix}$$

and

$$\alpha^2 = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_r \end{pmatrix}$$

$$\text{Par } C_1 = C_1(E) + \sum_{i=1}^{2} tr\alpha^i \deg[D_i]$$

where $C_1(E)$ is the first Chern number of E.

$$\begin{aligned} \text{Par } C_2 \;=\; & C_2(E) + \frac{1}{2}\Bigg(2\sum_{\alpha} \alpha \deg(E^{\alpha,0}|_{D_1}/E^{\alpha+,0}|_{D_1}) \\ & +2\sum_{\beta} \beta \deg(E^{0,\beta}|_{D_2}/E^{0,\beta+}|_{D_2}) + \sum_{i=1}^{2} tr(\alpha^i)^2 D_i^2 \\ & -2\sum_{i=1}^{2} tr\alpha^i \deg(E|_{D_i}) - \sum_{i,j=1}^{2} \left(tr\alpha^i tr\alpha^j D_i \cdot D_j\right)\Bigg). \end{aligned}$$

where $C_2(E)$ is the second Chern number of E, $D_i \cdot D_j = \int_X C_1([D_i]) \wedge C_1([D_j]$ is the intersection number of D_i and $D_j (i, j = 1, 2)$.

Set

$$C_1(E, g) = \int_X \frac{\sqrt{-1}}{2\pi} trF_g \wedge *\omega_0,$$

$$C_2(E, g) = -\int_X \frac{1}{8\pi^2}(trF_g \wedge trF_g - trF_g \wedge F_g).$$

Theorem 4.4 *Let u be the pluriharmonic map obtained in Theorem 2.1. Let $\theta = du$. Let*

$$j_* E = \cup_{\alpha \in R, \beta \in R} E^{\alpha,\beta}$$

be the filtration introduced by (g, θ). Then Par $C_1 = C_1(E, g)$ *and* Par $C_2 = C_2(E, g)$.

Proof: We choose a Hermitian metric g_0 on the bundle E. Set $g_0^{-1} g = g_1$, then tr $F_g = \mathrm{tr} F_{g_0} + \overline{\partial}\partial \log \det g_1$.

$$C_1(E, g) = \deg E + \frac{\sqrt{-1}}{2\pi} \int_X \overline{\partial}\partial \log \det g_1 \wedge *\omega$$

Set

$$S^1 = \begin{pmatrix} |\sigma_1|^{\alpha_1} & & \\ & \ddots & \\ & & |\sigma_1|^{\alpha_r} \end{pmatrix}$$

and

$$S^2 = \begin{pmatrix} |\sigma_2|^{\beta_1} & & \\ & \ddots & \\ & & |\sigma_2|^{\beta_r} \end{pmatrix}$$

We have

$$C_1(E, h) = \deg E + \frac{\sqrt{-1}}{2\pi} \int_X \overline{\partial}\partial \log \det(\prod_i (S^i)^2) \wedge *\omega$$

$$+ \frac{\sqrt{-1}}{2\pi} \int_X \overline{\partial}\partial \log \det(\prod_i (S^i)^{-2} g_1) \wedge *\omega$$

By the Poincaré-Lelong formula ([SABK], Ch.II, Section 1, Theorem 2), we have

$$C_1(E, g) = \text{Par } C_1(E) + \frac{\sqrt{-1}}{2\pi} \int_X \overline{\partial}\partial \log \det(\prod_i (S^i)^{-2} g) \wedge *\omega$$

It is not difficult to see that $\log \det(\prod_i (S^i)^{-2} g)$ can be extended smoothly to $\overline{X}$, so the last term on the right hand side of the identity vanishes. This proves the first part of the theorem.

We have

$$\begin{aligned}
& C_1(E, g) \wedge C_1(E, g) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 (trF_{g_0} + \overline{\partial}\partial \log \det g_1) \wedge (trF_{g_0} + \overline{\partial}\partial \log \det g_1) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \big((trF_{g_0})^2 + 2trF_{g_0} \wedge \overline{\partial}\partial \log \det g_1 \\
&\quad + \overline{\partial}\partial \log \det g_1 \wedge \overline{\partial}\partial \log \det g_1\big)
\end{aligned}$$

So,

$$\begin{aligned}
&\int_X C_1(E,g)\wedge C_1(E,g)\\
&=\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X (trF_{g_0})^2\\
&+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{g_0}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\\
&+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X trF_{g_0}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\\
&+\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^2\\
&+\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\\
&+2\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1).
\end{aligned}$$

Since $\log\det(\Pi_i(S^i)^{-2}g)$ can be extended smoothly to $\overline{X}$, we have

$$\begin{aligned}
&\int_X trF_{g_0}\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\\
&=\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\\
&=\int_X \overline{\partial}\partial\log\det(\Pi_i(S^i)^2)\wedge\overline{\partial}\partial\log\det(\Pi_i(S^i)^{-2}g_1)\\
&=0
\end{aligned}$$

We therefore have

$$\begin{aligned}
\int_X C_1(E,g)\wedge C_1(E,g) \;&=\; \int_X C_1(E)\wedge C_1(E)\\
&+2\sum_{i=1}^2 tr\alpha^i\deg(E|_{D_i})+\sum_{i,j=1}^2\left(tr\alpha^i tr\alpha^j D_i\cdot D_j\right).
\end{aligned}$$

