Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

Christian Fleischhack

Preprint no.: 70 2001

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Christian Fleischhack*

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22-26
04103 Leipzig, Germany

Institut für Theoretische Physik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig, Germany

September 21, 2001

Abstract

It is proven that the physical measure for the two-dimensional Yang-Mills theory is purely singular with respect to the kinematical Ashtekar-Lewandowski measure. For this, an explicit decomposition of the gauge orbit space into supports of these two measures is given. Finally, the results are extended to more general (e.g. confining) theories. Such a singularity implies, in particular, that the standard method of determining the physical measure via "exponential of minus the action times kinematical measure" is not applicable.

*e-mail: Christian.Fleischhack@itp.uni-leipzig.de *or* Christian.Fleischhack@mis.mpg.de

1 Introduction

The functional integral approach to quantum field theories consists of two basic steps: first the determination of a "physical" Euclidian measure on the configuration space and second the reconstruction of the quantum theory via an Osterwalder-Schrader procedure. The latter issue has been treated rigorously in several approaches – first by Osterwalder and Schrader [56, 57] for scalar fields, recently by Ashtekar et al. [14] for diffeomorphism invariant theories. However, in contrast to this, the former step kept a problem that has been solved completely only for some examples.

One of the most promising attempts to overcome this problem in a rather general context is the Ashtekar approach to gauge field theories. It is motivated by the observation that the first step above consists not only of the determination of the physical measure, but also of the preceding determination of the configuration space of the theory. Originally, in standard (pure) gauge field theories this space contains all smooth gauge fields modulo smooth gauge transformations. However, such a space has a very difficult mathematical structure – it is typically non-compact, non-affine, not finite-dimensional and not a manifold. This makes measure theory very complicated. How to get rid of this? First, Faddeev and Popov [31] tried to use gauge fixings to transfer the problem from the gauge orbit space to the much simpler affine space of all gauge fields. However, this failed because of the Gribov problem, i.e. the non-existence of global gauge fixings [40, 67]. Next, it is well-known that the quantization of a theory is typically accompanied with a loss of smoothness. This motivated the enlargement of the configuration space by Sobolev (i.e. non-smooth) gauge fields and gauge transforms [53, 55, 54]. This way, wide success has been made in the investigation of the geometry of the (enlarged) configuration space. It has been shown that the gauge transform action obeys a slice theorem which yields a stratification [46, 47]. Recently, all occurring gauge orbit types have been classified for certain models [61]. But, there is no nontrivial measure known on the total gauge orbit space. Third, the lattice theory has been developped. For this, one first reduces the degrees of freedom to a finite (floating) lattice and hopes for a reconstruction of the continuum theory by some continuum limit. Although several physical properties like confinement [81] have been explained within this approach, the full continuum limit remains in general an open problem.

The Ashtekar approach, in a sense, brings together the two last issues – the enlargement of the configuration space and the lattice theories. Its basic idea goes as follows: The continuum gauge theory is known as soon as its restrictions to all finite floating lattices are known. This means, in particular, that the expectation values of all observables that are sensitive only to the degrees of freedom of a certain lattice can be calculated by the corresponding integration over these finitely many degrees of freedom. Examples for those observables are the Wilson loop variables $\operatorname{tr} h_\beta$, where β is some loop in the space or space-time and h_β is the holonomy along that loop.

The above idea has been implemented rigorously for compact structure groups $\mathbf{G}$ as follows: First the original configuration space of all smooth gauge fields (modulo gauge transforms) has been enlarged by distributional ones [5]. This way the configuration space became compact and could now be regarded as a so-called projective limit of the lattice configuration spaces [7]. These, on the other hand, consist as in ordinary lattice gauge theories of all possible assignments of parallel transports to the edges of the considered floating lattices (again modulo gauge transforms). Since every parallel transport is an element of $\mathbf{G}$, the Haar measure on $\mathbf{G}$ yields a natural measure for the lattice theories. Now the so-called Ashtekar-Lewandowski measure μ_0 [6] is just that continuum measure whose restrictions to the lattice theories coincide with these natural lattice Haar measures. It serves as a canonical kinematical measure.

Due to the compactness both of the space $\overline{\mathcal{A}}$ of these generalized gauge fields (or, mathematically, connections) and of the group $\overline{\mathcal{G}}$ of generalized gauge transforms, the geometry of the factor space $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ is well-understood. As in the Sobolev case a slice theorem has been proven, a stratification has been found and the occurring gauge orbit types w.r.t. the action of $\overline{\mathcal{G}}$ have been determined (here completely for all space-times and all compact structure groups) [38]. Moreover, it has been shown that the so-called non-generic connections [38, 33] form a μ_0-zero subset of $\overline{\mathcal{A}}$. Additionally, as for smooth connections, typically (i.e. for $\mathbf{G} = SU(N)$ and some other groups) a Gribov problem arises in the sense that there is no continuous gauge fixing in $\overline{\mathcal{A}}$. However, here one can find a μ_0-zero subset in $\overline{\mathcal{A}}$ such that after its removal there is a continuous gauge fixing [33]. This implies that the Faddeev-Popov determinant equals 1 almost everywhere. Therefore no problems arise when integrating over $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ using such (almost complete) gauge fixings – at least on the kinematical level.

Problems Considered in this Article

In this article we are going to study the physical relevance of these rather mathematical structures. Our considerations are motivated by the following two, obviously connected problems.

Question 1 What is the impact of non-generic connections?

Question 2 How severe is the Gribov problem?

In this generality both questions, of course, can hardly be answered. Therefore we will first analyze them by means of a concrete example. Unfortunately, within the Ashtekar approach we have only two theories at our disposal that are investigated in detail: the quantum gravity (in particular, the canonical quantization [3, 12] and the quantum geometry [17, 59, 60, 8, 9, 10, 48, 4, 74, 77]) and the two-dimensional Yang-Mills theory [73, 13, 35, 36, 75]. Beyond these two there are only attempts for the treatment of matter fields [76], heat-kernel measures or measures coming from knot theory [7] or from Chern-Simons theory [18, 19]. Recently, the Fock space formulation has been connected to the Ashtekar framework [78, 79, 11, 80].

However, in the field of quantum gravity the problem is still a bit unclear. This is due to the canonical quantization used there [12, 3, 16, 15, 72]. Its starting point is a classical phase space (hence for quantum gravity a symplectic space whose position variables are just the Ashtekar connections) with certain constraints (here, e.g., Gauß constraint and diffeomorphism constraint). Afterwards, some algebra of functions on this phase space is associated an algebra of operators by naive quantization, such that Poisson brackets correspond to operator commutators, and then some Hilbert space is chosen where these operators are represented. For quantum gravity this Hilbert space is just the space $L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}})$ with the Ashtekar-Lewandowski measure μ_0 where $\mathbf{G} = SU(2)$. For μ_0, however, the Gribow problem and the impact of non-generic connections has already been investigated [38, 33]. Consequently, we can consider the questions above answered. But, of course, one can take the view that in any case $L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}})$ is only an auxiliary tool. Then these questions are not at issue because up to now it is not clear how the physical Hilbert space of quantum gravity looks like.

Therefore we will focus on the example of the two-dimensional quantum Yang-Mills theory (YM_2). As mentioned in the beginning, the central point here is the determination of a physical interaction measure μ_{YM} on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. Typically – neglecting mathematical problems – such a measure is defined by multiplying some kinematical measure with e^{-S}, where S is the action of the physical theory. The natural kinematical measure in the Ashtekar approach is the Ashtekar-Lewandowski measure; but the action $S(A) \equiv S_{\mathrm{YM}}(A) = \frac{1}{4}\int_M \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}x$ is only defined in the case $\mathcal{A}/\mathcal{G}$ and not for $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. This is obvious because products of space-time derivatives of distributional connections cannot be defined in general. This problem has been solved first by Thiemann [73] and Ashtekar et al. [13] for $SU(N)$ and $U(1)$: They used the fact that by the Riesz-Markov theorem the knowledge of all Wilson-loop expectation values is sufficient for the determination of μ_{YM} and calculated these expectation values by means of a lattice regularization of S_{YM}. More precisely, they chose on every quadratic lattice Γ the Wilson action $S_{\mathrm{YM,reg}\Gamma}(A) := \frac{N}{\mathrm{g}^2 a^2} \sum_\Box (1 - \frac{1}{N}\mathrm{Re}\ \mathrm{tr}\ h_\Box(A))$, where $\Box$ runs over all plaquettes of the lattice with lattice spacing a and side lengths L_x and L_y [81]. This function can be extended in a natural way to $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. Then the Wilson-loop expectation values are defined by exchanging limit and integral:

$$\langle \mathrm{tr}\ h_{\alpha_1} \cdots \mathrm{tr}\ h_{\alpha_n} \rangle := \lim_{a\to 0, L_x, L_y \to\infty} \frac{1}{Z_{a,L_x,L_y}} \int_{\overline{\mathcal{A}}/\overline{\mathcal{G}}} \mathrm{e}^{-S_{\mathrm{YM,reg}\Gamma}}\ \mathrm{tr}\ h_{\alpha_1} \cdots \mathrm{tr}\ h_{\alpha_n}\ \mathrm{d}\mu_0,$$

where Z_{a,L_x,L_y} only normalized $\langle 1 \rangle$ to 1. The usage of a fixed quadratic lattice remained a disadvantage because it only permitted the consideration of loops fitting in such a lattice; but this is per se not sufficient for a rigorous determination of μ_{YM}. This drawback has been removed in [36, 35] where not the loops are adapted to the regularization, but the regularization is adapted to the given loops. So for an arbitrary graph first the sum over all plaquettes has been replaced by the sum over all interior domains and second a^2 simply by the area of the corresponding domain. Moreover, the limiting process now instead of $a \to 0, L_x, L_y \to \infty$ consists of all possible refinements of the graph built by the α_i. This way, μ_{YM} has been defined rigorously.

However, properties of μ_{YM} are almost unknown. Only the invariance w.r.t. area-preserving diffeomorphisms has been shown [13]. Regarding to the two questions above there is a very interesting

Question 3 Is μ_{YM} absolutely continuous w.r.t. μ_0?

If we were able to answer this question with "yes", we would have proven that the set of all non-generic connections has not only Ashtekar-Lewandowski measure, but also Yang-Mills measure 0, and the Gribov problem remains harmless as well. Moreover, such an absolute continuity would guarantee the existence of a non-negative $L^1(\mu_0)$-function χ in $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ with $\mathrm{d}\mu_{\mathrm{YM}} = \chi\ \mathrm{d}\mu_0$. This function could be considered as $\mathrm{e}^{-S_{\mathrm{YM}}}$ for some generalized Yang-Mills action S_{YM}. However – μ_{YM} is *not* absolutely continuous w.r.t. μ_0. We will even be able to prove that μ_{YM} is *purely singular* w.r.t. μ_0, this means that the support of μ_{YM} is contained in a μ_0-zero subset. This, on the other hand, does *not* mean, that for instance the non-generic connections need have a Yang-Mills measure different from 0. This comes from the fact that despite of the singularity of μ_{YM} w.r.t. μ_0 on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ the corresponding lattice measures are always absolutely continuous w.r.t. the lattice Haar measures. Since both the non-generity and the almost global triviality of the generic stratum being responsible for the relevance of the Gribow-Problem can be described already on the level of graphs, we will get for the Yang-Mills measure similar answers to the first two questions as we did for the Ashtekar-Lewandowski measure. However, since we will observe a certain concentration of the Yang-Mills measure "near" non-generic connections, such strata should not simply be neglected.

