Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 58 2001

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Abstract

We show that nonlinear plate theory arises as a Γ-limit of three-dimensional nonlinear elasticity. A key ingredient in the proof is a sharp rigidity estimate for maps $v : (0,1)^3 \to \mathbb{R}^3$. We show that the L^2 distance of ∇v from a single rotation is bounded by a multiple of the L^2 distance from the set $SO(3)$ of all rotations.

Résumé Nous montrons que la théorie non linéaire des plaques émerge comme Γ-limite de la théorie de l'élasticité tridimensionnelle. La démonstration repose sur un résultat de rigidité pour des fonctions $v : (0,1)^3 \to \mathbb{R}^3$. Nous montrons que la distance L^2 de ∇v d'une rotation est bornée par un multiple de la distance L^2 à l'ensemble SO(3) des rotations.

Version française abrégée. Un problème fondamental dans la théorie de l'élasticité non linéaire est de comprendre la relation entre les théories tridimensionelles et bidimensionelles pour des domaines minces $\Omega_h = S \times (-\frac{1}{2}, \frac{1}{2})$. La dérivation de ces théories a une longue tradition. En l'absence de résultats rigoureux (voir [2] pour un résumé des exceptions) on s'est servi d'hypothèses a priori (petitesse, ansatz spéciaux) qui débouchent sur une variété de théories de plaques et de coques qui fréquemment ne sont pas mutuellement consistantes.

Le point de départ de notre approche rigoureuse est l'énergie élastique

$$E^h(v) = \int_{\Omega_h} W(\nabla v(z))dz$$

d'une déformation

$$v : \Omega_h = S \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3.$$

Heuristiquement on s'attend à ce que des déformations avec $E^h \sim h$ correspondant à une élongation du plan médian conduisent à une théorie de membrane, alors que des déformations avec $E^h \sim h^3$ correspondant à une flexion (sans élongation du plan médian) donnent une théorie de plaques non linéaire. La théorie de membrane a été justifiée rigoureusement dans [12, 13, 14], au sens de la Γ-convergence, [4, 3]. Dans cette note nous donnons une dérivation rigoureuse de la theorie de plaques non linéaire. Cela est plus difficile parce que le problème limite dépend des dérivées d'ordre supérieur et donc la limite $h \to 0$ correspond à une limite singulière.

Pour énoncer notre résultat, on se ramène à une région fixe par le changement de coordonnées $x = (z_1, z_2, \frac{z_3}{h})$, puisque $y : \Omega = S \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^3$. Introduisant la notation $x' = (x_1, x_2)$ et $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ pour les coordonnées et le gradient dans le plan, on a alors $\nabla v = (\nabla' y, \frac{1}{h} y_{,3})$ et

$$\frac{1}{h} E(v) = I^h(y) := \int_{\Omega} W(\nabla' y, \frac{1}{h} y_{,3}) dx.$$

Théorème 1 *Soit $S \subset \mathbb{R}^2$ un ouvert borné connexe lipschitzien. On suppose que la fonction W vérifie*

(i) $W(QF) = W(F) \qquad \forall Q \in SO(3),$

(ii) $W = 0 \qquad$ *on* $\quad SO(3),$

(iii) $W(F) \geq c\,\mathrm{dist}^2(F, SO(3)), \qquad c > 0,$

(iv) W *est* C^2 *dans une voisinage de* $SO(3)$.

Alors, lorsque $h \to 0$, *les fonctionnelles* $\frac{1}{h^2} I^h$ Γ*-convergent (dans la topologie faible ou forte de* $W^{1,2}(\Omega, \mathbb{R}^3)$*) vers* I^0 *définie par*

$$I^0(y) = \begin{cases} \frac{1}{24} \int\limits_S Q_2(II) dx' & \text{si } y \text{ est indépendant de } x_3 \text{ et } y \in \mathcal{A}, \\ +\infty & \text{sinon.} \end{cases}$$

Ici la classe de fonctions admissibles consiste en les isométries

$$\mathcal{A} = \{ y \in W^{2,2}(S; \mathbb{R}^3) : |y_{,1}| = |y_{,2}| = 1, \quad y_{,1} \cdot y_{,2} = 0 \}$$

et II *est la seconde forme fondamentale*

$$II_{ij} = \nu_{,i} \cdot y_{,j} = -\nu \cdot y_{,ij}, \quad \nu = y_{,1} \wedge y_{,2}.$$