It is clear that

$$F_g=F_{g_0}+\overline{\partial}(g_1^{-1}\partial_{g_0}g_1).$$

So,

$$F_g\wedge F_g=F_{g_0}\wedge F_{g_0}+2F_{g_0}\wedge\overline{\partial}(g_1^{-1}\partial_{g_0}g_1)+\overline{\partial}(g_1^{-1}\partial_{g_0}g_1)\wedge\overline{\partial}(g_1^{-1}\partial_{g_0}g_1)$$

Note that

$$\overline{\partial}(g_1^{-1}\partial_{g_0}g_1)=\overline{\partial}((Sg_1)^{-1}\partial_{g_0}(Sg_1))-\overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1)$$

where $S = \Pi_i (S^i)^{-2}$. Since (Sg) can be seen as an endomorphism of E, we have

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_g \wedge F_g) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{g_0} \wedge F_{g_0}) \\
&-2\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{g_0} \wedge \overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1)) \\
&+\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(\overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1) \wedge \overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1))
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{g_0} \wedge \overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1)) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(F_{g_0} \wedge \overline{\partial}(S^{-1}\partial_{g_0}S)) \\
&= 2\sum_\alpha \alpha \deg(E^{\alpha,0}|_{D_1}/E^{\alpha+,0}|_{D_1}) + 2\sum_\beta \beta \deg(E^{0,\beta}|_{D_2}/E^{0,\beta+}|_{D_2})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(\overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1) \wedge \overline{\partial}(g_1^{-1}(S^{-1}\partial_{g_0}S)g_1)) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \int_X tr(\overline{\partial}(S^{-1}\partial_{g_0}S) \wedge \overline{\partial}(S^{-1}\partial_{g_0}S)) \\
&= \sum_{i=1}^{2} tr(\alpha^i)^2 D_i^2.
\end{aligned}$$

This completes the proof of the theorem.

The Theorem clearly implies the following corollary.

Corollary 4.5 *Let (E, θ) be the Higgs bundle introduced by the harmonic map u obtained in Theorem 2.1. We have* Par $C_1 = 0$ *and* Par $C_2 = 0$.

Proof: Calculating directly, one gets

$$C_2(E,g) = -\frac{1}{8\pi^2}\int_X (trF_g \wedge trF_g - trF_g \wedge F_g) = -\frac{1}{8\pi^2}\int_X (trF_D \wedge trF_D - trF_D \wedge F_D).$$

Since $D = D_g + \theta$ is flat, we can see that Par $C_2 = C_2(E,g) = 0$. Similarly, one can also show that Par $C_1 = 0$. This proves the corollary.

5 The existence of flat connections

Let Λ be the contraction with respect to the complete Kähler metric ω. A Hermitian-Einstein (H-E) metric H is a Hermitian metric on E with the property that

$$\Lambda(F_H + [\theta, \theta^*_H]) = \mu I,$$

where F_H is the curvature of the Hermitian connection of H, and θ^* is defined by $(\theta u, v)_H = (u, \theta^*_H v)_H$, μ is a constant, I is the identity endomorphism of E.

We show, in Section 3 (Theorem 3.1), that there exist a Hermitian metric g and a Higgs field θ on E satisfying

$$|F_g|_\omega \leq C, \quad \text{and} \quad |\theta|_\omega \leq C$$

and the connection $D = D_g + \theta$ is flat. We consider the connection

$$D^\lambda = D_g + \lambda\theta \quad \text{for} \quad \lambda \neq 0.$$

It is clear that D^λ is a flat connection if and only if $|\lambda| = 1$. Using the heat flow method, we will deform a H-E metric H_0^λ on the Higgs bundle $(E, \lambda\theta)$ and show that $D_{H_0^\lambda} + \lambda\theta$ is a flat connection.

Theorem 5.1 *For any $\lambda \neq 0$, there is a metric H_0^λ on the Higgs bundle $(E, \lambda\theta)$ so that $D_0^\lambda = D_{H_0^\lambda} + \lambda\theta$ is flat. The parabolic structure introduced by H_0^λ is the same as that introduced by g. As $\lambda \to 0$, the connection $D_0^\lambda \to D_0$ in $C^1(X)$ and D_0 is a flat connection on E.*

Proof: We use the same notation as that in Section 3, set

$$D^1 = D = D_g + \theta \quad \text{and} \quad D^\lambda = D_g + \lambda\theta.$$

By Theorem 3.1, we can see that

$$|F_{D^\lambda}|_\omega \leq C,$$

where $C > 0$ is independent of λ.

For any Hermitian metric H on the bundle E, we set

$$D_H^\lambda = D_H + \lambda\theta \quad \text{and} \quad F_H^\lambda = D_H^\lambda \cdot D_H^\lambda.$$

Let $(F_H^\lambda)^\perp = F_H^\lambda - \frac{\mathrm{tr} F_H^\lambda}{\mathrm{rankE}} I$ be the trace free part of F_H^λ.

