Time-space discretization of the nonlinear hyperbolic system
\[ u_{tt} = \text{div}(\sigma(Du) + Du_t) \]

by

_Carsten Carstensen and Georg Dolzmann_

Preprint no.: 60

2001
TIME-SPACE DISCRETIZATION OF THE NONLINEAR HYPERBOLIC SYSTEM $u_{tt} = \text{div}(\sigma(Du) + Du_t)$

CARSTEN CARSTENSEN AND GEORG DOLZMANN

Abstract. The numerical treatment of the hyperbolic system of nonlinear wave equations with linear viscosity, $u_{tt} = \text{div}(\sigma(Du) + Du_t)$, is studied for a large class of globally Lipschitz continuous functions $\sigma$, including non-monotone stress-strain relations. The analyzed method combines an implicit Euler scheme in time with Courant (continuous and piecewise affine) finite elements in space for general time steps with varying meshes. Explicit a priori error bounds in $L^\infty(\Omega)$, $L^2(\Omega; W^{1,2})$, and $W^{1,2}(\Omega; L^2)$ are established for the solutions of the fully discrete scheme.

1. Introduction

In this paper we study the numerical treatment of the nonlinear hyperbolic system

$$u_{tt} = \text{div}(\sigma(Du) + Du_t) \text{ in } \Omega \times (0, T)$$

subject to the boundary and initial conditions

$$u = 0 \text{ on } \partial\Omega \times (0, T),$$

$$u = u_0 \text{ in } \Omega \times \{0\},$$

$$u_t = u_t \text{ in } \Omega \times \{0\}.$$

Here, $u$ is a vector-valued mapping from $\Omega \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ and the initial data satisfy $u_0 \in W^{1,2}_0(\Omega; \mathbb{R}^m)$ and $u_t \in L^2(\Omega; \mathbb{R}^m)$.

The physical interest in this equation lies in the fact that it describes for $m = n$ the evolution of a viscoelastic body with reference configuration $\Omega$. Non-monotone stress-strain relations, modeled by $\sigma = D\Phi$ for non-convex energy density functions $\Phi$, are of main interest in simulations of solid-solid phase transitions, see, e.g., [1, 9, 10, 4, 5]. This equation arises also in the two-dimensional scalar case ($n = 2$ and $m = 1$) for the out-of-plane displacement field of an anti-plane shear deformation [11]. Numerical experiments have been reported in [7].

Inspired by the uniqueness proof for Lipschitz continuous stresses $\sigma$ in [10, 6], we present in this paper the a priori error analysis for an approximating scheme for the system (1.1) that combines continuous finite elements in spaces with a discontinuous Galerkin approximation in time. We obtain estimates for the approximation error of

\small

\text{Date: August 7, 2001.}

1991 Mathematics Subject Classification. 65N12, 65N15, 35G25, 74S25.

Key words and phrases. Finite elements, a priori error estimates, non-linear wave equations.

The research of GD was partially supported by the California Institute of Technology through grants AFOSR/MURI (F 49620-98-1-0433), by the Max Planck Society and by the NSF through grant DMS0104118. CC was partially supported by the Powell Foundation at the California Institute of Technology and by the Max Planck Society.
the deformation $u$, the deformation gradient $Du$ and the velocity field $v = u_t$ under very general assumptions. In particular, the time step size $k_j$ has only to satisfy the condition $k_j \leq Qk_{j+1}$ for a global constant $Q > 0$ and the spatial triangulations are only assumed to be quasiuniform at each time step with a typical diameter $h_j$ of the elements. Our analysis reveals an interesting coupling of $k_j$ and $h_j$ through the quotient $h_j^3 / k_j$ which suggests that the time steps should not be too small compared to the square of the spatial discretization parameter $h_j$. Our general result, Theorem 4.1, implies immediately the following convergence estimates (see Section 2 for the precise definitions).

**Theorem 1.1.** Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^n$. Fix $T > 0$ and define discrete times $0 = t_0 < t_1 < \cdots < t_N = T$. Suppose that $T_j$ is a regular triangulation of $\Omega$ for $j = 0, \ldots, N$ such that $T_j$ is a refinement of $T_{j-1}$. Assume that $S_0(T_j)$ is the space of all continuous functions that vanish on $\partial \Omega$ and are affine on the elements in $T_j$. Let $U_j \in S_0(T_j)$ be the solution of the implicit Euler scheme defined in Section 2.3 and let

$$k_j = t_j - t_{j-1}, \quad k = \max_{j=1, \ldots, N} k_j, \quad Q = \max_{j=2, \ldots, N} \frac{k_j}{k_{j-1}}, \quad h = \max_{j=0, \ldots, N} h_j.$$ 

Assume, furthermore, that $\sigma$ is globally Lipschitz continuous and that the solution $u$ of the system (1.1) belongs to $W^{1, \infty}(W^{2,2}) \cap W^{2,2}(L^2)$. Finally, define the discretization errors $e_j$ and $\delta_j$ at the time step $t_j$ by

$$e_j = u(t_j) - U_j, \quad \delta_j = v(t_j) - \frac{1}{k_j}(U_j - U_{j-1}).$$

Then there exist constants $c_1$ and $c_2$ such that the following holds. If $c_1 k < 1$, then

$$\max_{v=1, \ldots, N} \|e_v\|^2 + \sum_{v=1}^N k_v \left( \|\delta_v\|^2 + \|Dv_v\|^2 \right) \leq c_2 (T + T^2 + h^4 + h^8) \exp(c_1 T) \|u\|_{1,k}^2,$$

where

$$\|u\|_{1,k}^2 = k^2 \|u_{xx}\| + \|Du_{xx}\| + \|Du_{xx}\|_{L^2}^2 + h^2 (1 + \max_{j=1, \ldots, N} \frac{k_j^2}{k_j}) \|u\|_{1,\infty}^2,$$

The constants $c_1$ and $c_2$ depend only on the Lipschitz constant of $\sigma$, the shape of the triangles in the triangulations $T_j$, and on $k_j$ via $Q$, but neither on $h_j$ nor on $u$.

**Remarks.** 1) The statement of the theorem assumes tacitly that the initial data can be approximated sufficiently well, see estimate (2.7) below.