La forme quadratique Q_2 *est définie par* $Q_2(G) = \min_{a \in \mathbb{R}^3} \quad Q_3(G + a \otimes e_3 + e_3 \otimes a)$ *où* $Q_3(F) = \frac{\partial^2 W}{\partial F^2}(Id)(F, F)$ *est la forme quadratique d'élasticité linéaire.*

Dans le cas de l'élasticité isotropique on a $Q_3(F) = 2\mu |\frac{F+F^T}{2}|^2 + \lambda (\mathrm{tr}\, F)^2$, $Q_2(G) = 2\mu |\frac{G+G^T}{2}|^2 + \frac{2\mu\lambda}{2\mu+\lambda} (\mathrm{tr}\, G)^2$. Donc I^0 est identique à l'expression proposée dans le travail original de Kirchhoff [9], équation (9.), mais pas à l'expression $\frac{1}{24} \int_S Q_2(D^2 y_3) dx'$ obtenue en supprimant la non-linéarité géométrique, qui, dans la littérature postérieure, a souvent été associée au nom de Kirchhoff.

Les points essentiels de la preuve sont les deux résultats suivants, qui permettent de linéariser le problème dans des directions transverses de SO(3). Le Théorème 2 est un résultat de rigidité pour des fonctions proches d'un mouvement rigide qui améliore les résultats classiques de John [7, 8]. Le Corollaire 3 est un résultat de compacité précis qui perd sa validité si le facteur $1/h^2$ avant l'intégrale I^h est remplacé par n'importe quel facteur convergent plus lentement vers l'infini.

THÉORÈME 2 *Soit* $n \geq 2$ *et soit* Q *un cube. Il existe alors une constante* C *telle que, pour toutes fonctions* $v \in W^{1,2}(Q, \mathbb{R}^n)$, *il existe une rotation* $R \in SO(n)$ *avec*

$$||\nabla v - R||_{L^2(Q)} \leq C || \mathrm{dist}(\nabla v, SO(n)) ||_{L^2(Q)}.$$

Corollaire 3 *On suppose que* $S \subset \mathbb{R}^2$ *est un ouvert borné connexe lipschitzien et que* $W(F) \geq c\,\mathrm{dist}^2(F, SO(3))$, $c > 0$. *On suppose aussi que* $y^{(h)} : \Omega = S \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^3$ *vérifie*

$$\limsup_{h \to 0} \frac{1}{h^2} I^h(y^{(h)}) < \infty.$$

Alors, pour une sous-suite,

$$\begin{aligned}
&y^{(h)} \to y \ \textit{dans}\ W^{1,2}(\Omega; \mathbb{R}^3), \quad \tfrac{1}{h} y^{(h)}_{,3} \to b \ \textit{dans}\ L^2(\Omega; \mathbb{R}^3),\\
&y \in W^{2,2}(\Omega; \mathbb{R}^3), \quad b \in W^{1,2}(\Omega; \mathbb{R}^3),\\
&|y_{,1}| = |y_{,2}| = 1, \ \ y_{,1} \cdot y_{,2} = 0, \ \ b = y_{,1} \wedge y_{,2}.
\end{aligned}$$

1 Introduction

A fundamental problem in nonlinear elasticity is to understand the relation between three-dimensional and two-dimensional theories for thin domains $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$. The derivation of such theories has a long history with major contributions from Euler, J. Bernoulli, Cauchy, Kirchhoff, Love, E. and F. Cosserat, von Kármán and a great many modern authors [16]. The derivations usually involve certain ansatzes and smallness assumptions leading to a variety of plate/shell theories which are often not consistent with each other. Thus there has been a great interest in precise convergence results comparing three-dimensional solutions and their two-dimensional counterparts (see [2] for a recent survey).

