Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The logarithmic tail of Néel walls in thin films

by

Christof Melcher

Preprint no.: 61 2001

The logarithmic tail of Néel walls in thin films

Christof Melcher

Max-Planck-Institut für Mathematik
in den Naturwissenschaften, Leipzig

Abstract

We study the multiple scale problem of a parametrized planar 180° rotation of magnetization states in a thin ferromagnetic film. In an appropriate scaling and when the film thickness is comparable to the Bloch line width the underlying variational principle has the form

$$Q \left\|m\right\|^2_{\dot{H}^1} + \left\|u\right\|^2_{L^2} + \left\langle u \middle| \mathcal{S}_Q \middle| u \right\rangle \to \min$$
$$m = (u,v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1$$

where the reduced strayfield operator $\mathcal{S}_Q$ approximates $(-\Delta)^{1/2}$ as the quality factor Q tends to zero. We show that the associated Néel wall profile u exhibits a very long logarithmic tail. The proof relies on limiting elliptic regularity methods on the basis of the associated Euler-Lagrange equation and symmetrization arguments on the basis of the variational principle. Finally we study the renormalized limit behaviour as Q tends to zero.

Contents

Introduction

Ferromagnetic materials show a complex variety of magnetic patterns, which involve many different length scales, see e.g. [6], [3]. Despite their multitude these patterns can often be understood through minimization of a simple energy functional, the micromagnetic energy. For a ferromagnetic body occupying a domain $\Omega \subset \mathbb{R}^3$ and in the absence of external fields it is given by

$$w^2 \int_\Omega |\nabla \mathbf{m}|^2 + Q \int_\Omega \varphi(\mathbf{m}) + \int_{\mathbb{R}^3} |\mathcal{H}(\mathbf{m})|^2 \quad \text{for} \quad \mathbf{m} : \Omega \to \mathbb{S}^2.$$

The Dirichlet part, the so called exchange part, comes from the quantum mechanical spin interaction. The function $\varphi : \mathbb{S}^2 \to [0, \infty)$ is an even polynomial and models crystalline anisotropy. Finally $\mathcal{H}(\mathbf{m}) : \mathbb{R}^3 \to \mathbb{R}^3$ is up to the sign the induced magnetostatic field, sometimes called the strayfield. In fact $\mathcal{H}(\mathbf{m})$ is the Helmholtz projection onto gradient fields of the magnetization field extended by 0 outside the ferromagnetic body. Formally it is given by $\mathcal{H}(\mathbf{m}) = \nabla \Delta^{-1} \nabla \cdot \mathbf{m}$. In particular this nonlocal term vanishes if $\nabla \cdot \mathbf{m} = 0$ in the distributional sense. There are two material parameters involved. The first is the quality factor $Q > 0$ which characterizes the strength of the anisotropy relative to the magnetostatic energy. The second is the exchange length $w > 0$ which measures (for fixed spatial units) the strength of the exchange energy relative the magnetostatic energy. The ratio $w/\sqrt{Q}$ defines the characteristic length scale of the Bloch wall transition to be introduced below.
A typical feature of minimizers of this energy is that they contain large

regions in which the magnetization changes slowly, the so called magnetic domains, separated by thin transition layers, known as domain walls. The simplest among them are 180° walls separating two domains of opposite direction. If a domain wall is considered to be plane and one-dimensional, there are two prototypical transition modes:

- The *Bloch wall* [1] where the transition proceeds perpendicular to the transition axis, is observed in bulk materials. The main feature is the avoidance of magnetic volume charges, i.e. $\nabla \cdot \mathbf{m} = 0$. It was first proposed and calculated by Landau and Lifshitz.

- The *Néel wall* [2] where the transition proceeds entirely in the plane spanned by the transition axis and the endstates, is observed in a suitable thin film regime due to the high penalty on out of plane magnetizations, see [4]. In contrast to the Bloch wall transition, volume charges have to be taken into account.

The domain wall can be described in both cases by the nontrivial component function which vanishes at the end states, the wall profile. In an infinitely extended material the Bloch wall can be computed explicitly. After a change of variable $x \mapsto \frac{\sqrt{Q}}{w} x$ the reduced variational principle for the 1d magnetization becomes

$$\mathcal{E}(m) = \|m\|^2_{\dot{H}^1} + \|u\|^2_{L^2} \to \min$$
$$m = (u, v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1.$$

and the resulting Euler-Lagrange equations can be solved, see [6]. The profile exhibits exponential decay beyond a transition zone of order unity. The situation for Néel walls is more complicated due to the contribution of the magnetostatic strayfield to the transition energy. The presence of three energy components with different scaling behaviour does not allow for a family of profiles which is generated by rescaling the transition parameter of a reference profile.

Instead of an infinite bulk material we consider an infinite ferromagnetic layer of thickness δ which we assume to be smaller than or at most comparable to the exchange length w. Rescaling $x \mapsto \frac{\delta}{Q} x$ we can eliminate one parameter by introducing the dimensionless aspect ratio $\kappa = w/\delta \geq 1$. Then

[1] named after Felix Bloch (1905-1983) who first conceived a continuous wall transition.

[2] named after Louis Néel (1904-2000) who first proposed an in-plane transition for domain walls in thin films.

the reduced variational principle becomes

$$\mathcal{E}_Q^\kappa(m) = \kappa^2\, Q\, \big\|m\big\|_{\dot H^1}^2 + \big\|u\big\|_{L^2}^2 + \Big\langle u \Big| \mathcal{S}_Q \Big| u \Big\rangle \to \min \qquad (0.1)$$
$$m = (u,v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1$$

where the reduced strayfield operator $\mathcal{S}_Q$ approximates $(-\Delta)^{1/2}$ as the quality factor Q tends to zero (see below for a more detailed description). Our goal is to find a universal profile together with an outer scaling law which approximates the Néel wall transition beyond a core region as Q tends to zero.

The main analytical feature of the variational problem (0.1) is that the energy gives only uniform control of the $H^{1/2}$-norm as Q tends to zero. Since $H^{1/2}(\mathbb{R})$ does not embed into $C^0(\mathbb{R})$, the pointwise constraint $u(0) = 1$ is delicate and one might expect a logarithmic singularity in a renormalized setting. Logarithmic tails of Néel walls have indeed been predicted by heuristic arguments and the resulting very long range interaction between different Néel walls has important consequences, see [6] pp. 242-245 and [8]. Logarithmic scaling for the energy was recently established in [2]. Here we prove for the first time that the full profile exhibits a logarithmic behaviour. More precisely we show:

(i) A Néel wall profile is nonnegative and symmetrically decreasing.

(ii) A Néel wall profile has logarithmic decay on the scale δ/Q:

$$u_Q^\kappa(x) \le c\, \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\, \log(1/x) \quad \text{for} \quad Q < x < 1/e$$

(iii) After a suitable renormalization the rescaled profiles converge to a multiple of the fundamental solution of the operator $(-\Delta)^{1/2} + \mathbf{1}$ as Q tends to zero.

We briefly discuss the main ideas and outline the proof. We consider the variational principle (0.1) as a singular perturbation problem with Q as the small parameter. We control the profile uniformly in $H^{1/2}(\mathbb{R})$ by the energy and in $L^\infty(\mathbb{R})$ by the saturation condition $|m| = 1$. The first step is to establish monotonicity which provides the link between mean values and the pointwise behaviour. In this context the nonlocal character of the energy enforces global symmetry of the profile. The main part is to derive L^p-bounds in terms of the energy. Since $H^{1/2}(\mathbb{R})$ does not embed into $L^\infty(\mathbb{R})$ such bounds necessarily diverge as p tends to infinity and it will be crucial to keep track of the precise dependence on p. Optimizing the choice of p

for a given point x we formally obtain the logarithmic decay. The bounds suggest to renormalize Néel wall profiles by the energy and to study their limit behaviour as Q tends to zero.

1 Mathematical framework for Néel walls

1.1 Mathematical notations and conventions

In these notes we consider mainly scalar functions $u : \mathbb{R} \to \mathbb{R}$ of a real variable x. We denote by $u' = \frac{du}{dx}$ the first derivative. In analogy to higher dimensions and in regard to our usage of Fourier multipliers we denote the second derivative by the Laplace operator symbol $\Delta = d^2/dx^2$. The integral symbol $\int$ with no specifications means integration over the whole real line. We denote by $\mathfrak{S}(\mathbb{R})$ the Schwartz class of rapidly decreasing smooth functions and for an open subset $\Omega \subseteq \mathbb{R}$ we indicate by $\mathfrak{D}(\Omega)$ the class of smooth functions which are compactly supported in Ω. $\mathfrak{S}'(\mathbb{R})$ and $\mathfrak{D}'(\Omega)$ are respectively their topological duals, the tempered and Schwartz distributions. Concerning the Fourier transform we make the convention that for $f \in \mathfrak{S}(\mathbb{R})$

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi}\, f(x)\, dx \in \mathfrak{S}(\mathbb{R})$$

with the usual extension to tempered distributions and in particular to a unitary operation on $L^2(\mathbb{R})$. Note that Parseval's formula $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ holds without additional constants. We denote the inverse Fourier transform by $\mathcal{F}^*$.

For a locally bounded function $\sigma \in C^1(\mathbb{R} \setminus \{0\})$ with at most polynomial growth at infinity let the Fourier multiplication operator $\mathcal{S}$ associated to the symbol (Fourier multiplier) $\mathcal{F}\mathcal{S} = \sigma$ be given by

$$\mathcal{S}f = \sigma(D)\, f = \mathcal{F}^*\big(\xi \mapsto \sigma(\xi)\, \hat{f}(\xi)\big) \in \mathfrak{S}'(\mathbb{R}) \quad \text{for} \quad f \in \mathfrak{S}(\mathbb{R}).$$

For $1 < p < \infty$ and $s \in \mathbb{R}$ the fractional Sobolev spaces (or Bessel potential spaces) are defined as follows, see e.g. [1] or [10]

$$H^s_p(\mathbb{R}) = \big\{ u \in \mathfrak{S}'(\mathbb{R}) \,|\, (1-\Delta)^{s/2} u \in L^p(\mathbb{R}) \big\},$$

where for $f \in \mathfrak{S}(\mathbb{R})$ the function $(1-\Delta)^{s/2} f \in \mathfrak{S}(\mathbb{R})$ is given in frequency space by $(1+|\xi|^2)^{s/2} \hat{f}(\xi)$ and the action of $(1-\Delta)^{s/2}$ on a tempered distribution $u \in \mathfrak{S}'(\mathbb{R})$ is defined by duality. For the Hilbert space case $p = 2$ we omit the index p for convenience. In the regime $s > 0$ the dot signifys the homogeneous Sobolev norm

$$\|u\|_{\dot{H}^s} = \left\|(-\Delta)^{s/2}u\right\|_{L^2} = \left\||\xi|^s\hat{u}(\xi)\right\|_{L^2}.$$

The notation carries over to vectorfields $m : \mathbb{R} \to \mathbb{R}^2$ by means of the Euclidian norm. In particular we denote the Dirichlet energy consistently by

$$\left\|m\right\|_{\dot{H}^1}^2 = \left\|m'\right\|_{L^2}^2.$$

The term *saturated* is common in mathematical micromagnetics to signify vectorfields of constant modulus 1.