Outline of the Article

The outline of the present artice is as follows:

- First after fixing the notations we will provide some theorems from the Fourier analysis on arbitrary compact Lie groups that will be needed for the investigation of the Radon-Nikodym derivatives $\mathrm{d}\mu_{\mathrm{YM},\Gamma}/\mathrm{d}\mu_{0,\Gamma}$ on the lattice levels and for the singularity theorem afterwards.

- Next we will review the construction of the Yang-Mills measure μ_{YM} [36, 13] in terms of loop-network states introduced by Thiemann [75] and give a proof for the well-definedness of μ_{YM} for arbitrary compact structure groups $\mathbf{G}$.
- Third we will investigate the lattice Radon-Nikodym derivatives and prove the inequivalence between the continuum Yang-Mills and the Ashtekar-Lewandowski measure by studying the support of the Yang-Mills measure. As by-products we get that the non-generic connections are contained in a μ_{YM}-zero subset, that the Gribov problem is again harmless, and that the regular (smooth) gauge orbits are again contained in a zero subset.
- Finally, we will indicate how these results can be generalized to other models [34]. We will see, e.g., that analogous support properties are shared typically by theories describing confinement.

2 Preliminaries

We recall the basic notations and results about generalized connections [5, 6, 51, 7, 38, 37, 39, 33].

Let M be some at least two-dimensional manifold, m be fixed in M and $\mathbf{G}$ be a connected compact (real) Lie group. $\mathcal{P}$ denotes the groupoid of all paths in M, $\mathcal{HG}$ the group of all paths starting and ending in m. The set $\overline{\mathcal{A}}$ of generalized connections $\overline{A}$ is defined by $\overline{\mathcal{A}} := \varprojlim_\Gamma \overline{\mathcal{A}}_\Gamma \equiv \varprojlim_\Gamma \mathbf{G}^{\#\mathbf{E}(\Gamma)} = \mathrm{Hom}(\mathcal{P}, \mathbf{G})$. Here Γ runs over all (finite) graphs in M. $\mathbf{E}(\Gamma)$ is the set of edges in Γ, $\mathbf{V}(\Gamma)$ will be that of all vertices. The canonical projections from $\overline{\mathcal{A}}$ to the spaces $\overline{\mathcal{A}}_\Gamma$ of lattice connections are denoted by π_Γ. Given $\overline{\mathcal{A}}$ the projective limit topology, it becomes compact Hausdorff. The group $\overline{\mathcal{G}}$ of generalized gauge transforms $\overline{g}$ is defined by $\overline{\mathcal{G}} := \varprojlim_\Gamma \overline{\mathcal{G}}_\Gamma \equiv \varprojlim_\Gamma \mathbf{G}^{\#\mathbf{V}(\Gamma)} = \mathrm{Maps}(M, \mathbf{G})$. It is compact as well and acts continuously on $\overline{\mathcal{A}}$ via $h_{\overline{A}\circ\overline{g}}(\gamma) = g_{\gamma(0)}^{-1} h_{\overline{A}}(\gamma) g_{\gamma(1)}$ where the path γ is in $\mathcal{P}$ and $h_{\overline{A}}$ is the homomorphism corresponding to $\overline{A}$. The projections are again denoted by π_Γ. Analogously to the definition of π_Γ we set $\pi_{\boldsymbol{\gamma}} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}_{\boldsymbol{\gamma}} \cong \mathbf{G}^{\#\boldsymbol{\gamma}}$, $h \longmapsto h(\boldsymbol{\gamma})$ etc. for all finite subsets $\boldsymbol{\gamma}$ of $\mathcal{P}$. The projections $\pi^{\Gamma_2}_{\Gamma_1} : \overline{\mathcal{A}}_{\Gamma_2} \longrightarrow \overline{\mathcal{A}}_{\Gamma_1}$ are given similarly for all $\Gamma_1 \leq \Gamma_2$, whereas the last notation means that every edge of Γ_1 is a product of edges in Γ_2. Moreover, we set $\overline{\mathcal{A}/\mathcal{G}} := \varprojlim_\Gamma \overline{\mathcal{A}}_\Gamma / \overline{\mathcal{G}}_\Gamma$. If the paths in $\mathcal{P}$ are restricted to the piecewise analytic category, there is a natural homeomorphism $\boldsymbol{\phi} : \overline{\mathcal{A}}/\overline{\mathcal{G}} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}$.

Every self-consistent family $(\mu_\Gamma)_\Gamma$ of normalized regular Borel measures on the $\overline{\mathcal{A}}_\Gamma$, i.e. $\mu_{\Gamma_1} = (\pi^{\Gamma_2}_{\Gamma_1})_* \mu_{\Gamma_2}$ for all $\Gamma_1 \leq \Gamma_2$, defines a unique normalized regular Borel measure μ on $\overline{\mathcal{A}}$, such that $\mu_\Gamma = (\pi_\Gamma)_* \mu$. Conversely, every such μ defines via $\mu_\Gamma := (\pi_\Gamma)_* \mu$ a self-consistent family. If one chooses for μ_Γ always the Haar measure on $\mathbf{G}^{\#\mathbf{E}(\Gamma)}$, one gets the Ashtekar-Lewandowski measure μ_0.

Finally, we call a generating system $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ of the fundamental group $\pi_1(\Gamma)$ of a connected graph Γ weak fundamental system iff there is a maximal tree T in Γ such that for every path $\alpha_i \in \boldsymbol{\alpha}$ there is an edge e_i in $\Gamma \setminus (T \cup \{e_1, \ldots, e_{i-1}\})$ such that α_i is a product of e_i and certain edges in $T \cup \{e_1, \ldots, e_{i-1}\}$. A weak fundamental system $\boldsymbol{\alpha}$ is always a free generating system and fulfills $(\pi_{\boldsymbol{\alpha}})_* \mu_0 = \mu_{\mathrm{Haar}}^{\#\boldsymbol{\alpha}}$.

3 Fourier Analysis

On compact Lie groups, integration is strongly related to Fourier analysis. The crucial connecting links are the integration formulae and the Peter-Weyl theorem. However, in contrast to the extensively investigated case of functions on $U(1)$ (or simply 2π-periodical functions on $\mathbb{R}$), general results about the convergence of Fourier series beyond the Peter-Weyl theorem are very rare and widespread. There are only few original articles such as

[70] or results for special cases like smooth functions (see [26]) or expansions of heat-kernels (see [68]). Some results presented in the sequel (in particular, in the subsections 3.4 till 3.6) seem to be folklore in part; however, we were not able to find the proofs in the literature. Therefore we briefly collect in this section the facts needed in the following, and provide the proofs if they are – to the best of our knowledge – unknown or non-standard. A more detailled treatment is given in [39].

3.1 Representations of Compact Lie Groups

For every connected compact Lie group there is [30] a simply connected semisimple compact Lie group $\mathbf{G}_{\mathrm{ss}}$, some natural number k and some finite Lie subgroup $\mathbf{N} \subseteq Z(\mathbf{G}_{\mathrm{ss}}) \times U(1)^k$, such that

$$\mathbf{G} \cong (\mathbf{G}_{\mathrm{ss}} \times U(1)^k)/\mathbf{N}.$$

Here, $Z(\mathbf{G}_{\mathrm{ss}})$ is the center of $\mathbf{G}_{\mathrm{ss}}$. We set l to be the rank of $\mathbf{G}_{\mathrm{ss}}$, i.e. the dimension of a maximal torus in its Lie algebra $\mathfrak{g}_{\mathrm{ss}}$. The set of all (equivalence classes of) irreducible unitary representations of $\mathbf{G}$ is denoted by $\mathcal{D}(\mathbf{G})$. It is well-known that every representation $\phi \in \mathcal{D}(\mathbf{G})$ of $\mathbf{G} = (\mathbf{G}_{\mathrm{ss}} \times U(1)^k)/\mathbf{N}$ can be identified with a uniquely determined irreducible representation of $\mathbf{G}_{\mathrm{ss}} \times U(1)^k$ and consequently [25] with a tensor product $\phi_{\mathrm{ss}} \otimes \phi_{\mathrm{ab}}$ of irreducible representations of $\mathbf{G}_{\mathrm{ss}}$ and of $U(1)^k$, respectively. Hence it can be viewed as an element $(\vec{n}, \vec{z}) \in \mathbb{N}^l \times \mathbb{Z}^k$. Here, $\vec{n} \in \mathbb{N}^l$ characterizes the heighest weight $\Lambda_{\vec{n}} := \sum n_i \Lambda_i$ of the representation ϕ_{ss} and $\vec{z} \in \mathbb{Z}^k$ identifies the representation $\vec{g} \longmapsto (g_i^{z_i})_i$ of the torus part. Typically, we will simply write $\vec{\mathbf{n}}$ instead of $(\vec{n}, \vec{z})$ and use $\phi_{\vec{\mathbf{n}}}$ or even simpler $\vec{\mathbf{n}}$ to denote the corresponding representation. Finally, we denote by $d_{\vec{\mathbf{n}}}$ (or d_ϕ) the dimension of the representation $\vec{\mathbf{n}}$ (or ϕ).

3.2 Peter-Weyl Theorem

For every irreducible representation ϕ of $\mathbf{G}$ we fix a basis on the corresponding representation space V. By $\phi(g) \in \mathrm{Gl}_{\mathbb{C}}(V)$ we can view every $\phi(g)$ as some matrix. In the following $\phi^{ij}(g) \in \mathbb{C}$ denotes the matrix element of $\phi(g)$ belonging to the i-th column and the j-th row.[1] We call the elements of $\mathcal{M} := \{\sqrt{\dim\phi}\,\phi^{ij} \mid [\phi] \in \mathcal{D}(\mathbf{G}), i,j = 1,\ldots,\dim\phi\} \subseteq C^\infty(\mathbf{G})$ elementary matrix functions. The set $\{\chi_\phi \mid [\phi] \in \mathcal{D}(\mathbf{G})\} \subseteq C^\infty_{\mathrm{Ad}}(\mathbf{G})$ of all characters χ_ϕ of irreducible representations is denoted by $\mathcal{M}_{\mathrm{Ad}}$.

Proposition 3.1 Let ϕ_1 and ϕ_2 be irreducible unitary representations of $\mathbf{G}$. Then [21]

$$\int_{\mathbf{G}} \overline{\phi_1^{i_1 j_1}(g)}\, \phi_2^{i_2 j_2}(g)\, \mathrm{d}\mu_{\mathrm{Haar}} \equiv (\phi_1^{i_1 j_1}, \phi_2^{i_2 j_2})_{\mathrm{Haar}} = \frac{1}{\dim\phi_1}\, \delta_{i_1 i_2}\delta_{j_1 j_2}\delta_{\phi_1\phi_2}.$$

Here, $\delta_{\phi_1\phi_2} = 1$, if $\phi_1 \cong \phi_2$, and $\delta_{\phi_1\phi_2} = 0$ else.