Our approach starts from the elastic energy

$$E^h(v) = \int_{\Omega_h} W(\nabla v(z))dz$$

of a deformation

$$v : \Omega_h = S \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3.$$

Heuristically one expects that deformations with $E^h \sim h$ correspond to a stretching of the midplane S leading to a membrane theory, while $E^h \sim h^3$ corresponds to a bending deformation (where S remains unstretched) leading to nonlinear plate theory. Membrane theory was rigorously justified in [12, 13, 14] in the sense of Γ-convergence [4, 3]. In this note we rigorously derive nonlinear plate theory. This is more delicate since the limit problem involves higher derivatives and hence the limit $h \to 0$ corresponds to a singular perturbation.

To state our result it is convenient to work in a fixed domain $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$, change variables $x = (z_1, z_2, \frac{z_3}{h})$ and rescale deformations according to $y(x) = v(z(x))$ so that $y : \Omega \to \mathbb{R}^3$. We abbreviate $x' = (x_1, x_2)$ and use the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the in-plane gradient so that

$$\nabla v = (\nabla' y, \frac{1}{h} y_{,3})$$

and

$$\frac{1}{h} E^h(v) = I^h(y) := \int\limits_{\Omega} W(\nabla' y, \frac{1}{h} y_{,3})\, dx.$$

Recall that a sequence of functionals $I^{(h)} : X \to \mathbb{R} \cup \{\infty\}$ on a Banach space X is said to be Γ-convergent to $I^0 : X \to \mathbb{R} \cup \{\infty\}$ with respect to the weak (respectively strong) topology on X if: (i) (Ansatz-free lower bound) For all sequences $y^{(h)}$ converging weakly (resp. strongly) to y, $\liminf_{h\to 0} I^{(h)}(y^{(h)}) \geq I^0(y)$, (ii) (Attainment of lower bound) For each $y \in X$ there exists a sequence y^h converging weakly (resp. strongly) to y such that $\lim_{h \to 0} I^{(h)}(y^{(h)}) = I^0(y)$.

Theorem 1 *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that the stored energy W satisfies*

(i) $W \in C^0(M^{3\times 3})$, $W \in C^2$ in a neighbourhood of $SO(3)$,

(ii) $W(QF) = W(F) \qquad \forall Q \in SO(3)$,

(iii) $W(F) \geq c \operatorname{dist}^2(F, SO(3))$ for some $c > 0$, $W = 0 \qquad$ on $\quad SO(3)$.

Then for $h \to 0$ the functionals $\frac{1}{h^2} I^h$ are Γ-convergent (with respect to the strong or the weak $W^{1,2}$ topology) to I given by

$$I^0(y) = \begin{cases} \frac{1}{24} \int\limits_S Q_2(II) dx' & \text{if } y \text{ is independent of } x_3 \text{ and } y \in \mathcal{A}, \\ +\infty & \text{else.} \end{cases}$$

Here the class $\mathcal{A}$ of admissible maps consist of isometries

$$\mathcal{A} = \{ y \in W^{2,2}(S; \mathbb{R}^3) : |y_{,1}| = |y_{,2}| = 1, \quad y_{,1} \cdot y_{,2} = 0 \}$$

and II is the second fundamental form

$$II_{ij} = \nu_{,i} \cdot y_{,j} = -\nu \cdot y_{,ij}, \quad \nu = y_{,1} \wedge y_{,2}.$$

The quadratic form Q_2 is defined by

$$Q_2(G) = \min_{a \in \mathbb{R}^3} \quad Q_3(G + a \otimes e_3 + e_3 \otimes a)$$

where

$$Q_3(F) = \frac{\partial^2 W}{\partial F^2}(Id)(F, F)$$

is twice the linearized energy.

Remarks 1. For isotropic elasticity we have

$$Q_3(F) = 2\mu|\frac{F + F^T}{2}|^2 + \lambda(\operatorname{tr} F)^2$$
$$Q_2(G) = 2\mu|\frac{G + G^T}{2}|^2 + \frac{2\mu\lambda}{2\mu + \lambda}(\operatorname{tr} G)^2$$

Thus I^0 agrees with the expression proposed in the original work of Kirchhoff [9], equation (9.), but not with the expression obtained by suppressing the geometric nonlinearity and replacing II_{ij} by $(y_3)_{,ij}$ which much of the subsequent literature associated with Kirchhoff's name.

2. In particular if $W(F) = \operatorname{dist}^2(F, SO(3))$ then the limit energy is

$$\frac{1}{12} \int_S |II|^2 dx$$

which agrees, up to a factor, with the Willmore functional [20].