1.2 Dimensional reduction and scaling

We consider an infinite uniaxial ferromagnetic layer $\Omega_\delta = \mathbb{R}^2 \times (-\delta, \delta)$ of small thickness. Recall that the micromagnetic energy density in the absence of external fields associated to some magnetization field $\mathbf{m} : \Omega_\delta \to \mathbb{S}^2$ is given by

$$w^2\, |\nabla \mathbf{m}|^2 + Q\, \varphi(\mathbf{m}) + |\mathcal{H}(\mathbf{m})|^2.$$

A parametrized in-plane rotation from $m(-\infty)$ to $m(\infty)$ is described by a reduced magnetization field $m : \mathbb{R} \to \mathbb{S}^1$ having in-plane components (u, v) which only depend on the x variable. Then integration of the resulting energy density over the whole space is clearly infinite. To find an appropriate renormalization we first consider the magnetostatic part. Here the geometry of the thin film configuration has an impact in terms of the film diameter due to the jump of the extended magnetization field at the boundary. Note that after continuation of $\mathbf{m}$ by zero outside the strip we have that $\mathcal{H}(\mathbf{m}) = \nabla\Delta^{-1}\nabla \cdot \mathbf{m}$. So if inside the strip $\mathbf{m} = (u, v, 0)$ only depends on the first variable then for fixed $\delta > 0$ the magnetostatic field $\mathcal{H}(\mathbf{m})$ is completely determined by the first component function u. Furthermore its second component vanishes, i.e. the vectorfield $\mathcal{H}(\mathbf{m})$ lives on the cutting plane along the transition axis, and there is no dependence on the y variable. We define the reduced strayfield operator

$$\mathcal{S}_\delta : u \mapsto \frac{1}{\delta} \fint_{-\delta}^{\delta} \mathcal{H}(\mathbf{m}) \cdot \hat{e}_1 \, dz$$

for any saturated in-plane extension $\mathbf{m} = (u, v, 0)$ of the profile u. The appearance of the additional factor $\frac{1}{\delta}$ in our definition will become clear when we study the scaling properties of $\mathcal{S}_\delta$. In fact the bilinear form $\langle u|\mathcal{S}_\delta|u\rangle$ is scaling invariant in the sense that for $u_\lambda(x) = u(\lambda\, x)$ we have that

$$\langle u_\lambda|\mathcal{S}_{(\lambda^{-1}\delta)}|u_\lambda\rangle = \langle u|\mathcal{S}_\delta|u\rangle.$$

Then by careful integration by parts (see section 5.2 in the Appendix) it is easy to see that a sensibly averaged description of the magnetostatic energy contribution is given by

$$\frac{1}{2\delta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{H}(\mathbf{m})|^2\; dx\; dz = \int_{-\infty}^{\infty} \fint_{-\delta}^{\delta} \mathbf{m}\cdot\mathcal{H}(\mathbf{m})\; dz\; dx = \delta\,\langle u|\mathcal{S}_\delta|u\rangle$$

The strayfield operator $\mathcal{S}_\delta$ is a Fourier multiplication operator. One finds that the operator $\mathcal{S}_\delta$ is given by the self-similarly rescaled symbol $\sigma_\delta(\xi) = \frac{1}{\delta}\,\sigma(\delta\,\xi)$ where the function $\sigma : \mathbb{R}\to\mathbb{R}$ reads like

$$\sigma(\xi) = 1 - \frac{1-\exp(-2|\xi|)}{2|\xi|} \sim \begin{cases} |\xi| & \text{for small } \xi \\ 1 & \text{for large } \xi. \end{cases}$$

The derivation of the symbol is done in section 5.2 in the Appendix. For the local components of the energy density, which only depend on the x variable, averaging along the y and z axis inside the strip is redundant. An appropriate description according to the one for the induced field energy is achieved by integration along the transition axis. Summing up all energy components, when we assume that for $\mathbf{m} = (u, v, 0)$ the anisotropy term can be expressed by $\varphi(\mathbf{m}) = |u|^2$, yields the following variational principle

$$\begin{gathered} \mathcal{F}^\delta_{Q,w}(m) = w^2\;\|m\|^2_{\dot{H}^1} + Q\;\|u\|^2_{L^2} + \delta\;\langle u|\mathcal{S}_\delta|u\rangle \\ m = (u, v) : \mathbb{R}\to\mathbb{S}^1 \quad\text{with}\quad u(0) = 1 \end{gathered} \tag{1.1}$$

where we imposed the constraint that a 90° rotation is achieved at the origin in order to get rid of the translation invariance of the profile and to exclude trivial solutions. The functional has the dimension of length due to integration.

In order to put this variational principle into a nondimensional framework we rescale the transition parameter. We distinguish between the thin film situation where the diameter δ is comparable or smaller than the exchange length w and the counterpart, i.e. the bulk situation. More specifically we rescale

$$x \mapsto \max\left(\frac{Q}{w}, \frac{Q}{\delta}\right)\, x.$$

We introduce the aspect ratio $\kappa = \frac{w}{\delta}$ and distinguish between the regimes $\kappa \geq 1$ and $0 < \kappa < 1$. In the thin film scaling the competition between the anisotropy and the induced field is emphasized whereas the bulk scaling is reminiscent of the Bloch wall scaling. Nevertheless there is always an interplay between the exchange and the induced field part competing with anisotropy. Then for large $\kappa \geq 1$ after a further renormalization by δ the variational principle turns into the following form for rescaled magnetization fields m

$$\mathcal{E}_Q^\kappa(m) = \kappa^2 \, Q \, \|m\|_{\dot{H}^1}^2 + \|u\|_{L^2}^2 + \left\langle u \middle| \mathcal{S}_Q \middle| u \right\rangle \to \min \qquad (1.2)$$
$$m = (u,v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1.$$

Remember that the Fourier multiplication operator $\mathcal{S}_Q$ is given by the Fourier symbol $\sigma_Q(\xi)$ which approaches the function $\xi \mapsto |\xi|$ as Q tends to zero. Therefore the family $\mathcal{S}_Q$ interpolates between zero and first order operators. Note that each appearing quantity is now dimensionless.
In the same manner we get in the regime $0 < \kappa < 1$ after renormalization by w for rescaled magnetization fields $\tilde{m}$ the following variational principle

$$\tilde{\mathcal{E}}_Q^\kappa(\tilde{m}) = Q \, \|\tilde{m}\|_{\dot{H}^1}^2 + \|\tilde{u}\|_{L^2}^2 + \left\langle \tilde{u} \middle| \mathcal{S}_Q^\kappa \middle| \tilde{u} \right\rangle \to \min \qquad (1.3)$$
$$\tilde{m} = (\tilde{u},\tilde{v}) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad \tilde{u}(0) = 1.$$

Here the aspect ratio $0 < \kappa < 1$ is attached to the strayfield operator $\mathcal{S}_Q^\kappa$ which is given by the symbol $\sigma_Q\left(\xi/\kappa\right)$.
Note that we did the renormalizations the way that $\inf \mathcal{E}_Q^\kappa / \inf \tilde{\mathcal{E}}_Q^\kappa = \kappa$ where the infima are taken over the set $\mathcal{M}$ of (saturated) magnetization fields with one point constraints at zero. We call the first component functions u and $\tilde{u}$ of a solution of either of the above variational principles a (rescaled) Néel wall profile.

1.3 A model problem related to Néel walls

To discuss some key ideas in our analysis we first consider a linear singular perturbation problem which is closely related to the analysis of Néel profiles. Note that the highest order term in the induced field energy corresponds to the square of the homogeneous $H^{1/2}$-norm. Furthermore we relax the saturation constraint $|m| = 1$ which agrees with the assumption that the second field component vanishes. Finally we get

$$\mathcal{I}_\varepsilon(u) = \varepsilon \, \|u\|_{\dot{H}^1}^2 + \|u\|_{\dot{H}^{1/2}}^2 + \|u\|_{L^2}^2 \to \min$$
$$u : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad u(0) = 1$$

where $\varepsilon > 0$ corresponds to $\kappa^2 Q$. This variational problem is well posed for each $\varepsilon > 0$ due to H^1-coercivity and the resulting a priori continuity of a minimizer. Note that there is no solution for $\varepsilon = 0$: due to the scaling invariance of the homogeneous $H^{1/2}$-norm each minimizing sequence tends to zero in $L^2(\mathbb{R})$. The operator associated with the quadratic form $\mathcal{I}_\varepsilon$ is the singular perturbed elliptic operator

$$\mathcal{L}_\varepsilon = \varepsilon\,(-\Delta) + (-\Delta)^{1/2} + \mathbf{1}$$

with Fourier symbol given by $P_\varepsilon(\xi) = \varepsilon\,|\xi|^2 + |\xi| + 1$. Note that the symbol $1/P_\varepsilon(\xi)$ of the inverse operator belongs to $L^1(\mathbb{R})$, but the norm explodes as ε tends to zero. In fact $\int \frac{d\xi}{P_\varepsilon(\xi)} \sim \log(1/\varepsilon)$ which turns out to be the inverse minimal energy. Now a minimizer u_ε solves the following linear Euler-Lagrange equation

$$\begin{aligned} \mathcal{L}_\varepsilon u_\varepsilon &= \Lambda(\varepsilon)\ \delta_0 \quad \text{in} \quad \mathfrak{D}'(\mathbb{R}) \\ \Lambda(\varepsilon) &= \inf\left(\mathcal{I}_\varepsilon(u)\ ,\ u(0) = 1\right). \end{aligned} \tag{1.4}$$

Using Fourier transform the minimizing profile u_ε can be calculated explicitly. Careful expansions suggest a division into three regions: the core region ε-close to the origin where $u(x) \sim 1$, the tail region up to $|x| \sim 1$ with logarithmic decay like $u(x) \sim \log(1/|x|)/\log(1/\varepsilon)$, and the far tail for large $|x|$ with algebraic decay. Nevertheless in view of more complicated nonlinear situations as the one we are going to study in these notes, we present a technique of getting an idea of the actual shape and parameter dependence of a minimizer by elliptic regularity theory. We have that

$$\|u_\varepsilon\|^2_{H^{1/2}} \le \Lambda(\varepsilon) = \left(\int \frac{d\xi}{P_\varepsilon(\xi)}\right)^{-1} \sim \frac{1}{\log(1/\varepsilon)} \to 0 \tag{1.5}$$

so energetics imply that u_ε tends to zero in $H^{1/2}(\mathbb{R})$ as ε tends to zero with a bound proportional to square root of the energy. But the computation of the actual energy parts shows that there is no equipartition: the L^2-norm tends to zero twice as fast as the rest, i.e. with a bound proportional to the energy. Actually by a simple elliptic regularity argument (compare Lemma 3.2) each L^p-norm for $p > 2$ is bounded by the energy and the control fades linearly in p. Moreover for each $1 < q < 2$ the $H^{1/2}_q$-norm has a bound which is proportional to the energy. Nonnegativity of the solution can be shown by means of symmetrization and implies an L^1-bound of the solution by integrating the equation (compare Proposition 1 in section 2.3 and the arguments in section 3.3). Finally we arrive at an estimate of the form

$$\|u_\varepsilon\|_{L^p} \le c\,\Lambda(\varepsilon)\ p \quad \text{for each} \quad p \ge 1$$

for some universal constant $c > 0$ independent of p and ε. Now the link to a pointwise estimate is given by the monotonicity of a minimizer for $x > 0$ which can be shown by some symmetrization procedure as in the proof of Proposition 1 in section 2.3. Clearly $u_\varepsilon(x) \leq ⨍_0^x u_\varepsilon$ for $x > 0$. Then by Hölder's inequality the L^p-estimate provide, as in the proof of the decay estimate in section 4.1, with an optimal choice of $p = p(x)$ the desired estimate

$$0 \leq u_\varepsilon(x) \leq c\,\Lambda(\varepsilon)\ \log(1/x) \quad \text{for} \quad 0 < x < 1/e$$

for some universal constant $c > 0$ independent of $\varepsilon > 0$.
To derive some reasonable limit problem we introduce the renormalized profiles

$$U_\varepsilon = \Lambda(\varepsilon)^{-1} u_\varepsilon \in H_q^{1/2}(\mathbb{R}) \quad \text{for each} \quad 1 < q < 2.$$

We choose some weakly converging sequence as ε tends to zero with weak limit $U_0 \in H_q^{1/2}(\mathbb{R})$. Combining equation (1.4) with the logarithmic energy bound (1.5) it is easy to see that U_0 agrees with the fundamental solution G of the operator $(-\Delta)^{1/2} + \mathbf{1}$. Each sequence must furthermore converge strongly in $L^p_{loc}(\mathbb{R})$ for each $1 < p < \infty$. Therefore we obtain that

$$u_\varepsilon(x) = \Lambda(\varepsilon)\,\Big\{G(x) + R_\varepsilon(x)\Big\}$$

where R_ε converges to zero in measure on each bounded zero neighbourhood. The fundamental solution G for the operator $(-\Delta)^{1/2} + \mathbf{1}$ is clearly given by the even Fourier transform (Cosine transform) of the symbol $1/(|\xi| + 1)$ of the inverse operator (see e.g. [11] p. 191) and reads as follows

$$G(x) = \frac{\cos x}{\pi} \int_x^\infty \frac{\cos t}{t}\,dt + \frac{\sin x}{\pi} \int_x^\infty \frac{\sin t}{t}\,dt, \quad \text{for} \quad x > 0.$$

Furthermore by the expansion for the sine and cosine integrals (see e.g. [11] p.126) the following asymptotic holds true

$$G(x) = \frac{1}{\pi}\left\{\log\left(\frac{1}{|x|}\right) - \gamma\right\} + r(x) \quad \text{as} \quad |x| \to 0$$

where $r(x) = \mathcal{O}(x)$ is nonnegative and $\gamma \sim 0.577$ denotes Euler's constant. In combination with the following refined energy asymptotics

$$\Lambda(\varepsilon) = \inf\big(\mathcal{I}_\varepsilon(u)\ ,\ u(0) = 1\big) = \frac{\pi}{\log(1/\varepsilon)}\big(1 + \mathcal{O}(\varepsilon)\big)$$

(compare section 4.2) we recover a logarithmic tail beyond a core region of order ε close to the origin.