2) The assumption $c_1 k \leq 1$ implies that $2k_j \text{Lip}(\sigma) < 1$ for $j = 1, \ldots, N$. This condition ensures that the discrete scheme has a unique solution, see Theorem 2.2.

The paper is organized as follows. We define the discrete scheme in Section 2 and prove existence and uniqueness of the discrete solution. Section 3 contains a series of estimates for the solutions of the approximating scheme which are used in Section 4 to prove the general convergence result in Theorem 4.1 which contains Theorem 1.1 as a special case.
2. The discrete scheme

In this section, we introduce the relevant notation and define the discrete scheme. Then we prove existence and uniqueness of the discrete solutions and derive an identity in the spirit of Galerkin orthogonality. This relation replaces in our convergence analysis the identity

\[ (u - \overline{u})_{tt} = \text{div} (\sigma(Du) - \sigma(D\overline{u}) + D(u - \overline{u})) \]

for the difference of two solutions \( u \) and \( \overline{u} \) of the system (1.1) and from which one easily deduces uniqueness of solutions for Lipschitz continuous \( \sigma \), see Section 2.2.

2.1. Notation. We assume that \( \Omega \subset \mathbb{R}^n \) is a polygonal domain with boundary \( \Gamma = \partial \Omega \) and exterior normal \( \nu \) to \( \Gamma \). We use the standard notation for the Lebesgue spaces \( L^p(\Omega; \mathbb{R}^m) \) with norm \( \| \cdot \|_p \), and we write \( \langle \cdot, \cdot \rangle \) for the inner product in \( L^2 \).

The Sobolev spaces \( W^{k,p}(\Omega; \mathbb{R}^m) \) are equipped with the standard norm \( \| \cdot \|_{k,p} \) and the seminorm \( | \cdot |_{k,p} \), respectively. We frequently abbreviate \( X(0,T;Y(\Omega)) \) by \( X(\cdot,Y) \).

If the corresponding domain \( \Omega \), time interval \( [0,T] \) and the range of the functions is clear from the context. Thus \( L^2(L^2) \) denotes, for example, both \( L^2(0,T;L^2(\Omega)) \) and \( L^2(0,T;L^2(\Omega,\mathbb{R}^m)) \). The space of all real \( m \times n \) matrices \( \mathbb{R}^{m \times n} \) is equipped with the Frobenius norm, \( \| A \|^2 = \text{tr}(A^T A) \), where \( A^T \) denotes the transpose of the matrix \( A \), and with the inner product \( F : G \) that is induced by the scalar product in \( \mathbb{R}^{mn} \).

Let \( 0 = t_0 < t_1 < \cdots < t_N = T \) be a partition of the time interval \( [0,T] \) into \( N \) subintervals \( I_j = (t_{j-1}, t_j) \) of length \( k_j = t_j - t_{j-1} \), \( j = 1, \ldots, N \). Suppose that \( \{\mathcal{T}_j\}_{j=0}^N \) is a family of regular triangulations in the sense of [2] with maximal mesh-size \( h_j \) and that the union of all elements in \( \mathcal{T}_j \) is equal to \( \Gamma \). We denote by \( \mathcal{S}_{0,j} = \mathcal{S}_{0,j}(\mathcal{T}_j) \) the finite element space of continuous functions \( u_h : \Omega \to \mathbb{R}^m \) that have zero boundary values on \( \Gamma \) and are affine on the elements in \( \mathcal{T}_j \). We use the interpolation operator \( \Pi_j \) onto \( \mathcal{S}_{0,j} \) due to Scott and Zhang [13] which satisfies the projection property

\[ \Pi_j v = v \quad \text{for all } v \in \mathcal{S}_{0,j}, \]

the stability estimate

\[ \| D\Pi_j u \| \leq c_s \| Du \| \quad \text{for all } u \in W^{1,2}_0(\Omega), \]

and the approximation estimate

\[ \| u - \Pi_j u \| + h_j \| Du - D\Pi_j u \| \leq c_A h_j^2 \| D^2 u \| \quad \text{for all } u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega). \]

Throughout the paper, \( u \) denotes the unique solution of the system (1.1) guaranteed by Theorem 2.1 below. We define the discrete solution \( U_j \in S_{0,j} \) at the time step \( t_j \) in Section 2.3 below and we use \( V_j = (U_j - U_{j-1})/k_j \) for \( j = 1, \ldots, N \) as an approximation for the discrete velocities.

The goal of our analysis is to estimate the errors in \( u \) and in \( u_t \). To simplify the notation, we set \( v = u_t \), \( u_j = u(t_j) \), \( v_j = v(t_j) \), and

\[ e_j = u_j - U_j = u(t_j) - U_j, \quad \delta_j = v_j - V_j = v(t_j) - V_j \quad \text{for } j = 1, \ldots, N. \]
2.2. Existence and uniqueness for Lipschitz continuous stress functions.

Our analysis relies on the existence result [6] for the system (1.1) which requires that \( \sigma(F) = \partial \Phi(F) / \partial F \) where the stored energy function \( \Phi \) has the following three properties (H1), (H2), and (H3):

(H1) \( \Phi \in C^2(\mathbb{M}^{m \times n}) \).
(H2) There exist constants \( \overline{c}, \overline{C} > 0 \) and \( p \geq 2 \) such that
\[
\overline{c} |F|^p - \overline{C} \leq \Phi(F) \leq \overline{C} (|F|^p + 1), \quad |\sigma(F)| \leq \overline{C}(|F|^{p-1} + 1)
\]
for all \( F \in \mathbb{M}^{m \times n} \).
(H3) There exists a constant \( K > 0 \) such that
\[
-K|F - G|^2 \leq (\sigma(F) - \sigma(G)) : (F - G) \quad \text{for all } F, G \in \mathbb{M}^{m \times n}.
\]

Hypothesis (H3) follows, for example, from monotonicity or global Lipschitz continuity of \( \sigma \). In the latter case one can choose \( K = \text{Lip}(\sigma) \).

In this situation, the following existence result holds (see [4, 5] for related results).