Set $X_\beta = \{x \in X \mid \log|\sigma|^2 > -\beta\}$. We solve the heat equation

$$\begin{cases} H^{-1}\frac{dH}{dt} = -\sqrt{-1}\Lambda(F_H^\lambda)^\perp \\ H|_{t=0} = g \\ \det H = \det g \end{cases} \tag{13}$$

on X_β with Dirichlet boundary condition $H|_{\partial X_\beta} = g$. By Lemma 6.1 in [S1] we have $(\triangle - \frac{\partial}{\partial t})|\Lambda(F^\lambda_{H_\beta})^\perp|^2_{H_\beta} = 2|\overline{\partial}\Lambda(F^\lambda_{H_\beta})^\perp|^2_{H_\beta} \geq 0$. So, $(\triangle - \frac{\partial}{\partial t})|\Lambda(F^\lambda_{H_\beta})^\perp|_{H_\beta} \geq 0$. The maximum principle implies

$$|\Lambda(F^\lambda_{H_\beta})^\perp|_{H_\beta} \leq C \tag{14}$$

for all $(x,t) \in X_\beta \times [0,\infty)$.

It is clear (cf. [D4] and [S1]) that the above equation has a solution H_β for all time and that a subsequence of $H_\beta(\cdot,t)$ converges in $C^2(\overline{X_\beta})$ to $H_\beta(\cdot)$ satisfying $\Lambda(F^\lambda_{H_\beta})^\perp = 0$, with $H_\beta|_{\partial X_\beta} = g$ and $\det H_\beta = \det g$. We set $H_\beta = g$ outside X_β.

Let $S(E)$ denote the real vector bundle of selfadjoint endomorphisms of E and $S(EndE)$ consists of the elements of $End(EndE)$ which are selfadjoint with respect to the metric $tr(AB^*)$. For two metrics K and $H = Ke^S$, we (c.f. [S1], Section 4) define the following functional

$$M(K,H) = \sqrt{-1}\int_X tr(S\Lambda F_K) + \int_X (\Psi(S)(D''S), D''S)_K,$$

where $\Psi : S(E) \longrightarrow S(EndE)$ is constructed as in [S1] from the function

$$\Psi(\lambda_1,\lambda_2) = \frac{e^{\lambda_2-\lambda_1} - (\lambda_2-\lambda_1) - 1}{(\lambda_2-\lambda_1)^2},$$

i.e., if we write $A = \sum_{i,j} A_{ij} e_i^* \otimes e_j$, then

$$\Psi(S)(A) = \sum_{i,j} \Psi(\lambda_i,\lambda_j) A_{ij} e_i^* \otimes e_j,$$

here $\{e_i\}$ is an orthonormal basis of eigenvectors of S with eigenvalues λ_i and $\{e_i^*\}$ is the dual basis of $\{e_i\}$ in E^*. For the details we refer the readers to [S1]. It is proved in [LW] (see also [S1]) that

$$M(g, H_\beta) \leq C.$$

We set

$$\omega_\alpha = \sqrt{-1}\sum_{i=1}^m \partial\overline{\partial}\log^{-\alpha}|\sigma_i|^{-2} + C_\alpha\omega_0$$

where $\alpha > 0$ ($i = 1,\cdots,m$). It is proved in [CL] (Theorem 1.1) that the Sobolev inequality holds on the manifold (X,ω_α). Let Λ_α be the contraction with respect to the Kähler metric ω_α, let $\bigtriangledown_\alpha$ be the gradient operator of the metric ω_α, and let $\triangle_\alpha$ be the Laplace operator of the metric ω_α.

Let $h_\beta = g^{-1}H_\beta$. In X_β, we have ([S1], Lemma 3.1 (c))

$$\triangle_\alpha \log\frac{1}{\mathrm{rankE}} trh_\beta \geq -(|\Lambda_\alpha(F^\lambda_g)^\perp|_g + |\Lambda_\alpha(F^\lambda_{H_\beta})^\perp|_{H_\beta}),$$

$$\log\frac{1}{\mathrm{rankE}} trh_\beta|_{\partial X_\beta} = 0,$$

so, we obtain,

$$\triangle_\alpha \frac{\log \frac{1}{\text{rankE}} trh_\beta}{(-\log|\sigma|)^\alpha} \geq -\frac{(|\Lambda_\alpha(F_g^\lambda)^\perp|_g + |\Lambda_\alpha(F_{H_\beta}^\lambda)^\perp|_{H_\beta})}{(-\log|\sigma|)^\alpha} + 2\bigtriangledown_\alpha \log \frac{1}{\text{rankE}} trh_\beta \cdot \bigtriangledown_\alpha \frac{1}{(-\log|\sigma|)^\alpha} - C_\alpha.$$

By Theorem 3.1 and (14), we have

$$\frac{|\Lambda_\alpha(F_g^\lambda)^\perp|_g + |\Lambda_\alpha(F_{H_\beta}^\lambda)^\perp|_{H_\beta}}{(-\log|\sigma|)^\alpha} \leq C.$$

