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by

Masato Kimura and Stephan Luckhaus

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A variational problem arising from Mullins-Sekerka problem

Masato Kimura,[*] Stephan Luckhaus[†]

Abstract

We consider a geometric minimizing problem which arises in time discretization of the Mullins-Sekerka problem. Some new geometric estimate for the shape of the global minimizer is presented. We also show that such a geometric estimate is useful to improve standard norm estimates.

1 Geometric variational problem

We suppose that Ω is a bounded Lipschitz domain of $\mathbf{R}^n$ ($n \geq 2$) throughout this paper. Let $\mathrm{BV}(\Omega)$ be the space of all functions in $L^1(\Omega)$ with bounded total variation. Then $\mathrm{BV}(\Omega)$ is a Banach space with the norm $\|f\|_{\mathrm{BV}(\Omega)} := \|f\|_{L^1(\Omega)} + \int_\Omega |\nabla f|$, where

$$\int_\Omega |\nabla f| := \sup\left\{ \int_\Omega f(\boldsymbol{x}) \mathrm{div}\boldsymbol{g}(\boldsymbol{x}) d\boldsymbol{x};\ \boldsymbol{g} \in C_0^1(\Omega)^n,\ |\boldsymbol{g}(\boldsymbol{x})| \leq 1\ (\boldsymbol{x} \in \Omega) \right\}.$$

A subset $E \subset \Omega$ is called a *Caccioppoli set* if its characteristic function $\chi_E \in \mathrm{BV}(\Omega)$, and its generalized perimeter in Ω is given by $\int_\Omega |\nabla \chi_E|$. In particular, if E has a Lipschitz boundary, $\int_\Omega |\nabla \chi_E|$ is equal to $(n-1)$-dimensional Hausdorff measure of ∂E. We refer [2], [6] and [10] for detail properties of $\mathrm{BV}(\Omega)$ and the Caccioppoli set, and the appendix of [9] for a quick review on them. In this paper, we are concerned with signed characteristic functions of Caccioppoli sets:

$$\mathcal{K} := \{\varphi \in \mathrm{BV}(\Omega);\ |\varphi(x)| = 1 \text{ for a.e. } x \in \Omega\}.$$

For $\delta > 0$, we consider the following functional

$$J_\delta(\varphi;\psi) := \delta \int_\Omega |\nabla \varphi| + \|\varphi - \psi\|^2_{H^{-1}(\Omega)} \quad (\varphi, \psi \in \mathcal{K}),$$

where the Sobolev space $H^{-1}(\Omega)$ is a dual space of $H_0^1(\Omega)$. The presice definitions of their norms are found in the beginning of § 3. Our problem is to find a global minimizer $\psi_\delta \in \mathcal{K}$ of $J_\delta(\cdot, \psi)$ for given $\delta > 0$ and $\psi \in \mathcal{K}$, i.e.:

$$\text{Find } \psi_\delta \in \mathcal{K} \text{ s.t. } J_\delta(\psi_\delta;\psi) = \min_{\varphi \in \mathcal{K}} J_\delta(\varphi;\psi). \tag{1.1}$$

[*]Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, JAPAN, e-mail: masato@math.sci.hiroshima-u.ac.jp

[†]Mathematisches Institute der Universität Leipzig, Augustusplatz 10/11, D-04109, Leipzig, Germany, e-mail: Luckhaus@mathematik.uni-leipzig.de

This minimizing problem appears in an implicit time discretization of the Mullins-Sekerka problem and enables us to construct its global weak solution ([5]). A similar technique can be applied to construct a global weak solution of the Stefan problem with the Gibbs-Thomson law ([4], [9]).

We remark that $\varphi \in \mathcal{K}$ is equivalent to $\varphi = 2\chi_E - 1$ a.e. in Ω for a Caccioppoli set $E \subset \Omega$ and that $\int_\Omega |\nabla\varphi| = 2\int_\Omega |\nabla\chi_E|$. Setting $\psi = 2\chi_{\Omega^+} - 1$ a.e. in Ω, $\Omega^+ \subset \Omega$, the minimizing problem (1.1) is equivalent to the problem to minimize the functional

$$\frac{\delta}{2}\int_\Omega |\nabla\chi_E| + \|\chi_E - \chi_{\Omega^+}\|^2_{H^{-1}(\Omega)},$$

among the all Caccioppoli set E.

In this paper, we use the following notation: For given $\psi \in \mathcal{K}$, we define open sets Ω^+ and Ω^- by

$$\Omega^\pm := \left\{ \boldsymbol{x} \in \Omega;\ {}^\exists r > 0,\ \mathcal{H}^n(\{\boldsymbol{y} \in B_r(\boldsymbol{x}) \cap \Omega;\ \psi(\boldsymbol{y}) = \mp 1\}) = 0 \right\}, \tag{1.2}$$

where $\mathcal{H}^m$ stands for the m-dimensional Hausdorff measure in $\mathbf{R}^n$ and $B_r(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbf{R}^n;\ |\boldsymbol{x} - \boldsymbol{y}| < r\}$. It follows that $\psi = \chi_{\Omega^+} - \chi_{\Omega^-}$ a.e. in Ω. The *essential boundary* of $\{\psi = 1\}$ or $\{\psi = -1\}$ in Ω is defined by

$$\Gamma := \left\{ \boldsymbol{x} \in \Omega;\ \begin{matrix} \mathcal{H}^n(\{\boldsymbol{y} \in B_r(\boldsymbol{x}) \cap \Omega;\ \psi(\boldsymbol{y}) = +1\}) > 0 \\ \mathcal{H}^n(\{\boldsymbol{y} \in B_r(\boldsymbol{x}) \cap \Omega;\ \psi(\boldsymbol{y}) = -1\}) > 0 \end{matrix}\ ({}^\forall r > 0) \right\}, \tag{1.3}$$

Then, it is known that $\Gamma = \Omega \setminus (\Omega^+ \cup \Omega^-) = \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega$. In the same way, we define $\Omega^\pm_\delta$ and Γ_δ from a global minimizer $\psi_\delta \in \mathcal{K}$.

Our main interest is to know the geometic property on the global minimizer ψ_δ, in other words, to know what shape Γ_δ has. The existence of a global minimizer is shown as follows.

Proposition 1.1 *We assume that $\Omega \subset \mathbf{R}^n$ is a bounded domain with Lipschitz boundary. For given $\psi \in \mathcal{K}$ and $\delta > 0$, there exists at least one global minimizer ψ_δ of (1.1).*

Proof. Let $\{\varphi_j\}_j$ be a sequence in $\mathcal{K}$ which attains $\inf_{\varphi\in\mathcal{K}} J_\delta(\varphi, \psi)$. Then $\{\varphi_j\}_j$ is bounded in $L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$. Since there exists compact imbedding of $\mathrm{BV}(\Omega)$ into $L^1(\Omega)$ (see [2], [6]), compact imbedding of $L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ into $L^2(\Omega)$ also exists, and a subsequence of $\{\varphi_j\}_j$ converges to some ψ_δ in $L^2(\Omega)$. From the lower semi-continuity of the perimeter functional with respect to $L^1(\Omega)$ (see [2], [6]), the lower semicontinuity of $J_\delta(\cdot, \psi)$ with respect to $L^2(\Omega)$ follows. Hence, ψ_δ belongs $\mathcal{K}$ and minimizes J_δ globally. □

From $J_\delta(\psi_\delta; \psi) \le J_\delta(\psi; \psi)$, we have

$$\|\psi_\delta - \psi\|^2_{H^{-1}(\Omega)} \le \delta \left(\int_\Omega |\nabla\psi| - \int_\Omega |\nabla\psi_\delta| \right), \tag{1.4}$$

which is a fundamental inequality in our analysis. The next proposition gives us some estimates for the global minimizer without any assumption for $\psi \in \mathcal{K}$.

Proposition 1.2 *There exist $C = C(\Omega) > 0$ such that any global minimizer ψ_δ of (1.1) for $\delta > 0$ and $\psi \in \mathcal{K}$ satisfies the following inequalities;*

$$\int_\Omega |\nabla \psi_\delta| \leq \int_\Omega |\nabla \psi|, \tag{1.5}$$

$$\|\psi_\delta - \psi\|_{H^{-1}(\Omega)} \leq \left(\int_\Omega |\nabla \psi|\right)^{\frac{1}{2}} \delta^{\frac{1}{2}},$$

$$\|\psi_\delta - \psi\|_{L^1(\Omega)} \leq C\left(1 + \int_\Omega |\nabla \psi|\right)^{\frac{1}{2}} \|\psi_\delta - \psi\|^{\frac{1}{2}}_{H^{-1}(\Omega)}.$$

Proof. The first two inequalities follow from (1.4) immediately. The last one is derived from Corollary 4.3. □

These known estimates guarantee the uniform boundedness of the perimeter $\int_\Omega |\nabla \psi_\delta|$, $O(\delta^{\frac{1}{2}})$-convergence to ψ in $H^{-1}(\Omega)$ and $O(\delta^{\frac{1}{4}})$-convergence to ψ in $L^1(\Omega)$. Another well known important property of ψ_δ is local Hölder regularity of Γ_δ when $2 \leq n \leq 7$ due to the theory of the almost minimal surfaces ([7], [9]). But we do not touch with this result in this paper. These estimates give us sufficient a priori estimates for time discretized solution to the Mullins-Sekerka problem and enable us to extract a subsequence which converges a weak solution using compactness.