2 Associated Operators and Shape of a Néel wall

2.1 Euler-Lagrange equations of Néel wall type

The starting point of our considerations is the derivation of Euler-Lagrange equations associated to variational principles for Néel walls (0.1). Paying attention to the boundedness and coercivity properties of the Néel wall energy, we will derive Euler-Lagrange equations for variational problems having a saturation constraint and a one point constraint in combination.
We consider bounded vectorfields $m = (u, v) : \mathbb{R} \to \mathbb{R}^2$ with $m' \in L^2(\mathbb{R}; \mathbb{R}^2)$ and $u \in L^2(\mathbb{R})$. These fields form a Banach space X equipped with the norm

$$\|m\|_X = \|m'\|_{\dot{H}^1} + \|u\|_{L^2} + \|v\|_{L^\infty}.$$

Clearly both component spaces $H^1(\mathbb{R})$ and $\dot{H}^1 \cap L^\infty(\mathbb{R})$ are multiplication algebras by means of the product rule and the Sobolev embedding. Let $\mathcal{E} : X \to \mathbb{R}$ be any continuously differentiable functional. We consider

$$\begin{aligned} \mathcal{E}(m) \to \min \quad \text{for} \quad m = (u, v) : \mathbb{R} \to \mathbb{R}^2 \in X \\ \text{with} \quad |m| = 1 \quad \text{and} \quad u(0) = 1. \end{aligned} \tag{2.1}$$

To encode the saturation condition $|m| = 1$ we introduce the smooth map

$$\mathcal{G}_1 : X \mapsto \dot{H}^1 \cap L^\infty(\mathbb{R}) \quad \text{defined by} \quad \mathcal{G}_1(m) = \frac{1}{2}\left(|m|^2 - 1\right)$$

and for the one point constraint we introduce the bounded affine functional

$$\mathcal{G}_2 : X \mapsto \mathbb{R} \quad \text{defined by} \quad \mathcal{G}_2(m) = u(0) - 1.$$

We need to check the nondegeneracy of $\mathcal{G}_1$ and $\mathcal{G}_2$ in the sense that for a solution $m = (u, v) \in X$ of the variational problem (2.1) the linear maps

$$d\mathcal{G}_1(m) : X \ni \phi \to m \cdot \phi \in \dot{H}^1 \cap L^\infty(\mathbb{R}),$$

$$d\mathcal{G}_2(m) : X \ni \phi \to \phi(0) \cdot \hat{e}_1 \in \mathbb{R}$$

are surjective. But for each function $f \in \dot{H}^1 \cap L^\infty(\mathbb{R})$ the field $\phi = f\, m$ belongs to X and solves $f = m \cdot \phi$ by the fact that $|m| = 1$. The nondegeneracy of $\mathcal{G}_2$ is obvious. According to the Lagrange multiplier rule as given in [12] a solution $m \in X$ of the variational problem (2.1) solves the following Euler-Lagrange equation

$$d\mathcal{E}(m) = \Lambda\, d\mathcal{G}_1(m) + \lambda\, d\mathcal{G}_2(m) \quad \text{in} \quad X' \tag{2.2}$$

for some distribution $\Lambda \in \left(\dot{H}^1 \cap L^\infty(\mathbb{R})\right)'$ and some number $\lambda \in \mathbb{R}$. It is clear that the saturation condition $|m| = 1$ and the one point constraint $u(0) = 1$ are related. The functional equation (2.2) admits the form

$$d\mathcal{E}(m) = \Lambda\, m + \lambda\, \delta_0\, \hat{e}_1. \tag{2.3}$$

The next step is to compute Λ. We multiply equation (2.3) by m, where for a vectorial distribution $T \in X'$ and a vectorfield $m \in X$ the product is well defined by $\langle T \cdot m, \phi\rangle = \langle T, m\,\phi\rangle$ for each $\phi \in \mathfrak{S}(\mathbb{R})$. Using the saturation condition $|m| = 1$ we find the relation $\Lambda = d\mathcal{E}(m) \cdot m - \lambda\, \delta_0$. Substituting this back into equation (2.3) gives

$$d\mathcal{E}(m) = d\mathcal{E}(m)\, m \otimes m - \lambda\, \delta_0\, m + \lambda\, \delta_0\, \hat{e}_1$$

and talking into account that $m(0) = \hat{e}_1$ the last two terms on the right hand side cancel. Thus we proved:

Lemma 2.1 (Euler-Lagrange equations). *Given a functional $\mathcal{E} : X \to \mathbb{R}$ which is supposed to be continuously differentiable. Furthermore suppose that the vectorfield $m = (u, v) : \mathbb{R} \to \mathbb{R}^2$ is a minimizer in X subject to the constraints that $|m| = 1$ and $u(0) = 1$. Then m solves the equation*

$$d\mathcal{E}(m) = d\mathcal{E}(m)\, m \otimes m \quad \text{in} \quad X'.$$

The lemma covers the the variational principle for Néel walls (0.1). A consequence of the saturation condition $|m| = 1$ is the following distributional identity

$$\big(-\Delta m \cdot m\big)\, m = |m'|^2\, m \quad \text{for} \quad m \in X.$$

Hence we obtain the following quasilinear system

$$\kappa^2\, Q\, (-\Delta) m + \begin{pmatrix} \mathcal{S}_Q u \\ 0 \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix} = \left\{\kappa^2\, Q\, |m'|^2 + |u|^2 + u\, \mathcal{S}_Q u\right\}\, m.$$

The projection onto the first component equation yields the Euler-Lagrange equation for a Néel wall profile

$$\kappa^2\, Q\, (-\Delta) u + \mathcal{S}_Q u + u = \left\{\kappa^2\, Q\, |m'|^2 + |u|^2 + u\, \mathcal{S}_Q u\right\}\, u \tag{2.4}$$

which holds true in $H^{-1}(\mathbb{R})$. Since by the saturation condition there holds $|m'|^2 = |u'|^2/(1 - u^2)$, the right hand side is completely determined by u.

2.2 Fourier representation and ellipticity properties

The following easy observations are crucial for all the subsequent estimates, which are achieved by solving linear equations (which are essentially elliptic of first order) and by asymptotic Sobolev embeddings. Recall that the Fourier symbol of the operator $\mathcal{S}_Q$ reads like

$$\mathcal{F}\mathcal{S}_Q(\xi) = \sigma_Q(\xi) = \frac{1}{Q}\left(1 - \frac{1-\exp(-2Q|\xi|)}{2Q|\xi|}\right) \quad \text{i.e.} \quad \sigma_Q(\xi) = \frac{1}{Q}\,\sigma(Q\,\xi).$$

The underlying scaling law, i.e. rescaling of the function followed be multiplication with the inverse scaling parameter, produces a family of self-similar functions which blow up as the scaling parameter decreases. To understand the operator $\mathcal{S}_Q$ we collect some particular properties of the function $\sigma(\xi)$, which imply important properties of the symbol $\mathcal{F}\mathcal{S}_Q$. First of all there holds

$$0 \le \sigma(\xi) = \left(1 - \frac{1-\exp(-2|\xi|)}{2|\xi|}\right) \le 1, \tag{2.5}$$

$$|\xi|\,|\sigma'(\xi)| = \Big|1 - \exp(-2|\xi|) - \sigma(\xi)\Big|. \tag{2.6}$$

Expanding the exponential we see that $\sigma(\xi) = |\xi| + \mathcal{O}\left(|\xi|^2\right)$ which can be sharpened for small ξ by estimating higher order terms, namely

$$\frac{1}{2}\min\left\{|\xi|, 1\right\} \le \sigma(\xi) \le \min\left\{|\xi|, 1\right\} \quad \text{for each} \quad \xi \in \mathbb{R}, \tag{2.7}$$

$$\sigma(\xi) + |\xi|^2 \ge |\xi| \quad \text{for each} \quad \xi \in \mathbb{R}. \tag{2.8}$$

By the expansion for the function σ and scaling rule we see that the symbols

$$\sigma_Q(\xi) = |\xi| + Q \cdot \mathcal{O}\big(|\xi|^2\big)$$

converge pointwise to $|\xi|$ as Q tends to zero and each function σ_Q could be described as the function $\xi \mapsto |\xi|$ truncated at the level $1/Q$. We conclude the following boundedness and convergence result for the operators $\mathcal{S}_Q$ and the corresponding quadratic form as the quality factor Q tends to zero.

Lemma 2.2. *The following inequalities hold for each $Q > 0$*

$$\langle u|\mathcal{S}_Q|u\rangle \le \|u\|^2_{\dot{H}^{1/2}} \quad \textit{and} \quad \|\mathcal{S}_Q u\|_{L^2} \le \|u\|_{\dot{H}^1}$$

and following asymptotics hold true as Q tends to zero:

(i) $\langle u|\mathcal{S}_Q|u\rangle$ *converges to* $\|u\|^2_{\dot H^{1/2}}$ *for each* $u \in H^{1/2}(\mathbb{R})$.

(ii) $\mathcal{S}_Q U_Q$ *converges to* $(-\Delta)^{1/2} U_0$ *in the sense of distributions for each sequence* (U_Q) *which converges weakly in* $L^2(\mathbb{R})$ *to* U_0.

Proof. Note that by the upper bound in (2.7) there holds $0 \le \sigma_Q(\xi) \le |\xi|$ for all $\xi \in \mathbb{R}$. Hence

$$\langle u|\mathcal{S}_Q|u\rangle = \int \sigma_Q(\xi)\ |\hat u(\xi)|^2\ d\xi \le \int |\xi|\ |\hat u(\xi)|^2\ d\xi = \|u\|^2_{\dot H^{1/2}}.$$

The pointwise convergence $\sigma_Q(\xi)|\hat u(\xi)|^2 \to |\xi||\hat u(\xi)|^2$ for almost every $\xi \in \mathbb{R}$ implies in addition the convergence of the quadratic form by majorized convergence. Similarly

$$\|\mathcal{S}_Q u\|^2_{L^2} = \int \sigma_Q^2(\xi)\ |\hat u(\xi)|^2\ d\xi \le \int |\xi|^2\ |\hat u(\xi)|^2\ d\xi = \|u\|^2_{\dot H^1}.$$

Clearly $\mathcal{S}_Q\phi$ converges to $(-\Delta)^{1/2}\phi$ in $L^2(\mathbb{R})$ for each $\phi \in \mathfrak{S}(\mathbb{R})$ by majorized convergence in frequency space and Planchrel's theorem. Hence

$$\left\langle \phi \,\middle|\, \mathcal{S}_Q \,\middle|\, U_Q \right\rangle \to \left\langle \phi \,\middle|\, (-\Delta)^{1/2} \,\middle|\, U_0 \right\rangle \quad \text{for each} \quad \phi \in \mathfrak{S}(\mathbb{R})$$

proving claim (ii). □

We introduce the second order Fourier multiplication operators which are associated to the Euler-Lagrange equation for the Néel wall profile (2.4): in the thin film regime for $\kappa \ge 1$ we define

$$\mathcal{L}_Q^\kappa = Q\,\kappa^2\,(-\Delta) + \mathcal{S}_Q + 1$$

with Fourier symbol given by $P_Q^\kappa(\xi) = \kappa^2\,Q\,\xi^2 + \sigma_Q(\xi) + 1$ and in the bulk regime for $0 < \kappa < 1$ we define

$$\tilde{\mathcal{L}}_Q^\kappa = Q\ \ (-\Delta) + \mathcal{S}_Q^\kappa + 1$$

with Fourier symbol $\tilde P_Q^\kappa(\xi) = Q\,\xi^2 + \sigma_Q\,(\xi/\kappa) + 1$. Now the important observation is that both operators considered in their proper parameter regimes share the same ellipticity properties. By the estimate (2.8) in both cases the exchange part compensates as the induced field part saturates. In the following we will restrict ourselves to the regime $\kappa \ge 1$. The other regime $0 < \kappa < 1$ can be treated analogously.

Lemma 2.3. *The operators $\mathcal{L}_Q^\kappa$ are elliptic of first order and the ellipticity is uniform in $Q > 0$ and $\kappa \geq 1$, i.e. for the symbol $P_Q^\kappa(\xi)$ we have*

$$|\xi| + 1 \leq P_Q^\kappa(\xi) \quad \textit{for each} \;\; Q > 0 \;\; \textit{and} \;\; \kappa \geq 1.$$

Furthermore there is a universal constant $c > 0$ such that

$$\left| \frac{\sigma_Q'(\xi)}{\sigma_Q(\xi)} \right| + \left| \frac{\big(P_Q^\kappa\big)'(\xi)}{P_Q^\kappa(\xi)} \right| \leq \frac{c}{|\xi|} \quad \textit{for each} \quad \kappa \geq 1 \quad \textit{and} \quad Q > 0.$$

Remark 2.1. *In the bulk regime $0 < \kappa < 1$ the analog assertion holds true for the corresponding operator $\tilde{\mathcal{L}}_Q^\kappa$ for all $Q > 0$.*

Proof. The first statement is a direct consequence of (2.8) writing

$$P_Q^\kappa(\xi) = \frac{1}{Q} \Big\{ (Q\,\xi)^2 + \sigma(Q\,\xi) \Big\} + 1 \geq |\xi| + 1.$$

For the second statement note that by (2.6) and (2.7) we have that

$$|Q\,\xi|\,|\sigma'(Q\,\xi)| \leq |\sigma(Q\,\xi)| \left(1 + \frac{1 - \exp(-2Q\xi)}{\sigma(Q\xi)} \right) \leq c\,\sigma(Q\,\xi)$$

but then we see that

$$\left| \frac{\sigma_Q'(\xi)}{\sigma_Q(\xi)} \right| = Q \left| \frac{\sigma'(Q\,\xi)}{\sigma(Q\,\xi)} \right| \leq \frac{c}{|\xi|} \quad \text{and} \quad \left| \frac{\big(P_Q^\kappa\big)'(\xi)}{P_Q^\kappa(\xi)} \right| \leq \frac{2}{|\xi|} + \left| \frac{\sigma_Q'(\xi)}{\sigma_Q(\xi)} \right| \leq \frac{c}{|\xi|}$$

for a constant which is independent of the parameters. □

Lemma 2.4. *The operators $\big(\mathcal{L}_Q^\kappa\big)^{-1}$ are uniformly regularizing of order one, that is they are bounded as*

$$\big(\mathcal{L}_Q^\kappa\big)^{-1} : H_q^s(\mathbb{R}) \to H_q^{s+1}(\mathbb{R}) \quad \textit{for each} \quad s \in \mathbb{R} \quad \textit{and} \quad 1 < q < \infty$$

where the bounds only depend on q.