It is clear that

$$|\bigtriangledown_\alpha \frac{1}{(-\log|\sigma|)^\alpha}| \leq C_\alpha.$$

Hence by Moser's iterative argument [Mo](see also [G-T] Ch.8), we have

$$\sup_{X_\beta} \frac{\log \frac{1}{\text{rankE}} trh_\beta}{(-\log|\sigma|)^\alpha} \leq C_\alpha(1 + (\int_{X_\beta} \log \frac{1}{\text{rankE}} \text{tr} h_\beta dV)).$$

Set $e^{s_\beta} = h_\beta = g^{-1}H_\beta$ in X_β and $s_\beta = 0$ outside X_β we have

$$\sup_X \frac{|s_\beta|_g}{(-\log|\sigma|)^\alpha} \leq C_1 + C_2 \|S_\beta\|_{L^2(X,\omega)}.$$

By an argument similar to the one used in the proof of Proposition 5.3 in [S1] (see also [LW]), we can see that

$$\sup_X \frac{|s_\beta|_g}{(-\log|\sigma|)^\alpha} \leq C,$$

because the parabolic Higgs bundle is stable. Therefore

$$|s_\beta|_g \leq C(-\log|\sigma|)^\alpha \quad \text{and} \quad |H_\beta|_g \leq Ce^{(-\log|\sigma|)^\alpha}. \tag{15}$$

Because ([S1], Lemma 3.1 (c))

$$\triangle trh_\beta \geq 2\sqrt{-1}tr(h_\beta \Lambda(F_g^\lambda)^\perp)) + 2|(D''h_\beta)h_\beta^{-\frac{1}{2}}|_g^2,$$

for any compact set $X_0 \subset X$, we have

$$\int_{X_0} |D''h_\beta|_K^2 dV \leq C(X_0).$$

By the diagonal argument, we obtain a sequence $\beta_i \to \infty$, such that $h_i = h_{\beta_i} \longrightarrow h_0$ in $L^2(X_0)$, for any $X_0 \subset\subset X$. Now set $h_{ij} = h_i^{-1}h_j$, then in X_0, there holds

$$\Delta \log(trh_{ij}) \geq 0$$

for sufficiently large i, j. By Moser's iterative argument again, we have for any $X_0 \subset\subset X$

$$\sup_{X_0} \log(tr(h_{ij})) \leq C(\int_X (\log(tr(h_{ij})))^2 dV)^{\frac{1}{2}}.$$

So $h_i \longrightarrow h$ in $C^0(X_0)$ for any $X_0 \subset\subset X$.

Clearly $H_i = gh_i \longrightarrow H_0' = gh_0$ in $C^0(X_0)$ and H_i are bounded locally in L_2^p for any $1 < p < \infty$ (see the remark following Lemma 6.4 in [S1]). So by going to a subsequence, we have $H_i \longrightarrow H_0'$ weakly in $L^p_{2,loc}$. Thus $F_{H_0'}$ is defined and $\Lambda(F^\lambda_{H_0'})^\perp = 0$. Elliptic regularity results imply that H_0' is smooth.

We set $H_0 = e^{-\phi} H_0'$, where ϕ is a smooth real valued function on X, we have

$$\sqrt{-1}\Lambda F^\lambda_{H_0} = \sqrt{-1}\Lambda F^\lambda_{H_0'} + \triangle\phi I = (\triangle\phi + \frac{\sqrt{-1}}{rankE} tr\Lambda F^\lambda_{H_0'})I.$$

It is clear that $f = \frac{\sqrt{-1}}{rankE\ Vol(X)} \int_X tr\Lambda F^\lambda_{H_0'} dV - \frac{\sqrt{-1}}{rankE} tr\Lambda F^\lambda_{H_0'} \in L^\infty(X, \omega)$, and we have $\int_{\overline{X}} f dV = 0$. We consider the closed subspace

$$L^2_0(X, \omega) = \{\varphi \in L^2(X, \omega) \mid \int_X \varphi dV = 0\}$$

of $L^2(X, \omega)$. $\triangle$ is a closed operator from $\mathcal{D}(\triangle) \cap L^2_0(X)$ to $L^2_0(X)$. Since the Poincaré inequality holds on (X, ω) ([Li] Theorem 2.4, see also [CL], Theorem 1.1), we know that 0 does not belong to the spectrum of $\triangle$ in $L^2_0(M)$. So we can get a smooth function ϕ such that $\triangle\phi = f$ in X. By an argument similar to the one used in the proof of (15) we have

$$|\phi| \leq C(-\log|\sigma|). \tag{16}$$

Hence

$$\Lambda F^\lambda_{H_0} = \mu I.$$

That is H_0 is a H-E metric. By (15) and (16), we have

$$|H_0|_g \leq e^{(-\log|\sigma|)^\alpha}. \tag{17}$$

By the estimate (17), note that

$$\int_{\partial X_\beta} \frac{1}{|\sigma|(-\log|\sigma|)^{1-\alpha}} ds = O(\frac{1}{\beta^{1-\alpha}}) \to 0 \quad \text{as} \quad \beta \to \infty,$$

we can see that the parabolic structure introduced by H_0 is the same as that introduced by g and that the metric H_0 gives the same parabolic first and second Chern numbers.