Corollary 3.2 Under the assumptions of the preceding proposition we have

$$(\chi_{\phi_1}, \chi_{\phi_2})_{\mathrm{Haar}} = \delta_{\phi_1\phi_2}.$$

Theorem 3.3 Peter-Weyl Theorem [21]

1. a) $\mathcal{M}$ is a complete orthonormal system in $L^2(\mathbf{G})$.
   b) $\mathrm{span}_{\mathbb{C}}\mathcal{M}$ is dense in $C(\mathbf{G})$.
2. a) $\mathcal{M}_{\mathrm{Ad}}$ is a complete orthonormal system in $L^2_{\mathrm{Ad}}(\mathbf{G})$.
   b) $\mathrm{span}_{\mathbb{C}}\mathcal{M}_{\mathrm{Ad}}$ is dense in $C_{\mathrm{Ad}}(\mathbf{G})$.

[1] In order to assign the same matrix element to equivalent representations, we choose the bases on the vector spaces V "consistently". More precisely, we fix in every equivalence class $[\phi] \in \mathcal{D}(\mathbf{G})$ some representation ϕ and choose on the corresponding representation space V a basis B_ϕ. Now, for the other $\phi' \in [\phi]$ there is an isomorphism $A : V \longrightarrow V'$ with $\phi'(g) = A\phi(g)A^{-1}$. We choose $B_{\phi'} := AB_\phi$ as a basis on V'.

Here, $L^2_{\mathrm{Ad}}(\mathbf{G})$ contains precisely the conjugation invariant L^2-functions on $\mathbf{G}$. Analogously, $C_{\mathrm{Ad}}(\mathbf{G})$ collects the conjugation invariant continuous functions on $\mathbf{G}$.

Corollary 3.4 For all $f \in L^2(\mathbf{G})$ we have

$$f = \sum_{[\phi]\in\mathcal{D}(\mathbf{G})} \sum_{i,j=1}^{\dim\phi} \dim\phi\, (\phi^{ij}, f)_{\mathrm{Haar}} \phi^{ij}.$$

Analogously, for all $f \in L^2_{\mathrm{Ad}}(\mathbf{G})$ we have

$$f = \sum_{[\phi]\in\mathcal{D}(\mathbf{G})} (\chi_\phi, f)_{\mathrm{Haar}} \chi_\phi.$$

3.3 Laplace-Beltrami and Casimir Operator

Let $\{X_i\}$ be a basis of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. The left-invariant vector field on $\mathbf{G}$ corresponding to X_i is denoted by $\widetilde{X}_i$.

Definition 3.1 Let $A = A^{ij} \in \mathbb{R}^{\dim\mathbf{G},\dim\mathbf{G}}$ be some matrix. Then $\Delta_A := A^{ij}\widetilde{X}_i\widetilde{X}_j : C^n(\mathbf{G}) \longrightarrow C^{n-2}(\mathbf{G})$, $n \in \mathbb{N}$, $n \geq 2$, is called **Laplace-Beltrami operator** for A and $\{X_i\}$.

Let $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \oplus \mathfrak{g}_{\mathrm{ab}}$ be the splitting of the Lie algebra $\mathfrak{g}$ into its semisimple and abelian part. Using the Killing form κ on $\mathfrak{g}_{\mathrm{ss}}$ and the pseudo-Killing form λ on $\mathfrak{g}_{\mathrm{ab}}$ we define by $\varkappa((X_{\mathrm{ss}}, X_{\mathrm{ab}}), (Y_{\mathrm{ss}}, Y_{\mathrm{ab}})) := \kappa(X_{\mathrm{ss}}, Y_{\mathrm{ss}}) + \lambda(X_{\mathrm{ab}}, Y_{\mathrm{ab}})$ a non-degenerate, symmetric, negative-definite bilinear form $\varkappa$ on $\mathfrak{g}$ – the so-called **natural** bilinear form. Here, the pseudo-Killing form is defined by $\lambda(e_i, e_j) := -\delta_{ij}$, where $e_j = (0, \ldots, 0, \mathrm{i}, 0, \ldots, 0)$ with i on the j-th slot gives the canonical basis of $\mathfrak{g}_{\mathrm{ab}} \cong \mathrm{i}\mathbb{R} \oplus \ldots \oplus \mathrm{i}\mathbb{R}$.

Definition 3.2 $\Delta := \Delta_{\varkappa^{-1}}$ is called **Casimir operator** on $\mathbf{G}$, where the matrix $\varkappa$ is defined by $\varkappa_{ij} := \varkappa(X_i, X_j)$. [68]

One immediately sees that Δ does not depend on the choice of the basis $\{X_i\}$. Moreover, Δ is symmetric on $C^2(\mathbf{G}) \subseteq L^2(\mathbf{G})$. [68] Now, we have

Proposition 3.5 For every irreducible representation ϕ of $\mathbf{G}$ there is a non-negative real number c_ϕ, such that $\Delta\phi^{ij} = c_\phi\, \phi^{ij}$ for every elementary matrix function ϕ^{ij} of ϕ. [68]

c_ϕ is also called Casimir eigenvalue. From $\chi_\phi(g) = \mathrm{tr}\,\phi(g) = \sum_i \phi^{ii}(g)$ we get

Corollary 3.6 The character χ_ϕ of ϕ fulfills the eigenvalue equation $\Delta\chi_\phi = c_\phi\, \chi_\phi$.

We will frequently use the following properties of the Casimir eigenvalues:

Proposition 3.7

- $c_{\vec{\mathbf{n}}} = 0 \Longleftrightarrow \vec{\mathbf{n}} = \vec{0} \Longleftrightarrow \phi_{\vec{\mathbf{n}}}$ is trivial.
- There are positive real numbers c_- and c_+, such that
$$c_- \|\vec{\mathbf{n}}\|^2 \leq c_{\vec{\mathbf{n}}} \leq c_+ \|\vec{\mathbf{n}}\|^2$$
for all $\vec{\mathbf{n}} \equiv (\vec{n}, \vec{z}) \in \mathbb{N}^l \times \mathbb{Z}^k$, whereas $\|\vec{\mathbf{n}}\|^2 := \|\vec{n}\|^2 + \|\vec{z}\|^2$ gives the standard norm on $\mathbb{R}^{l+k}$.

Here, $c_{\vec{\mathbf{n}}}$ is simply the Casimir eigenvalue for the irreducible representation $\phi_{\vec{\mathbf{n}}}$.

3.4 Dimension of Representations

We estimate the dimension of irreducible representations.

Proposition 3.8 For every $\mathbf{G}$ there are positive constants $\mathrm{const}_{\mathbf{G}}$ and $\mathrm{const}'_{\mathbf{G}}$, such that we have for all $\vec{n} \in \mathbb{N}^l$ and $\vec{z} \in \mathbb{Z}^k$:
1. $d_{\vec{n},\vec{z}} \leq \mathrm{const}'_{\mathbf{G}} \, (\|\vec{n}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)} + 1)$,
2. $d_{\vec{n},\vec{z}} \leq \mathrm{const}_{\mathbf{G}} \, \|\vec{n}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)}$ if $\vec{n} \neq \vec{0}$ and
3. $d_{\vec{n},\vec{z}} \leq \mathrm{const}_{\mathbf{G}} \, \|\vec{n},\vec{z}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)}$ if $(\vec{n},\vec{z}) \neq (\vec{0},\vec{0})$.

Here, $\dim \mathbf{G}_{\mathrm{ss}}$ equals the dimension of the semisimple part $\mathfrak{g}_{\mathrm{ss}}$ of $\mathfrak{g}$.

Proof
- By the Weyl formula [21, 27] the dimension $d_{\vec{n},\vec{z}}$ of the irreducible representation $\phi_{\vec{n},\vec{z}}$ equals
$$d_{\vec{n},\vec{z}} := \dim \phi_{\vec{n},\vec{z}} = \prod_{\alpha \in \Sigma^+} \frac{(\alpha, \Lambda_{\vec{n}} + \Lambda_{\vec{r}})^\kappa}{(\alpha, \Lambda_{\vec{r}})^\kappa}.$$
Here, Σ^+ denotes the system of positive roots of $\mathfrak{g}_{\mathrm{ss}}$, $\Lambda_{\vec{n}}$ is the highest weight of $\phi_{\vec{n}}$, $\Lambda_{\vec{r}} := \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$ is the so-called Weyl vector and $(\cdot,\cdot)^\kappa$ denotes the symmetric non-degenerate bilinear form on the root space induced by the Killing form. Using the standard properties of $(\cdot,\cdot)^\kappa$ we get
$$(\alpha, \Lambda_{\vec{n}} + \Lambda_{\vec{r}})^\kappa \quad \leq \quad \|\alpha\|^\kappa c'_+ (\|\vec{n}\| + \|\vec{r}\|),$$
for some constant c'_+, hence
$$d_{\vec{n},\vec{z}} = \prod_{\alpha \in \Sigma^+} \frac{(\alpha, \Lambda_{\vec{n}} + \Lambda_{\vec{r}})^\kappa}{(\alpha, \Lambda_{\vec{r}})^\kappa} \leq \Big(\prod_{\alpha \in \Sigma^+} c'_+ \frac{\|\alpha\|^\kappa}{(\alpha, \Lambda_{\vec{r}})^\kappa} \Big) \big(\|\vec{n}\| + \|\vec{r}\|\big)^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)}$$
because of $\#\Sigma^+ = \frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)$.
- Since for arbitrary $d \in \mathbb{R}$ and $s \in \mathbb{N}$ the function $x \longmapsto \frac{(x+d)^s}{x^s+1}$ is bounded on $[0,\infty)$, we have
$$\begin{aligned} d_{\vec{n},\vec{z}} \quad &\leq \quad \Big(\prod_{\alpha \in \Sigma^+} c'_+ \frac{\|\alpha\|^\kappa}{(\alpha, \Lambda_{\vec{r}})^\kappa} \Big) \frac{\big(\|\vec{n}\| + \|\vec{r}\|\big)^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)}}{\|\vec{n}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)} + 1} \big(\|\vec{n}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)} + 1\big) \\ &\leq \quad \mathrm{const}'_{\mathbf{G}} \big(\|\vec{n}\|^{\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l)} + 1\big) \end{aligned}$$
for all $\vec{n} \in \mathbb{N}^l$ and $\vec{z} \in \mathbb{Z}^k$.
- The remaining cases are proven analogously. **qed**

3.5 Asymptotic Behaviour of Fourier Coefficients

As we know from Corollary 3.4, the Fourier series $\sum_{[\phi]\in\mathcal{D}(\mathbf{G})} \sum_{i,j=1}^{\dim\phi} \dim\phi \; (\phi^{ij}, f)_{\mathrm{Haar}} \phi^{ij}$ of an arbitrary function $f \in L^2(\mathbf{G})$ converges to f in the L^2-sense. For studying when this series even converges in the space of continuous functions, i.e. uniformly, we need estimates about the asymptotic behaviour of the Fourier coefficients.