**Theorem 2.1** ([6, Theorem 4.1]). Under the foregoing assumptions, the system (1.1) has a weak solution
\[
u \in L^\infty(0, \infty; W_{loc}^{1,2}(\Omega; \mathbb{R}^m))) \cap L^\infty(0, \infty; W_{loc}^{1,2}(\Omega; \mathbb{R}^m)))
\]
\[
\cap L^\infty([0, \infty); W_{loc}^{2,2}(\Omega; \mathbb{R}^m))) \cap L^\infty([0, \infty); W^{-1,2}(\Omega; \mathbb{R}^m)),
\]
i.e., \( u(t, \cdot) = u_0, \ u_t(t, \cdot) = \nu_0, \) and, for all \( \zeta \in C_0^\infty(\Omega \times (0, \infty); \mathbb{R}^m) \),
\[
\int_0^\infty \int_\Omega (\sigma(\nabla u(t)) + \nabla \nu_t) : D\zeta - u_t : \zeta \, dx \, dt = 0.
\]

Moreover, \( u \) satisfies the dissipation inequality
\[
E[u(t), u_t(t)] - E[u, \nu_0] \leq - \int_0^t \int_\Omega |\nabla \nu_t|^2 \, dx \, ds
\]
for almost every \( t > 0 \) where the total energy is given by
\[
E[u, v] = \int_\Omega (\Phi(\nabla u) + \frac{1}{2} |v|^2) \, dx
\]

If \( \sigma \) is globally Lipschitz continuous, then the following inequalities imply uniqueness of the weak solution \( u \) (see [6, 10]). Suppose that \( u \) and \( \overline{\nu} \) are solutions with the same initial and boundary conditions. If we test the difference of the two equations
\[
u_t = \text{div} (\sigma(\nabla u) + \nabla \nu_t), \quad \overline{\nu}_t = \text{div} (\sigma(\overline{\nabla} \overline{\nu}) + \overline{\nabla} \overline{\nu}_t)
\]
by \( u - \overline{\nu} \) and integrate in space and time, then we obtain
\[
\int_0^T \int_\Omega |\nabla u - \nabla \overline{\nu}|^2 \, dx \, dt + \int_0^T \int_\Omega |u(t) - \overline{\nu}(t)|^2 \, dx \, dt
\]
\[
\leq \text{Lip}(\sigma) \int_0^T \int_\Omega (|\nabla u - \nabla \overline{\nu}|^2 + |u_t - \overline{\nu}_t|^2) \, dx \, dt.
\]

Similarly, if we use \( u_t - \overline{\nu}_t \) as a test function, we get
\[
\int_0^T \int_\Omega |u_t - \overline{\nu}_t|^2 \, dx \, dt \leq \frac{1}{4} \text{Lip}^2(\sigma) \int_0^T \int_\Omega |\nabla u - \nabla \overline{\nu}|^2 \, dx \, dt.
\]
The asserted uniqueness follows by applying Gronwall’s inequality (see, e.g., [12]) to the sum of the two inequalities. A discrete version of this Gronwall argument is used in Section 3 as the key ingredient in Theorem 4.1.

2.3. Definition of the implicit scheme. In order to define the discrete scheme, let $U_0, V_0 \in \mathcal{S}_0$ denote given approximations to $u_0$ and $v_0$. We assume that

$$\|e_0\| = \|u_0 - U_0\| \leq c_A h_0 \|Du_0\|,$$

and additionally for $u_0 \in W^{2,2}(\Omega; \mathbb{R}^m)$ and $v_0 \in W^{1,2}(\Omega; \mathbb{R}^m)$ that

$$(2.7) \quad \|e_0\| + h_0 \|Dv_0\| \leq c_A h_0 \|D^2u_0\|, \quad \|e_0\| = \|v_0 - V_0\| \leq c_A h_0 \|Dv_0\|.$$ 

We then define successively the discrete solution $U_j$ at time $t_j$ by minimizing the variational integral (2.8) below. Since we allow a variable step size in the time discretization, we cannot discretize the second derivatives with a second difference quotient, and we use a backwards difference quotient for the discrete velocities instead.

**Theorem 2.2.** Suppose that $\sigma = D\Phi$ is Lipschitz continuous with Lipschitz constant $\text{Lip}(\sigma)$ and that $k$ is small enough such that $2k\text{Lip}(\sigma) \leq 1$. Then there exists for $j = 1, \ldots, N$ a unique solution $U_j$ of the variational problem: Minimize

$$(2.8) \quad \int_{\Omega} \left( \Phi(DU) + \frac{1}{2k_j^2} |DU - DU_{j-1}|^2 + \frac{1}{2} k_j \left( |U - U_{j-1}| - |V_{j-1}| \right)^2 \right) dx$$

among all functions $U \in \mathcal{S}_{0,j}$. The minimizer $U_j$ is a solution of the corresponding Euler-Lagrange system in weak form, i.e., a solution of

$$(2.9) \quad \int_{\Omega} \left( (k_j \sigma(DU_j) + D(U_j - U_{j-1})) : D W_j + (V_j - V_{j-1}) \cdot W_j \right) dx = 0$$

for all $W_j \in \mathcal{S}_{0,j}$.

**Proof.** We only need to show that the variational integral has a convex integrand. Existence and uniqueness of solutions follows then from the direct method in the calculus of variations (see, e.g., [3]). By assumption,

$$0 \leq \frac{1}{k_j} |A - B|^2 - \text{Lip}(\sigma) |A - B|^2 \leq (\sigma(A) + \frac{1}{k_j} A - (\sigma(B) + \frac{1}{k_j} B)) : (A - B),$$

that is, $\sigma(F) + k^{-1} F$ is monotone and hence $\Phi(F) + |F|^2/(2k_j)$ convex. \hfill $\square$

**Remark.** The structural assumption $\sigma = D\Phi$ guarantees the existence of the solution $u$ and $U_j$ of the continuous and the discretized system. The error analysis below is entirely based on the Galerkin orthogonality (2.8) and does not rely on this assumption.

2.4. Discrete orthogonality. The following version of the Galerkin orthogonality is an important ingredient in the proof of Theorem 4.1.