We choose $f_\beta=\max\{0,1+\frac{\log|\sigma|^2}{\beta}\}$. By Theorem 7.3 in [S1], we have

$$\int_{X_\beta} f_\beta tr(F_{H_\beta}^\lambda\wedge F_{H_\beta}^\lambda)\wedge\omega^{n-2}\le\int_{X_\beta} f_\beta tr(F_g^\lambda\wedge F_g^\lambda)\wedge\omega^{n-2}+C\frac{1}{\beta}\int_{X_\beta}|s_\beta|_g|F_g^\lambda|_g dV.$$

Using (17), we get

$$\int_{X_\beta} f_\beta tr(F_{H_\beta}^\lambda\wedge F_{H_\beta}^\lambda)\wedge\omega^{n-2}\le\int_{X_\beta} f_\beta tr(F_g^\lambda\wedge F_g^\lambda)\wedge\omega^{n-2}+\frac{C}{\beta^{1-\alpha}}\int_{X_\beta}|F_g^\lambda|_g dV.$$

By the Riemman bilinear relations, one gets

$$tr(F_{H_\beta}^\lambda\wedge F_{H_\beta}^\lambda)\wedge\omega^{n-2}\ge -C|\Lambda F_{H_\beta}^\lambda|^2\omega^n.$$

Since

$$\sup_{X_\beta}|\Lambda(F_{H_\beta}^\lambda)^\perp|\le\sup_{X_\beta}|\Lambda(F_g^\lambda)^\perp|\le C$$

and $trF_{H_\beta}^\lambda=trF_g^\lambda$, we have

$$tr(F_{H_\beta}^\lambda\wedge F_{H_\beta}^\lambda)\wedge\omega^{n-2}\ge -C\omega^n.$$

Letting $\beta\to\infty$, using Fatou's lemma we obtain

$$\int_X tr(F_0^\lambda\wedge F_0^\lambda)\wedge\omega^{n-2}\le\int_X tr(F_g^\lambda\wedge F_g^\lambda)\wedge\omega^{n-2}=0.$$

Note that $C_1(E,H_0)=C_1(E,g)=0$, applying the Bogomolov-Gieseker inequality (c.f. [S1], Proposition 3.4), we get

$$\int_X tr(F_0^\lambda\wedge F_0^\lambda)\wedge\omega^{n-2}\ge 0,$$

therefore,

$$\int_X tr(F_0^\lambda\wedge F_0^\lambda)\wedge\omega^{n-2}=0,$$

which implies that $D_{H_0}^\lambda$ is a flat connection. Using the estimate (17), we can see that $D_{H_0}^\lambda$ converges to a flat connection as $\lambda\to 0$, in $C^2(U)$ where U is a small neighbourhood of D. We can see that the convergence also holds in $C_{loc}^2(X)$, otherwise, the second Chern number of the flat connection will decrease which contradicts the stability of the Higgs bundle. This proves the theorem.

Now we prove the uniqueness of the flat connection.

Theorem 5.2 *Suppose that H_1 and H_2 be two Hermitian-Einstein metrics on the bundle E with $|H_1|_g=O(e^{(-\log|\sigma|)^\alpha})$ and $|H_1|_g=O(e^{(-\log|\sigma|)^\alpha})$, then $H_1=UH_2$, where U is a unitary matrix.*

Proof: We set $h = H_1^{-1}H_2$, then we have

$$\log \operatorname{tr} h = O((-\log|\sigma|)^\alpha), \quad \text{and} \quad \triangle \log \operatorname{tr} h \geq 0.$$

Multiplying the last identity by $f_\beta^2 \log \operatorname{tr} h$ and integrating, we obtain:

$$\int_X f_\beta^2 |\bigtriangledown \log \operatorname{tr} h|^2 dV \leq \frac{C}{\beta^2} \int_X |\bigtriangledown \log|\sigma||^2 \log^2 \operatorname{tr} h dV$$

so,

$$\int_X f_\beta^2 |\bigtriangledown \log \operatorname{tr} h|^2 dV \leq \frac{C}{\beta^{1-2\alpha}}.$$

Letting $\beta \to \infty$ yields the theorem.

6 Applications to algebraic geometry

In this section, we consider applications of the previous results.

Theorem 6.1 *Let $f : X \to Y$ be a smooth morphism between quasi-projective manifolds X and Y, and let $\rho : \pi_1(X) \to G$ be a Zariski dense representation into an almost simple group G, which does not factor through f. If the restriction $\rho : \pi_1(f^{-1}(y)) \to G$ for some $y \in Y$ comes from a variation of Hodge structure, then ρ comes from a variation of Hodge structure.*

In other words, Theorem 6.1 just says that if the equivariant pluri-harmonic map $u : f^{-1}(y) \to G/K$ restricted to $f^{-1}(y)$ is holomorphic, then u itself is holomorphic. Before starting to prove Theorem 6.1 we need some preparations. Let G be an almost simple algebraic group and let $M(X, G)$ denote the moduli space of semi-simple representations of $\pi_1(X)$ into G.

Let $f : X \to Y$ be a smooth morphism from X to Y. Fix $y \in Y$ and let X_y denote the fibre of f over y. The inclusion $i : X_y \to X$ induces a morphism $i^* : M(X, G) \to M(X_y, G)$.

