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by

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Diffusion-Advection in Cellular Flows with Large Peclet Numbers

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Abstract

For periodic two-dimensional incompressible cellular flows we provide explicit upper and lower estimates of the effective diffusivity which have the correct scaling behavior for large Peclet numbers. We demonstrate that all allowed scaling laws can occur. The bounds prove that there is no residual diffusion in the infinite Peclet number limit for Hölder continuous flows, answering a problem posed by Kozlov.

1 Introduction

We consider the diffusion advection equation

$$\partial_t u(x,t) = \epsilon \Delta u(x,t) + b(x)\cdot \nabla u(x,t) \tag{1}$$

in $I\!R^2$ with a small diffusivity $\epsilon > 0$. The inverse $1/\epsilon$ is also called the Peclet number and is very large in applications. The vector field $b(x)$ is a given underlying incompressible flow field, 1-periodic in x_1 and x_2, with mean value zero. Introducing the periodic stream function $H(x)$ as $b(x) = \nabla^\perp H(x)$ we can rewrite (1) in divergence form

$$\partial_t u(x,t) = \nabla(A_\epsilon(x)\nabla u(x,t)). \tag{2}$$

with the non symmetric matrix

$$A_\epsilon(z) = \begin{pmatrix} \epsilon & H(z) \\ -H(z) & \epsilon \end{pmatrix}. \tag{3}$$

We consider cellular flows where the boundary of the unit cube $C = (-1/2, 1/2)^2$ consists of streamlines, i.e. we assume $H|_{\partial C} = 0$. The standard example is

$$H(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2). \tag{4}$$

In the large time, large distance scaling $t \to t/\delta^2$, $x \to x/\delta$ equation (1) becomes

$$\partial_t u(x,t) = \epsilon\Delta u(x,t) + \frac{1}{\delta} b(\frac{x}{\delta})\cdot\nabla u(x,t).$$

The theory of homogenization shows, that the limit $\delta \to 0$ is governed by an effective equation

$$\partial_t u(x,t) = \nabla(A^h_\epsilon \nabla u(x,t)).$$

In [10] it is shown that this is true even for stream functions $H \in L^2(C)$.

The constant effective diffusion matrix is computed from the cell problems in direction $k \in I\!R^2$

$$\epsilon\Delta\rho(x) + b(x)\cdot\nabla\rho(x) = 0, \tag{5}$$

where $\rho(x) - k\cdot x$ is 1-periodic in each direction x_i. For the uniqueness of ρ we assume

$$\int_C \rho(x)\, dx = 0.$$

The effective diffusivity matrix is then

$$A^h_\epsilon k = \int_C A_\epsilon(x)\nabla\rho(x)\, dx. \tag{6}$$

Another representation is

$$k_1 A_\epsilon^h k_2 = \int\limits_C \nabla\rho_1(x) A_\epsilon(x) \nabla\rho_2(x)\, dx. \tag{7}$$

where ρ_i is the solution corresponding to k_i.

We are interested in the dependence of the effective diffusivity A_ϵ^h on ϵ especially in the scaling as $\epsilon \to 0$.

Since we are also interested in non smooth stream functions the weak formulation of (6) will be used for $\rho(x) - k\cdot x \in H^1_{per}(C)$

$$\int\limits_C (\epsilon\nabla\rho - H\nabla^\perp\rho)\nabla\phi\, dx = 0 \tag{8}$$

for $\phi \in H^1_{per}(C)$. The weak formulation makes sense for $H \in L^\infty(C)$ and even for $H \in L^2(C)$ if ϕ is smooth.

For the standard example above Childress [3] calculated the asymptotic scaling of the effective diffusivity. By a formal asymptotic expansion he obtained

$$A_\epsilon^h \sim const\, \sqrt{\epsilon} I$$

and calculated the constant.

In [8], [9] this is made rigorous using an inf/sup argument on a suitable indefinite functional. The method in this paper gives only the asymptotic result without an estimate for nonzero ϵ. It seems difficult to use these variational principles for positive ϵ.

It is well known, that the effective diffusivity satisfies the following estimate

$$\epsilon \le \lambda(\epsilon) \le \epsilon + \frac{1}{\epsilon}\int\limits_C H(x)^2\, dx. \tag{9}$$

for a general stream function. Only for large ϵ this is a good approximation. This estimate has been improved in [2] through an asymptotic expansion for large ϵ. In this limit (5) becomes a regular perturbation problem. These estimates all diverge for small ϵ and do not give the correct scaling. We remark that for shear flows, i.e. $H(x)$ depends only on one variable, the effective diffusivity behaves as $1/\epsilon$ in the direction of the flow and as ϵ in the orthogonal direction. Hence these are the best possible estimates for an arbitrary flow. In cellular flows however the situation is quite different due to the nonexistence of unbounded streamlines. One expects an intermediate enhancement of the effective diffusivity. It is the purpose of this paper to give precise information on the enhancement depending on properties of the flow field.