On the other hand, nevertheless these estimates give us surprisingly few information on the shape of Γ which is our main interest. Actually, there exist many shapes which satisfy the above conditions (but which do not look like a solution of the Mullins-Sekerka problem). The aim of this paper is to improve the above norm estimates and to give a new estimate for the shape of Γ_δ under some regularity assumptions on Γ.

2 Regularity assumption and main result

In this section, we state our main result under some regularity assumptions on Γ.

We say an essential boundary Γ defined by (1.3) from $\psi \in \mathcal{K}$ is *Lipschitz in* $\overline{\Omega}$ if, for any point $\boldsymbol{x} \in \overline{\Gamma}$, there exist an open neighborhood $\tilde{U}$ of $\boldsymbol{x}$, an open set $\tilde{V} \subset \mathbf{R}^n$ and a bi-Lipschitz isomorphism $\boldsymbol{g} = (g_1, \cdots, g_n)$ from $U := \tilde{U} \cap \overline{\Omega}$ to $V := \{(y_1, \cdots, y_n) \in \tilde{V};\ y_n \geq 0\}$ such that $\boldsymbol{g}(\overline{\Omega^\pm} \cap U) = \{\boldsymbol{y} \in V;\ \pm(y_1 - g_1(\boldsymbol{x})) \geq 0\}$. We remark that if $\overline{\Gamma} \subset \Omega$ then this is equivalent to the standard definition of the locally Lipschitz boundary ([1]).

Let $\boldsymbol{n}_\Omega$ be the outward unit normal vector on $\partial\Omega$ and let $\boldsymbol{n}_\Gamma$ be the unit normal vector on Γ from Ω^- to Ω^+, where Γ is defined by (1.3) from $\psi \in \mathcal{K}$. The trace operators from $H^1(\Omega)$ into $L^2(\partial\Omega)$ and $L^2(\Gamma)$ (more precisely, onto $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\Gamma)$) are denoted by $\gamma_{\partial\Omega}$ and γ_Γ respectively. (See Theorem 1.5 of [3] etc.) We consider the following conditions for $\psi \in \mathcal{K}$:

$$\text{(A1)} \quad \begin{cases} \text{The essential boundary } \Gamma \text{ is Lipschitz in } \overline{\Omega}.\ {}^{\exists}\overline{\boldsymbol{n}} \in H^1(\Omega)^n \text{ s.t.} \\ \gamma_\Gamma(\overline{\boldsymbol{n}}) = \boldsymbol{n}_\Gamma \quad (\mathcal{H}^{n-1}\text{-a.e. on } \Gamma), \qquad |\overline{\boldsymbol{n}}(\boldsymbol{x})| \le 1 \quad (\boldsymbol{x} \in \overline{\Omega}), \\ \gamma_{\partial\Omega}(\overline{\boldsymbol{n}}) \cdot \boldsymbol{n}_\Omega = 0 \quad (\mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega), \qquad \mathrm{div}\overline{\boldsymbol{n}} \in H_0^1(\Omega). \end{cases}$$

$$\text{(A2)} \quad \overline{\Gamma} \subset \Omega.$$

The condition (A1) is a regularity assumption for Γ, the essential boundary of $\psi = 1$ and $\psi = -1$. We remark that, if Γ is a C^3-class hypersurface and satisfies (A2), then (A1) is fulfilled.

To give an estimate for the shape of Γ_δ, we consider the Hausdorff distance between Γ_δ and Γ. The Hausdorff distance between two compact sets in $\mathbf{R}^n$ is denoted by $\mathrm{dist_H}$. Under the above assumptions, we can get the main result of this paper:

Theorem 2.1 *We suppose (A1) and (A2). Then, as $\delta \to 0$, we have*

$$\|\psi_\delta - \psi\|_{H^{-1}(\Omega)} = O(\delta), \qquad \|\psi_\delta - \psi\|_{L^1(\Omega)} = O(\delta^{\frac{1}{2}+\frac{\alpha_n}{4}}), \tag{2.1}$$

$$\mathrm{dist_H}(\Gamma_\delta, \Gamma) \le \max\left(\mathrm{dist_H}(\overline{\Omega_\delta^+}, \overline{\Omega^+}), \mathrm{dist_H}(\overline{\Omega_\delta^-}, \overline{\Omega^-})\right) = O(\delta^{\alpha_n}), \tag{2.2}$$

where

$$\alpha_n := \begin{cases} \dfrac{1}{8} & (n = 2), \\ \dfrac{2}{4n-1} - \varepsilon & (3 \le n \le 5), \\ \dfrac{2}{(n+1)(n-2)} & (n \ge 6), \end{cases} \tag{2.3}$$

for any fixed $\varepsilon > 0$.

The notation $O(\delta^p)$ in the above theorem means that there exists a positive constants C and δ_0 which depend only on ψ, Ω and n (and ε if $n = 3, 4, 5$), such that the left hand side is bounded from above by $C\delta^p$ for $\delta \in (0, \delta_0)$. Particularly, although there is no uniqueness of the global minimizer ψ_δ in general, these constants are independent of the choice of ψ_δ.

A proof of this theorem is given in the last section after preparing some lemmas. In the rest of this section, we give a proof of $\|\psi_\delta - \psi\|_{H^{-1}(\Omega)} = O(\delta)$, which clarifies the meaning of the condition (A1).

Proposition 2.2 *Suppose the condition (A1). Then we have*

$$\|\psi_\delta - \psi\|_{H^{-1}(\Omega)} \le \|\mathrm{div}\overline{\boldsymbol{n}}\|_{H_0^1(\Omega)}\delta \qquad (\delta > 0). \tag{2.4}$$

Before the proof, we recall some useful fundamental properties of functions in $\mathrm{BV}(\Omega)$. It is known that $|\nabla f|$ and ∇f for $f \in \mathrm{BV}(\Omega)$ are regarded as scholar and vector valued Radon measures in Ω. The following Green's formula for functions in $\mathrm{BV}(\Omega)$ in known ([2] Theorem 2.10):

$$\int_\Omega \boldsymbol{g} \cdot \nabla f = -\int_\Omega \mathrm{div}\boldsymbol{g}(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{n}_\Omega \cdot \boldsymbol{g} f^* d\mathcal{H}^{n-1} \qquad (\boldsymbol{g} \in C^1(\overline{\Omega})^n,\ f \in \mathrm{BV}(\Omega)), \tag{2.5}$$

where $f^* \in L^1(\partial\Omega)$ represents the trace of f to $\partial\Omega$ in the sense of $\mathrm{BV}(\Omega)$. It is also known that there exists a unit vector field $\boldsymbol{n}_f(\boldsymbol{x}) \in \mathbf{R}^n$ for $|\nabla f|$-a.e. $\boldsymbol{x} \in \Omega$ such that $\nabla f = \boldsymbol{n}_f |\nabla f|$ (see [6] § 6). In particular, under the condition (A1), $|\nabla \psi| = 2\mathcal{H}^{n-1}|_\Gamma$ and $\boldsymbol{n}_\psi = \boldsymbol{n}_\Gamma$ $\mathcal{H}^{n-1}$-a.e. on Γ.

We consider a symmetric mollifier $\rho \in C_0^\infty(\mathbf{R}^n)$ such that $\rho(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathbf{R}^n$, $\int_{\mathbf{R}^n} \rho(\boldsymbol{x}) d\boldsymbol{x} = 1$, $\rho(-\boldsymbol{x}) = \rho(\boldsymbol{x})$ and $\mathrm{supp}(\rho) \subset B_1(0)$. For $u \in L^1(\Omega)$ and $\varepsilon > 0$, a regularization of u is defined by the convolution $u * \rho_\varepsilon(\boldsymbol{x}) := \int_\Omega u(\boldsymbol{y}) \rho_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}$, where $\rho_\varepsilon(\boldsymbol{x}) := \varepsilon^{-n} \rho(\boldsymbol{x}/\varepsilon)$.

Proof of Proposition 2.2. By a formal calculation, we have

$$\int_\Omega |\nabla \psi| - \int_\Omega |\nabla f| \leq \int_\Omega \overline{\boldsymbol{n}} \cdot \nabla \psi - \int_\Omega \overline{\boldsymbol{n}} \cdot \nabla f = \int_\Omega \mathrm{div}\overline{\boldsymbol{n}}(\boldsymbol{x})(f(\boldsymbol{x}) - \psi(\boldsymbol{x})) d\boldsymbol{x}, \tag{2.6}$$

for $f \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$. We apply this inequality to (1.4) with $f = \psi_\delta$. Then we have

$$\|\psi_\delta - \psi\|^2_{H^{-1}(\Omega)} \leq \delta \int_\Omega \mathrm{div}\overline{\boldsymbol{n}}(\boldsymbol{x})(\psi_\delta(\boldsymbol{x}) - \psi(\boldsymbol{x})) d\boldsymbol{x} \leq \delta \|\mathrm{div}\overline{\boldsymbol{n}}\|_{H^1_0(\Omega)} \|\psi_\delta - \psi\|_{H^{-1}(\Omega)},$$

and this yields (2.4).