Lemma 6.2 *Let $[\rho] \in M(X_y, G)$, with a Zariski-dense image in G. Then $i^{*-1}([\rho])$ consists of finitely many points.*

Proof: Consider the exact sequence of homotopy groups

$$\pi_1(X_y, x_0) \to \pi_1(X, x_0) \to \pi_1(Y, y) \to 1,$$

and take finitely many elements $\{\gamma_1, ..., \gamma_m\}$ from $\pi_1(X)$ such that together with $\pi_1(X_y)$ they generate $\pi_1(X)$. So, it is sufficient to show that there are only finitely many possibilities of values of $\rho'(\gamma_i) \in G$.

For any element $\alpha \in \pi_1(X_y, x_0)$, take the product $\gamma_i \alpha \gamma_i^{-1}$ which lies in $\pi_1(X_y, x_0)$. Fixing a $\rho'_0 : \pi_1(X, x_0) \to G$ such that $\rho'_0|_{\pi_1(X_y, x_0)} = \rho$, one has

$$\rho(\gamma_i\alpha\gamma_i^{-1}) = \rho_0'(\gamma_i\alpha\gamma_i^{-1}) = \rho_0'(\gamma_i)\rho(\alpha)\rho_0'(\gamma_i)^{-1}.$$

For any $\rho' : \pi_1(X, x_0) \to G$ with $\rho'|_{\pi_1(X_y,x_0)} = \rho$ we then have

$$(\rho'(\gamma_i)^{-1}\rho_0'(\gamma_i))\rho(\alpha)(\rho'(\gamma_i)^{-1}\rho_0'(\gamma_i))^{-1} = \rho(\alpha).$$

Note that $\rho(\pi_1(X_y, x_0))$ is Zariski dense in G, we see that the elements $\rho'(\gamma_i)^{-1}\rho_0'(\gamma_i)$ must lie in the center $\mathcal{Z}(G)$. Since G is almost simple, $\mathcal{Z}(G)$ consists of finitely many elements. This proves the lemma.

Proof of Theorem 6.1: Let $f : X \to Y$ be a smooth morphism between quasi-projective manifolds X and Y, and let $\overline{f} : \overline{X} \to \overline{Y}$ be a smooth compactification. Since ρ does not factor through f, the restriction $\rho : \pi_1(X_y) \to G$ is again Zariski dense. Since ρ is a semi-simple representation, by Theorem 2.1 there exists a harmonic metric h on the flat bundle V (with trivial filtrations of the local system), which makes V into a filtered Higgs bundle $\{(E, \theta)_\alpha\}$. Restricting everything to the fibre X_y, we obtain a harmonic metric $h|_{X_y}$ on the flat bundle $V|_{X_y}$ with trivial filtrations of the local system. The assumption in the theorem is that the corresponding Higgs bundle $(E, \theta)|_{X_y}$ is a system of Hodge bundles. Hence, its isomorphic class is fixed by the $\mathbb{C}^*-$action by Simpson's Theorem ([S2], Page 768, Theorem 8).

For each $t \in \mathbb{C}^*$ by constructing the Hermitian-Yang-Mills metric in Theorem 5.1 for the Higgs bundle $\{(E, t\theta)_\alpha\}$ we obtain a harmonic bundles (V_t, h_t) on X. By Theorem 5.1 $\{V_t, h_t\}$ converges to a harmonic bundle (V_0, h_0) as $t \to 0$.

Lemma 6.3 *The underling representation of V_0 is isomorphic to the original ρ.*

Proof: Since the isomorphic class of $(E, \theta)_\alpha|_{X_y}$ is fixed by the $\mathbb{C}^*-$action, the class of the underlying representation of $(E, t\theta)$ lies in the fibre $i^{*-1}(\rho|_{X_y})$. By Lemma 6.2 $i^{*-1}(\rho|_{X_y})$ consists of finitely many points. Note that the $\mathbb{C}^*-$action is continuous and contains the identity, $\rho_t \simeq \rho_1 = \rho$, $\forall t \in \mathbb{C}^*$. Let $t \to 0$, we obtain $\rho_0 \simeq \rho$. This proves the lemma.

Since the limiting point arising from this way is unique, for any $t' \in \mathbb{C}^*$ one has $\{(E_0, t'\theta_0)_\alpha\} \simeq \lim_{t\to 0}(E, t't\theta)_\alpha \simeq (E_0, \theta_0)_\alpha$. By Simpson's Theorem (E_0, θ_0) is a system of Hodge bundles. By Lemma 6.3, the underlying representation of (E_0, θ_0) is ρ. Thus, ρ comes from VHS. We complete the proof of Theorem 6.1.

Theorem 6.1 has the following consequence for the non-abelian Hodge $(p, q)-$type theorem. Let $f : X \to Y$ denote a smooth morphism between smooth quasi-projective manifolds X and Y. Let G be an almost simple algebraic group and let $M(X/Y, G)$ denote the the relative moduli space of the representations of the relative fundamental groups into G.