**Proposition 2.1.** Suppose that $u$ is the unique solution of the system (1.1) and that $\{U_j\}$ is the unique approximation constructed in (2.8). Then

$$(2.10) \quad (\delta_j - \delta_{j-1}, W_j) + \int_{I_j} (\sigma(Du) - \sigma(DU_j), DW_j) dt + (De_j - De_{j-1}, DW_j) = 0$$

for $j = 1, \ldots, N$ and for all $W_j \in \mathcal{S}_{0,j}$. 


Proof. The idea is to test the weak formulation (2.4) by \( \chi_j W_j \) in order to get an analogue of (2.9); \( \chi_j \) denotes the characteristic function of the time interval \( I_j \). Let \( W^j_\ell \in C^\infty_0(\Omega; \mathbb{R}^m) \) be a sequence of smooth functions with \( \| W^j_\ell - W^j \|_{W^{1,\infty}(\Omega)} \to 0 \) as \( \ell \to \infty \) and choose \( \psi_\mu \in C^\infty_0(I_j) \) with \( \psi_\mu \equiv 1 \) on \( (t_{j-1} + \mu, t_j - \mu) \). Then
\[
\int_0^\infty \int_\Omega \left( (\sigma(Du) + Du) \cdot D^\ell W^j_\ell(x) \psi_\mu(t) - u_t \cdot W^j_\ell(x) \frac{\partial}{\partial t} \psi_\mu(t) \right) \, dx \, dt = 0
\]
for \( \mu \in (0, k_j/4) \) and \( \ell \in \mathbb{N} \). This expression converges for \( \ell \to \infty \) and \( \mu \to 0 \) to
\[
\int_{I_j} \left( \sigma(Du(t, x)) + Du(t, x), DW_j \right) \, dt + (v_j - v_{j-1}, W_j) = 0.
\]
The assertion of the proposition follows by subtracting this equation from the Euler-Lagrange equation (2.9). \( \square \)

2.5. A discrete Gronwall inequality. Our convergence result is based on the following discrete Gronwall inequality.

**Lemma 2.1** ([8, Lemma 1.4.2]). Suppose that \( a > 0 \) and that \( \{b_\nu\}, \{\tau_\nu\} \) are sequences of non-negative real numbers. Assume that the sequence \( \{\varphi_\nu\} \) satisfies
\[
\varphi_0 \leq a \quad \text{and} \quad \varphi_\nu \leq a + \sum_{j=1}^\nu b_j + \sum_{j=0}^{\nu-1} \tau_j \varphi_j
\]
for \( n \geq 1 \). Then
\[
\varphi_\nu \leq (a + \sum_{j=1}^\nu b_j) \exp \left( \sum_{k=0}^{\nu-1} \tau_k \right). \quad \square
\]

3. Estimates for the solution of the discrete system

In our estimates, we will frequently express a difference \( (e_j - e_{j-1})/k_j \) of spatial errors as an error in velocities, \( \delta_j \). The resulting correction term \( K_j \) is characterized in the following lemma.

**Lemma 3.1**. Let \( K_j, j = 1, \ldots, N \), be given by
\[
K_j = \frac{1}{k_j} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_H(s) \, ds.
\]
Then, for \( j = 1, \ldots, N \),
\[
\frac{1}{k_j} (e_j - e_{j-1}) = \delta_j + K_j
\]
and
\[
\|K_j\|^2 \leq \frac{k_j}{3} \int_{t_{j-1}}^{t_j} \int_\Omega u^2_H \, dx \, dt, \quad \|DK_j\|^2 \leq \frac{k_j}{3} \int_{t_{j-1}}^{t_j} \int_\Omega |Du_H|^2 \, dx \, dt.
\]

**Proof.** It follows from Taylor’s formula that
\[
u(t, x) = u(t_{j-1}, x) + k_j u_t(t, x) - \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_H(s, x) \, ds.
\]
This allows us to estimate
\[ \frac{1}{k_j} (e_j - e_{j-1}) = \frac{U_j - U_{j-1}}{k_j} - \frac{u(t_j) - u(t_{j-1})}{k_j} = V_j - u(t_j) + K_j + \delta_j + K_j. \]

Finally, by Hölder’s inequality,
\[ ||K_j||^2 = \int_\Omega \left( \frac{1}{k_j} \int_{t_{j-1}}^{t_j} (s - t_j) u_H(s) \, ds \right)^2 \, dx \]
\[ \leq \frac{1}{k_j^2} \left( \int_{t_{j-1}}^{t_j} (s - t_j)^2 \, ds \right) \left( \int_{t_{j-1}}^{t_j} \int_\Omega u^2_H \, dx \, dt \right) \]
\[ = \frac{k_j}{3} \int_{t_{j-1}}^{t_j} \int_\Omega u^2_H \, dx \, dt. \]

This concludes the proof of the lemma. \( \square \)

The next proposition is a discrete analogue of (2.6). The idea is to use the orthogonality (2.10) with \( \Pi_j \delta_j = \delta_j + (\Pi_j \delta_j - \delta_j) \) and to recover the structure of (2.6) plus approximation errors.

**Proposition 3.1.** The following estimate holds for \( j = 1, \ldots, N \):
\[ \frac{1}{2} ||\delta_j||^2 - \frac{1}{2} ||\delta_{j-1}||^2 + \frac{1}{4} ||\delta_j - \delta_{j-1}||^2 \]
\[ \leq 2c_L^2 \text{Lip}^2(\sigma) k_j ||D \sigma_j||^2 - \frac{k_j}{2} ||D \delta_j||^2 + ||\delta_j - \Pi_j \delta_j||^2 \]
\[ + \frac{5k_j}{2} ||D \Pi_j \delta_j||^2 + \frac{5k_j}{2} ||D \Pi_j \delta_j||^2 + c_L^2 \text{Lip}^2(\sigma) k_j^2 ||D u_1||^2_{L^2(t_j, t_{j+1})}. \]