Proposition 3.9 Let $f \in C^{2s}(\mathbf{G})$ be a $2s$-times continuously differentiable function on $\mathbf{G}$. Then we have for all nontrivial irreducible representations ϕ and for all $i, j = 1, \ldots, \dim\phi$
$$|(\phi^{ij}, f)_{\mathrm{Haar}}| \leq \frac{1}{\sqrt{\dim\phi}} \frac{\mathrm{const}_{s,f}}{c_\phi^s}$$
and
$$|(\chi_\phi, f)_{\mathrm{Haar}}| \leq \frac{\mathrm{const}_{s,f}}{c_\phi^s}.$$
Here, $\mathrm{const}_{s,f} := \|\Delta^s f\|_{\mathrm{Haar}} < \infty$ does *not* depend on ϕ, but only on s and f.

Proof
- Let ϕ be some nontrivial representation. By Proposition 3.7 the eigenvalue c_ϕ of the Casimir operator is positive. Hence

$$\begin{aligned}(\phi^{ij}, f)_{\text{Haar}} &= c_\phi^{-s}\,(\Delta^s \phi^{ij}, f)_{\text{Haar}} && \text{(Proposition 3.5)}\\ &= c_\phi^{-s}\,(\phi^{ij}, \Delta^s f)_{\text{Haar}} && \text{(Symmetry of }\Delta).\end{aligned}$$

  Using the Schwarz inequality and $\|\phi^{ij}\|_{\text{Haar}} = (\dim\phi)^{-\frac{1}{2}}$ (cf. Proposition 3.1) we get

$$\begin{aligned}|(\phi^{ij}, f)_{\text{Haar}}| &= c_\phi^{-s}\,|(\phi^{ij}, \Delta^s f)_{\text{Haar}}|\\ &\leq c_\phi^{-s}\,\|\phi^{ij}\|_{\text{Haar}}\,\|\Delta^s f\|_{\text{Haar}}\\ &= \text{const}_{s,f}\,(\dim\phi)^{-\frac{1}{2}}\,c_\phi^{-s}\end{aligned}$$

  with $\text{const}_{s,f} = \|\Delta^s f\|_{\text{Haar}} < \infty$.
- The proof for the characters is completely analogous. Note only $\|\chi_\phi\|_{\text{Haar}} = 1$.

qed

3.6 Convergence Criterion for Fourier Series

For the proof of the uniform convergence we need the following lemmata:

Lemma 3.10 Let X be a metric space, Y a Banach space over $\mathbb{K}$ and let $f_\nu \in C(X,Y)$ for all $\nu \in \mathbb{N}$. Then we have:
If $\sum_{\nu\in\mathbb{N}} \|f_\nu\|_\infty$ converges, then $\sum_{\nu\in\mathbb{N}} f_\nu$ converges absolutely and uniformly on X to some $f \in C(X,Y)$.

Lemma 3.11 For all $\mu,\nu \in \mathbb{R}$ with $\mu \geq 0$ and $\frac{1}{2}(\dim \mathbf{G}_{\text{ss}} - l)\mu + 2\nu \leq -(k+l+1)$,

$$\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} d_{\vec{\mathbf{n}}}^{\mu}\, c_{\vec{\mathbf{n}}}^{\nu}$$

converges.

Proof By Proposition 3.7 and Proposition 3.8 we have for $\nu \geq 0$

$$\begin{aligned}\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} d_{\vec{\mathbf{n}}}^{\mu}\, c_{\vec{\mathbf{n}}}^{\nu} &\leq \sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} d_{\vec{\mathbf{n}}}^{\mu}\, c_{\vec{\mathbf{n}}}^{\nu}\\ &\leq \sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} \text{const}_{\mathbf{G}}^{\mu}\|\vec{\mathbf{n}}\|^{\frac{1}{2}(\dim\mathbf{G}_{\text{ss}}-l)\mu} c_+^{\nu}\|\vec{\mathbf{n}}\|^{2\nu}\\ &\leq \text{const}_{\mathbf{G}}^{\mu} c_+^{\nu} \sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} \|\vec{\mathbf{n}}\|^{-(k+l+1)}.\end{aligned}$$

In the last step the assumption $\frac{1}{2}(\dim \mathbf{G}_{\text{ss}} - l)\mu + 2\nu \leq -(k+l+1)$ and $\|\vec{\mathbf{n}}\| \geq 1$ have been used. The convergence is implied by Corollary A.4.
For $\nu < 0$ the argumentation is completely analogous. Just replace c_+ by the constant c_-.

qed

Proposition 3.12 Let $f \in C^{2s}(\mathbf{G})$ be a $2s$-times continuously differentiable function on $\mathbf{G}$ with $2s \geq \dim \mathbf{G} + 1$. Then the Fourier series

$$\sum_{[\phi]\in\mathcal{D}(\mathbf{G})} \sum_{i,j=1}^{\dim\phi} \dim\phi\,(\phi^{ij}, f)_{\text{Haar}}\phi^{ij}$$

of f converges absolutely and uniformly to f.[2]

[2] There is an even stronger result: Taylor [70] proved using Sobolev techniques that the Fourier series converges for every $f \in C^{2s}(\mathbf{G})$ if $s \in \mathbb{N}$ is larger than $\frac{1}{4}\dim\mathbf{G}$ (cf. the review [58]).

Remark Even

$$\sum_{[\phi]\in\mathcal{D}(\mathbf{G})}\sum_{i,j=1}^{\dim\phi}\dim\phi\,(\phi^{ij},f)_{\text{Haar}}D\phi^{ij}$$

converges absolutely and uniformly to Df for all smooth differential operators D, whose order is not larger than $2s-\dim\mathbf{G}+1$.

We remember that instead of $\sum_{[\phi]\in\mathcal{D}(\mathbf{G})}$ we can simply write $\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})\subseteq\mathbb{N}^l\times\mathbb{Z}^k}$ and replace ϕ correspondingly by $\phi_{\vec{\mathbf{n}}}$ or just $\vec{\mathbf{n}}$.

Proof First we show that $\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})\subseteq\mathbb{N}^l\times\mathbb{Z}^k}\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}} d_{\vec{\mathbf{n}}}(\phi^{ij}_{\vec{\mathbf{n}}},f)_{\text{Haar}}\phi^{ij}_{\vec{\mathbf{n}}}$ converges uniformly and absolutely.

- By the Schwarz inequality and the unitarity of $\phi_{\vec{\mathbf{n}}}$ we have $\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}}|\phi^{ij}_{\vec{\mathbf{n}}}(g)|\leq\sqrt{\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}}1}\sqrt{\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}}|\phi^{ij}_{\vec{\mathbf{n}}}(g)|^2}=d_{\vec{\mathbf{n}}}\sqrt{\operatorname{tr}\,\phi^+_{\vec{\mathbf{n}}}\phi_{\vec{\mathbf{n}}}}=d_{\vec{\mathbf{n}}}^{\frac{3}{2}}$ for all $g\in\mathbf{G}$. Hence by the asymptotics of the Fourier coefficients we get for $\vec{\mathbf{n}}\neq\vec{0}$

$$\begin{aligned}\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}}\|d_{\vec{\mathbf{n}}}(\phi^{ij}_{\vec{\mathbf{n}}},f)_{\text{Haar}}\phi^{ij}_{\vec{\mathbf{n}}}\|_\infty &= \sum_{i,j=1}^{d_{\vec{\mathbf{n}}}} d_{\vec{\mathbf{n}}}\,|(\phi^{ij}_{\vec{\mathbf{n}}},f)_{\text{Haar}}|\;\|\phi^{ij}_{\vec{\mathbf{n}}}\|_\infty\\ &\leq d_{\vec{\mathbf{n}}}\,d_{\vec{\mathbf{n}}}^{-\frac{1}{2}}\,\text{const}_{s,f}c_{\vec{\mathbf{n}}}^{-s}\,d_{\vec{\mathbf{n}}}^{\frac{3}{2}}\\ &= \text{const}_{s,f}\,d_{\vec{\mathbf{n}}}^2\,c_{\vec{\mathbf{n}}}^{-s}.\end{aligned}$$

  Moreover, $\|d_{\vec{0}}(\phi^{11}_{\vec{0}},f)_{\text{Haar}}\phi^{11}_{\vec{0}}\|_\infty\leq\|f\|_{\text{Haar}}<\infty$.
  Now, $\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})}\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}}\|d_{\vec{\mathbf{n}}}(\phi^{ij}_{\vec{\mathbf{n}}},f)_{\text{Haar}}\phi^{ij}_{\vec{\mathbf{n}}}\|_\infty$ converges by Lemma 3.11 and by $\frac{1}{2}(\dim\mathbf{G}_{\text{ss}}-l)\cdot 2+2\cdot(-s)\leq\dim\mathbf{G}_{\text{ss}}-l-\dim\mathbf{G}-1=-(k+l+1)$.
- By Lemma 3.10, $\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})\subseteq\mathbb{N}^l\times\mathbb{Z}^k}\sum_{i,j=1}^{d_{\vec{\mathbf{n}}}} d_{\vec{\mathbf{n}}}(\phi^{ij}_{\vec{\mathbf{n}}},f)_{\text{Haar}}\phi^{ij}_{\vec{\mathbf{n}}}$ converges absolutely and uniformly to some $\widehat{f}\in C(\mathbf{G})$.

By the Peter-Weyl theorem $\widehat{f}$ and f coincide as L^2-functions. The continuity yields $\widehat{f}\equiv f$. **qed**

Corollary 3.13 Let $f\in C^{2s}_{\text{Ad}}(\mathbf{G})$ be a $2s$-times continuously differentiable conjugation invariant function on $\mathbf{G}$ and let $2s\geq\dim\mathbf{G}+1$. Then

$$\sum_{[\phi]\in\mathcal{D}(\mathbf{G})}(\chi_\phi,f)_{\text{Haar}}\chi_\phi$$

converges absolutely and uniformly to f.

The proof is straightforward.

3.7 Fourier Series on $\mathbf{G}^n$ and $\mathbf{G}^n/\text{Ad}$

Later on we are mostly concerned not with functions on $\mathbf{G}$, but on $\mathbf{G}^n$. To treat them we need complete orthonormal systems on $\mathbf{G}^n$ and $\mathbf{G}^n/\text{Ad}$. The first case is simple; one gets such systems due to $L^2(\mathbf{G}^n)=\bigotimes^n L^2(\mathbf{G})$ by tensoring orthonormal bases on $L^2(\mathbf{G})$. Ad-invariant functions are more complicated, since not every Ad-invariant function on $\mathbf{G}^n$ can be written as a tensor product of Ad-invariant functions on $\mathbf{G}$. The solution of this problem comes from the theory of the so-called loop-network states introduced by Thiemann [75].