To show (2.6), we define $\overline{\boldsymbol{n}}_\varepsilon := \overline{\boldsymbol{n}} * \rho_\varepsilon$, then $|\overline{\boldsymbol{n}}_\varepsilon| \leq 1$, $\overline{\boldsymbol{n}}_\varepsilon \in C^\infty(\overline{\Omega})^n$. Let $f \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$. Since $|\overline{\boldsymbol{n}}_\varepsilon \cdot \boldsymbol{n}_f| \leq 1$ $|\nabla f|$-a.e. in Ω, we have

$$\int_\Omega |\nabla f| \geq \int_\Omega \overline{\boldsymbol{n}}_\varepsilon \cdot \boldsymbol{n}_f |\nabla f| = \int_\Omega \overline{\boldsymbol{n}}_\varepsilon \cdot \nabla f.$$

From $|\nabla \psi| = 2\mathcal{H}^{n-1}|_\Gamma$, we also have

$$\begin{aligned} \int_\Omega |\nabla \psi| &= \int_\Omega \overline{\boldsymbol{n}}_\varepsilon \cdot \nabla \psi + \int_\Omega (\boldsymbol{n}_\Gamma - \overline{\boldsymbol{n}}_\varepsilon) \cdot \boldsymbol{n}_\Gamma |\nabla \psi| \\ &\leq \int_\Omega \overline{\boldsymbol{n}}_\varepsilon \cdot \nabla \psi + 2\int_\Omega |\gamma_\Gamma(\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon)| d\mathcal{H}^{n-1}, \end{aligned}$$

$$\begin{aligned} \int_\Omega |\nabla \psi| - \int_\Omega |\nabla f| &\leq \int_\Omega \overline{\boldsymbol{n}}_\varepsilon \cdot \nabla(\psi - f) + 2\int_\Omega |\gamma_\Gamma(\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon)| d\mathcal{H}^{n-1} \\ &= \int_\Omega \mathrm{div}\overline{\boldsymbol{n}}_\varepsilon (f - \psi) d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{n}_\Omega \cdot (\overline{\boldsymbol{n}}_\varepsilon - \gamma_{\partial\Omega}(\overline{\boldsymbol{n}}))(\psi - f)^* d\mathcal{H}^{n-1} \\ &\quad + 2\int_\Omega |\gamma_\Gamma(\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon)| d\mathcal{H}^{n-1}. \end{aligned}$$

Since, as $\varepsilon \to 0$,

$$\int_\Omega \mathrm{div}\overline{\boldsymbol{n}}_\varepsilon (f - \psi) d\boldsymbol{x} \to \int_\Omega \mathrm{div}\overline{\boldsymbol{n}} (f - \psi) d\boldsymbol{x},$$

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{n}_\Omega \cdot (\overline{\boldsymbol{n}}_\varepsilon - \gamma_{\partial\Omega}(\overline{\boldsymbol{n}}))(\psi - f)^* d\mathcal{H}^{n-1} &\leq \|\gamma_{\partial\Omega}(\overline{\boldsymbol{n}}_\varepsilon - \overline{\boldsymbol{n}})\|_{L^2(\partial\Omega)} \|(\psi - f)^*\|_{L^2(\partial\Omega)} \\ &\leq C\|\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon\|_{H^1(\Omega)} \|(\psi - f)^*\|_{L^2(\partial\Omega)} \to 0,\end{aligned}$$

$$\int_\Omega |\gamma_\Gamma(\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon)| d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\Gamma)^{\frac{1}{2}} \|\gamma_\Gamma(\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon)\|_{L^2(\Gamma)} \leq C\|\overline{\boldsymbol{n}} - \overline{\boldsymbol{n}}_\varepsilon\|_{H^1(\Omega)} \to 0,$$

we obtain (2.6). □

3 Estimates for H^{-1}-norm

In this section, we give some useful lemmas to estimate the difference $\|\varphi_2 - \psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1 - \psi\|^2_{H^{-1}(\Omega)}$ under the condition (3.3) below. These lemmas will be used in the proof of the main theorem.

Before stating the lemmas, we fix our notation in this paper concerning the Sobolev spaces $H^1_0(\Omega)$ and its normed dual space $H^{-1}(\Omega)$. (See [1] and [8] for their definitions. Convenient brief reviews on Sobolev spaces are found also in [3], [9] etc.) They are both Hilbert spaces and, in this paper, the following inner product of $H^1_0(\Omega)$ is adopted:

$$(u, v)_{H^1_0(\Omega)} := \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} \qquad (u, v \in H^1_0(\Omega)).$$

The duality pairing ${}_{H^1_0(\Omega)}\langle u, v\rangle_{H^{-1}(\Omega)}$ is chosen as the standard way, i.e. it is given by $\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x}$ if $v \in L^2(\Omega)$. Then, it is known that the Laplace operator is an isomorphism from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$ and that $(u, v)_{H^{-1}(\Omega)} = {}_{H^1_0(\Omega)}\langle -\Delta_D^{-1} u, v\rangle_{H^{-1}(\Omega)}$, where Δ_D is the Laplacian with zero Dirichlet boundary condition (see [8] Theorem 23.1).

The inner product of $H^{-1}(\Omega)$ for L^2-functions is represented in term of the Green function $G(\boldsymbol{x}, \boldsymbol{y})$ for $-\Delta_D$:

$$(u, v)_{H^{-1}(\Omega)} = \int_\Omega \int_\Omega G(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{x}) v(\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} \quad (u, v \in L^2(\Omega)). \tag{3.1}$$

The symmetricity $G(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x})$ and the positivity $G(\boldsymbol{x}, \boldsymbol{y}) > 0$ ($\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $\boldsymbol{x} \neq \boldsymbol{y}$) are well-known properties. The following proposition is a simple consequence of the maximum principle.

Proposition 3.1 *Let $n \geq 2$. Then the Green function in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ satisfies*

$$\frac{1}{n\omega_n} \int_{|\boldsymbol{x} - \boldsymbol{y}|}^{\mathrm{dist}(\boldsymbol{x}, \partial\Omega)} s^{1-n} ds \leq G(\boldsymbol{x}, \boldsymbol{y}) \leq \frac{1}{n\omega_n} \int_{|\boldsymbol{x} - \boldsymbol{y}|}^{\mathrm{diam}(\Omega)} s^{1-n} ds \qquad (\boldsymbol{x} \in \Omega,\ \boldsymbol{y} \in \overline{\Omega},\ \boldsymbol{x} \neq \boldsymbol{y}),$$

where ω_n denotes the n-dimensional volume of a unit ball of $\mathbf{R}^n$

Proof. The fundamental solution of $-\Delta$ in $\mathbf{R}^n$ is given by $f_n(|\boldsymbol{x}|)$, where $f_n(s) := -(n\omega_n)^{-1}\int s^{1-n}ds$ for $s>0$. For a fixed $\boldsymbol{x}\in\Omega$, $G(\boldsymbol{x},\boldsymbol{y})-f_n(|\boldsymbol{x}-\boldsymbol{y}|)$ is harmonic in Ω with respect to $\boldsymbol{y}$ and is equal to $-f_n(|\boldsymbol{x}-\boldsymbol{y}|)$ for $\boldsymbol{y}\in\partial\Omega$. From the maximum principle for the harmonic function, we have

$$G(\boldsymbol{x},\boldsymbol{y})-f_n(|\boldsymbol{x}-\boldsymbol{y}|)\le \max_{\boldsymbol{y}\in\partial\Omega}(-f_n(|\boldsymbol{x}-\boldsymbol{y}|))=-f_n(\max_{\boldsymbol{y}\in\partial\Omega}|\boldsymbol{x}-\boldsymbol{y}|)\le -f_n(\mathrm{diam}(\Omega)),$$

$$G(\boldsymbol{x},\boldsymbol{y})-f_n(|\boldsymbol{x}-\boldsymbol{y}|)\ge \min_{\boldsymbol{y}\in\partial\Omega}(-f_n(|\boldsymbol{x}-\boldsymbol{y}|))=-f_n(\min_{\boldsymbol{y}\in\partial\Omega}|\boldsymbol{x}-\boldsymbol{y}|)=-f_n(\mathrm{dist}(\boldsymbol{x},\Omega)),$$

and these inequalities yield the proposition. □

For $A\subset\Omega$ and $\varepsilon>0$, ε-neighborhood of A is defined by

$$N^{\varepsilon}(A):=\{\boldsymbol{x}\in\Omega;\ \mathrm{dist}(\boldsymbol{x},A)<\varepsilon\}. \tag{3.2}$$

We assume the following condition and notation in this section:

$$\begin{cases} \psi\in\mathcal{K},\quad \Omega^{\pm}\text{ and }\Gamma\text{ are defined by (1.2) and (1.3)},\\[2mm] F_2\subset F_1\subset\Omega,\quad E=F_1\setminus F_2,\quad \mathcal{H}^n(E)>0,\\[2mm] \varphi_1,\ \varphi_2\in\mathcal{K},\quad \varphi_i=(1-2\chi_{F_i})\psi\ \text{a.e. in }\Omega\quad (i=1,2),\\[2mm] \varepsilon_1:=\inf\{\varepsilon;\ \mathcal{H}^n(E\cap N^{\varepsilon}(\Gamma))>0\},\ \varepsilon_2:=\inf\{\varepsilon;\ \mathcal{H}^n(E\cap N^{\varepsilon}(\partial\Omega))>0\}. \end{cases} \tag{3.3}$$

Lemma 3.2 *Under the condition (3.3), we suppose that there exist* $\beta\ge 0$ *and* $q\in L^1(E;H_0^1(\Omega))$ $(q=q(\boldsymbol{x},\boldsymbol{y}),\ \boldsymbol{x}\in E,\ \boldsymbol{y}\in\Omega)$*, such that*

$$q(\boldsymbol{x},\boldsymbol{y})=G(\boldsymbol{x},\boldsymbol{y})\qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in F_1,\ \psi(\boldsymbol{x})\ne\psi(\boldsymbol{y})), \tag{3.4}$$
$$q(\boldsymbol{x},\boldsymbol{y})\le G(\boldsymbol{x},\boldsymbol{y})\qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in F_2,\ \psi(\boldsymbol{x})=\psi(\boldsymbol{y})), \tag{3.5}$$
$$q(\boldsymbol{x},\boldsymbol{y})\le \beta G(\boldsymbol{x},\boldsymbol{y})\qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in E,\ \psi(\boldsymbol{x})=\psi(\boldsymbol{y})). \tag{3.6}$$