**Proof.** It follows from (2.10) with \( W_j = \Pi_j \delta_j \) that
\[ \frac{1}{2} ||\delta_j||^2 - \frac{1}{2} ||\delta_{j-1}||^2 + \frac{1}{4} ||\delta_j - \delta_{j-1}||^2 = (\delta_j, \delta_j - \delta_{j-1}) \]
\[ = (\delta_j - \delta_{j-1}, \delta_j - \Pi_j \delta_j) - (D \sigma_j - D \sigma_{j-1}, \Pi_j \delta_j) \]
\[ - \int_{I_j} (\sigma(D u) - \sigma(D u_j), \Pi_j \delta_j) \, dt. \]

We denote the three terms on the right-hand side by \( T_1, T_2, \) and \( T_3. \) Then
\[ T_1 \leq \frac{1}{4} ||\delta_j - \delta_{j-1}||^2 + ||\delta_j - \Pi_j \delta_j||^2, \]
and by (3.1)
\[ T_2 = -k_j (D \delta_j + D \Pi_j \delta_j, D \delta_j - D \delta_{j-1} + D \Pi_j \delta_j) \]
\[ \leq k_j \left( -\frac{3}{4} ||D \delta_j||^2 + \frac{5}{2} ||D \Pi_j \delta_j||^2 + \frac{5}{2} ||D \delta_j - D \Pi_j \delta_j||^2 \right). \]
We have by Hölder’s inequality, Young’s inequality, and the stability estimate (2.2) that
\[
|T_3| \leq \int_{I_j} \|\sigma(Du) - \sigma(DU_j)\| \|D\Pi j\delta_j\| dt
\leq \sqrt{k_j} \|\sigma(Du) - \sigma(DU_j)\|_{L^2(I_j; L^2)} \|D\Pi j\delta_j\|
\leq c^2_5 \text{Lip}^2(\sigma) \int_{I_j} \int_{\Omega} |Du - DU_j|^2 dx \, dt + \frac{k_j}{4c_5^2} \|D\Pi j\delta_j\|^2
\leq 2c^2_5 \text{Lip}^2(\sigma) \int_{I_j} \int_{\Omega} (|Du - DU_j|^2 + |Du_j - DU_j|^2) dx \, dt + \frac{k_j}{4} \|D\delta_j\|^2.
\]
Since
\[
|Du - DU_j|^2 = \int_{I_j}^t \int_{I_j} \|Du_j(s)\|^2 ds \leq (t - t) \int_{I_j} \|Du_j(s)\|^2 ds
\]
and
\[
\int_{I_j}^t \int_{I_j} \|Du_j(s)\|^2 ds \, dt \leq \int_{I_j}^t \int_{I_j} \|Du_j\|^2 \, ds \, dt = \frac{k_j^2}{2} \int_{I_j}^t \int_{I_j} \|Du\|^2 \, dx \, dt,
\]
we obtain
\[
\int_{I_j} \int_{\Omega} |Du - DU_j|^2 dx \, dt \leq \frac{k_j^2}{2} \int_{I_j} \int_{\Omega} |Du_j|^2 dx \, dt.
\]
The assertion of the proposition follows easily from the foregoing estimates. \qed

In the next proposition we derive the analogue of the estimate (2.5) for the discrete scheme.

**Proposition 3.2.** The following estimate holds for \( j = 1, \ldots, N \):
\[
\frac{1}{2} \|De_j\|^2 - \frac{1}{2} \|De_{j-1}\|^2 + \frac{1}{4} \|De_j - De_{j-1}\|^2
\leq (c^2_5 \text{Lip}^2(\sigma) + \frac{1}{4} k_j) \|De_j\|^2 - (\delta_j - \delta_{j-1}, e_j) + \frac{k_j}{2} \|D\delta_j\|^2
+ \frac{1}{4} \|\delta_j - \delta_{j-1}\|^2 + \|e_j - \Pi_je_j\|^2 + k_j \|De_j - D\Pi je_j\|^2
+ \frac{k_j}{2} \|DK_j\|^2 + \frac{1}{2} c^2_5 \text{Lip}^2(\sigma) k_j^2 \|Du_j\|^2_{Z(\text{sgn}(\delta_j), \Omega)}.
\]

**Proof.** We obtain from (2.10) with \( W_j = \Pi_je_j \) that
\[
\frac{1}{2} \|De_j\|^2 - \frac{1}{2} \|De_{j-1}\|^2 + \frac{1}{2} \|De_j - De_{j-1}\|^2 = (De_j - De_{j-1}, De_j - D\Pi je_j)
- (\delta_j - \delta_{j-1}, \Pi_je_j) - \int_{I_j} (\sigma(Du) - \sigma(DU_j), D\Pi je_j) dt.
\]
We denote the three terms on the right-hand side by \( T_1, T_2, \) and \( T_3 \). By Lemma 3.1,
\[
T_1 = k_j (D\delta_j + DK_j, De_j - D\Pi je_j)
\leq k_j \left( \frac{1}{4} \|D\delta_j\|^2 + \frac{1}{2} \|DK_j\|^2 + \|De_j - D\Pi je_j\|^2 \right).
\]
and by Young’s inequality
\[ T_2 = (\delta_j - \delta_{j-1}, e_j - \Pi_j e_j) - (\delta_j - \delta_{j-1}, e_j) \leq -\frac{1}{4}||\delta_j - \delta_{j-1}||^2 + ||e_j - \Pi_j e_j||^2. \]

Finally, \( T_3 \) can be estimated as in the proof of Proposition 3.1 by
\[ T_3 \leq \sqrt{\frac{k_j}{2j} \text{Lip}(\sigma)} ||Du - D\Pi_j||^2_{L^2(I_j; L^2)} ||D\Pi_j e_j|| \]
\[ \leq \frac{1}{2} c^2 \text{Lip}^2(\sigma) ||Du - D\Pi_j||^2_{L^2(I_j; L^2)} + \frac{k_j}{2c^2} ||D\Pi_j e_j||^2 \]
\[ \leq \frac{1}{2} c^2 \text{Lip}^2(\sigma) k_j^2 ||Du||^2_{L^2(I_j; L^2)} + \frac{1}{2} k_j ||De_j||^2. \]

The assertion of the proposition follows from the foregoing inequalities. \( \square \)

We now combine the estimates in Propositions 3.1 and 3.2 to obtain an estimate on the time interval \( I_j \).