We will provide two proofs for an explicit upper bound under different assumptions on the regularity and symmetry properties of the stream function.

For symmetric $H(x)$ a lower estimate, which also has the same scaling as the upper bound, will be given in the second part. All estimates have the correct scaling behavior for $\epsilon \to 0$ which is the most interesting case in applications, compare the review article [13].

Our method relies on the use of appropriate test functions which gives automatically the correct size of the boundary layer near the level set $H(x) = 0$ and the scaling of the effective diffusivity. Since our bounds involve explicit constants we have an estimate for the range of validity of the scaling behavior for large Peclet numbers.

The upper estimates also answer a problem posed by Kozlov in [12], i.e. if it is possible to have a nonzero limit for the effective diffusivity as ϵ tends to zero for cellular flows. This is called residual or turbulent diffusion. It is known, that for a discontinuous stream function $H(x)$ nonzero residual diffusion is possible [11] page 23 & 36 and [14]. The flow field v has to be interpreted as a distribution in this case. Our bounds exclude this possibility for general C^1 and for symmetric Hölder continuous vector fields $b(x)$.

We also mention a series of related papers [4]-[7], where effective properties of a Rayleigh-Bernard convection problem are derived using also suitable test functions.

2 Upper Bounds

2.1 General Case

We consider the cell problem for an arbitrary direction k with $|k| = 1$. We assume $b \in C^1(C)$ with $\nu \cdot b = 0$ on ∂C. In the estimate below the following constant appears

$$M = \sup_{x \in C} \sup_{|\eta|=1} \eta^T \, D_x b(x) \, \eta = \sup_{x \in C} \left(\frac{1}{4} (\partial_{11} H - \partial_{22} H)^2 + (\partial_{12} H)^2 \right)^{1/2} (x). \tag{10}$$

Theorem 1 *The effective diffusivity is estimated by*

$$k A_\epsilon^h k \le \sqrt{M\epsilon} \, \coth \sqrt{\frac{M}{4\epsilon}} \tag{11}$$

for all $\epsilon > 0$.

Proof: Let $v - k \cdot x \in H^1_{per}(C)$ and use $\phi = \rho - v$ as a test function in (1). This gives

$$\epsilon \int_C |\nabla \rho|^2 \, dx = \epsilon \int_C (\nabla \rho \nabla v - b \cdot \nabla \rho \; v + \frac{1}{2} b \cdot \nabla \rho^2) \, dx$$

$$\le \frac{\epsilon}{4} \int_C |\nabla \rho|^2 \, dx + \epsilon \int_C |\nabla v|^2 \, dx + + \frac{\alpha}{2} \int_C v^2 \, dx + \frac{1}{2\alpha} \int_C |b \cdot \nabla \rho|^2 \, dx,$$

where $\nabla \cdot b = 0$ and $b \cdot \nu = 0$ has been used. As a second test function we take $b \cdot \nabla \rho$:

$$\int_C |b \cdot \nabla \rho|^2 \, dx = -\epsilon \int_C (b \cdot \nabla \rho) \Delta \rho \, dx$$

$$= \frac{\epsilon}{2} \int_C b \cdot \nabla |\nabla \rho|^2 \, dx + \epsilon \int_C \nabla \rho D_x b \nabla \rho \, dx$$

$$\le \epsilon M \int_C |\nabla \rho|^2 \, dx$$

where M is defined in (10). Again $\nabla \cdot b = 0$ has been used. Boundary terms do not occur, since the derivatives of ρ are periodic. Combining the two estimates we get with $\alpha = 2M$

$$\frac{\epsilon}{2} \int_C |\nabla \rho|^2 \, dx \le \epsilon \int_C |\nabla v|^2 \, dx + M \int_C v^2 \, dx. \tag{12}$$

The minimum is achieved for v satisfying:

$$\epsilon \Delta v = M v,$$

s.t. $v - k \cdot x$ is periodic. This can be solved explicitly

$$v(x) = \frac{k_1}{2} \frac{\sinh(x_1 \sqrt{\frac{M}{\epsilon}})}{\sinh(\sqrt{\frac{M}{4\epsilon}})} + \frac{k_2}{2} \frac{\sinh(x_2 \sqrt{\frac{M}{\epsilon}})}{\sinh(\sqrt{\frac{M}{4\epsilon}})}$$

for $x \in C$. A calculation implies for the right hand side in (12)

$$k A_\epsilon^h k \le \sqrt{M\epsilon} \, \coth \sqrt{\frac{M}{4\epsilon}}.$$

This completes the proof.