Proof. We write the inverse operator in the following form

$$\big(\mathcal{L}_Q^\kappa\big)^{-1} = \Big\{ (\mathbf{1} - \Delta)^{1/2} \big(\mathcal{L}_Q^\kappa\big)^{-1} \Big\} \, (\mathbf{1} - \Delta)^{-1/2}.$$

Since $(\mathbf{1} - \Delta)^{-1/2} : H_q^s(\mathbb{R}) \to H_q^{s+1}(\mathbb{R})$ is bounded we only need to check (parameter independent) L^q-boundedness of the remainder $(\mathbf{1} - \Delta)^{1/2}\big(\mathcal{L}_Q^\kappa\big)^{-1}$.

By Lemma 2.3 its Fourier multiplier $(1+\xi^2)^{1/2}/P_Q^\kappa(\xi)$ is uniformly bounded by 1 independently of the parameters. Furthermore it is smooth away from the origin and its derivative is bounded by

$$\left|\frac{\left(P_Q^\kappa\right)'(\xi)}{P_Q^\kappa(\xi)}\right| + \frac{|\xi|}{1+\xi^2} \le \frac{c}{|\xi|} \quad \text{for each} \quad \xi \neq 0$$

where the constant $c > 0$ is independent of the parameters. We conclude by the multiplier theorem (see e.g. [10] p. 96) that the corresponding operator $(\mathbf{1}-\Delta)^{1/2}\left(\mathcal{L}_Q^\kappa\right)^{-1}$ is bounded on $L^q(\mathbb{R})$ for each $1 < q < \infty$. □

2.3 Kernel representation and rearrangement properties

Qualitative properties like symmetry, monotonicity and positivity are easier seen in real space representations of operators and functions. Computation of the inverse Fourier transform yields the convolution kernel representation of the operator $\mathcal{S}_Q$

$$\mathcal{S}_Q u = \frac{1}{Q}\left(\delta_0 - \mathcal{K}_Q\right) * u = \frac{1}{Q}\left(u - \mathcal{K}_Q * u\right).$$

Explicitly we have the following kernel function which can be found e.g. in [2] and [8]

$$\mathcal{K}(x) = \frac{1}{4\pi}\log\left(1+\frac{4}{x^2}\right) \quad , \quad \mathcal{K}_Q(x) = \frac{1}{Q}\,\mathcal{K}\left(\frac{x}{Q}\right).$$

It is positive and absolutely integrable with $\int \mathcal{K}\, dx = 1$. The kernel representation of the operator leads to the following kernel representation for the reduced strayfield energy

$$\begin{aligned}\left\langle u \middle| \mathcal{S}_Q \middle| u \right\rangle &= \frac{1}{Q}\int u^2(x) - u(x)\int \mathcal{K}_Q(x-y)\, u(y)\, dy\, dx \qquad (2.9)\\ &= \frac{1}{2Q}\int\int \mathcal{K}_Q(x-y)\, \big(u(x)-u(y)\big)^2\, dx\, dy. \qquad (2.10)\end{aligned}$$

From the last formula we can read off a density function for the magnetostatic energy in real space

$$k_Q[u](x) = \frac{1}{2Q}\int \mathcal{K}_Q(x-y)\, \big(u(x)-u(y)\big)^2\, dy. \qquad (2.11)$$

Using (2.10) it follows readily a multiplication inequality.

Lemma 2.5. *There holds for each pair of functions* $u, \phi \in L^2 \cap L^\infty(\mathbb{R})$

$$\frac{1}{2} \langle u\,\phi \,|\mathcal{S}_Q|\, u\,\phi \rangle \le \|\phi\|^2_{L^\infty} \,\langle u \,|\mathcal{S}_Q|\, u \rangle + \|u\|^2_{L^\infty} \,\langle \phi \,|\mathcal{S}_Q|\, \phi \rangle.$$

In particular we have $\langle u^2 \big|\mathcal{S}_Q\big| u^2 \rangle \le 4\, \|u\|^2_{L^\infty} \,\langle u \big|\mathcal{S}_Q\big| u \rangle$.

Remark 2.2. *Since* $\big||a| - |b|\big| \le |a - b|$ *and the kernel function* $\mathcal{K}$ *is non-negative we immediately see by (2.10) that*

$$\Big\langle |u| \,\Big|\mathcal{S}_Q\Big|\, |u| \Big\rangle \le \Big\langle u \,\Big|\mathcal{S}_Q\Big|\, u \Big\rangle.$$

Rearrangement. Let us recall a concept of symmetrization. Let $u : \mathbb{R} \to \mathbb{R}$ be a measurable, nonnegative function vanishing at infinity, i.e. all super level sets have finite measure. Then the symmetrically decreasing rearrangement of u is defined as

$$u^*(x) = \int_0^\infty \chi_{\{u>t\}^*}(x)\; dt$$

where for a Borel set $A \subset \mathbb{R}$ the rearranged set A^* is the centered interval with measure $|A|$. An immediate property is equimeasurability, i.e. for all $t > 0$ there holds $|\{u > t\}| = |\{u^* > t\}|$ which implies that $\|u^*\|_{L^p} = \|u\|_{L^p}$ for each $1 \le p \le \infty$. Consult the monograph [7] section 3 for further information. One can find the following rearrangement property as a special case of Theorem 3.9 in [7].

Lemma 2.6 (Strict rearrangement inequality). *Let* u *and* $\mathcal{K}$ *be non-negative measurable functions both vanishing at infinity. Suppose that* $\mathcal{K}$ *is strictly symmetrically decreasing. Then*

$$\big\langle u, \mathcal{K} * u \big\rangle \le \big\langle u^*, \mathcal{K} * u^* \big\rangle$$

and equality holds true only if u *is a translation of its rearrangement* u^*.

Now the kernel function $\mathcal{K}$ we just introduced is strictly decreasing. Thus we conclude by the strict rearrangement inequality applied to (2.9) a strict rearrangement inequality for the magnetostatic part of the energy:

Lemma 2.7. *Suppose that the function* $u \in L^2(\mathbb{R})$ *is nonnegative and let* u^* *be its symmetrically decreasing rearrangement. Then there holds*

$$\Big\langle u^* \Big|\mathcal{S}_Q\Big| u^* \Big\rangle \le \Big\langle u \Big|\mathcal{S}_Q\Big| u \Big\rangle$$

with equality only if u *is a translation of a symmetrically decreasing function.*

Next we need a rearrangement inequality for the exchange part of the energy, a slight improvement of the rearrangement inequality for the Dirichlet integral $\|u^*\|_{\dot{H}^1} \leq \|u\|_{\dot{H}^1}$ (see [7] Lemma 7.17) adapted to saturated vectorfields:

Lemma 2.8. *Let $u \in H^1(\mathbb{R})$ be nonnegative and $v \in \dot{H}^1(\mathbb{R})$ be bounded such that $u^2 + v^2 = 1$ in $\mathbb{R}$. Then there holds for any function $v_* \in \dot{H}^1(\mathbb{R})$ with $v_*^2 = 1 - (u^*)^2$ the inequality $\|v_*\|_{\dot{H}^1} \leq \|v\|_{\dot{H}^1}$.*

Proof. Note that each function $v_* \in \dot{H}^1(\mathbb{R})$ which satisfies $v_*^2 = 1 - (u^*)^2$ has the same Dirichlet energy. We consider one special representative and define for $z \geq 0$ the nonnegative nondecreasing function

$$\phi(z) = 1 - \big(1 - \min(z,1)^2\big)^{1/2} \quad \text{and} \quad v_* = 1 - \phi(u^*).$$

Since $u \in L^2(\mathbb{R})$ the function $\phi(u)$ is vanishing at infinity we can conclude using the composition property of the rearrangement operation

$$\|v_*\|_{\dot{H}^1} = \|\phi(u^*)\|_{\dot{H}^1} = \|(\phi \circ u)^*\|_{\dot{H}^1} \leq \|\phi \circ u\|_{\dot{H}^1} = \|v\|_{\dot{H}^1}$$

by the rearrangement inequality for the Dirichlet energy. □

Putting together the above considerations we arrive at the following monotonicity result:

Proposition 1 (Monotonicity). *The profile of a 1d Néel wall is nonnegative and symmetrically decreasing.*

Proof. Consider a minimizing profile $u : \mathbb{R} \to \mathbb{R}$ for our variational problem

$$\kappa^2\, Q\, \Big(\|u\|_{\dot{H}^1}^2 + \|v\|_{\dot{H}^1}^2\Big) + \langle u|\mathcal{S}_Q|u\rangle + \|u\|_{L^2}^2 \to \min$$
$$u^2 + v^2 = 1 \quad \text{and} \quad u(0) = 1.$$

The existence of a minimizing profile $u \in H^1(\mathbb{R})$ is provided by the coercivity and convexity properties of the functional and the continuity of the constraints along a minimizing sequence. Furthermore we can assume that u is continuous. By the invariance of the L^2-norm under rearrangement, the strict rearrangement inequality in Lemma 2.7 and the rearrangement inequality for the Dirichlet energy in combination with Lemma 2.8 each nonnegative profile u is a translation of a symmetrically decreasing function. Since u attains its maximum at zero it is actually symmetrically decreasing. Thus we only need to show nonnegativity. Suppose that this is not the case. Then (using Remark 2.2) $|u|$ is also a minimizing profile and not symmetrically decreasing. This is a contradiction. □

3 Elliptic estimates and integrability

We begin with an a priori L^p-estimate for profiles of finite Néel wall energy which can be obtained from the functional. It will be sharpened for a minimizer, which solves the Euler-Lagrange equations. We will make use of regularity techniques for elliptic equations. The aim is to control higher L^p-norms by the energy and to show that the control fades linearly in p. Then the argument is finished in section 4.1 by combining estimates for the mean value with the monotonicity property of the profile established in the last section to get the desired pointwise estimate. Our general assumptions from now on are to be in the regime of thin films, i.e. $\kappa \geq 1$, and nonvanishing anisotropy, i.e. $Q > 0$.

Lemma 3.1. *Suppose that the magnetization field $m = (u, v) : \mathbb{R} \to \mathbb{S}^1$ has finite Néel wall energy. Then $u \in L^p(\mathbb{R})$ for each $2 < p < \infty$ and we have for some universal constant $c > 0$*

$$\|u\|_{L^p} \leq c\, p^{1/2}\, \mathcal{E}_Q^\kappa(m)^{1/2} \quad \textit{for each} \quad 2 < p < \infty.$$

Proof. Let $q < 2$ be given by $1/q = 1 - 1/p$, then we have by Hölder's inequality using the ellipticity property established in Lemma 2.3

$$\|\hat{u}\|_{L^q} = \left\| \left\{ P_Q^\kappa(\xi)\, |\hat{u}(\xi)|^2 \right\}^{1/2} \left\{ P_Q^\kappa(\xi) \right\}^{-1/2} \right\|_{L^q} \leq \mathcal{E}_Q^\kappa(m)^{1/2}\, \left\| (1 + |\xi|)^{-1/2} \right\|_{L^r}$$

for $1/q = 1/r + 1/2$. Now integration shows

$$\left\| (1 + |\xi|)^{-1/2} \right\|_{L^r} = \left(\frac{p-2}{2} \right)^{\frac{1}{2} - \frac{1}{p}} \quad \text{for} \quad 1/r = 1/2 - 1/p,$$

and the result follows from the Hausdorff-Young inequality $\|u\|_{L^p} \leq \|\hat{u}\|_{L^q}$.

□

3.1 Bounds for the nonlinearity

This section is devoted to the analysis of the Euler-Lagrange equation for Néel wall profiles. We identify the fundamental analytical shape of the nonlinearity arising from the saturation condition $|m| = 1$. This provides the requisite structure that allows to combine a priori bounds, as stated in Lemma 3.1, with linear elliptic regularity theory.