3. Similar results can be proved if one imposes suitable boundary conditions which are consistent with $\mathcal{A}$. The effect of more general boundary conditions can be rather subtle, see [1, 6] for an example related to the formation of blisters in thin films under compression for which one can prove $\min I^h \sim h$.

4. Condition (i) can be relaxed. It suffices that W has a second order Taylor expansion at the identity and that W is continuous in a neighbourhood of $SO(3)$ (such as $\{F \in M^{3\times 3} \mid \det F > 0\}$), taking the value $+\infty$ elsewhere.

In independent work Pantz [18] establishes an interesting partial Γ-convergence result using strong a priori assumptions on the 3D deformations similar to those made by John [7, 8] (see below) which do not follow from smallness of the elastic energy I^h.

2 Geometric rigidity

The key ingredient in the proof of Theorem 1 is the following rigidity result which guarentees that low energy maps are close to a rigid motion.

Theorem 2 *Let $n \geq 2$ and let $Q \subset \mathbb{R}^n$ be a cube. Then there exists a constant C such that for all $v \in W^{1,2}(Q;\mathbb{R}^n)$ there exists a rotation $R \in SO(n)$ such that*

$$||\nabla v - R||_{L^2(Q)} \leq C|| \operatorname{dist}(\nabla v, SO(n))||_{L^2(Q)}. \tag{1}$$

Remarks. 1. The celebrated results of John [7, 8] establish Theorem 2 under the additional restrictions that $y \in C^1(Q;\mathbb{R}^n)$ and $\operatorname{dist}(\nabla y(x), O(n)) \leq \delta$ for all $x \in Q$ and some sufficiently small $\delta > 0$ (see also [10]). For the application to plate theory it is crucial to remove these restrictions, since they do not follow from smallness of the elastic energy.

2. Likewise, it is important to obtain an estimate which is linear in $\varepsilon = || \operatorname{dist}(\nabla v, SO(n))||_{L^2(Q)}$ since we need to sum over many small cubes of size h. An estimate in terms of $\sqrt{\varepsilon} + \varepsilon$ is much easier to prove but useless for the present purpose.

3. For $\varepsilon = 0$ we recover the Liouville theorem which states that $\nabla v \in SO(n)$ implies $\nabla v = \text{const}$. In the $W^{1,\infty}$ setting this was first proved by Reshetnyak [19]. More generally, he proved that a sequence $\nabla v^{(k)}$ bounded in L^2 that converges to SO(3) in measure in fact converges strongly to a single matrix on SO(3). The latter, however, is not sufficient to prove strong convergence of $(\nabla' y, \frac{1}{h} y_{,3})$ for a sequence for which $\frac{1}{h^2} I^h$ is uniformly bounded. The stronger estimate of Theorem 2 is needed because of the explicit presence of the scales $\frac{1}{h^2}$ and $\frac{1}{h}$.

4. The result can be extended to general Lipschitz domains.

5. The estimate holds also in L^p, for $1 < p < \infty$.

Sketch of proof. Let $\varepsilon = || \operatorname{dist}(\nabla v, SO(n)||_{L^2(Q)}$. By scaling we may assume that Q is the unit cube and it suffices to consider $\varepsilon \leq 1$. We first prove an interior estimate on a concentric subcube Q'.

Step 1 (truncation). A truncation argument [15, 21, 5] shows that there exists a constant M such that if the result holds for v with $|\nabla v| \leq M$, then it is true in general.

Step 2 (bound by $\sqrt{\varepsilon}$). For a matrix $F \in M^{n \times n}$ let $\operatorname{cof} F$ denote the matrix of $(n-1) \times (n-1)$ minors such that $F^T \operatorname{cof} F = \det F Id$. Then for all $v \in W^{1,\infty}$ we have $\operatorname{div} \operatorname{cof} \nabla v = 0$. Since $\operatorname{cof} F - F$ vanishes on $SO(n)$ we deduce that

$$-\Delta v = \operatorname{div} f, \quad |f| \leq C \operatorname{dist}(\nabla v, SO(n)).$$