This estimate behaves as $\sqrt{M\epsilon}$ for small ϵ and is linear in ϵ for large ϵ, i.e. it is qualitatively correct for all $\epsilon > 0$.

2.2 The Symmetric Case

We will derive an upper bound also for nonsmooth flow fields $b(x)$. In this way we will produce a ϵ^α scaling of the effective diffusivity for any α between 0 and 1 depending on the behavior of the flow field near the boundary of the unit cell. That is why we consider the weak formulation (8). We require for the stream function $H \in W_0^{1,1}(D) \cap L^\infty(D)$. Furthermore we make the following symmetry assumption:

$$H(x) \text{ is odd in } x_1 \text{ and } x_2. \tag{13}$$

Both assumptions are satisfied e.g. for

$$H(x) = h(x)|h(x)|^{p-1}$$

with $p > 0$, where $h(x)$ is the standard example (4). Also in [8], [9] symmetry assumptions on H are used crucially, but only the case $p = 1$ is considered there.

Let ρ_1, ρ_2 be the solutions of the cell problems corresponding to $k = (1, 0)$ and $k = (0, 1)$. The oddness of H implies that ρ_1 is odd in x_1 and even in x_2 and vice versa for ρ_2. From the definition of A_ϵ^h it follows now easily that A_ϵ^h is a diagonal matrix.

Now let ρ be the solution of (5) corresponding to $k = (1, 0)$. Using the symmetry we can restrict the cell problem to the quarter cell $D = (0, 1/2)^2$

$$\int_D \nabla\phi(\epsilon\nabla\rho - H\nabla^\perp\rho)\, dx = 0 \tag{14}$$

for all $\phi \in H^1(D)$, $\phi(1/2, x_2) = \phi(0, x_2) = 0$ with boundary conditions

$$\rho(1/2, x_2) = 1/2, \quad \rho(0, x_2) = 0.$$

The weak maximum principle implies for all $x \in D$

$$0 < \rho(x) < 1/2.$$

This is the main difference to the case without symmetry where a L^∞ estimate, which is independent of ϵ, is not available. We will prove an estimate of

$$\lambda(\epsilon) := kA_\epsilon^h k = \epsilon \int_C |\nabla\rho|^2\, dx.$$

For $\delta < 1/2$ take $\phi(x) = \rho(x) - v(x_1)$ with

$$v(x_1) = \begin{cases} 0, & 0 < x_1 < 1/2 - \delta \\ \frac{1}{2\delta}(x_1 - 1/2 + \delta), & 1/2 - \delta < x_1 < 1/2 \end{cases}$$

as a test function in (14).

$$\frac{\lambda(\epsilon)}{4} = \epsilon \int_D |\nabla\rho|^2\, dx = \int_D \nabla v(\epsilon\nabla\rho - H\nabla^\perp\rho)\, dx = \frac{1}{2\delta} \int_{1/2-\delta}^{1/2} \int_0^{1/2} (\epsilon\partial_1\rho - \partial_2 H\rho)\, dx_2\, dx_1$$

$$\leq \frac{\epsilon}{8\delta} + \frac{1}{4\delta} \int_{1/2-\delta}^{1/2} \int_0^{1/2} \max(0, -\partial_2 H)\, dx_2\, dx_1.$$

If $\partial_2 H$ has some decay rate near $x_1 = 1/2$ then the right hand side can be minimized with respect to δ. We state the result for an algebraic decay. Assume that for some $p > 0$ and some function $\gamma(x_2) \geq 0$ we have the following estimate

$$-\partial_2 H(x_1, x_2) \leq \gamma(x_2)(1/2 - x_1)^p$$

for all $(x_1, x_2) \in C$. This gives

$$\lambda(\epsilon) \leq \frac{\epsilon}{2\delta} + \frac{\delta^p \overline{\gamma}}{p+1} \tag{15}$$

with $\overline{\gamma} = \int_0^{1/2} \gamma(x_2)\, dx_2$. Minimizing over $0 < \delta < 1/2$ implies the following theorem.