Let $n\in\mathbb{N}_+$ be fixed and denote by $\vec{\phi}\in\mathcal{D}(\mathbf{G})^n$ some n-tuple of irreducible representations. Analogously, $\vec{\imath}$ and $\vec{\jmath}$ are n-tuples of natural numbers. We will call the functions

$$T^{\vec{\imath}\vec{\jmath}}_{\vec{\phi}}\equiv T^{i_1\ldots i_n\,j_1\ldots j_n}_{\phi_1\ldots\phi_n}:=\bigotimes_{\nu=1}^n\sqrt{\dim\phi_\nu}\;\phi_\nu^{i_\nu j_\nu}:\mathbf{G}^n\longrightarrow\mathbb{C}$$

with $i_\nu,j_\nu=1,\ldots,\dim\phi_\nu$ **elementary n-matrix functions** and the Ad-invariant functions

$$T_{\vec{\phi},\phi} := \frac{1}{\sqrt{\dim \phi}} \sum_{\vec{\imath},\vec{\jmath}} T_{\vec{\phi}}^{\vec{\imath}\vec{\jmath}} \mathcal{C}_{\vec{\phi},\phi}^{\vec{\jmath}\vec{\imath}} : \mathbf{G}^n \longrightarrow \mathbb{C}$$

n-**characters**. Here, ϕ is an irreducible representation contained in $\bigotimes_\nu \phi_\nu$ and $\mathcal{C}_{\vec{\phi},\phi} : \bigotimes V_{\phi_\nu} \longrightarrow V_\phi \subseteq \bigotimes V_{\phi_\nu}$ is the corresponding projection matrix. The set of all elementary n-matrix functions is denoted by $\mathcal{M}^n$, that of n-characters by $\mathcal{M}^n_{\mathrm{Ad}}$.

Now we have the generalized Peter-Weyl theorem

Theorem 3.14
1. a) $\mathcal{M}^n$ is a complete orthonormal system in $L^2(\mathbf{G}^n)$.
   b) $\mathrm{span}_{\mathbb{C}} \mathcal{M}^n$ is dense in $C(\mathbf{G}^n)$.
2. a) $\mathcal{M}^n_{\mathrm{Ad}}$ is a complete orthonormal system in $L^2_{\mathrm{Ad}}(\mathbf{G}^n)$.
   b) $\mathrm{span}_{\mathbb{C}} \mathcal{M}^n_{\mathrm{Ad}}$ is dense in $C_{\mathrm{Ad}}(\mathbf{G}^n)$.

The proof is not very difficult, but quite technical and is therefore skipped here. It can be found in [39].

4 Determination of the Yang-Mills Measure

In this section we review the definition of the measure μ_{YM} for the two-dimensional quantum Yang-Mills theory, first proposed by Thiemann [73] and Ashtekar et al. [13] for loops in a quadratic lattice and later extended to the general case [36]. However, in the last reference only Wilson loops have been used for the calculation. Although this is sufficient for unitary $\mathbf{G}$, for arbitrary groups we have to resort to the loop networks of Thiemann [75] that will be introduced in a slightly modified version in the next subsection. Afterwards we describe the chosen regularization and quote the basic results from [36] about flag worlds. Next we review the definition of the Yang-Mills measure in a formulation that (after proving the existence of a certain limit) can directly be reused for other models. Finally, the necessary expectation values of the Yang-Mills measure are given.

From now on M equals[3] $\mathbb{R}^2$ and we restrict ourselves to the case of piecewise analytic paths. Moreover, until the end of Section 6 we mean by "graphs" always connected simple graphs, i.e. graphs whose interior domains are bounded by Jordan curves only. This is not a severe restriction, since every graph can be refined to such a graph [36, 35]. Simple domains are just domains enclosed by Jordan curves. Finally we denote the Ashtekar-Lewandowski measure pushed forward by the homeomorphism $\phi : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}$ from $\overline{\mathcal{A}/\mathcal{G}}$ to $\overline{\mathcal{A}/\mathcal{G}}$ again by μ_0.

4.1 Loop-Network States

For the determination of a regular Borel measure on a compact Hausdorff space X it is by the Riesz-Markov theorem sufficient to define a positive, linear and continuous functional F on $C(X)$. Since even this is quite difficult in general, one only determines the restriction of F to some (if possible, easily controllable) dense subset D of $C(X)$ and gets F then by continuous extension. In the case of $X = \overline{\mathcal{A}/\mathcal{G}}$ one typically [13, 36] chooses for D the so-called holonomy algebra $\mathcal{HA}$ generated by the Wilson loops $T_\alpha = \mathrm{tr}\, h_\alpha : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \mathbb{C}$. However, here we will use the loop-network states introduced by Thiemann [75, 73]. Those indeed span (in a certain interpretation) a dense subalgebra in $C(\overline{\mathcal{A}/\mathcal{G}})$ for *every* $\mathbf{G}$. For the holonomy algebra such a result is only known for $SU(N)$, $U(N)$, $SO(2N+1)$ and $O(N)$.

[3] Our considerations can quite easily be transferred to the case of an arbitrary (compact) Riemannian surface. In the classical approach this has been performed by Fine [32], Witten [82] and Sengupta [64, 66]. Within the Ashtekar approach Ashtekar et al. [13] were able to compute at least certain expectation values for $M = S^2$ or – with $\mathbf{G} = U(1)$ – for $M = S^1 \times S^1$ as well. The general case has been discussed by Aroca and Kubyshin [1, 2].

For connected structure groups this problem has not been solved; for non-connected ones there are counterexamples mentioned in the paper of Sengupta [65] based on investigations of Burnside.

Before we come to the loop-network states, we recall the definition of cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$.

Definition 4.1 A function $f \in C(\overline{\mathcal{A}/\mathcal{G}})$ is called **cylindrical function**, if there is a graph Γ and a function $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ such that $f = f_\Gamma \circ \pi_\Gamma$.
The set of all cylindrical functions is denoted by $\mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$.

Since we deal here with piecewise analytic graphs only, we have [6]

Lemma 4.1 $\mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$ is a dense $*$-subalgebra in $C(\overline{\mathcal{A}/\mathcal{G}})$.

Definition 4.2
- The triple $(\boldsymbol{\alpha}, \vec{\phi}, \phi)$ is called **loop-network** iff
  - there is a connected graph Γ with $m \in \mathbf{V}(\Gamma)$ which $\boldsymbol{\alpha}$ is a weak fundamental system for,
  - $\vec{\phi} = (\phi_1, \ldots, \phi_{\#\boldsymbol{\alpha}})$ is a $\#\boldsymbol{\alpha}$-tuple of (equivalence classes of) irreducible representations of $\mathbf{G}$ and
  - ϕ is some irreducible representation contained in $\bigotimes_{i=1}^{\#\boldsymbol{\alpha}} \phi_i$.
- Every loop-network $(\boldsymbol{\alpha}, \vec{\phi}, \phi)$ is assigned a function
$$T_{(\boldsymbol{\alpha},\vec{\phi},\phi)} := T_{\vec{\phi},\phi} \circ \pi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbb{C},$$
where $T_{\vec{\phi},\phi} : \mathbf{G}^{\#\boldsymbol{\alpha}} \longrightarrow \mathbb{C}$ is the $\#\boldsymbol{\alpha}$-character to $(\vec{\phi}, \phi)$.
$T_{(\boldsymbol{\alpha},\vec{\phi},\phi)}$ is called **loop-network state**.

We note that our definition makes that of Thiemann [75] a bit more general because here the impact of the choice of the generating system is taken into account.

Lemma 4.2 Let $(\boldsymbol{\alpha}, \vec{\phi}, \phi)$ be a loop-network and Γ be the graph spanned by $\boldsymbol{\alpha}$. Then there is a unique continuous function $t_{(\boldsymbol{\alpha},\vec{\phi},\phi)} \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ with
$$t_{(\boldsymbol{\alpha},\vec{\phi},\phi)} \circ \pi_\Gamma \circ \boldsymbol{\phi} \circ \pi = T_{(\boldsymbol{\alpha},\vec{\phi},\phi)}.$$

Proof We set
$$t_{(\boldsymbol{\alpha},\vec{\phi},\phi)} := (\iota_{\boldsymbol{\alpha}}^\Gamma)^* \bigl((\pi_{\mathrm{Ad}}^*)^{-1} T_{\vec{\phi},\phi}\bigr),$$
whereas π_{Ad} denotes the canonical projection from $\mathbf{G}^n$ to $\mathbf{G}^n/\mathrm{Ad}$ and $\iota_{\boldsymbol{\alpha}}^\Gamma$ is the homeomorphism $[h] \longmapsto [h(\boldsymbol{\alpha})]_{\mathrm{Ad}}$ between $\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma$ and $\mathbf{G}^{\dim \pi_1(\Gamma)}/\mathrm{Ad}$. $t_{(\boldsymbol{\alpha},\vec{\phi},\phi)}$ is well-defined by the Ad-invariance of $T_{\vec{\phi},\phi}$ and we have
$$\begin{aligned} t_{(\boldsymbol{\alpha},\vec{\phi},\phi)} \circ \pi_\Gamma \circ \boldsymbol{\phi} \circ \pi &= \bigl((\pi_{\mathrm{Ad}}^*)^{-1} T_{\vec{\phi},\phi}\bigr) \circ \iota_{\boldsymbol{\alpha}}^\Gamma \circ \pi_\Gamma \circ \boldsymbol{\phi} \circ \pi \\ &= \bigl((\pi_{\mathrm{Ad}}^*)^{-1} T_{\vec{\phi},\phi}\bigr) \circ \iota_{\boldsymbol{\alpha}}^\Gamma \circ \pi_{\overline{\mathcal{G}}_\Gamma} \circ \pi_\Gamma \\ &= \bigl((\pi_{\mathrm{Ad}}^*)^{-1} T_{\vec{\phi},\phi}\bigr) \circ \pi_{\mathrm{Ad}} \circ \pi_{\boldsymbol{\alpha}} \;=\; T_{\vec{\phi},\phi} \circ \pi_{\boldsymbol{\alpha}} \;=\; T_{(\boldsymbol{\alpha},\vec{\phi},\phi)}. \end{aligned}$$
Here, $\pi_{\overline{\mathcal{G}}_\Gamma}$ is the canonical projection $\overline{\mathcal{A}}_\Gamma \longrightarrow \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma$. **qed**

Sometimes we call $t_{(\boldsymbol{\alpha},\vec{\phi},\phi)}$ loop-network state as well.

Definition 4.3 The set of all functions $t_{(\boldsymbol{\alpha},\vec{\phi},\phi)}$ belonging to some $\boldsymbol{\alpha}$ is denoted by $\mathcal{L}_{\boldsymbol{\alpha}}$.

Proposition 4.3 Let Γ be a graph and $\boldsymbol{\alpha}$ be some of its weak fundamental systems. Then $\mathrm{span}_{\mathbb{C}} \mathcal{L}_{\boldsymbol{\alpha}}$ is dense in $C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$.