Then

$$\|\varphi_2-\psi\|^2_{H^{-1}(\Omega)}-\|\varphi_1-\psi\|^2_{H^{-1}(\Omega)}\le 4\|q\|_{L^1(E;H_0^1(\Omega))}\|\varphi_1-\psi\|_{H^{-1}(\Omega)}+4(2\beta-1)\|G\|_{L^1(E\times E)}.$$

Proof. Since $\chi_{F_1}=\chi_{F_2}+\chi_E$, we have

$$\begin{aligned} \|\varphi_2-\psi\|_{H^{-1}(\Omega)^2}-\|\varphi_1-\psi\|_{H^{-1}(\Omega)^2} &= \|2\chi_{F_2}\psi\|_{H^{-1}(\Omega)^2}-\|2\chi_{F_1}\psi\|_{H^{-1}(\Omega)^2}\\ &= -4\left((\chi_E+2\chi_{F_2})\psi,\ \chi_E\psi\right)_{H^{-1}(\Omega)}. \end{aligned}$$

Using the relation $\chi_{E\cup F_2}\psi=(\psi-\varphi_1)/2$, we obtain, for $\boldsymbol{x}\in E$,

$$\begin{aligned} &(\chi_E+2\chi_{F_2})G(\boldsymbol{x},\cdot)\psi\\ =\ & \chi_E G(\boldsymbol{x},\cdot)\psi+2\left(\chi_{F_2}(G(\boldsymbol{x},\cdot)-q(\boldsymbol{x},\cdot))\psi-\chi_E q(\boldsymbol{x},\cdot)\psi+q(\boldsymbol{x},\cdot)\frac{\psi-\varphi_1}{2}\right)\\ =\ & \chi_E(G(\boldsymbol{x},\cdot)-2q(\boldsymbol{x},\cdot))\psi+2\chi_{F_2}(G(\boldsymbol{x},\cdot)-q(\boldsymbol{x},\cdot))\psi+q(\boldsymbol{x},\cdot)(\psi-\varphi_1). \end{aligned}$$

From (3.1), we have

$$
\begin{aligned}
& \|\varphi_2-\psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1-\psi\|^2_{H^{-1}(\Omega)} \\
= & -4\int_E \psi(\boldsymbol{x})\int_\Omega \left(\chi_E(\boldsymbol{y})+2\chi_{F_2}(\boldsymbol{y})\right) G(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} \\
= & \ 4\int_E\int_E (2q(\boldsymbol{x},\boldsymbol{y})-G(\boldsymbol{x},\boldsymbol{y}))\psi(\boldsymbol{x})\psi(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} \\
& +8\int_E\int_{F_2}(q(\boldsymbol{x},\boldsymbol{y})-G(\boldsymbol{x},\boldsymbol{y}))\psi(\boldsymbol{x})\psi(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} \\
& +4\int_E\int_\Omega q(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{x})(\varphi_1(\boldsymbol{y})-\psi(\boldsymbol{y}))d\boldsymbol{y}d\boldsymbol{x}.
\end{aligned}
$$

To estimate the first and second integrals, we apply the following inequalities:

$$
\begin{aligned}
& (2q(\boldsymbol{x},\boldsymbol{y})-G(\boldsymbol{x},\boldsymbol{y}))\psi(\boldsymbol{x})\psi(\boldsymbol{y}) \le (2\beta-1)G(\boldsymbol{x},\boldsymbol{y}) \qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in E), \\
& (q(\boldsymbol{x},\boldsymbol{y})-G(\boldsymbol{x},\boldsymbol{y}))\psi(\boldsymbol{x})\psi(\boldsymbol{y}) \le 0 \qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in F_2),
\end{aligned}
$$

which are directly shown by the assumptions. Hence, we obtain

$$
\begin{aligned}
& \|\varphi_2-\psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1-\psi\|^2_{H^{-1}(\Omega)} \\
\le & \ 4(2\beta-1)\int_E\int_E G(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} + 4_{H^1_0(\Omega)}\left\langle \int_E q(\boldsymbol{x},\cdot)\psi(\boldsymbol{x})d\boldsymbol{x},\ \varphi_1-\psi\right\rangle_{H^{-1}(\Omega)} \\
\le & \ 4(2\beta-1)\|G\|_{L^1(E\times E)}. + 4\|q\|_{L^1(E;H^1_0(\Omega))}\|\varphi_1-\psi\|_{H^{-1}(\Omega)}.
\end{aligned}
$$

□

Choosing some suitable $q(\boldsymbol{x},\boldsymbol{y})$, we have the following lemmas:

Lemma 3.3 *We suppose (3.3) and $\varepsilon_1>0$. Then there exists $C>0$ which depends only on ε_1 and Ω such that*

$$
\|\varphi_2-\psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1-\psi\|^2_{H^{-1}(\Omega)} + 4\|G\|_{L^1(E\times E)} \le C\mathcal{H}^n(E)\|\varphi_1-\psi\|_{H^{-1}(\Omega)}.
$$

Proof. We define

$$
q(\boldsymbol{x},\boldsymbol{y}) := \begin{cases} \max\left(1-\dfrac{2}{\varepsilon_1}\mathrm{dist}(\boldsymbol{y},\Omega^-),\ 0\right)G(\boldsymbol{x},\boldsymbol{y}) & (\boldsymbol{x}\in E\cap\Omega^+,\ \boldsymbol{y}\in\Omega), \\ \max\left(1-\dfrac{2}{\varepsilon_1}\mathrm{dist}(\boldsymbol{y},\Omega^+),\ 0\right)G(\boldsymbol{x},\boldsymbol{y}) & (\boldsymbol{x}\in E\cap\Omega^-,\ \boldsymbol{y}\in\Omega), \end{cases}
$$

and apply Lemma 3.2 with $\beta=0$. Since

$$
|\nabla_y q(\boldsymbol{x},\boldsymbol{y})| \le 2\varepsilon_1^{-1}G(\boldsymbol{x},\boldsymbol{y}) + |\nabla_y G(\boldsymbol{x},\boldsymbol{y})| \le 2\varepsilon_1^{-1}\gamma_0+\gamma_1 \qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in\Omega),
$$

$$
\gamma_0 := \sup\{G(\boldsymbol{x},\boldsymbol{y});\ |\boldsymbol{x}-\boldsymbol{y}|>\varepsilon_1/2\}, \qquad \gamma_1 := \sup\{|\nabla_y G(\boldsymbol{x},\boldsymbol{y})|;\ |\boldsymbol{x}-\boldsymbol{y}|>\varepsilon_1/2\},
$$

we have $\|q\|_{L^1(E;H^1_0(\Omega))} \le \left(2\varepsilon_1^{-1}\gamma_0+\gamma_1\right)\mathcal{H}^n(\Omega)^{\frac{1}{2}}\mathcal{H}^n(E)$ and the assertion follows. □

Lemma 3.4 *Under the condition (3.3), we suppose that* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$. *If* $\beta > 0$ *satisfies*

$$\int_{\varepsilon_1}^{\mathrm{diam}(\Omega)} s^{1-n} ds \le \beta \int_{\mathrm{diam}(E)}^{\varepsilon_2} s^{1-n} ds, \tag{3.7}$$

then we have

$$\begin{aligned} & \|\varphi_2 - \psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1 - \psi\|^2_{H^{-1}(\Omega)} \\ \le \;& 4\left(\frac{1}{n\omega_n}\int_{\varepsilon_1}^{\mathrm{diam}(\Omega)} s^{1-n} ds\right)^{\frac{1}{2}} \mathcal{H}^n(E)\|\varphi_1 - \psi\|_{H^{-1}(\Omega)} + 4(2\beta - 1)\|G\|_{L^1(E\times E)}. \end{aligned}$$

Proof. We define

$$q_0 := \left(\frac{1}{n\omega_n}\int_{\varepsilon_1}^{\mathrm{diam}(\Omega)} s^{1-n} ds\right)^{\frac{1}{2}}, \qquad q(\boldsymbol{x},\boldsymbol{y}) := \min(G(\boldsymbol{x},\boldsymbol{y}), q_0^2) \qquad (\boldsymbol{x}\in E,\ \boldsymbol{y}\in\Omega),$$

and apply Lemma 3.2. For $\boldsymbol{x} \in E$, $\boldsymbol{y} \in F_1$, $\psi(\boldsymbol{x}) \ne \psi(\boldsymbol{y})$, from Proposition 3.1, we have

$$G(\boldsymbol{x},\boldsymbol{y}) \le \frac{1}{n\omega_n}\int_{|\boldsymbol{x}-\boldsymbol{y}|}^{\mathrm{diam}(\Omega)} s^{1-n} ds \le q_0^2,$$

and (3.4) follows. The condition (3.5) also follows from the definition of q. For $\boldsymbol{x} \in E$, $\boldsymbol{y} \in E$, $\psi(\boldsymbol{x}) = \psi(\boldsymbol{y})$, from Proposition 3.1, we have

$$q(\boldsymbol{x},\boldsymbol{y}) \le q_0^2 \le \frac{\beta}{n\omega_n}\int_{\mathrm{diam}(E)}^{\varepsilon_2} s^{1-n} ds \le \beta G(\boldsymbol{x},\boldsymbol{y}).$$