Suppose that the magnetization field $m = (u, v) : \mathbb{R} \to \mathbb{S}^1$ minimize the functional

$$\mathcal{E}_Q^\kappa(m) = \kappa^2\, Q\, \|m\|_{\dot{H}^1}^2 + \|u\|_{L^2}^2 + \langle u\, | \mathcal{S}_Q |\, u \rangle \tag{3.1}$$

among all vectorfields which are saturated with a one point constraint at the origin. As we already saw in the end of section 2.1 the associated profile u solves the following Euler-Lagrange equation, which is defined for magnetization fields of finite Néel wall energy in the sense of distributions

$$\kappa^2 \, Q \, (-\Delta) u + \mathcal{S}_Q u + u = \left\{ \kappa^2 \, Q \, |m'|^2 + |u|^2 + u \, \mathcal{S}_Q u \right\} \, u. \tag{3.2}$$

For fixed u it can be written as the linear distributional equation

$$\mathcal{L}_Q^\kappa u = \delta_Q^\kappa[u] \quad \text{in} \quad \mathfrak{S}'(\mathbb{R}) \quad \text{or more precisely in} \quad H^{-1}(\mathbb{R}) \tag{3.3}$$

with the already studied singular perturbed elliptic operator operator

$$\mathcal{L}_Q^\kappa u = \kappa^2 \, Q \, (-\Delta) u + \mathcal{S}_Q u + u \tag{3.4}$$

and the following distribution in dependence of a profile u of finite Néel wall energy

$$\delta_Q^\kappa[u] : \phi \mapsto \left\langle \kappa^2 \, Q \, |m'|^2 + |u|^2, \phi \, u \right\rangle + \left\langle u \, | \mathcal{S}_Q | \, u^2 \, \phi \right\rangle. \tag{3.5}$$

Note that by the saturation condition $|m| = 1$ the integrable quantity $|m'|^2 = |u'|^2/(1-u^2)$ is completely determined by the profile u. We consider separately the local part arising from the exchange and anisotropy terms

$$e_Q^\kappa[u] \, u : \phi \mapsto \left\langle \kappa^2 \, Q \, |m'|^2 + |u|^2, u \, \phi \right\rangle \in H^{-1}(\mathbb{R}). \tag{3.6}$$

which is clearly bounded in $L^1(\mathbb{R})$ by the energy and the nonlocal part

$$r_Q[u] : \phi \mapsto \left\langle u \, | \mathcal{S}_Q | \, u^2 \, \phi \right\rangle \in H^{-1}(\mathbb{R}), \tag{3.7}$$

which needs to be decomposed once more. The goal is to extract an integrable portion which is correlated with the energy density. We define the distributions

$$\begin{aligned} f_Q[u] : \phi \quad &\mapsto \quad \Big\langle \mathcal{S}_Q u, (-\Delta)^{-1/4} \big[(-\Delta)^{1/4}, u^2 \big] \phi \Big\rangle, \\ g_Q[u] : \phi \quad &\mapsto \quad \Big\langle \mathcal{S}_Q u, (-\Delta)^{-1/4} \big\{ u^2 (-\Delta)^{1/4} \phi \big\} \Big\rangle. \end{aligned}$$

Then it is easy to see that $r_Q[u] = f_Q[u] + g_Q[u]$ in the sense of distributions. By Plancherel and Cauchy-Schwarz (using the fact that $0 \le \sigma_Q(\xi)/|\xi| \le 1$ is uniformly bounded) we see that

$$|\langle g_Q[u], \phi \rangle| \le \left\| \sigma_Q^{1/2}(\xi) \, \hat{u}(\xi) \right\|_{L^2} \, \left\| u^2 (-\Delta)^{1/4} \phi \right\|_{L^2}.$$

Note that $\left\|\sigma_Q^{1/2}(\xi)\,\hat{u}(\xi)\right\|_{L^2}$ agrees with the square root of the strayfield energy. Hence by Hölder's inequality there holds for $1/2 = 1/p + 1/r$

$$|\langle g_Q[u], \phi\rangle| \leq \mathcal{E}_Q^\kappa(m)^{1/2}\,\|u\|_{L^\infty}\,\|u\|_{L^p}\,\left\|(-\Delta)^{1/4}\phi\right\|_{L^r}.$$

Since $(-\Delta)^{1/4}\phi$ differs from $(1-\Delta)^{1/4}\phi$ only by a convolution with a finite measure (see [10] p.133) the estimate $\|(-\Delta)^{1/4}\phi\|_{L^r} \leq c\,\|\phi\|_{H_r^{1/2}}$ holds true with a constant which is independent of r.
As above we use Plancherel, Cauchy-Schwarz and the bounds on $\sigma_Q(\xi)$ to estimate

$$|\langle f_Q[u], \phi\rangle| \leq \left\|\sigma_Q^{1/2}(\xi)\,\hat{u}(\xi)\right\|_{L^2}\,\left\|\left[(-\Delta)^{1/4}, u^2\right]\phi\right\|_{L^2}$$

while

$$\left\|\left[(-\Delta)^{1/4}, u^2\right]\phi\right\|_{L^2} \leq \left\|u\left[(-\Delta)^{1/4}, u\right]\phi\right\|_{L^2} + \left\|\left[(-\Delta)^{1/4}, u\right](u\phi)\right\|_{L^2}.$$

Hence using the commutator estimate for fractional derivatives (see [5] Corollary 1.2) and Hölder there is a constant $c > 0$ such that

$$\left\|\left[(-\Delta)^{1/4}, u^2\right]\phi\right\|_{L^2} \leq c\left\|(-\Delta)^{1/4}u\right\|_{L^2}\|u\|_{L^\infty}\,\|\phi\|_{L^\infty} \tag{3.8}$$

Finally using Lemma 2.3 we can estimate

$$\left\|(-\Delta)^{1/4}u\right\|_{L^2}^2 \leq \int(1+|\xi|^2)^{1/2}|\hat{u}(\xi)|^2\,d\xi \leq \int P_Q^\kappa(\xi)\,|\hat{u}(\xi)|^2\,d\xi \leq \mathcal{E}_Q^\kappa(m).$$

Thus we proved the following decomposition result:

Proposition 2. *Let the profile u correspond to a magnetization field m of finite Néel wall energy. Then there is a decomposition of the nonlocal distribution $r_Q[u]$ defined in (3.7)*

$$\langle r_Q[u], \phi\rangle = \langle f_Q[u], \phi\rangle + \langle g_Q[u], \phi\rangle \quad \textit{for all} \quad \phi \in \mathfrak{S}(\mathbb{R}),$$

such that $f_Q[u]$ is an absolutely continuous measure and $g_Q[u]$ is a singular distribution. Furthermore we have the estimates

$$|\langle f_Q[u], \phi\rangle| \leq c\,\mathcal{E}_Q^\kappa(m)\;\;\|\phi\|_{L^\infty}\,,$$
$$|\langle g_Q[u], \phi\rangle| \leq c\,\mathcal{E}_Q^\kappa(m)^{1/2}\;\;\|u\|_{L^p}\;\;\|\phi\|_{H_r^{1/2}}$$

for each $p > 2$ with $1/2 = 1/p + 1/r$ and some universal constant $c > 0$.

Remark 3.1. *By the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow H_r^{1/2}(\mathbb{R})$ for $r \geq 2$ the bounds remain valid for testfunctions in $H^1(\mathbb{R})$.*

Our result up to now is the following representation of the Euler-Lagrange equation for a Néel wall profile

$$\mathcal{L}_Q^\kappa u = e_Q^\kappa[u]\, u + f_Q[u] + g_Q[u] \quad \text{in} \;\; H^{-1}(\mathbb{R}) \tag{3.9}$$

with estimates established in Proposition 2. The next section addresses the inversion of the operator $\mathcal{L}_Q^\kappa$ and the resulting L^p-estimates.

3.2 Inversion of the operator $\mathcal{L}_Q^\kappa$

The inversion of a first order elliptic operator in one dimension where the right hand side is only supposed to be integrable, just fails to provide control on the continuity of the solution in terms of the integrability of the right hand side. Nevertheless we can control the blow up of higher L^p-norms as p tends to infinity.

Lemma 3.2. *Let $f \in L^1(\mathbb{R})$ and suppose that u solves the equation $\mathcal{L}_Q^\kappa u = f$. Then there holds*

$$\|u\|_{L^p} \leq c\, p\, \|f\|_{L^1} \quad \textit{for each} \;\; 2 < p < \infty \tag{3.10}$$

for each $Q > 0$ and $\kappa \geq 1$ and for some universal constant $c > 0$.

Remark 3.2. *The same estimate holds true for finite measures.*

Proof. Let $P_Q^\kappa(\xi)$ be the symbol of the operator $\mathcal{L}_Q^\kappa$. We have in frequency space $\hat{u}(\xi) = P_Q^\kappa(\xi)^{-1}\, \hat{f}(\xi)$ and using the ellipticity property established in Lemma 2.3 we infer that for $1 < q < 2$

$$\left\| P_Q^\kappa(\xi)^{-1}\, \hat{f} \right\|_{L^q}^q \leq c\, \|\hat{f}\|_{L^\infty}^q \int_0^\infty \frac{d\xi}{(1+\xi)^q} \leq \frac{c}{q-1}\, \|f\|_{L^1}^q .$$

We conclude by the Hausdorff-Young inequality that

$$\|u\|_{L^p} \leq c\,(p-1)^{1-1/p}\, \|f\|_{L^1} \quad \text{for} \quad 1/p + 1/q = 1$$

which immediately implies the assertion. □

Remark 3.3. *By the embedding $L^1(\mathbb{R}) \hookrightarrow H_q^{-1/2}(\mathbb{R})$ for each $q \in (1,2)$ the regularization property of the operator $\left(\mathcal{L}_Q^\kappa\right)^{-1}$ stated in Lemma 2.4 implies that in the context of Lemma 3.2*

$$\|u\|_{H_q^{1/2}} \leq c(q)\, \|f\|_{L^1} \quad \textit{for each} \;\; 1 < q < 2$$

for some constant which only depends on q and the same estimate holds true for finite measures.

As a direct consequence of Lemma 3.2 and the bounds in Proposition 2 of the last section there holds for a profile u which corresponds to a magnetization field m of finite Néel wall energy

$$\left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} \left\{ e_Q^\kappa[u]\, u \right\} \right\|_{L^p} + \left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} f_Q[u] \right\|_{L^p} \leq c\, p\, \mathcal{E}_Q^\kappa(m) \tag{3.11}$$

for each $p > 2$ and for some universal constant $c > 0$. Moreover by Remark 3.3 we have for each $1 < q < 2$ and for some constant $c(q) > 0$ which only depends on q

$$\left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} \left\{ e_Q^\kappa[u]\, u \right\} \right\|_{H_q^{1/2}} + \left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} f_Q[u] \right\|_{H_q^{1/2}} \leq c(q)\, \mathcal{E}_Q^\kappa(m). \tag{3.12}$$

The same type of estimate holds true for the singular part. Indeed by the bound in Proposition 2 the distribution $g_Q[u]$ extends for $u \in L^p(\mathbb{R})$ to an element in the dual space of $H_r^{1/2}(\mathbb{R})$ known as $H_q^{-1/2}(\mathbb{R})$ for $1/q + 1/r = 1$. Moreover we have the bounds

$$\left\| g_Q[u] \right\|_{H_q^{-1/2}} \leq c\, \mathcal{E}_Q^\kappa(m)^{1/2}\, \|u\|_{L^p} \quad \text{for} \quad 1/q = 1/p + 1/2 \tag{3.13}$$

and we conclude utilizing the a priori L^p-bounds in Lemma 3.1 and the regularization property of the operator $\left(\mathcal{L}_Q^\kappa\right)^{-1}$ stated in Lemma 2.4 that for all $1 < q < 2$

$$\left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} g_Q[u] \right\|_{H_q^{1/2}} \leq c(q)\, p^{1/2}\, \mathcal{E}_Q^\kappa(m) \quad \text{for} \quad 1/q = 1/p + 1/2. \tag{3.14}$$

Using interpolation we can substitute the constant $c(q)$ in the regime $p \geq 4$ by the universal constant $C = \max\{c(4/3), c(2)\}$. Finally by the Sobolev inequality below to be proved in the appendix we see that there is a universal constant $c > 0$ such that

$$\left\| \left(\mathcal{L}_Q^\kappa\right)^{-1} g_Q[u] \right\|_{L^p} \leq c\, p\, \mathcal{E}_Q^\kappa(m) \quad \text{for each} \quad p \geq 4. \tag{3.15}$$

Lemma 3.3. *There is universal constant $c > 0$ such that for $p \geq 4$*

$$\|f\|_{L^p} \leq c\, p^{1/2}\, \|f\|_{H_q^{1/2}} \quad \textit{for} \quad 1/q = 1/p + 1/2.$$