Hence $v = w + z$ where w is harmonic, $z \in W_0^{1,2}(Q;\mathbb{R}^n)$ and

$$||\nabla z||_{L^2(Q)} = ||f||_{L^2(Q)} \leq C\varepsilon \tag{2}$$

$$\Delta(|\nabla w|^2 - n) = \nabla w \cdot \Delta \nabla w + |\nabla^2 w|^2 = |\nabla^2 w|^2. \tag{3}$$

Let $Q' \subset Q'' \subset Q$ be strictly increasing concentric cubes. Multiplying (3) by $\varphi \in C_0^\infty(Q)$ with $\varphi = 1$ on Q'' and $\varphi \geq 0$ and using the fact that $|F|^2 - n = 0$ on $SO(n)$ and (2) we deduce

$$\int_{Q''} |\nabla^2 w|^2 dx \leq C\varepsilon.$$

Since w is harmonic this yields $\sup_{Q'} |\nabla^2 w| \leq C\sqrt{\varepsilon}$ and

$$\sup_{Q'} |\nabla w - R| \leq C\sqrt{\varepsilon} \quad \text{for some} \quad R \in SO(n). \tag{4}$$

Step 3 (linearization of SO(n)). We may suppose that $R = Id$ and we use the expansion

$$\operatorname{dist}(G, SO(n)) = |\frac{G + G^T}{2} - Id| + O(|G - Id|^2).$$

This yields

$$||\frac{\nabla w + (\nabla w)^T}{2} - Id||_{L^2(Q')} \leq C\varepsilon$$

and by Korn's inequality

$$||\nabla w - Id - W||_{L^2(Q')} \leq C\varepsilon, \quad W^T = -W. \tag{5}$$

Moreover (4) with $R = Id$ implies $|W| \leq C\sqrt{\varepsilon}$, $\operatorname{dist}(Id + W, SO(n)) \leq C\varepsilon$ and this together with (2) finishes the proof of the interior estimate $||\nabla v - R||_{L^2(Q')} \leq C\epsilon$.

Step 4 (global estimates). Cover Q by cubes $Q(a_i, r_i) = a_i + r_i(-\frac{1}{2}, \frac{1}{2})^n$ such that

$$2r_i \leq \operatorname{dist}(a_i, \partial Q) \leq C r_i \tag{6}$$

and such that each $x \in Q$ is contained in at most N enlarged cubes $Q(a_i, 4r_i)$. Application of the interior estimate to w and harmonicity of w yield

$$\int_{Q(a_i,r_i)} {r_i}^2 |\nabla^2 w|^2 dx \leq C \int_{Q(a_i,2r_i)} |\nabla w - R_i|^2 dx \leq C \int_{Q(a_i,4r_i)} \operatorname{dist}^2(\nabla w, SO(n)) dx.$$

The desired assertion now follows from (6) and the weighted Poincaré inequality (see [17], Thm. 1.5 or [11], Thm. 8.8)

$$\int_Q |g|^2 dx \le C_Q \int_Q (|g|^2 + |\nabla g|^2) \operatorname{dist}^2(x, \partial Q) dx$$

applied with $g = |\nabla w - R|$.

3 Compactness and Γ-convergence

Corollary 3. Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that

$$W(F) \ge c \operatorname{dist}^2(F, SO(3)), \quad c > 0.$$

Suppose further that $y^{(h)} : \Omega = S \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^3$ satisfies

$$\limsup_{h \to 0} \frac{1}{h^2} I^h(y^{(h)}) < \infty. \tag{7}$$

Then, for a subsequence

$$\begin{aligned} & y^{(h)} \to y \text{ in } W^{1,2}(\Omega; \mathbb{R}^3), \quad \frac{1}{h} y^{(h)}_{,3} \to b \quad \text{in} \quad L^2(\Omega; \mathbb{R}^3), \\ & y \in W^{2,2}(\Omega; \mathbb{R}^3), \quad b \in W^{1,2}(\Omega; \mathbb{R}^3) \\ & |y_{,1}| = |y_{,2}| = 1, \quad y_{,1} \cdot y_{,2} = 0, \quad b = y_{,1} \wedge y_{,2}. \end{aligned}$$

Remark. The scaling in (7) is optimal. There exists a sequence $y^{(h)}$ such that $h^{-\beta} I(y^{(h)}) \to 0$ for all $\beta < 2$ such that $y^{(h)} \rightharpoonup y$ weakly in $W^{1,2}$ but strong convergence fails.