Hence, (3.6) is fulfilled and the assertion follows from Lemma 3.2 and the equality:

$$\|q(\boldsymbol{x},\cdot)\|_{H_0^1(\Omega)} = q_0 \qquad (\boldsymbol{x}\in E).$$

This equality is shown as follows. We define $D := \{\boldsymbol{y} \in \Omega;\ G(\boldsymbol{x},\boldsymbol{y}) > q_0^2\}$ for fixed $\boldsymbol{x} \in E$, and let $\boldsymbol{n}_D$ be the inner unit normal on ∂D. Using the property $-\Delta_y G(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{y}-\boldsymbol{x})$ (Dirac's delta distribution) in the sense of $\mathcal{D}'(\Omega)$, we have

$$\begin{aligned} \|q(\boldsymbol{x},\cdot)\|^2_{H_0^1(\Omega)} &= \int_{\Omega\setminus D} |\nabla_y G(\boldsymbol{x},\boldsymbol{y})|^2 d\boldsymbol{y} = \int_{\partial D} \boldsymbol{n}_D(\boldsymbol{y})\cdot\nabla_y G(\boldsymbol{x},\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y})d\sigma_y \\ &= q_0^2 \int_{\partial D} \boldsymbol{n}_D(\boldsymbol{y})\cdot\nabla_y G(\boldsymbol{x},\boldsymbol{y})d\sigma_y = q_0^2. \end{aligned}$$

□

Corollary 3.5 *Let* $n = 2$. *Under the condition (3.3), we suppose that* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$. *If*

$$\mathrm{diam}(E) \le \frac{\varepsilon_1^2\varepsilon_2}{\mathrm{diam}(\Omega)^2}, \tag{3.8}$$

then we have

$$\|\varphi_2 - \psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1 - \psi\|^2_{H^{-1}(\Omega)} \le 4\left(\frac{1}{2\pi}\log\frac{\mathrm{diam}(\Omega)}{\varepsilon_1}\right)^{\frac{1}{2}} \mathcal{H}^2(E)\|\varphi_1 - \psi\|_{H^{-1}(\Omega)}.$$

Proof. Since the condition (3.8) is equivalent to (3.7) with $\beta = 1/2$, the assertion follows by Lemma 3.4. □

Corollary 3.6 *Let $n \geq 3$. Under the condition (3.3), we suppose that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. If*

$$\operatorname{diam}(E) \leq 5^{-\frac{1}{n-2}} \min(\varepsilon_1, \varepsilon_2), \tag{3.9}$$

then we have

$$\begin{aligned} & \|\varphi_2 - \psi\|^2_{H^{-1}(\Omega)} - \|\varphi_1 - \psi\|^2_{H^{-1}(\Omega)} + 2\|G\|_{L^1(E\times E)} \\ \leq \ & 4\left(n(n-2)\omega_n \varepsilon_1^{n-2}\right)^{-\frac{1}{2}} \mathcal{H}^n(E)\|\varphi_1 - \psi\|_{H^{-1}(\Omega)}. \end{aligned}$$

Proof. We apply Lemma 3.4 with $\beta = 1/4$. The condition (3.7) is shown as follows;

$$\begin{aligned} \frac{1}{4}\int_{\operatorname{diam}(E)}^{\varepsilon_2} s^{1-n} ds \ & \geq \ \frac{1}{4}\int_{\operatorname{diam}(E)}^{5^{\frac{1}{n-2}}\operatorname{diam}(E)} s^{1-n} ds = \frac{\operatorname{diam}(E)^{2-n}}{5(n-2)} \\ & \geq \ \frac{\varepsilon_1^{2-n}}{n-2} = \int_{\varepsilon_1}^{+\infty} s^{1-n} ds \geq \int_{\varepsilon_1}^{\operatorname{diam}(\Omega)} s^{1-n} ds. \end{aligned}$$

□

4 Estimates for L^1-norm

In this section, we derive an interpolation inequality for L^1-norm, which is estimated in terms of the total variation and H^{-1}-norm (Lemma 4.2). Coefficients in the inequality is explicitly given and, in particular, the dependence on the support of the function is clarified. This will enable us to use a geometric information on the support of the function in § 5.

Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be a symmetric mollifier as in § 2. We define

$$C_0 := \int_{\mathbf{R}^n} |\boldsymbol{x}|\rho(\boldsymbol{x})d\boldsymbol{x}, \quad C_1 := \left(\sum_{i=1}^n \|\partial_i \rho\|^2_{L^1(\mathbf{R}^n)}\right)^{\frac{1}{2}},$$

where $\partial_i = \frac{\partial}{\partial x_i}$. We also define $\Omega^\varepsilon := \Omega \setminus N^\varepsilon(\partial\Omega)$ for $\varepsilon > 0$, where $N^\varepsilon(\cdot)$ is defined by (3.2). Then we have the following lemmas.

Lemma 4.1

$$\|u - u * \rho_\varepsilon\|_{L^1(\Omega^\varepsilon)} \leq C_0 \left(\int_\Omega |\nabla u|\right)\varepsilon \qquad (u \in \mathrm{BV}(\Omega),\ \varepsilon > 0).$$

Proof. Let $w \in C^1(\overline{\Omega}) \cap \mathrm{BV}(\Omega)$ and let $\beta \geq \varepsilon > 0$ be fixed. For $\boldsymbol{z} \in \mathbf{R}^n$, $|\boldsymbol{z}| < \varepsilon$, we have

$$\int_{\Omega^\beta} |w(\boldsymbol{x}) - w(\boldsymbol{x} - \boldsymbol{z})| d\boldsymbol{x} = \int_{\Omega^\beta} \left| \int_0^1 \boldsymbol{z} \cdot \nabla w(\boldsymbol{x} - t\boldsymbol{z}) dt \right| d\boldsymbol{x} \leq |\boldsymbol{z}| \int_{\Omega^{\beta-\varepsilon}} |\nabla w(\boldsymbol{x})| d\boldsymbol{x}.$$

Hence, we have the following estimate:

$$\begin{aligned} \|w - w * \rho_\varepsilon\|_{L^1(\Omega^\beta)} &= \int_{\Omega^\beta} \left| \int_{B_\varepsilon(0)} (w(\boldsymbol{x}) - w(\boldsymbol{x} - \boldsymbol{z})) \rho_\varepsilon(\boldsymbol{z}) d\boldsymbol{z} \right| d\boldsymbol{x} \\ &\leq \int_{B_\varepsilon(0)} \left(|\boldsymbol{z}| \int_{\Omega^{\beta-\varepsilon}} |\nabla w(\boldsymbol{x})| d\boldsymbol{x} \right) \rho_\varepsilon(\boldsymbol{z}) d\boldsymbol{z} \\ &= C_0 \left(\int_{\Omega^{\beta-\varepsilon}} |\nabla w(\boldsymbol{x})| d\boldsymbol{x} \right) \varepsilon. \end{aligned} \tag{4.1}$$

For $u \in \mathrm{BV}(\Omega)$, we can choose $\eta_k > 0$ ($k \in \mathbf{N}$) such that $\lim_{k\to\infty} \eta_k = 0$ and

$$\int_{\partial\Omega^{\eta_k}} |\nabla u| = 0 \qquad (k \in \mathbf{N}), \tag{4.2}$$

as a consequence of the coarea formula ([2], Theorem 1.23). For a fixed $\varepsilon > 0$, we define $\beta_k := \varepsilon + \eta_k$, and we apply (4.1) for $w = u * \rho_\sigma \in C^\infty(\overline{\Omega})$ ($\sigma > 0$). Then we have

$$\begin{aligned} & \|u - u * \rho_\varepsilon\|_{L^1(\Omega^{\beta_k})} \\ \leq\ & \|u * \rho_\sigma - (u * \rho_\sigma) * \rho_\varepsilon\|_{L^1(\Omega^{\beta_k})} + \|u - u * \rho_\sigma\|_{L^1(\Omega^{\beta}_k)} + \|(u - u * \rho_\sigma) * \rho_\varepsilon\|_{L^1(\Omega^{\beta_k})} \\ \leq\ & C_0 \left(\int_{\Omega^{\eta_k}} |\nabla (u * \rho_\sigma)(\boldsymbol{x})| d\boldsymbol{x} \right) \varepsilon + 2\|u - u * \rho_\sigma\|_{L^1(\Omega)}. \end{aligned}$$

From (4.2), $\lim_{\sigma\to+0} \int_{\Omega^{\eta_k}} |\nabla (u * \rho_\sigma)(\boldsymbol{x})| d\boldsymbol{x} = \int_{\Omega^{\eta_k}} |\nabla u|$ is derived (see [2] Proposition 1.15). Taking $\sigma \to 0$, we have

$$\|u - u * \rho_\varepsilon\|_{L^1(\Omega^{\beta_k})} \leq C_0 \left(\int_{\Omega^{\eta_k}} |\nabla u| \right) \varepsilon \leq C_0 \left(\int_\Omega |\nabla u| \right) \varepsilon,$$

and we have the assertion of the lemma by taking the limit $k \to \infty$. □

Since Ω is a bounded domain with a Lipschitz boundary, We can define

$$C_\Omega := \sup_{\varepsilon>0} \mathcal{H}^n(N^\varepsilon(\partial\Omega) \cap \Omega)\varepsilon^{-1} < \infty.$$

Lemma 4.2 *For $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and $\varepsilon > 0$, we have*

$$\begin{aligned} \|u\|_{L^1(\Omega)} \leq\ & \left(C_0 \int_\Omega |\nabla u| + C_\Omega \|u\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)} \right) \varepsilon \\ & + C_1 \sqrt{\mathcal{H}^n(N^\varepsilon(\mathrm{supp}(u)) \cap \Omega^\varepsilon)} \|u\|_{H^{-1}(\Omega)} \varepsilon^{-1}. \end{aligned}$$