Theorem 6.4 *(Non-abelian Hodge $(p, q)-$type theorem) Let s be a flat section in the space $M(X/Y, G)$. Suppose $s(y_0)$ is Zariski-dense and comes from a VHS from some $y_0 \in Y$. Then $s(y)$ comes from a VHS for all $y \in Y$.*

Proof: Let $\mathcal{Z}$ denote the center of G, which is a finite group. We show first that s lifts to a global representation $\rho : \pi_1(X, x_0) \to G/\mathcal{Z}$.

Since s is a flat section in $M(X/Y, G)$, the class $s(y_0) = [\rho_{y_0}] \in M(X_{y_0}, G)$ is fixed by the $\pi_1(Y, y_0)-$action, i.e. for each $\beta \in \pi_1(Y, y_0)$ there exists some $g(\beta) \in G$ such that $\rho_{y_0}(\beta\alpha\beta^{-1}) = g(\beta)\rho_{y_0}(\alpha)g(\beta)^{-1}$ (in general, such $g(\beta)$ is not unique). By the same argument as in the proof of Lemma 6.2, for any two $g(\beta)$ and $g(\beta)'$ one has $g(\beta)g(\beta)'^{-1} \in \mathcal{Z}$. Thus, $g(\beta)$ is well defined in $G/\mathcal{Z}$. By taking $\rho(\beta) = g(\beta) \in G/\mathcal{Z}$ we lift ρ_{y_0} to a representation $\rho : \pi_1(X) \to G/\mathcal{Z}$.

Now we are in the position to apply Theorem 6.1. Since $\rho|_{X_{y_0}}$ is Zariski dense and comes from a VHS, ρ comes from a VHS. In particular, $\rho|_{X_y}$ comes from a VHS for all $y \in Y$. Since $\mathcal{Z}$ is finite, $s(y)$ comes from a VHS. This completes the proof of Theorem 6.4.

The following Theorem 6.5 is an arithmetic analogue of Theorem 6.4.

Theorem 6.5 *Let s be a flat section in $M(X/Y, G)$. Suppose that $s(y_0)$ takes values in the ring of algebraic integers $\mathcal{O}_K$. Then s lifts to a global representation $\rho : \pi_1(X) \to G/\mathcal{Z}$ that takes values in the ring of algebraic integers $\mathcal{O}_K$.*

Remark. If $\pi_1(Y)$ is replaced by the Galois group $Gal(\overline{L}/L)$ of a number field L then the above statement is just Simpson's theorem on representations from $\pi_1^{alg}(X/L)$ into $G(\mathcal{O}_K)$.

Proof: By Theorem 6.4, we may lift $s(y_0)$ to a global representation $\rho : \pi_1(X) \to G/\mathcal{Z}(K)$. Theorem 6.5 will follow if we can show that, for any prime ideal p of $\mathcal{O}_K$, the induced representation $\rho : \pi_1(X) \to G/\mathcal{Z}(K_p)$ is $p-$bounded.

Let Δ denote the Bruhat-Tits building of $G(K_p)$. By Theorem 2.1, there exists a $\rho-$equivariant pluriharmonic map

$$u_\rho : \tilde{X} \to \Delta.$$

Since $\rho|_{X_{y_0}}$ is $p-$bounded, $u_\rho|_{\tilde{X}_{y_0}}$ is constant. Since any X_y is homotopic to X_{y_0}, their fundamental groups have the same image in X. Hence, $u_\rho|_{\tilde{X}_y}$ is constant for any $y \in Y$. That implies that u_ρ factors to pluriharmonic map $v_\rho : \tilde{Y} \to \Delta$ by the map $\tilde{f} : \tilde{X} \to \tilde{Y}$.

We claim that u_ρ is constant: otherwise $v_\rho : \tilde{Y} \to \Delta$ is unbounded. Let $\Delta(v_\rho)$ denote the convex subcomplex generated by the image of v_ρ. Then $\Delta(v_\rho)$ is fixed by $\rho(\pi_1(X_{y_0})$, because a point $z \in v_\rho(Y)$ is just the image $u_\rho(X_y)$, which is fixed by $\rho(\pi_1(X_y)) = \rho(\pi_1(X_{y_0}))$. Since $\Delta(u_\rho)$ contains at least one geodesic line, $\rho(\pi_1(X_{y_0}))$ fixes this line. Thus $\rho(\pi_1(X_{y_0}))$ is contained in a parabolic subgroup $P \subset G/\mathcal{Z}$. By the exact sequence of the homotopy groups

$$\pi_1(X_{y_0}) \to \pi_1(X) \to \pi_1(Y) \to 1$$

one shows that $\overline{\rho}(\pi_1(X_{y_0})) \subset \overline{\rho}(\pi_1(X) = G/\mathcal{Z}$ where $\overline{\rho}(\pi_1(X_{y_0}))$ is normal and contained in P. Since $G/\mathcal{Z}$ is a simple algebraic group, $\rho(\pi_1(X_{y_0})) = \{1\}$. Thus, $s(y_0)$ is contained in $\mathcal{Z}$. Hence, it is finite. This contradicts the Zariski density of $s(y_0)$.

Since u_ρ is constant, ρ is $p-$bounded. This proves Theorem 6.5.

We have the following application on deformations of families of abelian varieties.