**Corollary 3.1.** Let \( A_j \) denote the approximation errors,
\[ A_j = ||\delta_j - \Pi_j \delta_j||^2 + \frac{5}{2} k_j ||D\delta_j - D\Pi_j \delta_j||^2 + ||e_j - \Pi_j e_j||^2 + k_j ||De_j - D\Pi_j e_j||^2, \]
and \( R_j \) the terms depending on the regularity of \( u \),
\[ R_j = \frac{k_j}{2} ||K_j||^2 + 3k_j ||DK_j||^2 + \frac{3}{2} c^2 \text{Lip}^2(\sigma) k_j^2 ||Du||^2_{L^2(I_j; L^2)}. \]

Then the following estimate holds for \( j = 1, \ldots, N \):
\[ \frac{1}{2} ||\delta_j||^2 - \frac{1}{2} ||\delta_{j-1}||^2 + \frac{1}{2} ||De_j||^2 - \frac{1}{2} ||De_{j-1}||^2 + \frac{1}{4} ||De_j - De_{j-1}||^2 \]
\[ \leq \left( 3c^2 \text{Lip}^2(\sigma) + \frac{1}{2} \right) k_j ||De_j||^2 - (\delta_j - \delta_{j-1}, e_j) + A_j + R_j - \frac{k_j}{2} ||K_j||^2. \] \( \square \)

Remark. The discrete Gronwall inequality will be applied to the sum of (3.3) in \( j \). The sum of the terms \(- (\delta_j - \delta_{j-1}, e_j)\) on the right-hand side is estimated by a discrete summation by parts which leads to the (discrete) time integral of \(||\delta_j||^2\).

**Proposition 3.3.** Let
\[ a_j = \frac{k_j}{k_{j-1}}, \quad Q_{\nu} = \max_{j=2, \ldots, \nu} a_j, \quad c_3 = \max \left\{ 3c^2 \text{Lip}^2(\sigma) + \frac{1}{2} Q_N + \frac{1}{2} \right\}. \]

Then the following estimate holds for \( \nu = 1, \ldots, N \):
\[ \frac{1}{2} ||\delta_{\nu}||^2 + \frac{1}{2} ||De_{\nu}||^2 \leq c_3 \sum_{j=1}^{\nu} k_j (||\delta_j||^2 + ||De_j||^2) - (\delta_0, e_1) \]
\[ + \sum_{j=1}^{\nu} (A_j + R_j) + \frac{1}{2} ||\delta_0||^2 + \frac{1}{2} ||De_0||^2. \]
Proof. We take the sum of the inequality (3.3) for \(j = 1, \ldots, \nu\) and obtain
\[
\frac{1}{2}||\delta_\nu||^2 + \frac{1}{2}||De_\nu||^2 \leq (3\kappa^2 L^2(\sigma) + \frac{1}{2}) \sum_{j=1}^\nu k_j||De_j||^2 - \sum_{j=1}^\nu (\delta_j - \delta_{j-1}, e_j)
\]
\[
+ \sum_{j=1}^\nu (A_j + R_j - \frac{k_j}{2}||K_j||^2) + \frac{1}{2}||\delta_0||^2 + \frac{1}{2}||De_0||^2.
\]

With a discrete summation by parts in the second term on the right-hand side, we deduce that
\[
-\sum_{j=1}^\nu (\delta_j - \delta_{j-1}, e_j) = -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1} (\delta_j, e_{j+1} - e_j)
\]
\[
= -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1} k_{j+1} (\delta_j, \delta_{j+1} + K_{j+1})
\]
\[
\leq -(\delta_\nu, e_\nu) + (\delta_0, e_1) + \sum_{j=1}^{\nu-1} k_{j+1} (||\delta_j||^2 + \frac{1}{2}||\delta_{j+1}||^2 + \frac{1}{2}||K_{j+1}||^2).
\]

The assertion follows from this inequality since \(k_{j+1}||\delta_j||^2 = q_{j+1}k_j||\delta_j||^2\). \(\square\)

In order to apply Gronwall’s inequality, we need a further summation of the inequality in Proposition 3.3. The term \(-(\delta_\nu, e_\nu)\) corresponds to the spatial integral of \(u_\nu u = \partial_t ||u||^2/2\) which fits naturally in the formulation of Gronwall’s inequality in Section 2.2. This is not the case in the implicit time discretization used here.

**Proposition 3.4.** Let
\[
(3.5) \quad \varphi_\nu = ||e_\nu||^2 + \sum_{j=1}^\nu k_j (||\delta_j||^2 + ||De_j||^2).
\]

Then
\[
\frac{1}{2} \varphi_N \leq (c_3 + \frac{1}{2}) \sum_{\nu=1}^N k_\nu \varphi_\nu + T(\delta_0, e_1) + \frac{1}{2}||e_0||^2
\]
\[
+ \frac{T}{2} (||\delta_0||^2 + ||De_0||^2) + 2 \sum_{\nu=1}^N k_\nu \sum_{j=1}^\nu (A_j + R_j).
\]

Proof. We multiply the inequality in the assertion of Proposition 3.3 by \(k_\nu\) and take the sum of the resulting inequalities from \(\nu = 1\) to \(N\). This leads to
\[
\frac{1}{2} \sum_{\nu=1}^N k_\nu (||\delta_\nu||^2 + ||De_\nu||^2) \leq c_3 \sum_{\nu=1}^N k_\nu \sum_{j=1}^\nu k_j (||\delta_j||^2 + ||De_j||^2)
\]
\[
- \sum_{\nu=1}^N k_\nu (\delta_\nu, e_\nu) + T(\delta_0, e_1) + \sum_{\nu=1}^N k_\nu \sum_{j=1}^\nu (A_j + R_j) + \frac{T}{2} (||\delta_0||^2 + ||De_0||^2).
\]
In view of Lemma 3.1,

\[- \sum_{\nu=1}^{N} k_{\nu}(\delta_{\nu}, e_{\nu}) = - \sum_{\nu=1}^{N} (e_{\nu} - e_{\nu-1} - k_{\nu}K_{\nu}, e_{\nu}) \]

\[= - \frac{1}{2} \sum_{\nu=1}^{N} (||e_{\nu}||^2 - ||e_{\nu-1}||^2 + ||e_{\nu} - e_{\nu-1}||^2) + \sum_{\nu=1}^{N} k_{\nu}(K_{\nu}, e_{\nu}) \]

\[\leq - \frac{1}{2} ||e_{N}||^2 + \frac{1}{2} ||e_{0}||^2 + \frac{1}{2} \sum_{\nu=1}^{N} k_{\nu}(||e_{\nu}||^2 + ||K_{\nu}||^2). \]