Proof. Setting $\mathrm{sgn}(s) := \pm 1$ $(\pm s > 0)$ and $\mathrm{sgn}(0) := 0$, we have

$$\begin{aligned}
\|u * \rho_\varepsilon\|_{L^1(\Omega^\varepsilon)} &= \int_{\Omega^\varepsilon} (u * \rho_\varepsilon)(\boldsymbol{x}) \mathrm{sgn}((u * \rho_\varepsilon)(\boldsymbol{x})) d\boldsymbol{x} \\
&= \int_{\Omega^\varepsilon} {}_{H^{-1}(\Omega)}\langle u, \rho_\varepsilon(\boldsymbol{x} - \cdot)\rangle_{H_0^1(\Omega)} \mathrm{sgn}((u * \rho_\varepsilon)(\boldsymbol{x})) d\boldsymbol{x} \\
&= {}_{H^{-1}(\Omega)}\left\langle u, \int_{\Omega^\varepsilon} \rho_\varepsilon(\boldsymbol{x} - \cdot) \mathrm{sgn}((u * \rho_\varepsilon)(\boldsymbol{x})) d\boldsymbol{x} \right\rangle_{H_0^1(\Omega)} \\
&= {}_{H^{-1}(\Omega)}\langle u, (\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)) * \rho_\varepsilon \rangle_{H_0^1(\Omega)} \\
&\le \|u\|_{H^{-1}(\Omega)} \|(\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)) * \rho_\varepsilon\|_{H_0^1(\Omega)}
\end{aligned}$$

Using the equality $\|\partial_i \rho_\varepsilon\|_{L^1(\mathbf{R}^n)} = \varepsilon^{-1} \|\partial_i \rho\|_{L^1(\mathbf{R}^n)}$ $(i = 1, 2, \cdots, n)$, we have

$$\begin{aligned}
\|(\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)) * \rho_\varepsilon\|^2_{H_0^1(\Omega)} &= \sum_{i=1}^n \|\partial_i \left((\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)) * \rho_\varepsilon\right)\|^2_{L^2(\Omega)} \\
&\le \sum_{i=1}^n \|\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)\|^2_{L^2(\Omega)} \|\partial_i \rho_\varepsilon\|^2_{L^1(\mathbf{R}^n)} \\
&= C_1^2 \|\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)\|^2_{L^2(\Omega)} \varepsilon^{-2}.
\end{aligned}$$

From these inequalities and Lemma 4.1, we have

$$\begin{aligned}
\|u\|_{L^1(\Omega)} &\le \|u - u * \rho_\varepsilon\|_{L^1(\Omega^\varepsilon)} + \|u * \rho_\varepsilon\|_{L^1(\Omega^\varepsilon)} + \|u\|_{L^1(\Omega \setminus \Omega^\varepsilon)} \\
&\le C_0 \left(\int_\Omega |\nabla u|\right) \varepsilon + \|u\|_{H^{-1}(\Omega)} C_1 \|\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)\|_{L^2(\Omega)} \varepsilon^{-1} \\
&\quad + C_\Omega \|u\|_{L^\infty(\Omega \setminus \Omega^\varepsilon)} \varepsilon.
\end{aligned}$$

Hence, the assertion follows from

$$\|\chi_{\Omega^\varepsilon} \mathrm{sgn}(u * \rho_\varepsilon)\|^2_{L^2(\Omega)} = \mathcal{H}^n(\mathrm{supp}(u * \rho_\varepsilon) \cap \Omega^\varepsilon) \le \mathcal{H}^n(N^\varepsilon(\mathrm{supp}(u)) \cap \Omega^\varepsilon).$$

□

Setting $\varepsilon = (\int_\Omega |\nabla u| + \|u\|_{L^\infty(\Omega)})^{-\frac{1}{2}} \|u\|^{\frac{1}{2}}_{H^{-1}(\Omega)}$ in this lemma, We also have the standard interpolation inequality:

Corollary 4.3 *For $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, we have*

$$\|u\|_{L^1(\Omega)} \le C \left(\int_\Omega |\nabla u| + \|u\|_{L^\infty(\Omega)}\right)^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{H^{-1}(\Omega)},$$

where $C := \max(C_0, C_\Omega) + C_1 \mathcal{H}^n(\Omega)^{\frac{1}{2}}$.

5 Estimates for the global minimizer

Some estimates for the global minimizer ψ_δ of (1.1) are proved in this section. A proof of the main theorem (Theorem 2.1) is given at the end. We start from the following geometric lemma which is based on the isoperimetric inequality. We remark that, in the statement of the next lemma, if $\overline{A} \subset B_r(\boldsymbol{x}_0)$ then (5.1) becomes a usual isoperimetric inequality.

Lemma 5.1 *Let $n \geq 2$. Then there exist positive constants R_0 and σ_0 depending only on n such that, for $A \subset \mathbf{R}^n$ and $\boldsymbol{x}_0 \in \mathbf{R}^n$ satisfying $\chi_A \in \mathrm{BV}(\mathbf{R}^n)$, $\mathcal{H}^n(A) > 0$ and $\mathcal{H}^n(A \cap B_\varepsilon(\boldsymbol{x}_0)) > 0$ $({}^\forall \varepsilon > 0)$, there exists $r \in (0, R_0 \mathcal{H}^n(A)^{\frac{1}{n}})$ and*

$$0 < \sigma_0 \mathcal{H}^n(A \cap B_r(\boldsymbol{x}_0))^{\frac{n-1}{n}} \leq \int_{\mathbf{R}^n} |\nabla \chi_A| - \int_{\mathbf{R}^n} |\nabla \chi_{A \setminus B_r(\boldsymbol{x}_0)}|. \tag{5.1}$$

Proof. We define

$$f(r) := \mathcal{H}^n(A \cap B_r(\boldsymbol{x}_0)) \qquad (r > 0).$$

For a fixed $\theta \in (0,1)$, we define $r_0 := (\mathcal{H}^n(A)/(\theta \omega_n))^{\frac{1}{n}}$. Since

$$0 < f(r) \leq \mathcal{H}^n(A) \leq \theta \mathcal{H}^n(B_{r_0}(\boldsymbol{x}_0)) \leq \theta \mathcal{H}^n(B_r(\boldsymbol{x}_0)) \qquad (r > r_0),$$

from the isoperimetric inequality in a ball (see [2] Corollary 1.29, [10] Theorem 5.4.3), we have

$$2\sigma_0 f(r)^{\frac{n-1}{n}} \leq \int_{B_r(\boldsymbol{x}_0)} |\nabla \chi_A| \qquad (r > r_0), \tag{5.2}$$

where $\sigma_0 > 0$ depends only on θ and n.

We have the following equality from Remark 2.13 and 2.14 of [2];

$$f'(r) = \int_{B_r(\boldsymbol{x}_0)} |\nabla \chi_A| + \int_{\mathbf{R}^n} |\nabla \chi_{A \setminus B_r(\boldsymbol{x}_0)}| - \int_{\mathbf{R}^n} |\nabla \chi_A| \qquad (\text{a.e. } r \in (0, \infty)). \tag{5.3}$$

We define

$$R_0 := \frac{n}{\sigma_0} + (\theta \omega_n)^{-\frac{1}{n}}, \qquad r_1 := R_0 \mathcal{H}^n(A)^{\frac{1}{n}}.$$

If we assume, contrary to the lemma, that

$$\sigma_0 \mathcal{H}^n(A \cap B_r(\boldsymbol{x}_0))^{\frac{n-1}{n}} > \int_{\mathbf{R}^n} |\nabla \chi_A| - \int_{\mathbf{R}^n} |\nabla \chi_{A \setminus B_r(\boldsymbol{x}_0)}| \qquad ({}^\forall r \in (0, r_1)),$$

then, by (5.2) and (5.3), we have

$$\sigma_0 f(r)^{\frac{n-1}{n}} > 2\sigma_0 f(r)^{\frac{n-1}{n}} - f'(r) \qquad (\text{a.e. } r \in (r_0, r_1)).$$

This is equivalent to $(f(r)^{\frac{1}{n}})' > \sigma_0/n$ a.e. $r \in (r_0, r_1)$, and integrating this inequality by r over the interval (r_0, r_1), we have

$$f(r_1)^{\frac{1}{n}} - f(r_0)^{\frac{1}{n}} > \frac{\sigma_0}{n}(r_1 - r_0) = \mathcal{H}^n(A)^{\frac{1}{n}}.$$

This contradicts to $f(r_1) \leq \mathcal{H}^n(A)$. □

If we assume the condition (A1), then we can define the constants

$$M_1 := \sup_{\varepsilon > 0} \mathcal{H}^n(N^\varepsilon(\Gamma) \cap \Omega)\varepsilon^{-1} < \infty, \qquad M_2 := \|\mathrm{div}\overline{\boldsymbol{n}}\|_{H_0^1(\Omega)}. \tag{5.4}$$

From Proposition 1.2 and 2.2, we have

$$\|\psi_\delta - \psi\|_{L^1(\Omega)} \leq C(\Omega) \left(1 + \int_\Omega |\nabla \psi|\right)^{\frac{1}{2}} M_2^{\frac{1}{2}} \delta^{\frac{1}{2}}, \tag{5.5}$$

where $C(\Omega)$ is the constant which appears in Proposition 1.2. Since $\|\psi_\delta - \psi\|_{L^1(\Omega)} = 2\mathcal{H}^n(\mathrm{supp}(\psi_\delta - \psi))$, (5.5) gives us a decay order $O(\delta^{\frac{1}{2}})$ of the volume of symmetric difference between $\Omega_\delta^\pm$ and $\Omega^\pm$. If we have a geometric information on $\mathrm{supp}(\psi_\delta - \psi)$, then we can improve (5.5) as follows.