Idea of proof. Application of Theorem 2 to slightly overlapping cubes yields an L^2 difference quotient estimate for $\nabla_h y := (\nabla' y, \frac{1}{h} y_{,3})$. This implies compactness and $W^{2,2}$ regularity of the limit.

Proof of Theorem 1(sketch). The main point is to establish a lower bound on $\liminf_{h \to 0} \; h^{-2} I^h(y^{(h)})$ for $y^{(h)} \rightharpoonup y$. Using Theorem 1 we can find a map $Q^{(h)} : S \to SO(3)$, piecewise constant on squares of side h, such that

$$(Q^{(h)})^T \quad \frac{(\nabla' y^{(h)}, \frac{1}{h} y_{,3}) - Q^{(h)}}{h} \rightharpoonup G \quad \text{in} \quad L^2(\Omega),$$

and with the help of Corollary 3 one can conclude that

$$G_{ij}(x', x_3) = G^0_{ij}(x') + x_3 II_{ij}(x'), \quad i, j \in \{1, 2\}.$$

Careful expansion of the energy W, the definition of Q_3 and Q_2 and minimization over G^0 then yield the desired lower bound in terms of $I^0(y)$.

4 Acknowledgement

RDJ thanks AFOSR/MURI (F49620–98–1–0433), ARO (DAAG55–98–1–0335), NSF (DMS–0074043) and ONR (N00014–99–1–0925) for supporting his work. GF and SM were partially supported by the TMR network FMRX–CT98–0229.

References

[1] H. Ben Belgacem, S. Conti, A. DeSimone, and S. Müller, *Rigorous bounds for the Föppl-von Kármán theory of isotropically compressed plates*, J. Nonlinear Science **10** (2000), 661–685.

[2] P. G. Ciarlet, *Mathematical elasticity II – theory of plates*, Elsevier, Amsterdam, 1997.

[3] G. DalMaso, *An introduction to* Γ*-convergence*, Birkhäuser, 1993.

[4] E. DeGiorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58** (1975), 842–850.

[5] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[6] W. Jin and P. Sternberg, *Energy estimates for the von Kármán model in thin-film blistering*, J. Math. Phys. **42** (2001), 192–199.

[7] F. John, *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.

[8] ______, *Bounds for deformations in terms of average strains*, in: O. Shisha (ed.), Inequalities III (1972), 129–144.

[9] G. Kirchhoff, *Über das Gleichgewicht und die Bewegung einer elastischen Scheibe*, J. Reine Angew. Math. **40** (1850), 51–88.

[10] R. V. Kohn, *New integral estimates for deformations in terms of their nonlinear strains*, Arch. Rat. Mech. Analysis **78** (1982), 131–172.

[11] A. Kufner, *Weighted Sobolev spaces*, Wiley, New York, 1985.

[12] H. LeDret and A. Raoult, *Le modéle de membrane non linéaire comme limite variationelle de l'élasticité non linéaire tridimensionelle*, C. R. Acad. Sci. Paris **317** (1993), 221–226.

[13] ______, *The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity*, J. Math. Pures Appl. **73** (1995), 549–578.

[14] ______, *The membrane shell model in nonlinear elasticity: a variational asymptotic derivation*, J. Nonlinear Science **6** (1996), 59–84.

[15] F.-C. Liu, *A Lusin property of Sobolev functions*, Indiana U. Math. J. **26** (1977), 645–651.

[16] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4^{th} ed., Cambridge University Press, Cambridge, 1927.

[17] J. Nečas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 305–326.

[18] O. Pantz, *Une justification partielle du modèle de plaque en flexion par* Γ*-convergence*, C. R. Acad. Sci. Paris Série I **332** (2001), 587–592.

[19] Y. G. Reshetnyak, *Liousville's theory on conformal mappings under minimal regularity assumptions*, Sibirskii Math. **8** (1967), 69–85.

[20] T. J. Willmore, *Total curvature in Riemannian geometry*, John Wiley & Sons, New York, 1982.

[21] W. Ziemer, *Weakly Differentiable Functions*, Springer, 1989.