3.3 Integrability of Néel wall profiles

Formal integration of the Euler-Lagrange equation yields an L^1-bound by the nonnegativity of the profile shown in Proposition 1 in section 2.3. To make this rigorous let $0 \le \varphi \le 1$ be a smooth symmetrically decreasing function with compact support and $\varphi(0) = 1$. Then $\varphi_\varepsilon(x) = \varphi(\varepsilon\, x)$ converges monotonically to 1 as ε tends to zero for each $x \in \mathbb{R}$. We multiply the profile u by the rescaled bump function φ_ε and integrate

$$\int (\varphi_\varepsilon\, u)\; dx = \langle \varphi_\varepsilon, \mathcal{L}_Q^\kappa u\rangle + \big\langle \big(1 - \mathcal{L}_Q^\kappa\big)\, \varphi_\varepsilon, u\big\rangle = \langle \varphi_\varepsilon, \mathcal{L}_Q^\kappa u\rangle + \mathcal{O}\big(\varepsilon^{1/2}\big).$$

To see this just note that we can estimate $\big|\big\langle \big(1 - \mathcal{L}_Q^\kappa\big)\varphi_\varepsilon, u\big\rangle\big|$ by

$$\kappa^2\, Q |\langle (-\Delta)^{1/2}\varphi_\varepsilon, (-\Delta)^{1/2} u\rangle| + |\langle \mathcal{S}_Q \varphi_\varepsilon, u\rangle| \le c\; \mathcal{E}_Q^\kappa(m)^{1/2}\; \|\varphi_\varepsilon\|_{\dot{H}^1} = \mathcal{O}\big(\varepsilon^{1/2}\big)$$

where we used Lemma 2.2. So we see that by monotone convergence the L^1-norm of a nonnegative profile u corresponding to a magnetization field m of finite energy is given by

$$\|u\|_{L^1} = \lim_{\varepsilon\downarrow 0} \int (\varphi_\varepsilon\, u)\; dx = \lim_{\varepsilon \downarrow 0} \langle \mathcal{L}_Q^\kappa \varphi_\varepsilon, u\rangle \tag{3.16}$$

independently of the choice of the bump function φ. Now let $u = u_Q^\kappa$ be a Néel wall profile. The quantity I_Q^κ to be introduced below can be identified with the integral of the right hand side of the Euler-Lagrange equation (3.3). We define

$$I_Q^\kappa = \lim_{\varepsilon\downarrow 0} \left\{ \int e_Q^\kappa[u]\; (u\, \varphi_\varepsilon)\; dx + \Big\langle u \Big| \mathcal{S}_Q \Big| u^2\, \varphi_\varepsilon \Big\rangle \right\} \tag{3.17}$$

where as in section 2.1 $e_Q^\kappa[u]$ is the local portion of the energy density given by

$$e_Q^\kappa[u] = \kappa^2\, Q\, |m'|^2 + |u|^2.$$

According to the monotone convergence theorem there holds

$$\lim_{\varepsilon\downarrow 0} \int e_Q^\kappa[u]\; (u\, \varphi_\varepsilon)\; dx = \int e_Q^\kappa[u]\, u\, dx \le \kappa^2\, Q\, \|m\|_{\dot{H}^1}^2 + \|u\|_{L^2}^2.$$

Likewise by the boundedness of u in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ the functions $u^2\varphi_\varepsilon$ converge to u^2 in $L^2(\mathbb{R})$. Hence by the L^2-boundedness of the operator $\mathcal{S}_Q$ for positive Q we infer that

$$\lim_{\varepsilon\downarrow 0} \Big\langle u \Big| \mathcal{S}_Q \Big| u^2\, \varphi_\varepsilon \Big\rangle = \Big\langle u \Big| \mathcal{S}_Q \Big| u^2 \Big\rangle$$

while by Cauchy-Schwarz and Lemma 2.5

$$\left|\left\langle u\middle|\mathcal{S}_Q\middle|u^2\right\rangle\right| \le \left\langle u\middle|\mathcal{S}_Q\middle|u\right\rangle^{1/2}\left\langle u^2\middle|\mathcal{S}_Q\middle|u^2\right\rangle^{1/2} \le 2\left\langle u\middle|\mathcal{S}_Q\middle|u\right\rangle$$

Thus I_Q^κ is well defined by (3.17), i.e. independently of the choice of φ and

$$I_Q^\kappa \le \kappa^2\, Q\,\|m'\|^2_{\dot H^1} + \|u\|^2_{L^2} + 2\,\langle u|\mathcal{S}_Q|u\rangle \le 2\,\mathcal{E}_Q^\kappa(m). \tag{3.18}$$

According to the Euler-Lagrange equation (3.2) for a Néel wall profile there holds

$$\langle \mathcal{L}_Q^\kappa\,\varphi_\varepsilon, u\rangle = \int e_Q^\kappa[u]\,(u\,\varphi_\varepsilon)\,dx + \left\langle u\middle|\mathcal{S}_Q\middle|u^2\,\varphi_\varepsilon\right\rangle \tag{3.19}$$

In the limit $\varepsilon \downarrow 0$ we conclude by (3.16), (3.17) and (3.18) the following integrability estimate:

Lemma 3.4. *A Néel wall profile is integrable with the bound*

$$\|u_Q^\kappa\|_{L^1} \le 2\,\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa$$

where $\mathcal{M}$ denotes the set of admissible magnetization fields.

We finish this section with an integral representation formula for $\langle u|\mathcal{S}_Q|u^2\rangle$ using the kernel representation in section 2.3.

Lemma 3.5. *Let $\mathcal{S}u = u - \mathcal{K} * u$ where $\mathcal{K} \in L^1(\mathbb{R})$ is symmetric with $\int \mathcal{K} = 1$. Then there holds for $u \in L^2 \cap L^\infty(\mathbb{R})$ the formula*

$$\int \big(u\,\mathcal{S}u\big)\,u\,dx = \int u(x) \int \mathcal{K}(x-y)\big(u(x)-u(y)\big)^2\,dy\,dx.$$

Proof. Writing down the convolution we see elementarily that

$$\begin{aligned} u(x)\,\big(u\,\mathcal{S}u\big)(x) &= u(x)\int\Big\{u^2(x) - u(x)\,\mathcal{K}(x-y)\,u(y)\Big\}\,dy \\ &= u(x)\int \mathcal{K}(x-y)\,\big(u(x)-u(y)\big)^2\,dy \\ &\quad+ \int \mathcal{K}(x-y)\Big\{u^2(x)u(y) - u^2(y)u(x)\Big\}\,dy. \end{aligned}$$

Carrying out the x integration, the last integral cancels by symmetry. □

Thus for a Néel wall profile u_Q^κ we have the additional formula for the integral $I_Q^\kappa = \int u_Q^\kappa(x)\,dx$ in terms of densities

$$I_Q^\kappa = \int \big\{ e_Q^\kappa[u](x) + 2\,k_Q[u](x) \big\}\; u(x)\; dx \quad \text{for} \quad u = u_Q^\kappa \tag{3.20}$$

where $e_Q^\kappa[u]$ is the sum of the exchange and anisotropy density and $k_Q[u]$ is the magnetostatic density function introduced in section 2.3 given by

$$k_Q[u](x) = \frac{1}{2\,Q} \int \mathcal{K}_Q(x-y)\big(u(x)-u(y)\big)^2\; dy.$$

4 Pointwise bounds and singular limits

4.1 L^p-bounds and logarithmic decay

We summarize the results of the previous sections. Let u_Q^κ be a Néel wall profile for some pair of parameters $Q>0$ and $\kappa \geq 1$. In the context of the results of section 3.1 the profile u_Q^κ solves the Euler-Lagrange equation

$$\mathcal{L}_Q^\kappa u_Q^\kappa = e_Q^\kappa[u_Q^\kappa]\, u_Q^\kappa + f_Q[u_Q^\kappa] + g_Q[u_Q^\kappa] \quad \text{in} \quad H^{-1}(\mathbb{R})$$
$$\text{which means that} \quad u_Q^\kappa = \big(\mathcal{L}_Q^\kappa\big)^{-1} \big\{ e_Q^\kappa[u_Q^\kappa]\, u_Q^\kappa + f_Q[u_Q^\kappa] + g_Q[u_Q^\kappa] \big\}\,.$$

Then (3.11) and (3.15) imply

$$\big\| u_Q^\kappa \big\|_{L^p} \leq c\, p\, \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \quad \text{for} \quad p \geq 4$$

where $\mathcal{M}$ denotes the set of admissible magnetizations. Now a complete L^p-estimate is easily proved by filling the gap between the L^1-bound, proved in Lemma 3.4, and the L^4-bound by interpolation. In addition we have bounds in (subcritical) Sobolev spaces using (3.12) and (3.14).

Proposition 3 (Regularity bounds). *Let u_Q^κ be a rescaled Néel wall profile. Then*

$$\big\| u_Q^\kappa \big\|_{L^p} \leq c\, p\, \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \quad \textit{for each} \quad p \geq 1 \tag{4.1}$$

for some universal constant $c>0$ and moreover

$$\big\| u_Q^\kappa \big\|_{H_q^{1/2}} \leq c(q)\; \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \quad \textit{for each} \quad 1<q<2 \tag{4.2}$$

where the constant $c(q)>0$ only depends on q.

Now the following statement is a combination of the monotonicity property established in section 2.3 and the L^p-bounds in Proposition 3:

Theorem 1 (Logarithmic decay). *There holds the following decay estimate*

$$0 \leq u_Q^\kappa(x) \leq c \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \, \log\left(\frac{1}{|x|}\right) \quad \textit{for} \quad 0 < |x| < 1/e \tag{4.3}$$

for some universal constant $c > 0$.

Proof. By the monotonicity result of Proposition 1 in section 2.3, Hölder's inequality and the L^p-bounds in Proposition 3 we conclude for $u = u_Q^\kappa$ and $x > 0$

$$0 \leq u(x) \leq \fint_0^x u \, dy \leq \left(\fint_0^x |u|^p \, dy\right)^{1/p} \leq c \, p \, \left(\frac{1}{x}\right)^{1/p} \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa$$

for each $p \geq 1$. We optimize this estimate with respect to p that is we set $p = \log(1/x)$ and note that $|z|^{1/\log(|z|)} = 1/e$. Thus we get the desired logarithmic decay estimate. □

Remark 4.1. *As mentioned in the beginning we obtain with the same analysis the analogous result in the bulk regime $0 < \kappa < 1$ keeping in mind the different scaling.*

4.2 Review of the energy bounds

In this section we recall some results concerning the energy. We denote by $\mathcal{M}$ the set of admissible magnetizations, i.e. the set of saturated vectorfields with one point constraint at the origin.
Since $\langle u|\mathcal{S}_Q|u\rangle$ approaches $\|u\|^2_{\dot{H}^{1/2}}$ as Q tends to zero as shown in Lemma 2.2, we can approximate the Néel wall energy $\mathcal{E}_Q^\kappa(m)$ in (0.1) by the functional

$$\mathcal{I}_\varepsilon(m) = \varepsilon \, \|m\|^2_{\dot{H}^1} + \|u\|^2_{\dot{H}^{1/2}} + \|u\|^2_{L^2} \quad \text{with} \quad \varepsilon = Q\kappa^2. \tag{4.4}$$

But as we already saw in Lemma 2.2 $\langle u|\mathcal{S}_Q|u\rangle \leq \|u\|^2_{\dot{H}^{1/2}}$ for each function $u \in H^{1/2}(\mathbb{R})$. Hence we have for each admissible magnetization field m the relation

$$\mathcal{E}_Q^\kappa(m) \leq \mathcal{I}_\varepsilon(m) \quad \text{and therefore} \quad \inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \leq \inf_{\mathcal{M}} \mathcal{I}_\varepsilon \quad \text{with} \quad \varepsilon = \kappa^2 \, Q$$

for each $Q > 0$ and $\kappa \geq 1$. A comparison argument in [2] shows that

$$\inf_{\mathcal{M}} \mathcal{I}_\varepsilon \leq \frac{C}{\log(1/\varepsilon)}$$

as ε tends to zero. This can be improved (see [3]) to a bound

$$\inf_{\mathcal{M}} \mathcal{I}_\varepsilon = \frac{\pi}{\log(1/\varepsilon)} + \mathcal{O}\left(\frac{(\log\log(1/\varepsilon))^2}{(\log(1/\varepsilon))^2} \right) \tag{4.5}$$

as ε tends to zero. In fact one can prove asymptotics to the relaxed problem from section 1.3 from an energetic point of view. For our subsequent analysis we need a much weaker lower bound on the energy.