Proof Let $n := \#\boldsymbol{\alpha}$. By the generalized Peter-Weyl theorem (Theorem 3.14) the set $\mathcal{M}^n_{\mathrm{Ad}}$ of all n-characters $T_{\vec{\phi},\phi}$ spans a dense subspace in the space of all Ad-invariant continuous functions in $C(\mathbf{G}^n)$. The assertion now comes from the fact that $(\pi^*_{\mathrm{Ad}})^{-1} : C_{\mathrm{Ad}}(\mathbf{G}^n) \longrightarrow C(\mathbf{G}^n/\mathrm{Ad})$ and $(\iota^\Gamma_{\boldsymbol{\alpha}})^* : C(\mathbf{G}^n/\mathrm{Ad}) \longrightarrow C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ are norm-preserving isomorphisms. **qed**

Proposition 4.4 Choose for every graph Γ some weak generating system $\boldsymbol{\alpha}(\Gamma)$ and put

$$\mathcal{L} := \bigcup_\Gamma \pi^*_\Gamma(\mathcal{L}_{\boldsymbol{\alpha}(\Gamma)}).$$

Then $\mathrm{span}_{\mathbb{C}}\mathcal{L}$ is dense in $C(\overline{\mathcal{A}/\mathcal{G}})$.

Proof Let $f \in C(\overline{\mathcal{A}/\mathcal{G}})$. By the denseness of $\mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$ in $C(\overline{\mathcal{A}/\mathcal{G}})$ (see Lemma 4.1) there is a graph Γ and an $f''_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ with $\|f''_\Gamma \circ \pi_\Gamma - f\|_\infty < \frac{1}{2}\varepsilon$. By Proposition 4.3 there is some $f'_\Gamma \in \mathrm{span}_{\mathbb{C}}\mathcal{L}_{\boldsymbol{\alpha}(\Gamma)}$ with $\|f'_\Gamma - f''_\Gamma\|_\infty < \frac{1}{2}\varepsilon$, hence with $\|f - f'_\Gamma \circ \pi_\Gamma\|_\infty \leq \|f - f''_\Gamma \circ \pi_\Gamma\|_\infty + \|f''_\Gamma \circ \pi_\Gamma - f'_\Gamma \circ \pi_\Gamma\|_\infty \leq \|f - f''_\Gamma \circ \pi_\Gamma\|_\infty + \|f''_\Gamma - f'_\Gamma\|_\infty < \varepsilon$. **qed**

Remark In contrast to [75] we do not claim that the set $\mathcal{L}$ (after removing all loop-network states being pull-backs of other ones) is an orthonormal system for $L^2(\overline{\mathcal{A}/\mathcal{G}})$.

By means of the proposition above every regular Borel measure on $\overline{\mathcal{A}/\mathcal{G}}$ can be reconstructed from the corresponding expectation values of loop-network states. However, this does not mean that every assignment of real numbers to loop-networks indeed corresponds to expectation values of some measure.

4.2 Continuum Limit and Regularization of the Yang-Mills Action

First we define our version of the continuum limit $\Gamma \to \mathbb{R}^2$. (Actually, this definition includes both the continuum limit and the thermodynamical limit.)

Definition 4.4
1. We say that a sequence (Γ_n) of graphs converges to $\mathbb{R}^2$ (shortly $\Gamma_n \to \mathbb{R}^2$) iff for $n \to \infty$
   - the supremum of the diameters of all interior domains of Γ_n goes to zero and
   - the supremum of the diameters of all circles in $\mathbb{R}^2$ having center m and being disjoint to the exterior domain of Γ_n goes to infinity.
2. Let Γ be a graph and $z_{\Gamma'} \in \mathbb{C}$ for all $\Gamma' \geq \Gamma$.
   We say $\lim_{\Gamma \leq \Gamma' \to \mathbb{R}^2} z_{\Gamma'} = z$ iff $\lim_{n\to\infty} z_{\Gamma_n} = z$ for all sequences (Γ_n) of graphs with $\Gamma_n \geq \Gamma$ and $\Gamma_n \to \mathbb{R}^2$.

Now we come to the regularization of the Yang-Mills action. Neglecting convergence problems we have for all sufficiently regular partitions $\{G_\alpha\}$ of $\mathbb{R}^2$:

$$\begin{aligned} S_{\mathrm{YM}}(A) &= \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}\, \mathrm{d}x \\ &= \frac{1}{2}\sum_\alpha \int_{G_\alpha} \mathrm{tr}\, F_{12}F^{12}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &\approx \frac{1}{2}\sum_\alpha |G_\alpha|\mathrm{tr}\, \frac{(h_A(\alpha) - \mathbf{1})^2}{|G_\alpha|^2} \qquad \text{(Proposition B.1)} \\ &\approx \sum_\alpha \frac{N}{|G_\alpha|}\big(1 - \frac{1}{N}\mathrm{Re}\ \mathrm{tr}\, h_A(\alpha)\big) \\ &\qquad (\mathrm{tr}(g-\mathbf{1})^2 \approx -2\mathrm{Re}\ \mathrm{tr}(g-\mathbf{1}) \text{ for small } \|g-\mathbf{1}\|_\bullet). \end{aligned}$$

Including the coupling constant $\mathrm{g} \in (0,\infty)$ of the theory we have (cf. [81, 13, 36, 1, 2])

Definition 4.5 For every graph Γ the **regularized Yang-Mills action** $S_{\text{YM},\Gamma} : \overline{\mathcal{A}} \longrightarrow \mathbb{R}$ is defined by

$$S_{\text{YM},\Gamma}(\overline{A}) = \sum_{G_I \in L_{\text{innen}}(\Gamma)} \frac{N}{\text{g}^2} \frac{1}{|G_I|} \Big(1 - \frac{1}{N} \text{Re tr } h_{\overline{A}}(\alpha_I)\Big).$$

Here, α_I is some boundary loop of the domain G_I in Γ, N is the natural number given by the embedding $\mathbf{G} \subseteq U(N)$ and $L_{\text{innen}}(\Gamma)$ collects the interior domains of Γ.

Obviously, $S_{\text{YM},\Gamma}$ is well-defined and a gauge-invariant function. This way, we get by $(\pi^*)^{-1} S_{\text{YM},\Gamma} \circ \boldsymbol{\phi}^{-1} : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \mathbb{R}$ a regularized Yang-Mills action on $\overline{\mathcal{A}/\mathcal{G}}$ that will be denoted by $S_{\text{YM},\Gamma}$ as well.

One could now try to define via $S_{\text{YM,reg}}(\overline{A}) := \lim_{m \leq \Gamma \to \mathbb{R}^2} S_{\text{YM},\Gamma}(\overline{A})$ a generalized Yang-Mills action on $\overline{\mathcal{A}}$. Although then – as indicated above – we would have $S_{\text{YM,reg}}(A) = S_{\text{YM}}(A)$ for smooth connections (if necessary under additional regularity assumptions); for generalized connections the limit $S_{\text{YM,reg}}$ however does typically *not* exist. It is easy to see that $S_{\text{YM},\Gamma}(\overline{A})$ still converges in $[0, \infty]$ for every limiting process $\Gamma_n \to \mathbb{R}^2$ where Γ_{n+1} is a refinement of Γ_n, but this limit depends crucially on the considered limiting process. There are even connections that yield sometimes 0 and sometimes ∞ in the limit [36].

4.3 Flag Worlds

In this section we recall the most important facts about the flag worlds and adapt them to our formulation. Their introduction has been necessary because the Wilson regularization has been extended from quadratic to floating lattices. For a detailed discussion we refer to [36, 39].

Lemma 4.5 For every simple domain G in Γ and every $v \in \mathbf{V}(\Gamma) \cap \partial G$ there is a unique loop $\alpha_{G,v}$ in Γ with base point v and without self-intersections, such that

- im $\alpha_{G,v}$ equals the boundary ∂G of G and
- $\alpha_{G,v}$ surrounds G counterclockwise.

We call $\alpha_{G,v}$ boundary loop of G with base point v.

Definition 4.6 Let G be a simple domain in a graph Γ.
A closed path $f_{G,v}$ in Γ is called **flag** for the domain G iff

- $f = \gamma_{mv} \alpha_{G,v} \gamma_{mv}^{-1}$,
- $\alpha_{G,v}$ is the boundary loop of G with base point v and
- γ_{mv} is a path in $\Gamma \setminus \Gamma_G$ from m to v.

Here, Γ_G denotes the subgraph of Γ built by the boundary of G. If we later say a path to be a flag, we will simply assume the existence of some graph Γ that contains this path as a flag. For instance, every flag itself spans a graph with exactly one interior domain. One easily sees that for every simple domain G in Γ and every $v \in \mathbf{V}(\Gamma_G)$ there is a flag $f_{G,v}$. Additionally, two flags in a simple graph Γ are called non-overlapping iff the domains enclosed by them are disjoint.

The most important examples of sets of non-overlapping flags are the flag worlds.

Definition 4.7 Let Γ be a simple graph.

- A set $\mathcal{F}$ of flags in Γ is called **flag world** for Γ iff $\mathcal{F} = \{f_G \mid G \text{ interior domain of } \Gamma\}$, where f_G in each case is some flag enclosing the (simple) interior domain G.
- A flag world of Γ is called **moderately independent** iff it is a weak fundamental system for Γ.

Proposition 4.6 In every simple graph there is a moderately independent flag world.

For the investigation of the limiting process in the calculation of the Wilson-loop expectation values that runs over all refinements of a given graph, we have to study the behaviour of flag worlds under refining the graph under consideration.

Definition 4.8 Let Γ and Γ' be simple graphs with $\Gamma' \geq \Gamma$ and $\mathcal{F}$ and $\mathcal{F}'$ be flag worlds for Γ and Γ', respectively.
$\mathcal{F}'$ is called **refinement** of $\mathcal{F}$ iff for every interior domain G_I of Γ the flag $f_I \in \mathcal{F}$ corresponding to G_I can be written as a product $f_{I,1} \cdots f_{I,\lambda_I}$ of just those flags $f_{I,i_I} \in \mathcal{F}'$ that belong to the interior domains $G_{I,i_I} \subseteq G_I$ of Γ'.

Obviously the refinement relation is transitive.

Proposition 4.7 Let Γ and Γ' be simple graphs with $\Gamma' \geq \Gamma$.
Then for every moderately independent flag world $\mathcal{F}$ of Γ there is a moderately independent flag world $\mathcal{F}'$ of Γ' being a refinement of $\mathcal{F}$.

The rather technical proof is analogous to that given in [36] for a similar proposition.

Finally we express the Yang-Mills action in terms of flag worlds.

Lemma 4.8 For every graph Γ and every moderately independent flag world $\mathcal{F}$ in Γ there is a unique continuous function $S^{\mathcal{F}}_{\mathrm{YM},\Gamma} : \mathbf{G}^{n(\Gamma)} \longrightarrow \mathbb{R}$ with $S^{\mathcal{F}}_{\mathrm{YM},\Gamma} \circ \pi_{\mathcal{F}} = S_{\mathrm{YM},\Gamma}$.
Then we have

$$S^{\mathcal{F}}_{\mathrm{YM},\Gamma}(\vec{g}) = \sum_{G_I \in L_{\mathrm{innen}}(\Gamma)} \frac{N}{\mathrm{g}^2} \frac{1}{|G_I|} \Bigl(1 - \frac{1}{N} \mathrm{Re}\ \mathrm{tr}\, g_I\Bigr).$$

$n(\Gamma)$ denotes the rank of the fundamental group of Γ.