This implies

\[\frac{1}{2} \left\{ ||e_{N}||^2 + \sum_{\nu=1}^{N} k_{\nu}(||\delta_{\nu}||^2 + ||De_{\nu}||^2) \right\} \]

\[\leq (c_{3} + \frac{1}{2}) \sum_{\nu=1}^{N} k_{\nu} \left\{ ||e_{\nu}||^2 + \sum_{j=1}^{\nu} k_{j} (||\delta_{j}||^2 + ||De_{j}||^2) \right\} \]

\[+ T(\delta_{0}, e_{1}) + \frac{1}{2} ||e_{0}||^2 + \frac{T}{2} \left\{ ||\delta_{0}||^2 + ||De_{0}||^2 \right\} + 2 \sum_{\nu=1}^{N} k_{\nu} \sum_{j=1}^{\nu} (A_{j} + R_{j}). \]

This implies the assertion of the proposition. \( \square \)

It only remains to estimate the term \( T(\delta_{0}, e_{1}) \) on the right-hand side. This is done in the next lemma.

**Lemma 3.2.** Let \( \varphi_{\nu} \) be the quantity defined in (3.5). Then

\[|T(\delta_{0}, e_{1})| \leq k_{1} \sum_{\nu=1}^{N} k_{\nu}\varphi_{\nu} + 2 k_{1}TR_{1} + \frac{3T}{4} ||\delta_{0}||^2 + T||e_{0}||^2. \]

**Proof.** It follows from (3.1) that

\[|\delta_{0}, e_{1}| \leq |\delta_{0}, e_{1} - e_{0}| + ||\delta_{0}, e_{0}|| \leq |k_{1}(\delta_{0}, \delta_{1} + K_{1}) + ||\delta_{0}, e_{0}|| \]

\[\leq \frac{1}{4} ||\delta_{0}||^2 + k_{1}||\delta_{1}||^2 + \frac{1}{4} ||\delta_{0}||^2 + k_{1}||K_{1}||^2 + \frac{1}{2} ||\delta_{0}||^2 + ||e_{0}||^2. \]

By definition, \( k_{1}||K_{1}||^2 \leq 2R_{1} \) and \( k_{1}||\delta_{1}||^2 \leq \varphi_{\nu} \) for \( \nu = 1, \ldots, N \). Thus

\[T(\delta_{0}, e_{1}) \leq \frac{3T}{4} ||\delta_{0}||^2 + T||e_{0}||^2 + k_{1} \sum_{\nu=1}^{N} k_{\nu}\varphi_{\nu} + 2 k_{1} \sum_{\nu=1}^{N} k_{\nu}R_{1}. \]

This implies the assertion of the lemma. \( \square \)

**4. General convergence result**

We are now in a position to state and prove the convergence result for the approximation of solutions of the system (1.1) by the implicit Euler scheme defined in Section 2.3. Recall that \( A_{j}, R_{j} \) and \( \varphi_{\nu} \) have been defined in Proposition 3.1 and Corollary 3.4, respectively. Moreover, we set

\[A(T) = 8(1 + k_{1}) \sum_{\nu=1}^{N} k_{\nu} \sum_{j=1}^{\nu} A_{j}, \quad R(T) = 8(1 + k_{1}) \sum_{\nu=1}^{N} k_{\nu} \sum_{j=1}^{\nu} R_{j}. \]
and
\begin{equation}
\alpha = 2(2T + 1)||e_0||^2 + 2T||De_0||^2 + 5T||\delta_0||^2.
\end{equation}
Finally, we define \( c_4 = c_4(\text{Lip}(\sigma), Q_N) \) and \( c_5 = c_5(\text{Lip}(\sigma), Q_N) \) by
\begin{equation}
\begin{aligned}
c_4 &= 4(c_3 + \frac{1}{2} + k_1), \\
\alpha &= \max \left\{ \frac{3}{2} c_3^2 \text{Lip}^2(\sigma), 1 \right\},
\end{aligned}
\end{equation}
respectively.

**Theorem 4.1.** Suppose that \( \Omega, T_j \) and \( S_{0,j} \) satisfy the assumptions in Section 2.1, and that there exists a \( Q_N > 0 \) such that \( k_j \leq Q_N k_j \) for \( j = 2, \ldots, N \). Assume, furthermore, that \( \sigma \) is globally Lipschitz continuous, and that the unique solution of the system (1.1) belongs to \( W^{1,2}(W^{1,2}) \). Suppose that \( k < 1/c_4 \). Then
\[
\max_{\nu = 1, \ldots, N} ||e_{\nu}||^2 + \sum_{\nu = 1}^N k_\nu (||\delta_\nu||^2 + ||De_\nu||^2) \leq (\alpha + A(T) + R(T)) \exp(\alpha T).
\]
Moreover, if \( u \in W^{1,\infty}(W^{2,2}) \cap W^{2,2}(L^2) \), then
\[
\mathcal{R}(T) \leq 12c_5 T^2 k^2 \left( ||u||_{L^2(\Omega)}^2 + ||Du||_{L^2(\Omega)}^2 + ||Du||_{L^2(\Omega)}^2 \right),
\]
and
\[
\mathcal{A}(T) \leq 60c_3^2 T^2 k^2 \max_{\nu = 1, \ldots, N} \left\{ \frac{h_j^2}{k_j} + 1 \right\} \|u\|_{W^{1,\infty}(W^{2,2})} + \mathcal{C}(T),
\]
where \( \mathcal{C}(T) \) is the coarsening error,
\[
\mathcal{C}(T) = 12T \sum_{\nu = 1}^N \left\{ \frac{2}{k_j} ||U_{\nu,j-1} - \Pi_j U_{\nu,j-1}|| + \frac{5}{k_j} ||U_{\nu,j-1} - \Pi_j U_{\nu,j-1}||_{L^2} \right\}
\]
which vanishes if \( T_j \) is a refinement of \( T_{j-1} \). Finally, if the approximation estimate (2.7) holds, then
\[
\alpha \leq c_4^2 h_0^2 ((4T + 2) h_0^2 + 4T) \left( ||D^2 u_0||^2 + ||Dv_0||^2 \right).
\]