Lemma 5.2 *Under the assumption (A1), we suppose that*

$$\psi_\delta = \psi \qquad \text{a.e. in } \Omega \setminus N^{a_\delta}(\Gamma) \qquad (0 < \delta < \delta_0), \tag{5.6}$$

where $a_\delta := Q\delta^\alpha$ and $Q > 0$, $\alpha \in (0, 2/3)$ and $\delta_0 > 0$ are constants which depends only on ψ, Ω and n. Then there exists $R = R(\psi, \Omega, n, Q, \alpha, \delta_0) > 0$ such that

$$\|\psi_\delta - \psi\|_{L^1(\Omega)} \le R\delta^{\frac{1}{2}+\frac{\alpha}{4}} \qquad (0 < \delta < \delta_0). \tag{5.7}$$

Proof. Setting $\varepsilon = \delta^{\frac{1}{2}+\frac{\alpha}{4}}$, we apply Lemma 4.2. Using the inequalities (1.5) and (2.4), for $\delta \in (0, \delta_0)$, we have

$$\begin{aligned}
\|\psi_\delta - \psi\|_{L^1(\Omega)} &\le 2\left(C_0 \int_\Omega |\nabla \psi| + C_\Omega\right)\varepsilon + C_1 M_1^{\frac{1}{2}} (a_\delta + \varepsilon)^{\frac{1}{2}} M_2 \delta \varepsilon^{-1} \\
&= 2\left(C_0 \int_\Omega |\nabla \psi| + C_\Omega\right)\delta^{\frac{1}{2}+\frac{\alpha}{4}} + C_1 M_1^{\frac{1}{2}} M_2 \left(\delta^{\frac{6-\alpha}{4}} + Q\delta^{1+\frac{\alpha}{2}}\right)^{\frac{1}{2}} \\
&\le 2\left(C_0 \int_\Omega |\nabla \psi| + C_\Omega\right)\delta^{\frac{1}{2}+\frac{\alpha}{4}} + C_1 M_1^{\frac{1}{2}} M_2 \left(\delta_0^{\frac{2-3\alpha}{4}} + Q\right)^{\frac{1}{2}} \delta^{\frac{1}{2}+\frac{\alpha}{4}}.
\end{aligned}$$

□

As shown in Lemma 5.2, a geometric estimate for the shape of $\Omega_\delta^\pm$, such as (5.6), helps us even in quantitative estimate, such as $L^1(\Omega)$-estimate.

We first prove the following theorem which is slightly weaker than Theorem 2.1.

Theorem 5.3 *Under the assumptions (A1) and (A2), there exist $\delta_0 = \delta_0(\psi, \Omega, n) > 0$ and $Q = Q(\psi, \Omega, n) > 0$ such that*

$$\psi_\delta = \psi \qquad \text{a.e. in } \Omega \setminus N^{a_\delta}(\Gamma) \qquad (0 < \delta < \delta_0),$$

where $a_\delta := Q\delta^{\alpha_n}$ and

$$\alpha_n := \begin{cases} \dfrac{1}{8} & (n = 2), \\[2ex] \dfrac{1}{2n} & (3 \le n \le 5), \\[2ex] \dfrac{2}{(n+1)(n-2)} & (n \ge 6). \end{cases} \tag{5.8}$$

To prove this theorem, we need the following lemma, whose proof will be given at the end of this section. We remark that the condition (A2) is equivalent to

$$b_0 := \inf_{\boldsymbol{x} \in \Gamma} \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > 0.$$

Lemma 5.4 *Under the assumptions (A1) and (A2), for a fixed* $a \in (0, b_0)$*, there exists* $\delta_0 = \delta_0(a, \psi, \Omega, n) > 0$ *such that*

$$\psi_\delta = \psi \qquad \text{a.e. in } \Omega \setminus N^a(\Gamma) \qquad (0 < \delta < \delta_0).$$

Proof of Theorem 5.3. Let $a \in (0, b_0)$ and let $\delta_0 > 0$ be as given in Lemma 5.4. For a sufficiently large $Q > 0$, we define

$$A^\pm := \{\boldsymbol{x} \in \Omega \setminus N^{a_\delta}(\Gamma);\ \psi_\delta(\boldsymbol{x}) \neq \psi(\boldsymbol{x}) = \pm 1\}.$$

On the contrary to the assertion of the theorem, let us assume that $\mathcal{H}^n(A^+ \cup A^-) > 0$. Without loss of generality, we assume that $\mathcal{H}^n(A^+) > 0$. We define

$$A := \{\boldsymbol{x} \in \Omega;\ \psi_\delta(\boldsymbol{x}) \neq \psi(\boldsymbol{x}) = 1\} \supset A^+,$$

and fix $\boldsymbol{x}_0 \in A^+$ such that $\mathcal{H}^n(A \cap B_\varepsilon(\boldsymbol{x}_0)) > 0$ ($^\forall\varepsilon > 0$). We remark that, from (5.5),

$$\mathcal{H}^n(A) \leq \frac{1}{2}\|\psi_\delta - \psi\|_{L^1(\Omega)} \leq R_1 \delta^{\frac{1}{2}} \qquad (\delta > 0), \tag{5.9}$$

where $R_1 := \frac{1}{2}C(\Omega)\left(1 + \int_\Omega |\nabla\psi|\right)^{\frac{1}{2}} M_2^{\frac{1}{2}}$. By Lemma 5.1, there exists $r \in (0, R_0\mathcal{H}^n(A)^{\frac{1}{n}})$ such that (5.1) holds. From (5.9), we have

$$r < R_0\mathcal{H}^n(A)^{\frac{1}{n}} \leq R_0 R_1^{\frac{1}{n}} \delta^{\frac{1}{2n}} \qquad (\delta > 0). \tag{5.10}$$

Since

$$\alpha_n \leq \frac{1}{2n} \tag{5.11}$$

taking enough large $Q = Q(\psi, \Omega, n) > 0$ in advance (and changing $\delta_0 > 0$ smaller if necessary), we can assume that $2r \leq Q\delta^{\alpha_n} = a_\delta \leq a$ $(0 < \delta < \delta_0)$.

We define $E := A \cap B_r(\boldsymbol{x}_0)$ and $\varphi_E := (1 - 2\chi_E)\psi_\delta$. We remark that (3.3) is fulfilled by $F_1 = \{\boldsymbol{x} \in \Omega;\ \psi_\delta(\boldsymbol{x}) \neq \psi(\boldsymbol{x})\}$, $F_2 = F_1 \setminus E$, $\varphi_1 = \psi_\delta$ and $\varphi_2 = \varphi_E$. In particular, we have

$$r \leq \frac{a_\delta}{2} \leq \min(\varepsilon_1, \varepsilon_2) \qquad (0 < \delta < \delta_0). \tag{5.12}$$

Then, from (5.1) and (5.12), we have

$$0 < \sigma_0 \mathcal{H}^n(E)^{\frac{n-1}{n}} \leq \int_{\mathbf{R}^n} |\nabla\chi_A| - \int_{\mathbf{R}^n} |\nabla\chi_{A\setminus E}| = \frac{1}{2}\left(\int_\Omega |\nabla\psi_\delta| - \int_\Omega |\nabla\varphi_E|\right).$$

Since ψ_δ is a global minimizer, hence we have

$$0 < \sigma_0\mathcal{H}^n(E)^{\frac{n-1}{n}}\delta \leq \frac{1}{2}\left(\|\varphi_E - \psi\|^2_{H^{-1}(\Omega)} - \|\psi_\delta - \psi\|^2_{H^{-1}(\Omega)}\right) \qquad (0 < \delta < \delta_0). \tag{5.13}$$

Let $n = 2$. We apply Corollary 3.5. Taking enough large Q, the condition (3.8) is satisfied as follows. From (5.10), (5.12) and $\varepsilon_2 > b_0 - a$, we have

$$\operatorname{diam}(E) \leq 2r \leq 2R_0 R_1^{\frac{1}{2}} \delta^{\frac{1}{4}} \leq \frac{b_0 - a}{\operatorname{diam}(\Omega)^2}\left(\frac{a_\delta}{2}\right)^2 \leq \frac{\varepsilon_1^2 \varepsilon_2}{\operatorname{diam}(\Omega)^2}.$$

Hence, from Corollary 3.5 and Proposition (2.2), we have

$$\|\varphi_E - \psi\|^2_{H^{-1}(\Omega)} - \|\psi_\delta - \psi\|^2_{H^{-1}(\Omega)} \le 4\left(\frac{1}{2\pi}\log\frac{\mathrm{diam}(\Omega)}{\varepsilon_1}\right)^{\frac{1}{2}} \mathcal{H}^2(E) M_2 \delta.$$

Together with (5.12) and (5.13), for $\delta \in (0, \delta_0)$, we have

$$\sigma_0 \le 2M_2\left(\frac{1}{2\pi}\log\frac{\mathrm{diam}(\Omega)}{\varepsilon_1}\right)^{\frac{1}{2}} \mathcal{H}^2(E)^{\frac{1}{2}} \le 2M_2\left(\frac{1}{2\pi}\log\frac{2\mathrm{diam}(\Omega)}{Q\delta^{\alpha_2}}\right)^{\frac{1}{2}} \mathcal{H}^2(E)^{\frac{1}{2}}.$$

This contradicts to $\mathcal{H}^n(E) \le \mathcal{H}^n(A) \le R_1\delta^{\frac{1}{2}}$ if δ is small.