Lemma 4.1 (Lower bound). *There is a universal constant $c > 0$ such that*

$$\left(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \right)^{-1} \leq C \, \log(1/Q) \quad \textit{as} \quad Q \to 0.$$

Proof. It is enough to derive a bound for the inverse energy of the relaxed problem where the saturation constraint is omitted. The associated Euler-Lagrange equation becomes linear. Note that the second component of a minimizer necessarily vanishes. Like in section 1.3 the inverse energy is given by

$$\left(\inf_{u(0)=1} \mathcal{E}_Q^\kappa \right)^{-1} = \int \frac{d\xi}{P_Q^\kappa(\xi)} = \int \frac{d\eta}{\kappa^2\eta^2 + \sigma(\eta) + Q} \leq \int\limits_{|\eta|\leq 1} \frac{d\eta}{|\eta| + Q} + \int\limits_{|\eta|>1} \frac{d\eta}{\eta^2}$$

with at most logarithmic divergence as Q tends to zero. □

4.3 Convergence and the limit shape

In the following we will always suppose that the aspect ratio $\kappa \geq 1$ is fixed. We already know that each sequence of (rescaled) Néel wall profiles u_Q converges to zero in $L^p(\mathbb{R})$ for each $p < \infty$ and uniformly away from the origin as Q tends to zero. In order to derive a nontrivial limit equation we consider renormalized Néel wall profiles

$$U_Q = \left(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa \right)^{-1} u_Q : \mathbb{R} \to \mathbb{R}. \tag{4.6}$$

By Proposition 3 the functions U_Q are bounded in each $H_q^{1/2}(\mathbb{R})$ with $1 < q < 2$ by some constant which only depends on q. By the decay estimate in section 4.1 they are furthermore uniformly bounded on each closed subset excluding the origin. In particular U_Q has a weak cluster point U_0 for each sequence $Q \to 0$. Therefore we conclude by the uniqueness of a distributional limit, that if for some sequence $Q \to 0$ the corresponding renormalized profiles U_Q converge to U_0 in the sense of distributions they necessarily converge weakly in $H^{1/2}_q(\mathbb{R})$ for $1 < q < 2$ and weakly in $L^p(\mathbb{R})$ for $1 < p < \infty$. We even have strong L^p-compactness on bounded subsets.

Remark 4.2. *For small $Q > 0$ the set $\{U_Q\}$ is precompact in $L^p_{loc}(\mathbb{R})$ for each $1 \le p < \infty$.*

Proof. Note that $(\mathbf{1} - \Delta)^{-1/4} f = k * f$ with $k \in L^r(\mathbb{R})$ for all $r \in [1, 2)$, see e.g. [10], 5.3.1. Now the operator $Tf = k * f$ is compact from $L^q(\mathbb{R})$ to $L^p_{loc}(\mathbb{R})$ for $1 + 1/p = 1/q + 1/r$. To see this it suffices to note that the convolution with a mollified function k_ε defines by Rellich's theorem a compact operator $T_\varepsilon : L^q(\mathbb{R}) \to L^p_{loc}(\mathbb{R})$ and

$$\|T - T_\varepsilon\|_{\mathcal{L}(L^q;L^p)} \le C \, \|k - k_\varepsilon\|_{L^r} \quad \text{as} \quad \varepsilon \to 0.$$

This proves the remark for $p \in (1, \infty)$ since U_Q is uniformly bounded in $H^{1/2}_q(\mathbb{R})$, which means that $(\mathbf{1} - \Delta)^{1/4} U_Q$ is bounded in $L^q(\mathbb{R})$ for each $q \in (1, 2)$. The case $p = 1$ follows from Hölder's inequality. □

We identify such a limit by the following sequence of arguments. A renormalized profile U_Q solves the renormalized Euler-Lagrange equation

$$\mathcal{L}^\kappa_Q U_Q = \Big(\inf_{\mathcal{M}} \mathcal{E}^\kappa_Q \Big)^{-1} \Big\{ e^\kappa_Q[u_Q] \, u_Q + f_Q[u_Q] + g_Q[u_Q] \Big\} \quad \text{in} \quad H^{-1}(\mathbb{R}). \tag{4.7}$$

The energy bound in Lemma 4.1 shows that the left hand side converges to $(-\Delta)^{1/2} U_0 + U_0$. We show that the limit distribution on the right hand side is supported at the origin. Then we conclude by the estimates in section 3.1, which are essentially improved by the regularity results from section 4.1 that it is indeed a finite measure. This shows that U_0 agrees with a multiple of the fundamental solution of the operator $(-\Delta)^{1/2} + \mathbf{1}$ (compare section 1.3). Finally we show that the renormalized integrals $\int U_Q$ converge to $\int U_0$ which turns out to agree with the (nonnegative) multiplicity. By the results from section 3.3 it is bounded by 2.

It is convenient to consider the energy functional in the following equivalent form

$$\mathcal{E}^\kappa_Q(m) = \kappa^2 \, Q \int \frac{(u')^2}{1 - u^2} \, dx + \|u\|^2_{L^2} + \Big\langle u \Big| \mathcal{S}_Q \Big| u \Big\rangle$$

for $m = (u, v) : \mathbb{R} \to \mathbb{S}^1$. Then it is easy to see that for a Néel wall profile u_Q and $\phi \in C_0^\infty(\mathbb{R})$ with compact support in the set $\{u_Q \neq 1\}$ the variation $u_Q + \varepsilon\phi$ is admissible for small ε. Carrying out the differentiation of the exchange part we arrive at the nonlinear singularly perturbed operator $\mathcal{R}_Q^\kappa$ defined by

$$\mathcal{R}_Q^\kappa(u) = \kappa^2\, Q \left\{ -\frac{d}{dx}\left(\frac{u'}{1-u^2}\right) + \left(\frac{u'}{1-u^2}\right)^2 u \right\}.$$

Then a Néel wall profile u_Q solves the following Euler-Lagrange equation

$$\mathcal{R}_Q^\kappa(u_Q) + \mathcal{S}_Q u_Q + u_Q = 0 \quad \text{in} \quad \mathfrak{D}'\left(\{u \neq 1\}\right). \tag{4.8}$$

Lemma 4.2. *For each $\varepsilon > 0$ there is a $Q(\varepsilon) > 0$ such that the distribution*

$$\mathcal{R}_Q^\kappa(u_Q) \in \mathfrak{D}'\big(\mathbb{R} \setminus [-\varepsilon, \varepsilon]\big) \qquad \textit{for each} \quad Q < Q(\varepsilon)$$

is well defined. As Q tends to zero the renormalization $\left(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\right)^{-1} \mathcal{R}_Q^\kappa(u_Q)$ convergences to zero in $\mathfrak{D}'\big(\mathbb{R} \setminus [-\varepsilon, \varepsilon]\big)$.

Proof. Let for given $\varepsilon > 0$ the function ϕ be in $C_0^\infty\left(\mathbb{R} \setminus [-\varepsilon, \varepsilon]\right)$. Take $Q(\varepsilon) > 0$ so small that $1 - u_Q^2$ in uniformly bigger then zero on the support of ϕ for $Q < Q(\varepsilon)$. Note that this is possible by the pointwise decay estimate. Then Hölder's inequality yields for $Q < Q(\varepsilon)$

$$\frac{|\langle \mathcal{R}_Q^\kappa(u_Q), \phi\rangle|}{\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa} \leq \left(\frac{Q\,\kappa^2}{\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa} \int \frac{|\phi'|^2}{1-u_Q^2}\,dx\right)^{1/2} + \sup\left|\frac{\phi\, u_Q}{1-u_Q^2}\right|. \tag{4.9}$$

The first term tends to zero by the lower energy bound in Lemma 4.1. For the second term note that as a direct consequence of the pointwise decay estimate in section 4.1 and the upper energy bounds in section 4.2 the profiles u_Q converge uniformly to zero on each compact subset excluding the origin. □

Proposition 4. *Let for some sequence $Q \to 0$ the corresponding sequence of the renormalized Néel wall profiles U_Q converge to U_0 in the sense of distributions. Then the following assertions hold true.*

(i) U_0 solves the equation $(-\Delta)^{1/2} U_0 + U_0 = 0$ in $\mathfrak{D}'\left(\mathbb{R} \setminus \{0\}\right)$.

(ii) U_0 is integrable and $\int U_0$ is the limit of the integrals $\int U_Q$.

Proof. Let ϕ be any testfunction in $\mathfrak{D}(\mathbb{R}\setminus\{0\})$. Then by Lemma 4.2 there is a number $Q_0 > 0$ such that ϕ is an admissible testfunction for (4.8) for all $Q < Q_0$. We may assume that U_Q converges weakly in $L^2(\mathbb{R})$. Then assertion (i) follows from the convergence properties of the operator $\mathcal{S}_Q$ stated in Lemma 2.2 and the second claim in Lemma 4.2.
Note that the integrability of U_0 follows from the uniform bound of U_Q in $L^1 \cap L^2(\mathbb{R})$ and the lower semicontinuity of the total mass.
Let ϕ be a smooth cutoff function such that $\phi(x) = 0$ for $|x| < 1/2$ and $\phi(x) = 1$ for $|x| > 1$. Using the cutoff function $\phi_\varepsilon(x) = \phi(\varepsilon x)$ we decompose

$$\int U_Q \, dx = \int U_Q \left(1 - \phi_\varepsilon\right) dx + \int U_Q \, \phi_\varepsilon \, dx$$

As an immediate consequence of the weak L^p-convergence and the integrability of U_0 we infer that as the sequence $Q \to 0$

$$\int U_Q \left(1 - \phi_\varepsilon\right) dx \to \int U_0 \, dx + o(1)$$

where $o(1)$ goes to zero as ε tends to zero. It is easy to see that ϕ_ε is an admissible test function for (4.8) and there holds

$$\int U_Q \, \phi_\varepsilon \, dx \le \left(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\right)^{-1} \left|\left\langle \mathcal{R}_Q^\kappa(u_Q), \phi_\varepsilon \right\rangle\right| + \left|\left\langle U_Q, \mathcal{S}_Q \phi_\varepsilon \right\rangle\right|$$

But by (4.9) and the subsequent arguments we see that the first term converges to zero as Q tends to zero and the convergence is uniform in $\varepsilon > 0$. For the second term we use Cauchy-Schwarz and the bounds for the operator $\mathcal{S}_Q$ to see that $|\langle U_Q, \mathcal{S}_Q \phi_\varepsilon\rangle| \le \|U_Q\|_{L^2} \|\phi_\varepsilon\|_{\dot{H}^1} = \mathcal{O}\big(\varepsilon^{1/2}\big)$. Now the claim follows as Q and ε tend to zero. ☐

Lemma 4.3. *Let for some sequence $Q \to 0$ the sequence renormalized Néel wall profiles U_Q converge to U_0 in the sense of distributions. Then $\mathcal{L}_Q^\kappa U_Q$ converges to $(-\Delta)^{1/2} U_0 + U_0$ in the sense of distributions.*

Proof. We may assume that U_Q converges weakly in $L^2(\mathbb{R})$. By Lemma 2.2 we only need to check the convergence of the second order term, but Lemma 4.1 shows that

$$Q \left|\left\langle U_Q \middle| (-\Delta) \middle| \phi \right\rangle\right| \le Q^{1/2} \left(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\right)^{-1/2} \|\phi\|_{\dot{H}^1} \to 0 \quad \text{for each} \quad \phi \in C_0^\infty(\mathbb{R}),$$

hence $Q(-\Delta) U_Q$ converges to zero in the sense of distributions. ☐

Theorem 2. *For any sequence $Q \to 0$ such that the corresponding sequence of renormalized Néel wall profiles U_Q converges in the sense of distributions, the weak limit U_0 is a multiple of the fundamental solution of the operator $(-\Delta)^{1/2} + \mathbf{1}$. The multiplicity is given by the limit of the of the integrals $\int U_Q$.*

Proof. Let U_Q converges to U_0 in the sense of distributions. At first U_Q solves the renormalized Euler-Lagrange equation (4.7). By Lemma 4.3 the distribution $\mathcal{L}_Q^\kappa U_Q$ converges weakly to $(-\Delta)^{1/2} U_0 + U_0$. By claim (i) in Proposition 4 the right hand side divided by the minimal energy converges to a distribution which is supported at the origin. Using the bound (3.13) from section 3.1 and the (improved) L^p-bounds in Proposition 3 from section 4.1, it is a measure. Indeed

$$g_Q[u_Q] \sim \big(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\big)^{3/2} \quad \text{in} \quad H_q^{-1/2}(\mathbb{R}) \quad \text{for} \quad 4/3 \le q < 2.$$

We conclude that U_0 is a multiple of the fundamental solution for the operator $(-\Delta)^{1/2} + \mathbf{1}$. The multiplicity μ_0 agrees with the integral of the function U_0. Indeed $(-\Delta)^{1/2} U_0 + U_0 = \mu_0\, \delta_0$ and by Proposition 4 U_0 is integrable. For the scaled bump function φ_ε which we already used in section 3.3 we have that

$$\int U_0\, \varphi_\varepsilon\, dx = \mu_0 - \big\langle (-\Delta)^{1/2} \varphi_\varepsilon, U_0 \big\rangle$$

with $|U_0\, \varphi_\varepsilon| \le |U_0| \in L^1(\mathbb{R})$ and $|\langle (-\Delta)^{1/2} \varphi_\varepsilon, U_0\rangle| \le \|U_0\|_{L^2} \|\varphi_\varepsilon\|_{\dot{H}^1} = \mathcal{O}\big(\varepsilon^{1/2}\big)$. Letting ε tend to zero we conclude by majorized convergence that $\mu_0 = \int U_0$. But claim (ii) in Proposition 4 implies that

$$\text{multiplicity} = \int U_0(x)\; dx = \lim \int U_Q(x)\; dx = \lim \Big\{ I_Q^\kappa \, \big(\inf_{\mathcal{M}} \mathcal{E}_Q^\kappa\big)^{-1} \Big\} \le 2$$

where I_Q^κ is the integral of the profile u_Q^κ given by (3.20). □

5 Appendix

5.1 A Sobolev inequality and the proof of Lemma 3.3

The following result is a special case of the well known Hardy-Littlewood-Sobolev inequality. We follow the proof given in [9] keeping track of the growth of constants. Optimal constants for fractional integration have been found by E. H. Lieb, see [7] and the literature cited therein.