**Proof.** Based on the estimates in Section 3, we first show that \( \varphi_N \) satisfies the assumptions in the discrete Gronwall inequality of Lemma 2.1. It follows from Proposition 3.4 and Lemma 3.2 that
\[
\begin{aligned}
\frac{1}{2} \varphi_N &\leq \left( c_3 + \frac{1}{2} + k_1 \right) \sum_{\nu = 1}^N k_\nu \varphi_N + 2(1 + k_1) \sum_{\nu = 1}^N k_\nu \sum_{j = 1}^N (A_j + \mathcal{R}_j) \\
&+ (T + \frac{1}{2}) ||e_0||^2 + \frac{T}{2} ||De_0||^2 + \frac{5T}{4} ||\delta_0||^2.
\end{aligned}
\]
By assumption, \( k_N \leq k \) and thus \( c_4 k_N = 4(c_3 + 1/2 + k_1)k_N \leq 1 \). This allows us to absorb the term \( (c_3 + \frac{1}{2} + k_1)k_N \varphi_N \leq \varphi_N/4 \) on the left-hand side. We obtain
\[
\begin{aligned}
\varphi_N &\leq c_4 \sum_{\nu = 1}^N k_\nu \varphi_N + 8(1 + k_1) \sum_{\nu = 1}^N k_\nu \sum_{j = 1}^N (A_j + \mathcal{R}_j) \\
&+ 2(2T + 1) ||e_0||^2 + 2T ||De_0||^2 + 5T ||\delta_0||^2.
\end{aligned}
\]
It follows that the assumptions in the discrete Gronwall inequality in Lemma 2.1 are satisfied with $a$ as defined in (4.1) and
\[ b_{\nu} = 8(1 + k_1)k_\nu \sum_{j=1}^{\nu} (A_j + R_j), \quad \tau_{\nu} = c_4 k_{\nu}, \quad \nu = 1, \ldots, N. \]
Thus\[ \varphi_N \leq (a + \sum_{\nu=1}^{N} b_{\nu}) \exp(c_4 T) = (a + A(T) + R(T)) \exp(c_4 T). \]
We now estimate $A(T)$ and $R(T)$. By definition of $R_j$ in Corollary 3.1 and by (3.2) we infer\[ R_j \leq c_5 k_j^2 \left( \|u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} \right). \]
This implies\[ \sum_{j=1}^{N} k_j \sum_{\nu=1}^{\nu} R_j \leq c_5 T k^2 \left( \|u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} \right). \]
Since $k c_4 \leq 1$ implies $k_1 \leq 1/2$ we obtain\[ R(T) \leq 12 c_5 T k^2 \left( \|u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} + \|D u_{t_{\nu}}\|^2_{L^2(I_j;L^2_{\nu})} \right). \]
It remains to estimate the approximation error. By definition of $e_j$ and (2.1),\[ e_j - \Pi_j e_j = u(t_j) - U_j - \Pi_j (u(t_j) - U_j) = u(t_j) - \Pi_j u(t_j). \]
The approximation estimate (2.3) implies\[ \|e_j - \Pi_j e_j\|^2 + k_j \|D e_j - D \Pi_j e_j\|^2 \leq 3 c_4 h_j^2 (h_j^2 + \delta_j) \|D^2 u_{t_j}\|^2. \]
Similarly,\[ \delta_j - \Pi_j \delta_j = v(t_j) - \Pi_j v(t_j) + \frac{1}{k_j} (U_{j-1} - U_j U_{j-1}) \]
This estimate and (2.2)-(2.3) yield\[ \|\delta_j - \Pi_j \delta_j\|^2 + \frac{5}{2} k_j \|D \delta_j - D \Pi_j \delta_j\|^2 \leq 2 c_4 h_j^2 (h_j^2 + \frac{5}{2} k_j) \|D^2 u_{t_j}\|^2 \]
\[ + \frac{2}{k_j} \|U_{j-1} - \Pi_j U_{j-1}\|^2 + \frac{5}{k_j} \|U_{j-1} - \Pi_j U_{j-1}\|^2. \]
We conclude from the foregoing estimates that\[ \sum_{j=1}^{N} A_j \leq 5 \sum_{j=1}^{N} h_j^2 c_4 \left( h_j^2 + \frac{5}{2} k_j \right) \left( \|D^2 u_{t_j}\|^2 + \|D^2 u_{t_j}\|^2 + \frac{1}{12 T} C(T) \right) \]
\[ \leq 5 c_4^2 T h^2 \max_{j=1,\ldots,N} \left( \frac{h_j^2}{k_j} + 1 \right) \|u\|^2_{W^{1,\infty}(W_{2,2})} + \frac{1}{12 T} C(T). \]
Hence\[ A(T) \leq 12 T \sum_{j=1}^{N} A_j \leq 60 c_4^2 T h^2 \max_{j=1,\ldots,N} \left( 1 + \frac{h_j^2}{k_j} \right) \|u\|^2_{W^{1,\infty}(W_{2,2})} + C(T). \]
Clearly, if $T_j$ is a refinement of $T_{j-1}$, then, by (2.1), $C(T) = 0$. Finally, if (2.7) holds, then

$$a \leq c_A^2 \left( 2(2T + 1)b_0^1 + 2T b_0^2 \right) \|D^2 w_0\|^2 + 4T b_0^2 \|D w_0\|^2.$$

This proves the assertion of the theorem. \qed

\textbf{Proof of Theorem 1.1.} This is a special case of Theorem 4.1. \qed

\begin{thebibliography}{99}


\end{thebibliography}