Theorem 2 *The effective diffusivity is estimated by*

$$\lambda(\epsilon) \leq \epsilon^{\frac{p}{p+1}} \left(\frac{p+1}{2p}\right)^{\frac{p}{p+1}} \overline{\gamma}^{1/(1+p)}, \qquad \text{for } \epsilon \leq \frac{p\overline{\gamma}}{2^p(p+1)}. \tag{16}$$

Remarks:

Instead of the estimate near $x_1 = 1/2$ the same bound can be obtained near $x_1 = 0$ but with a possibly different scaling exponent p. For small ϵ the larger p determines the upper bound for the effective diffusivity. This scaling behavior has been conjectured by Avellaneda [1].

For the standard example (4) we have $p = 1$, $\gamma = 4\pi^2 \max(-\cos(2\pi x_2), 0)$, $\overline{\gamma} = 2\pi$. This gives

$$\lambda(\epsilon) \leq \begin{cases} (2\pi\epsilon)^{1/2}, & \epsilon < 0.08 \\ \sqrt{\epsilon^2 + 1/2}, & \epsilon > 0.08 \end{cases}. \tag{17}$$

The second inequality is proved in [11], page 45. The two estimates are equal for ϵ approximately 0.08. This value limits the validity of the $\sqrt{\epsilon}$ behavior of the effective diffusivity.

Choosing another decay rate for $\partial_2 H$ in (15), e.g. exponential, one can produce arbitrary small scaling laws for the enhancement of the effective diffusivity. We state the extreme case where $H(x) = 0$ for $1/2 - a \leq x_1 \leq 1/2$. Choosing $\delta = a$ and $\gamma(x_2) = 0$ in (15) implies

$$\epsilon \int_C |\nabla \rho|^2\, dx = \epsilon \int_{-1/2}^{1/2} \int_{1/2-a}^{1/2} \partial_1 \rho\, dx_1\, dx_2 \leq \frac{\epsilon}{2a}.$$

Hence there is no significant enhancement in this case.

On the other hand the result shows that for an odd $C^{1,p}$ stream function H the effective diffusivity always satisfies

$$\lim_{\epsilon \to 0} \lambda(\epsilon) = 0.$$

Hence there is no residual diffusion for symmetric Hölder continuous flow fields $b(x)$. This answers a problem posed by Kozlov in [12].

3 Lower Bound

The assumptions on the stream function H are the same as in the previous section. We will use the general duality formula for the homogenization of a 2×2 matrix $A(x)$ (see [11]):

$$\frac{A_\epsilon^h}{\det A_\epsilon^h} = \left(\frac{A_\epsilon}{\det A_\epsilon}\right)^h.$$

Since A_ϵ^h is diagonal we have for $\lambda(\epsilon) = e_1 A_\epsilon^h e_1$

$$\frac{1}{\lambda(\epsilon)} = \frac{e_2 A_\epsilon^h e_2}{\det A_\epsilon^h} = e_2 \left(\frac{A_\epsilon}{\det A_\epsilon}\right)^h e_2$$

with $e_1 = (1,0), e_2 = (0,1)$. Therefor we consider the weak formulation of the cell problem associated to $\frac{A_\epsilon}{\det A_\epsilon}$ in direction e_2

$$\int_C \frac{\nabla \phi (\epsilon \nabla \psi - H \nabla^\perp \psi)}{\epsilon^2 + H^2}\, dx = 0 \tag{18}$$

for all $\phi \in H^1_{per}(C)$. The solution ψ satisfies $\psi(x) - x_2 \in H^1_{per}(C)$. The inverse of the effective diffusivity is given by

$$\frac{1}{\lambda(\epsilon)} = \epsilon \int_C \frac{|\nabla\psi|^2}{\epsilon^2 + H^2}\, dx \tag{19}$$

Again ψ is symmetric and (18) can be restricted to $D := (0, 1/2)^2$

$$\int_D \frac{\nabla\phi(\epsilon\nabla\psi - H\nabla^\perp\psi)}{\epsilon^2 + H^2}\, dx = 0 \tag{20}$$

for $\phi \in H^1(D)$, $\phi(1/2, x_2) = \phi(0, x_2) = 0$ with boundary conditions

$$\psi(x_1, 1/2) = 1/2, \quad \psi(x_1, 0) = 0.$$

The maximum principle implies $0 < \psi < 1/2$ on D.