Let $n \ge 3$. We apply Corollary 3.6. From (5.10) and (5.12), taking enough large Q, we have

$$\mathrm{diam}(E) \le 2r < 2R_0 R_1^{\frac{1}{n}} \delta^{\frac{1}{2n}} \le 5^{-\frac{1}{n-2}}\frac{a_\delta}{2} \le 5^{-\frac{1}{n-2}} \min(\varepsilon_1, \varepsilon_2) \qquad (0 < \delta < \delta_0),$$

and the condition (3.9) is fulfilled for $\delta \in (0, \delta_0)$. To estimate the term $\|G\|_{L^1(E\times E)}$, we define

$$G_0 := \inf\{G(\boldsymbol{x}, \boldsymbol{y});\ \boldsymbol{x} \in N^a(\Gamma),\ \boldsymbol{y} \in N^a(\Gamma)\} > 0.$$

Then $\|G\|_{L^1(E\times E)} \ge G_0\mathcal{H}^n(E)^2$ for $\delta \in (0, \delta_0)$ since $E \subset N^a(\Gamma)$. From Corollary 3.6, Proposition 2.2, (5.12) and (5.13), for $\delta \in (0, \delta_0)$, we have

$$\begin{aligned}\sigma_0 \mathcal{H}^n(E)^{\frac{n-1}{n}}\delta + G_0\mathcal{H}^n(E)^2 &\le 2\left(n(n-2)\omega_n\varepsilon_1^{n-2}\right)^{-\frac{1}{2}} \mathcal{H}^n(E) M_2\delta \\ &\le R_3 Q^{-\frac{n-2}{2}} \mathcal{H}^n(E)\delta^{1-\frac{(n-2)\alpha_n}{2}},\end{aligned}$$

where $R_3 := 2^{\frac{n}{2}}(n(n-2)\omega_n)^{\frac{1}{2}} M_2$. Hence, we have

$$\sigma_0\delta^{\frac{(n-2)\alpha_n}{2}} \le R_3 Q^{-\frac{n-2}{2}}\mathcal{H}^n(E)^{\frac{1}{n}}, \qquad G_0\mathcal{H}^n(E) \le R_3 Q^{-\frac{n-2}{2}}\delta^{1-\frac{(n-2)\alpha_n}{2}},$$

and they yield the following inequality for $\delta \in (0, \delta_0)$;

$$\begin{aligned}\sigma_0\delta^{\frac{(n-2)\alpha_n}{2}} &\le R_3 Q^{-\frac{n-2}{2}}\left(\frac{R_3 Q^{-\frac{n-2}{2}}\delta^{1-\frac{(n-2)\alpha_n}{2}}}{G_0}\right)^{\frac{1}{n}} \\ &= \left(\frac{R_3^{n+1}}{G_0}\right)^{\frac{1}{n}} Q^{-\frac{(n+1)(n-2)}{2n}}\delta^{\frac{1}{n}\left(1-\frac{(n-2)\alpha_n}{2}\right)}.\end{aligned} \tag{5.14}$$

We remark that

$$\frac{(n-2)\alpha_n}{2} \le \frac{1}{n}\left(1 - \frac{(n-2)\alpha_n}{2}\right),$$

is equivalent to

$$\alpha_n \le \frac{2}{(n+1)(n-2)}. \tag{5.15}$$

Since

$$\alpha_n = \min\left(\frac{1}{2n},\ \frac{2}{(n+1)(n-2)}\right) \qquad (n \geq 3), \tag{5.16}$$

if Q is large enough, we have contradiction from the inequality (5.14). □

Hence, we have obtained a geometric estimate in Theorem 5.3 and simultaneously a sharper L^1-estimate as in Lemma 5.2. Let us check the proof of Theorem 5.3 in detail. The decay rate α_n for $n \geq 3$ is given by (5.16), which is required from (5.11) and (5.15). The condition (5.11) is also required from (5.9), but now, we have a better estimate (5.7) than (5.9). A recursive argument for α_n between the geometric estimate and L^1-estimate leads us the following proof of Theorem 2.1.

Proof of Theorem 2.1. Let $n \geq 3$ and let α_n^{old} be as in (5.8). From Lemma 5.2, (5.9) in the proof of Theorem 5.3 can be replaced by

$$\mathcal{H}^n(A) \leq \frac{1}{2}\|\psi_\delta - \psi\|_{L^1(\Omega)} \leq \tilde{R}_1 \delta^{\frac{1}{2}+\frac{\alpha_n^{\mathrm{old}}}{4}} \qquad (0 < \delta < \delta_0),$$

for some $\tilde{R}_1 > 0$, and then, the condition (5.11) can be replaced by

$$\alpha_n \leq \frac{1}{n}\left(\frac{1}{2} + \frac{\alpha_n^{\mathrm{old}}}{4}\right).$$

Hence, the assertion of Theorem 5.3 is valid even for

$$\alpha_n = \min\left(\frac{1}{n}\left(\frac{1}{2} + \frac{\alpha_n^{\mathrm{old}}}{4}\right),\ \frac{2}{(n+1)(n-2)}\right) \qquad (n \geq 3).$$

If $3 \leq n \leq 5$, this new exponent α_n is greater than the old one. Repeating this procedure recursively, it is shown that the assertion of Theorem 5.3 is valid even for α_n of (2.3). The norm estimate (2.1) follows from Proposition 2.2 and Lemma 5.2.

To prove (2.2), we note that, since Γ is Lipschitz and $\Gamma = \overline{\Gamma} \subset \Omega$, there exist $C_* \geq 1$ and $\varepsilon_* > 0$ such that

$$\Omega \subset N^{C_*\varepsilon}(\Omega \setminus N^\varepsilon(\Gamma)) \qquad (0 < \varepsilon < \varepsilon_*).$$

For simplicity we define $d(\delta) := \max\left(\mathrm{dist}_{\mathrm{H}}(\overline{\Omega_\delta^+}, \overline{\Omega^+}), \mathrm{dist}_{\mathrm{H}}(\overline{\Omega_\delta^-}, \overline{\Omega^-})\right)$. From the inequalities;

$$\mathrm{dist}(\boldsymbol{x}, \Gamma) = \mathrm{dist}(\boldsymbol{x}, \overline{\Omega^\mp}) \leq d(\delta) \quad (\boldsymbol{x} \in \Gamma_\delta \cap \overline{\Omega^\pm}),$$

$$\mathrm{dist}(\boldsymbol{x}, \Gamma_\delta) = \mathrm{dist}(\boldsymbol{x}, \overline{\Omega_\delta^\mp}) \leq d(\delta) \quad (\boldsymbol{x} \in \Gamma \cap \overline{\Omega_\delta^\pm}),$$

we have

$$\mathrm{dist}_{\mathrm{H}}(\Gamma_\delta, \Gamma) \leq d(\delta),$$

for small δ. It is clear that $\mathrm{dist}(\boldsymbol{x}, \overline{\Omega^{\pm}}) \le a_\delta$ ($\boldsymbol{x} \in \overline{\Omega_\delta^{\pm}}$). We also have

$$\mathrm{dist}(\boldsymbol{x}, \overline{\Omega_\delta^{\pm}}) \le \mathrm{dist}(\boldsymbol{x}, \overline{\Omega_\delta^{\pm}} \setminus N^{a_\delta}(\Gamma)) = \mathrm{dist}(\boldsymbol{x}, \Omega^{\pm} \setminus N^{a_\delta}(\Gamma)) \le C_* a_\delta \qquad (\boldsymbol{x} \in \overline{\Omega^{\pm}}).$$

Hence we have $d(\delta) \le C_* a_\delta$ for small δ and we obtain the desired result. □

Finally, we give a proof of Lemma 5.4 and then complete the proof of our main theorem.

Proof of Lemma 5.4. For fixed $a \in (0, b_0)$, we also fix $b \in (0, b_0 - a)$. First we note that

$$\psi_\delta = \psi \qquad \text{a.e. in } \Omega \setminus (N^a(\Gamma) \cup N^b(\partial\Omega)),$$

for small δ. We omit its proof, since it is proved by an argument similar to (but rather simpler than) the proof of Theorem 5.3.

Then the assertion of the Lemma is shown as follows. Let $E := \Omega \cap N^b(\partial\Omega)$ and let $\varphi_E := (1 - 2\chi_E)\psi_\delta \in \mathcal{K}$. From the isoperimetric inequality in Ω (see [10] Theorem 5.4.3 and 5.11.1), there exists $\sigma_0 > 0$ which does not depend on δ, such that

$$\sigma_0 \mathcal{H}^n(E)^{\frac{n-1}{n}} \le \int_\Omega |\nabla \chi_E|,$$

and (5.13) holds. We apply Lemma 3.3 with $\varphi = \psi_\delta$ and $h = (b_0 - b)/2$. There exists $C > 0$ which depends only on b, ψ, Ω and n, such that

$$\sigma_0 \mathcal{H}^n(E)^{\frac{n-1}{n}} \delta \le C \mathcal{H}^n(E) \|\psi_\delta - \psi\|_{H^{-1}(\Omega)} \le C M_2 \mathcal{H}^n(E) \delta.$$

This contradicts to $\mathcal{H}^n(E) \le \|\psi_\delta - \psi\|_{L^1(\Omega)}/2$ if δ is small. □

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