Theorem 6.6 *Let $\pi : Z \to B$ be a family of quasi-projective manifolds. Suppose that over one fiber Z_b there is a family of polarized abelian varieties $f_b : A_b \to Z_b$ of dimension g, and such that the monodromy representation $\rho_b : \pi_1(Z_b) \to Sp(g, \mathbb{Z})$ lies in a finite orbit of the $\pi_1(B, b)$-action on $M(Z_b, Sp(g, \mathbb{Z})$. Then the family $f_b : A_b \to Z_b$ can be extended to a family of polarized abelian varieties $f : A \to Z$ after a base change.*

Remark 1) Let $\mathcal{M}_g$ denote the moduli space of abelian varieties of dimension g and with fixed polarization. If the induced map $\pi_1(Z_b) \to \pi_1(\mathcal{M}_g)$ is an isomorphism, by Weil's local rigidity theorem for lattices in Lie groups of rank ≥ 2 the monodromy representation ρ_b satisfies the condition stated above. 2) Theorem 6.6 should be considered as a statement on variation of Hodge structure. Namely, if a representation restricted to one fibre comes from a geometric situation then the whole representation comes from a geometric situation.

Proof: Since $\rho_b \in M(Z_b, Sp(g, \mathbb{R}))$ lies in a finite orbit of the $\pi_1(B, b)-$action, after taking a finite etale cover of B we may assume ρ_b is fixed by the $\pi_1(B, b)-$action. Since ρ_b comes from a $\mathbb{Z}-$VHS of weight 2, by Theorem 6.5 and Theorem 6.6 ρ_b lifts to a representation $\rho : \pi_1(Z) \to Sp(g, \mathbb{R})$ and comes from a $\mathbb{Z}-$VHS of weight 2. (Here we may assume ρ_b is Zariski dense. In general, we shall work on the Zariski closure $\rho_b(\pi_1(Z_b)) \subset Sp(g, \mathbb{R})$.) The period map u_ρ corresponding to ρ descends to a morphism $\phi : Z \to \mathcal{M}_g$, which extends $\phi_b : Z_b \to \mathbb{M}_g$ induced by $f_b : A_b \to Z_b$. After a base changing $Z' \to Z$ ϕ induces a family $f : A \to Z'$, which extends $f'_b : A'_b \to Z'_b$. This proves Theorem 6.6.

Theorem 6.7 *Let G be an algebraic group of rank ≥ 2 that is not of Hodge type. Then any lattice $\Gamma \subset G$ can't be the fundamental group of a quasi-compact Kähler manifold.*

This theorem generalizes Simpson's Theorem in the compact case. For example, $SL(n, \mathbb{Z})$ $(n \geq 3)$ can't be the fundamental group of a quasi-compact Kähler manifold.

Proof of Theorem 6.7: Suppose there is a quasi-compact Kähler manofold X such that $\pi_1(X) = \Gamma \subset G$. Let $\rho : \pi_1(X) \to G$ denote the induced representation. Since ρ is Zariski-dense in G where G is almost simple, ρ is, in particular, semi-simple. By Theorem 2.1 there exists a harmonic metric h on the flat bundle V with trivial filtrations on the local system. Let $\{(E, \theta)_\alpha\}_{-\infty<\alpha<\infty}$ denote the filtered Higgs sheaf corresponding to the harmonic bundle (V, h). Now note that for each $t \in \mathbb{C}^*$ the filtered Higgs sheaf $\{(E, t\theta)_\alpha\}_{-\infty<\alpha<\infty}$, $t \in \mathbb{C}^*$ is again parabolic poly-stable, by constructing the Hermitian-Yang-Mills metric for $\{(E, t\theta)_\alpha\}_{-\infty<\alpha<\infty}$ in Theorem 5.1 we obtain harmonic bundles (V_t, h_t) on X. As $t \to 0$, by Theorem 5.1, we can see that the harmonic bundles (V_t, h_t) converge to a harmonic bundle (V_0, h_0). Let $\{(E_0, \theta_0)_\alpha\}_{-\infty<\alpha<\infty}$ denote the corresponding filtered Higgs sheaf.

Note that the continuous $\mathbb{C}^*-$action on the representation space $M(X, G)$ via Higgs sheaves contains the identity and all the deformations of $\rho : \Gamma \subset G$ are obtained by conjugations by Weil's local rigidity theorem, $\rho_t \simeq \rho_1 = \rho$, $\forall t \in \mathbb{C}^*$. This implies that $\rho_0 \simeq \rho$. Since the limiting point $\lim_{t\to 0}(V_t, h_t)$ is unique, $\{(E_0, \theta_0)_\alpha\}$ is fixed by the $\mathbb{C}^*-$action. Again by Simpson's Theorem $(E_{0,\alpha}, \theta_{0,\alpha})$ is decomposed into a system of Higgs bundles $E_{0,\alpha} = \oplus E^{p,q}_{0,\alpha}$ with $\theta_{0,\alpha} : E^{p,q}_{0,\alpha} \to E^{p-1,q+1}_{0,\alpha}\Omega^1_{\overline{X}}(\log D)$, i.e ρ comes from a variation of Hodge structures. Thus the Zariski closure $\overline{\rho} = G$ is of Hodge type. But, this is a contradiction to the assumption.

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