Theorem 3 *Assume, that there exists a set $J \subset (0, 1/2)$ and constants $\beta, q > 0$, s.t. the stream function satisfies $H(x)$*

$$|H(x_1, x_2)| \geq \beta \left\{|x_1|^q, (1/2 - |x_1|)^q\right\} \tag{21}$$

for all $(x_1, x_2) \in (0, 1/2) \times J$. Furthermore assume that $\partial_1 H(x_1, x_2)$ has for fixed x_2 exactly one zero $x_1 = \Gamma(x_2)$ in $(0, 1/2)$. Then the effective diffusivity is estimated by

$$\lambda(\epsilon) \geq \epsilon^{\frac{q}{q+1}} \frac{|J|}{\alpha} \left(\frac{2(q+1)}{q} + \frac{\epsilon^{\frac{1}{q+1}}}{4q^2|J|^2} \right)^{-1} \tag{22}$$

with $\alpha = \left(\frac{2q|J|}{\beta}\right)^{\frac{1}{q+1}}$ for

$$\epsilon < \beta \min\left(\frac{1}{4^{\frac{1}{q+1}} q|J|}, (q|J|)^q \right).$$

Proof: Let $\phi(x) = \psi(x) - v(x_2)$ in (20), where $v(x_2)$ grows linearly on J from 0 to 1/2 and is constant otherwise. We get

$$2\epsilon|J| \int_D \frac{|\nabla\psi|^2}{\epsilon^2 + H^2}\, dx = \int_J \int_0^{1/2} \frac{\epsilon\partial_2\psi - H\partial_1\psi}{\epsilon^2 + H^2}\, dx_1\, dx_2 \tag{23}$$

For $\delta \leq 1/4$ let

$$A_\delta = \{x \in D |\ \delta < x_1 < 1/2 - \delta, x_2 \in J\}$$
$$B_\delta = ((0, 1/2) \times J) \setminus A_\delta$$

Decompose the right hand side in (23) into integrals I_1 and I_2 over A_δ and B_δ.

$$|I_2| \leq \left(\int_{B_\delta} \frac{|\nabla\psi|^2}{\epsilon^2 + H^2}\, dx \right)^{1/2} |B_\delta|^{1/2} \leq \epsilon|J| \int_{B_\delta} \frac{|\nabla\psi|^2}{\epsilon^2 + H^2}\, dx + \frac{\delta}{2\epsilon}$$

On A_δ we get

$$|I_1| \leq \epsilon \left(\int_{A_\delta} \frac{|\partial_2\psi|^2}{\epsilon^2 + H^2}\, dx \right)^{1/2} \left(\int_{A_\delta} \frac{1}{\epsilon^2 + H^2}\, dx \right)^{1/2} + \int_{A_\delta} \frac{H}{\epsilon^2 + H^2} \partial_1\psi\, dx$$

$$\leq \epsilon|J| \int_{A_\delta} \frac{|\nabla\psi|^2}{\epsilon^2 + H^2}\, dx + \frac{\epsilon}{4|J|} \int_{A_\delta} \frac{1}{\epsilon^2 + H^2}\, dx + \int_{A_\delta} \frac{H}{\epsilon^2 + H^2} \partial_1\psi\, dx$$

Combinig the estimates for I_1 and I_2 we obtain

$$\epsilon|J| \int_D \frac{|\nabla\psi|^2}{\epsilon^2+H^2}\,dx \le \frac{\delta}{2\epsilon} + \frac{\epsilon}{4|J|}\int_{A_\delta} \frac{1}{\epsilon^2+H^2}\,dx + \int_{A_\delta} \frac{H}{\epsilon^2+H^2}\partial_1\psi\,dx \tag{24}$$

$$\le \frac{\delta}{2\epsilon} + \frac{\epsilon}{4\beta^2\delta^{2q}} + \int_{A_\delta} \frac{H}{\epsilon^2+H^2}\partial_1\psi\,dx$$

where we used the assumption (21) on H. This implies for all $x_2 \in J$

$$\begin{array}{ll} H(x) \ge \beta x_1^q \ge \beta\delta^q, & \delta < x_1 < 1/4 \\ H(x) \ge \beta(1/2-x_1)^q \ge \beta\delta^q, & 1/4 < x_1 < 1/2-\delta. \end{array} \tag{25}$$

Now we use partial integration in the last integral in (24)

$$\int_{A_\delta} \frac{H}{\epsilon^2+H^2}\partial_1\psi\,dx = -\int_{A_\delta} \psi\partial_1\left(\frac{H}{\epsilon^2+H^2}\right)\,dx + \int_J \left(\frac{H\psi}{\epsilon^2+H^2}(1/2-\delta,x_2) - \frac{H\psi}{\epsilon^2+H^2}(\delta,x_2)\right)\,dx_2 \tag{26}$$