4.4 Expectation Values

Since the limit $\lim_{m \leq \Gamma \to \mathbb{R}^2} S_{\mathrm{YM},\Gamma}$ on $\overline{\mathcal{A}/\mathcal{G}}$ is known to be not well-defined, a definition of the expectation values via

$$\frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{e}^{-\lim_{m \leq \Gamma \to \mathbb{R}^2} S_{\mathrm{YM},\Gamma}} f \, \mathrm{d}\mu_0$$

is not meaningful. Now, the trick of Thiemann was simply to exchange limit and integration. A priori it is unclear whether this is possible, in particular, because of the customary non-existence of the limit $S_{\mathrm{YM,reg}}$. However, just this (mathematically not justifiable) operation enables the calculation of the expectation values. Moreover, the results become completely independent of the choice of the limiting process.

Now we

1. set for all graphs Γ, all continuous functions $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ and all refinements Γ' of Γ

$$\mathbf{E}^{\Gamma}_{\Gamma'}(f_\Gamma) \;:=\; \frac{\int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{e}^{-S_{\mathrm{YM},\Gamma'}} f_\Gamma \circ \pi_\Gamma \, \mathrm{d}\mu_0}{\int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{e}^{-S_{\mathrm{YM},\Gamma'}} \, \mathrm{d}\mu_0}, \tag{1}$$

2. set for all graphs Γ and all continuous functions $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$

$$\mathbf{E}^{\Gamma}(f_\Gamma) \;:=\; \lim_{\Gamma \leq \Gamma' \to \mathbb{R}^2} \mathbf{E}^{\Gamma}_{\Gamma'}(f_\Gamma), \tag{2}$$

3. set for all cylindrical functions $f \in \mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$

$$\mathbf{E}(f) \;:=\; \mathbf{E}^{\Gamma}(f_\Gamma), \tag{3}$$

where Γ is some graph and $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ some function fulfilling $f = f_\Gamma \circ \pi_\Gamma$, and

4. extend $\mathbf{E} : \mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}}) \longrightarrow \mathbb{C}$ to a functional

$$\mathbf{E} \ : \ C(\overline{\mathcal{A}/\mathcal{G}}) \longrightarrow \mathbb{C}. \tag{4}$$

Finally, this functional $\mathbf{E}$ determines the desired regular Borel measure μ_{YM} on $\overline{\mathcal{A}/\mathcal{G}}$ via the Riesz-Markov theorem. The most (and only) important step in this procedure is the proof of the following

Proposition 4.9 For every graph Γ, for every moderately independent flag world $\mathcal{F}$ in Γ and for every loop-network state $t_{(\mathcal{F},\vec{\phi},\phi)} \in \mathcal{L}_{\mathcal{F}}$ the limit (2) exists.

Namely, this proposition implies

Theorem 4.10 The functional $\mathbf{E} : C(\overline{\mathcal{A}/\mathcal{G}}) \longrightarrow \mathbb{C}$ is well-defined, linear, continuous and positive.
Hence there is a unique normalized regular Borel measure μ_{YM} on $\overline{\mathcal{A}/\mathcal{G}}$ with $\mathbf{E}(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} f \, \mathrm{d}\mu_{\mathrm{YM}}$ for all $f \in C(\overline{\mathcal{A}/\mathcal{G}})$.

Proof 1. Well-definedness of $\mathbf{E}^{\Gamma}$
The linearity of $\mathbf{E}^{\Gamma}_{\Gamma'}$ in (1) implies the existence of the limit $\mathbf{E}^{\Gamma}$ in (2) for all $f_\Gamma \in \mathrm{span}_{\mathbb{C}}\mathcal{L}_{\mathcal{F}}$.
By $|\mathbf{E}^{\Gamma}_{\Gamma'}(f_\Gamma)| \leq \|f_\Gamma \circ \pi_\Gamma\|_\infty \leq \|f_\Gamma\|_\infty$ for all $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ we have $|\mathbf{E}^{\Gamma}(f_\Gamma)| \leq \|f_\Gamma\|_\infty$ on $\mathrm{span}_{\mathbb{C}}\mathcal{L}_{\mathcal{F}}$. Hence $\mathbf{E}^{\Gamma}$ is a linear, continuous and (as $\mathbf{E}^{\Gamma}_{\Gamma'}$) positive functional on $\mathrm{span}_{\mathbb{C}}\mathcal{L}_{\mathcal{F}}$.
Since $\mathrm{span}_{\mathbb{C}}\mathcal{L}_{\mathcal{F}}$ is dense in $C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$ (Proposition 4.3), we can extend $\mathbf{E}^{\Gamma}$ continuously to a linear, continuous and positive functional $\overline{\mathbf{E}}^{\Gamma}$ on the whole $C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$. Moreover, obviously $\overline{\mathbf{E}}^{\Gamma}(f_\Gamma) = \lim_{\Gamma\leq\Gamma'\to\mathbb{R}^2} \mathbf{E}^{\Gamma}_{\Gamma'}(f_\Gamma) =: \mathbf{E}^{\Gamma}(f_\Gamma)$ for all $f_\Gamma \in C(\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma)$. Hence $\overline{\mathbf{E}}^{\Gamma} = \mathbf{E}^{\Gamma}$.

2. Well-definedness of $\mathbf{E}$ on $\mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$
Let $\Gamma \geq \Gamma_0$ and $f_{\Gamma_0} \in C(\overline{\mathcal{A}/\mathcal{G}}_{\Gamma_0})$. We have $\mathbf{E}^{\Gamma}_{\Gamma'}(f_{\Gamma_0} \circ \pi^{\Gamma}_{\Gamma_0}) = \mathbf{E}^{\Gamma_0}_{\Gamma'}(f_{\Gamma_0})$ for all $\Gamma' \geq \Gamma$ by (1) and $\pi_{\Gamma_0} = \pi^{\Gamma}_{\Gamma_0} \circ \pi_\Gamma$. Thus,

$$\begin{aligned}
\mathbf{E}^{\Gamma}(f_{\Gamma_0} \circ \pi^{\Gamma}_{\Gamma_0}) &= \lim_{\Gamma\leq\Gamma'\to\mathbb{R}^2} \mathbf{E}^{\Gamma}_{\Gamma'}(f_{\Gamma_0} \circ \pi^{\Gamma}_{\Gamma_0}) \\
&= \lim_{\Gamma\leq\Gamma'\to\mathbb{R}^2} \mathbf{E}^{\Gamma_0}_{\Gamma'}(f_{\Gamma_0}) \\
&= \lim_{\Gamma_0\leq\Gamma'\to\mathbb{R}^2} \mathbf{E}^{\Gamma_0}_{\Gamma'}(f_{\Gamma_0}) \\
&= \mathbf{E}^{\Gamma_0}(f_{\Gamma_0}),
\end{aligned}$$

because the limit of a subsequence of a convergent sequence equals the limit of the total sequence.
Let now Γ_1 and Γ_2 be two graphs and f_{Γ_1} and f_{Γ_2} two continuous functions on $\overline{\mathcal{A}/\mathcal{G}}_{\Gamma_1}$ and $\overline{\mathcal{A}/\mathcal{G}}_{\Gamma_2}$, resp., with $f_{\Gamma_1} \circ \pi_{\Gamma_1} = f = f_{\Gamma_2} \circ \pi_{\Gamma_2}$. For every refinement Γ of Γ_1 and Γ_2 we have $f = f_\Gamma \circ \pi_\Gamma$ with $f_{\Gamma_1} \circ \pi^{\Gamma}_{\Gamma_1} = f_\Gamma = f_{\Gamma_2} \circ \pi^{\Gamma}_{\Gamma_2}$. ($f_\Gamma$ is well-defined by the surjectivity of π_Γ.) As just seen, we have

$$\mathbf{E}^{\Gamma_1}(f_{\Gamma_1}) = \mathbf{E}^{\Gamma}(f_{\Gamma_1} \circ \pi^{\Gamma}_{\Gamma_1}) = \mathbf{E}^{\Gamma}(f_{\Gamma_2} \circ \pi^{\Gamma}_{\Gamma_2}) = \mathbf{E}^{\Gamma_2}(f_{\Gamma_2}),$$

i.e., $\mathbf{E}$ is well-defined.

3. Linearity, positivity and continuity of $\mathbf{E}$ on $\mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$
Linearity and positivity follow immediately from the corresponding properties of $\mathbf{E}^{\Gamma}$. For the continuity let $f = f_\Gamma \circ \pi_\Gamma \in \mathrm{Cyl}(\overline{\mathcal{A}/\mathcal{G}})$. Then $\mathbf{E}(f) = \mathbf{E}^{\Gamma}(f_\Gamma) \leq \|f_\Gamma\|_\infty = \|f_\Gamma \circ \pi_\Gamma\|_\infty = \|f\|_\infty$ by the surjectivity of π_Γ and by $\|\mathbf{E}^{\Gamma}\| \leq 1$.

Since the cylindrical functions form a dense subspace of $C(\overline{\mathcal{A}/\mathcal{G}})$, the extension of $\mathbf{E}$ to $C(\overline{\mathcal{A}/\mathcal{G}})$ is well-defined, linear, continuous and positive. Hence, by the Riesz-Markow theorem there is a unique regular Borel measure μ_{YM} on $\overline{\mathcal{A}/\mathcal{G}}$ with $\mathbf{E}(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} f \, \mathrm{d}\mu_{\mathrm{YM}}$ for all $f \in C(\overline{\mathcal{A}/\mathcal{G}})$. The normalization comes from $\mu_{\mathrm{YM}}(\overline{\mathcal{A}/\mathcal{G}}) = \mathbf{E}(1) = 1$. **qed**

Remark In the proof above we implicitely used the fact, that any graph can be refined to a simple connected graph. This way we can find, in particular, for every cylindrical function f a simple connected graph Γ such that $f = f_\Gamma \circ \pi_\Gamma$ for some continuous f_Γ.

Note finally, that in this subsection it actually does not matter whether we speak about two-dimensional Yang-Mills theory or any other gauge theory. We could simply substitute the regularized Yang-Mills actions $S_{\mathrm{YM},\Gamma}$ by some other actions. The above theorem remains valid for all theories; we have to guarantee "only" that Proposition 4.9 is valid.

4.5 Determination of the Expectation Values

Let us now take care of Proposition 4.9 (of course, only in the YM_2-case).