Lemma 5.1. *Suppose that $p \geq 4$. Then there is a universal constant $c > 0$ such that for each function $f \in \mathfrak{S}(\mathbb{R})$*

$$\left\|(-\Delta)^{-1/4} f\right\|_{L^p} \leq c\,(p-2)^{\frac{1}{2}\,\frac{p-2}{p+2}}\,\left\|f\right\|_{L^q} \quad \textit{for} \quad 1/q = 1/p + 1/2 \tag{5.1}$$

Proof. The operator $(-\Delta)^{-1/4}$ is associated to the locally integrable kernel function $k(x) = |x|^{-1/2}$ which just fails to be square integrable. We split $k = k \cdot \chi_R + k \cdot (1-\chi_R)$ where χ_R denotes the characteristic function of the interval $(-R, R)$. Then we have for the first term the pointwise estimate

$$|f * (k \cdot \chi_R)|(x) \leq 4\,R^{1/2}\,(Mf)(x), \tag{5.2}$$

where Mf denotes the Hardy-Littlewood maximal function [3] of f. We estimate the second term by Hölder's inequality for $q = \frac{2p}{p+2}$

$$\left|f * (k \cdot (1-\chi_R))\right|(x) \leq \|f\|_{L^q} \|k \cdot (1-\chi_R)\|_{L^{q'}}.$$

Now by definition $q' = \frac{2p}{2+p}$ and $\frac{2}{q'-2} = \frac{p-2}{2}$. Hence there holds

$$\left\|k \cdot (1-\chi_R)\right\|_{L^{q'}} = 2^{1/q'} \left(\frac{2}{q'-2}\right)^{1/q'} R^{1/q'-1/2} = (p-2)^{1/q'}\,R^{1/q'-1/2},$$

that is we have the pointwise estimate

$$\left|f * (k \cdot (1-\chi_R))\right|(x) \leq (p-2)^{1/q'}\,\|f\|_{L^q}\,R^{1/q'-1/2} \tag{5.3}$$

Now we choose $R = R(x)$ such that the terms on the right hand side in (5.2) and (5.3) agree. An easy computation shows that this is the case for

$$R(x) = \left\{\frac{(p-2)^{1/q'}\,\|f\|_{L^q}}{4\,(Mf)(x)}\right\}^{\frac{2p}{p+2}}$$

which we substitute into (5.2) to get the final pointwise estimate

$$|f * k| \leq 2\,|f * (k \cdot \chi_R)| \leq c\,(p-2)^{\frac{1}{2}\,\frac{p-2}{p+2}}\,\|f\|_{L^q}^{\frac{p}{p+2}}\,(Mf)^{1-\frac{p}{p+2}},$$

The Hardy-Littlewood maximal theorem (see [10]) completes the proof. □

Now the proof of Lemma 3.3 easily follows by the well known fact that the action of the operator $(-\Delta)^{1/4}(1-\Delta)^{-1/4}$ is given by the convolution with a finite measure, which implies uniform boundedness on all $L^p(\mathbb{R})$ spaces including the endpoint $p = \infty$. See e.g. [10], p. 133.

[3] The Hardy-Littlewood maximal function of a locally integrable function f is defined by $Mf(x) = \sup_{r>0} ⨍_{-r}^{r} f(x-y)\,dy$, see [10] at the very beginning.

5.2 Derivation of the reduced strayfield operator

The derivation of the kernel representation of the reduced magnetostatic energy can be found e.g. in [2] by means of magnetostatic potentials. Here we present for the convenience of the reader the derivation of the Fourier representation by computing Fourier integrals. We recall the definition of the reduced strayfield operator acting on profiles $u : \mathbb{R} \to \mathbb{R}$ which are coming from a planar magnetization fields

$$\mathbf{m} : \mathbb{R}^3 \ni (x, y, z) \mapsto (u(x), v(x), 0)\ \chi_\delta(z) \in \mathbb{S}^2$$

where χ_δ denotes the characteristic function of the interval $(-\delta, \delta)$. It is defined by the following renormalized average of the first component of the (negative) magnetic field $\mathcal{H}(\mathbf{m}) : \mathbb{R}^3 \to \mathbb{R}^3$ induced by $\mathbf{m} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathcal{S}_\delta : u \mapsto \frac{1}{\delta} \fint_{-\delta}^{\delta} \mathcal{H}(\mathbf{m}) \cdot \hat{e}_1\ dz. \tag{5.4}$$

In fact $\mathcal{H}(\mathbf{m}) = \nabla\Delta^{-1}\nabla \cdot \mathbf{m}$ is the Helmholtz projection and has values in $\mathbb{R} \times \{0\} \times \mathbb{R}$. Now the first goal is to clarify the link to the magnetostatic energy. Intergration of $|\mathcal{H}(\mathbf{m})|^2$ over the relevant space directions x and z and averaging by the film thickness leads to the averaged magnetostatic energy

$$\mathcal{E}_{mag}(\mathbf{m}) = \frac{1}{2\delta} \int\int |\mathcal{H}(\mathbf{m})|^2\, dx\, dz$$

Since the volume charge $\nabla \cdot \mathbf{m}$ only depends on x and z we can apply Green's formula in two dimensions to write the energy as

$$\begin{aligned} \mathcal{E}_{mag}(\mathbf{m}) &= \frac{1}{2\delta} \int\int \nabla \cdot \mathbf{m}\ (-\Delta)^{-1}\nabla \cdot \mathbf{m}\ dx\, dz \\ &= \frac{1}{2\delta} \int\int \nabla \cdot \mathbf{m}\ (-\Delta_{\mathbb{R}^2})^{-1}\nabla \cdot \mathbf{m}\ dx\, dz. \end{aligned} \tag{5.5}$$

The first step is to show that the magnetostatic energy can be expressed in terms of the reduced strayfield operator as defined in (5.4). A direct consequence of the definition (5.4) and the cancellation of the second and third component functions is the identity

$$\delta \left\langle u \middle| \mathcal{S}_\delta \middle| u \right\rangle = \int_{-\infty}^{\infty} \fint_{-\delta}^{\delta} \mathbf{m} \cdot \mathcal{H}(\mathbf{m})\ dz\ dx. \tag{5.6}$$

By (5.5) and integration by parts we see that the projection property of the Helmholtz transform is carried over to our reduced ansatz

$$\int_{-\infty}^{\infty} \fint_{-\delta}^{\delta} \mathbf{m} \cdot \mathcal{H}(\mathbf{m})\, dz\, dx = \frac{1}{2\delta} \int \int |\mathcal{H}(\mathbf{m})|^2\, dx\, dz. \tag{5.7}$$

Equation (5.7) shows by (5.6) that with our definition of the reduced stray-field operator the averaged magnetostatic energy is given by

$$\frac{1}{2\delta} \int \int |\mathcal{H}(\mathbf{m})|^2\, dx\, dz = \delta \left\langle u \middle| \mathcal{S}_\delta \middle| u \right\rangle. \tag{5.8}$$

The next step is to compute the Fourier representation of $\mathcal{S}_\delta$. For this purpose we first compute the Fourier representation of the magnetostatic energy. Starting from (5.5) with $\nabla \cdot \mathbf{m} = u'(x)\chi_\delta(z)$, Parseval's formula yields

$$\begin{aligned}
\mathcal{E}_{mag}(\mathbf{m}) &= \frac{1}{2\delta} \int \int \nabla \cdot \mathbf{m}\, (-\Delta_{\mathbb{R}^2})^{-1} \nabla \cdot \mathbf{m}\, dx\, dz \\
&= \frac{1}{2\delta} \int \int \frac{|\xi\, \hat{u}(\xi)|^2}{\xi^2 + \eta^2} \left(\sqrt{\frac{2}{\pi}} \frac{\sin(\delta\eta)}{\eta} \right)^2 d\xi\, d\eta \\
&= \frac{1}{\delta\pi} \int \int \left\{ \frac{\sin^2(\delta\eta)}{\eta^2} - \frac{\sin^2(\delta\eta)}{\xi^2 + \eta^2} \right\} d\eta\, |\hat{u}(\xi)|^2\, d\xi.
\end{aligned}$$

We have to compute the inner integrals. The first is given by Parseval's formula

$$\int \frac{\sin^2(\delta\eta)}{\eta^2}\, d\eta = \frac{\pi}{2} \int |\mathcal{F}(\chi_\delta)(\eta)|^2\, d\eta = \frac{\pi}{2} \int |\chi_\delta(z)|^2\, dz = \delta\, \pi$$

whereas the second splits once more into two standard integrals,

$$\begin{aligned}
&\int \frac{\sin^2(\delta\eta)}{\xi^2 + \eta^2}\, d\eta = \frac{1}{2} \int \frac{1 - \cos(2\delta\eta)}{\xi^2 + \eta^2}\, d\eta \\
&= \frac{1}{2|\xi|} \left\{ \int \frac{d\tau}{1 + \tau^2} - \int \frac{\cos(2\delta|\xi|\tau)}{1 + \tau^2}\, d\tau \right\} = \frac{\pi}{2|\xi|} \left(1 - \exp(-2\delta|\xi|)\right)
\end{aligned}$$

(see e.g. [11] p. 191). Putting everything together yields the following Fourier representation for the averaged magnetostatic energy

$$\frac{1}{2\delta} \int \int |\mathcal{H}(\mathbf{m})|^2\, dx\, dz = \int \left\{ 1 - \frac{1 - \exp(-2\delta|\xi|)}{2\delta|\xi|} \right\} |\hat{u}(\xi)|^2\, d\xi. \tag{5.9}$$

From this we can read off by (5.4) and (5.6) our final result, the Fourier representation for the magnetostatic energy:

$$\delta \left\langle u \middle| \mathcal{S}_\delta \middle| u \right\rangle = \int \left\{ 1 - \frac{1 - \exp(-2\delta|\xi|)}{2\delta|\xi|} \right\} |\hat{u}(\xi)|^2 \, d\xi$$

and thus the Fourier representation $\mathcal{F}\mathcal{S}_\delta(\xi)$ for the reduced strayfield operator

$$\mathcal{F}\mathcal{S}_\delta(\xi) = \frac{1}{\delta}\, \sigma(\delta\xi) = \frac{1}{\delta} \left(1 - \frac{1 - \exp(-2\delta|\xi|)}{2\delta|\xi|} \right) \quad \text{for each} \quad \delta > 0.$$

Acknowledgements

I am indebted to Stefan Müller for suggesting me this topic and for many fruitful and stimulating discussions. Moreover I thank the Max-Planck-Institute for Mathematics in the Sciences for providing excellent working conditions.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces, an Introduction*, Springer-Verlag, New York, 1976.

[2] C.J. Gracia-Cervera, *Magnetic Domains and Magnetic Domain Walls*, PhD thesis, New York University, 1999.

[3] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Magnetic microstructure - a paradigm of multiscale problems, Proc. ICIAM 99 (Edinburgh), Eds. J.M. Ball and J.C.R. Hunt, 1999, pp. 175-199

[4] G. Gioia and R. D. James, Micromagnetics of very thin films, Proc. R. Soc. Lond. A **453** (1997), pp. 213-223.

[5] S. Hofmann, An Off-Diagonal $T(1)$ Theorem and Applications, J. Funct. Analysis **160** (1998), pp. 581-622.

[6] A. Hubert and R. Schäfer, *Magnetic Domains, The Analysis of Magnetic Microstructures*, Springer-Verlag, Berlin-Heidelberg, 1998.

[7] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics Vol.14 American Mathematical Society, 1996.

[8] H. Riedel and A. Seeger, Micromagnetic Treatment of Néel Walls, Phys. Stat. Sol. (B) **46** (1971), pp. 377-384

[9] E.M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, Vol. 43, Princeton University Press, 1993.

[10] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[11] *Teubner-Taschenbuch der Mathematik*, Ed. E. Zeidler, Teubner, Stuttgart-Leipzig , 1996.

[12] E. Zeidler, *Applied Functional Analysis, Main Principles and their Application*, AMS 109, Springer, New York, 1995.