We examine the monotonicity intervals of $\frac{H}{\epsilon^2+H^2}$ in the x_1 direction:

$$\partial_1\left(\frac{H}{\epsilon^2+H^2}\right) = \partial_1 H \frac{\epsilon^2-H^2}{(\epsilon^2+H^2)^2}$$

We restrict δ to

$$\beta\delta^q > \epsilon \tag{27}$$

which implies

$$\epsilon^2 - H(x)^2 < 0$$

for all $x \in A_\delta$. Let $\Gamma(x_2)$ be the unique point, s.t.

$$\partial_1 H(\Gamma(x_2), x_2) = 0.$$

For fixed x_2 it is a maximum point for H and therefor a local minimum for $\frac{H}{\epsilon^2+H^2}$, since $H \mapsto H/(\epsilon^2+H^2)$ is decreasing for $H > \epsilon$. Furthermore this is the only extremum in A_δ. This implies

$$\partial_1\left(\frac{H}{\epsilon^2+H^2}\right)(x_1,x_2) < 0, \qquad \text{for } \delta < x_1 < \Gamma(x_2)$$

$$\partial_1\left(\frac{H}{\epsilon^2+H^2}\right)(x_1,x_2) > 0, \qquad \text{for } \Gamma(x_2) < x_1 < 1/2-\delta.$$

Using $0 < \psi < 1/2$ we can now estimate (26) by

$$\int_{A_\delta} \frac{H}{\epsilon^2+H^2}\partial_1\psi\,dx \le \frac{1}{2}\int_J \frac{H}{\epsilon^2+H^2}(\delta,x_2)\,dx_2 + \frac{1}{2}\int_J \frac{H}{\epsilon^2+H^2}(1/2-\delta,x_2)\,dx_2 \tag{28}$$

$$\le \frac{|J|\beta\delta^q}{\epsilon^2+\beta^2\delta^{2q}} \le \frac{|J|}{\beta\delta^q}.$$

We have estimated all terms in (24):

$$\epsilon|J|\int_D \frac{|\nabla\psi|^2}{\epsilon^2+H^2}\,dx \le \frac{\delta}{2\epsilon} + \frac{|J|}{\beta\delta^q} + \frac{\epsilon}{4\beta^2\delta^{2q}}.$$

Since the last term will be of lower order we minimize only the first two terms with respect to δ. We obtain for $\delta = \left(\frac{2\epsilon q|J|}{\beta}\right)^{\frac{1}{q+1}}$ with $\alpha = \left(\frac{2q|J|}{\beta}\right)^{\frac{1}{q+1}}$:

$$\epsilon \int_D \frac{|\nabla \psi|^2}{\epsilon^2 + H^2}\, dx_2 \leq \epsilon^{-\frac{q}{q+1}} \frac{\alpha}{|J|} \left(\frac{(q+1)}{2q} + \frac{\epsilon^{\frac{1}{q+1}}}{16q^2|J|^2} \right)$$

or

$$\lambda(\epsilon) \geq \epsilon^{\frac{q}{q+1}} \frac{|J|}{\alpha} \left(\frac{2(q+1)}{q} + \frac{\epsilon^{\frac{1}{q+1}}}{4q^2|J|^2} \right)^{-1} .$$

The restrictions for δ are $(\epsilon/\beta)^{1/q} < \delta < 1/4$ which implies for ϵ

$$\epsilon \leq \beta \min \left(\frac{1}{4^{\frac{1}{q+1}} q|J|}, (q|J|)^q \right) .$$

This completes the proof.

Remarks: The upper and lower bounds have the same asymptotic growth for small ϵ. Under appropriate assumptions on $H(x)$ a similar lower bound is expected to hold also in the general case.

For the standard example we choose $J = (1/4, 3/4)$ which gives $\beta = \sqrt{8}$. For $\lambda(\epsilon)$ we get approximately

$$\lambda(\epsilon) \geq \begin{cases} 0.16\frac{\sqrt{\epsilon}}{1+\sqrt{\epsilon}}, & \epsilon < 0.38 \\ \epsilon, & \epsilon > 0.38 \end{cases} .$$

For ϵ approximately 0.38 the two bounds agree. This is within the allowed range of the theorem above, which is $\epsilon < \frac{1}{\sqrt{2}}$.

Refining the arguments in the proof above this estimate can be further improved.

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