Let Γ be some graph and $\mathcal{F}$ be a moderately independent flag world in Γ. Furthermore, let Γ' be a refinement of Γ and $\mathcal{F}'$ be a corresponding refinement of $\mathcal{F}$ according to Proposition 4.7. Finally, $(\mathcal{F}, \vec{\phi}, \phi)$ be some loop-network. Then

$$t_{(\mathcal{F},\vec{\phi},\phi)} \circ \pi_\Gamma \circ \boldsymbol{\phi} \circ \pi = T_{(\mathcal{F},\vec{\phi},\phi)} = T_{\vec{\phi},\phi} \circ \pi_{\mathcal{F}} = T_{\vec{\phi},\phi} \circ \pi_{\mathcal{F}}^{\mathcal{F}'} \circ \pi_{\mathcal{F}'}.$$

Thus – remember that μ_0 and $S_{\mathrm{YM},\Gamma'}$ denote both objects on $\overline{\mathcal{A}}$ and on $\overline{\mathcal{A}/\mathcal{G}}$ –

$$\begin{aligned}
& \int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{e}^{-S_{\mathrm{YM},\Gamma'}} \; t_{(\mathcal{F},\vec{\phi},\phi)} \circ \pi_\Gamma \, \mathrm{d}\mu_0 \\
= & \int_{\overline{\mathcal{A}}} \mathrm{e}^{-S_{\mathrm{YM},\Gamma'}} \; t_{(\mathcal{F},\vec{\phi},\phi)} \circ \pi_\Gamma \circ \boldsymbol{\phi} \circ \pi \, \mathrm{d}\mu_0 \\
= & \int_{\overline{\mathcal{A}}} \mathrm{e}^{-S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'} \circ \pi_{\mathcal{F}'}} \; T_{\vec{\phi},\phi} \circ \pi^{\mathcal{F}'}_{\mathcal{F}} \circ \pi_{\mathcal{F}'} \, \mathrm{d}\mu_0 \qquad (S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'} \circ \pi_{\mathcal{F}'} = S_{\mathrm{YM},\Gamma'}, \text{ see Lemma 4.8}) \\
= & \int_{\mathbf{G}^{n(\Gamma')}} \mathrm{e}^{-S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'}} \; T_{\vec{\phi},\phi} \circ \pi^{\mathcal{F}'}_{\mathcal{F}} \, \mathrm{d}\mu^{n(\Gamma')}_{\mathrm{Haar}}.
\end{aligned}$$

In the last step we used that per definitionem every moderately independent flag world is a weak fundamental system and that therefore [33] the projection of μ_0 w.r.t. $\pi_{\mathcal{F}'}$ equals the Haar measure.

Let us denote by G_I, $I = 1, \ldots, n(\Gamma)$, the interior domains of Γ and by G_{I,i_I} those of Γ'. Here we assume that every G_I is just refined into the set $\{G_{I,1}, \ldots, G_{I,\lambda_I}\}$. Then we have

$$S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'}(\vec{g}) = \sum_{G_{I,i_I} \in L_{\mathrm{innen}}(\Gamma')} \frac{N}{\mathrm{g}^2} \frac{1}{|G_{I,i_I}|} \Big(1 - \frac{1}{N} \mathrm{Re}\;\mathrm{tr}\; g_{I,i_I} \Big)$$

and due to the special relation between flags in $\mathcal{F}$ and flags in $\mathcal{F}'$ (cf. Definition 4.8)

$$\begin{aligned}
(T_{\vec{\phi},\phi} \circ \pi^{\mathcal{F}'}_{\mathcal{F}})(\vec{g}) &= \frac{1}{\sqrt{\dim \phi}} \sum_{\vec{p},\vec{q}} \mathcal{C}^{\vec{q}\vec{p}}_{\vec{\phi},\phi} \prod_{I=1}^{n(\Gamma)} \sqrt{\dim \phi_I} \; \phi_I^{p_I q_I}(g_{I,1} \cdots g_{I,\lambda_I}) \\
&= \frac{1}{\sqrt{\dim \phi}} \sum_{\vec{p},\vec{q}} \mathcal{C}^{\vec{q}\vec{p}}_{\vec{\phi},\phi} \prod_{I=1}^{n(\Gamma)} \sqrt{\dim \phi_I} \; \delta^{p_I}_{r_{I,0}} \delta^{q_I}_{r_{I,\lambda_I}} \prod_{i_I=1}^{\lambda_I} \phi_I^{r_{I,i_I-1}\, r_{I,i_I}}(g_{I,i_I}).
\end{aligned}$$

Since $\mathrm{e}^{-S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'}} \; T_{\vec{\phi},\phi} \circ \pi^{\mathcal{F}'}_{\mathcal{F}}$ is a product of functions in the g_{I,i_I}, the integral over $\mathbf{G}^{n(\Gamma')}$ factorizes into the single $\mathbf{G}$-integrations. Additionally,

$$\mu_{\mathrm{Haar},|G|} := \mathrm{e}^{-\frac{N}{\mathrm{g}^2} \frac{1}{|G|} (1 - \frac{1}{N} \mathrm{Re}\;\mathrm{tr}\; \cdot)} \odot \mu_{\mathrm{Haar}}$$

is an Ad-invariant regular Borel measure on $\mathbf{G}$; hence

$$\int_{\mathbf{G}^{n(\Gamma')}} \mathrm{e}^{-S^{\mathcal{F}'}_{\mathrm{YM},\Gamma'}}\; T_{\vec{\phi},\phi} \circ \pi^{\mathcal{F}'}_{\mathcal{F}}\; \mathrm{d}\mu^{n(\Gamma')}_{\mathrm{Haar}}$$

$$= \frac{1}{\sqrt{\dim\phi}} \sum_{\vec{p},\vec{q}} \mathcal{C}^{\vec{q}\vec{p}}_{\vec{\phi},\phi} \prod_{I>n(\Gamma)} \left(\int_{\mathbf{G}} \mathrm{d}\mu_{\mathrm{Haar},|G_{I,1}|} \right) \prod_{I=1}^{n(\Gamma)} \left(\sqrt{\dim\phi_I}\; \delta^{p_I}_{r_{I,0}} \delta^{q_I}_{r_{I,\lambda_I}} \prod_{i_I=1}^{\lambda_I} \int_{\mathbf{G}} \phi_I^{r_{I,i_I-1}\, r_{I,i_I}}(g_{I,i_I})\; \mathrm{d}\mu_{\mathrm{Haar},|G_{I,i_I}|}(g_{I,i_I}) \right)$$

$$= \sqrt{\dim\phi} \prod_{I>n(\Gamma)} \left(\int_{\mathbf{G}} \mathrm{d}\mu_{\mathrm{Haar},|G_{I,1}|} \right) \prod_{I=1}^{n(\Gamma)} \left(\sqrt{\dim\phi_I} \prod_{i_I=1}^{\lambda_I} \int_{\mathbf{G}} \frac{1}{\dim\phi_I} \chi_{\phi_I}(g_{I,i_I})\; \mathrm{d}\mu_{\mathrm{Haar},|G_{I,i_I}|}(g_{I,i_I}) \right).$$

In the last step we used the fact that first the integrals over matrix functions can be reduced to the integrations of the corresponding characters [13] and second $\mathcal{C}_{\vec{\phi},\phi}$ is the projection of $\bigotimes_I V_{\phi_I}$ to V_ϕ and hence has trace $\dim\phi$.

Hence, we have

$$\mathbf{E}^{\Gamma}_{\Gamma'}(t_{(\mathcal{F},\vec{\phi},\phi)}) = \sqrt{\dim\phi} \prod_{I=1}^{n(\Gamma)} \left(\sqrt{\dim\phi_I} \prod_{i_I=1}^{\lambda_I} \frac{\int_{\mathbf{G}} \frac{1}{\dim\phi_I}\chi_{\phi_I}\; \mathrm{d}\mu_{\mathrm{Haar},|G_{I,i_I}|}}{\int_{\mathbf{G}} \mathrm{d}\mu_{\mathrm{Haar},|G_{I,i_I}|}} \right). \tag{5}$$

We see, in particular, that integrals for domains outside of Γ ($I > n(\Gamma)$) do not contribute by the normalization. Thus, the only terms in (5) depending on Γ' are the products $\prod_{i_I} \frac{\cdots}{\cdots}$ over all refinements of the interior domains of Γ into interior domains of Γ'. Consequently, to prove convergence of $\lim_{\Gamma\leq\Gamma'\to\mathbb{R}^2} \mathbf{E}^{\Gamma}_{\Gamma'}(t_{(\mathcal{F},\vec{\phi},\phi)})$ we only have to show that

$$\prod_{i=1}^{\lambda} \frac{\int_{\mathbf{G}} \frac{1}{\dim\phi_I}\chi_{\phi_I}\; \mathrm{d}\mu_{\mathrm{Haar},|G_i|}}{\int_{\mathbf{G}} \mathrm{d}\mu_{\mathrm{Haar},|G_i|}} \tag{6}$$

for $\{G_i\}$ with $\sum_{i=1}^{\lambda} |G_i| = |G| =$ const always goes to one and the same value if $\sup |G_i|$ goes to zero. This proof is not very difficult, but technically strenuous. Therefore we simply refer to [35]. The proof given there for $\mathbf{G} = SU(N)$ and $\mathbf{G} = U(1)$ can be quite easily extended to general compact $\mathbf{G}$.[4] It gives immediately

Proposition 4.11 Let Γ be a graph having interior domains G_I and $\mathcal{F}$ be a moderately independent flag world in Γ. Moreover, let $(\mathcal{F},\vec{\phi},\phi)$ be a loop-network and c_{ϕ_I} be the corresponding eigenvalue of the Casimir operator of the representation ϕ_I. Then we have

$$\mathbf{E}(t_{(\mathcal{F},\vec{\phi},\phi)} \circ \pi_\Gamma) = \mathbf{E}^{\Gamma}(t_{(\mathcal{F},\vec{\phi},\phi)}) = \sqrt{\dim\phi} \prod_{I=1}^{n(\Gamma)} \left(\sqrt{\dim\phi_I}\; \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_I}|G_I|} \right).$$

If Γ has precisely one interior domain G (i.e. Γ is a flag β), we have for all irreducible representations ϕ

$$\mathbf{E}(t_{(f,\phi,\phi)} \circ \pi_\Gamma) = \dim\phi\; \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_\phi |G|}.$$

This concludes the proof of the existence of the physical measure μ_{YM} for the two-dimensional Euclidian quantum Yang-Mills theory within the Ashtekar approach. Moreover, all Wilson-loop expectation values given here coincide with those of other approaches [24, 49, 50, 44, 52, 28, 41, 45].

[4] Similar proofs are already contained in articles about "ordinary" lattice Yang-Mills theory (see, e.g., [23]) written before the Ashtekar approach was born. However, they were restricted to the case of quadratic lattices.

5 Radon-Nikodym Derivatives

In this section we show that the push-forwards of the Yang-Mills measure to the lattice theories are absolutely continuous w.r.t. the lattice Haar measures and study the properties of the corresponding Radon-Nikodym derivatives, in particular, of their Fourier expansions.

By means of the homeomorphism $\boldsymbol{\phi}$ between $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ and $\overline{\mathcal{A}/\mathcal{G}}$ we can regard μ_{YM} as a measure on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. Moreover, it is well-known that the canonical projection $\pi : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ yields a natural bijection between the $\overline{\mathcal{G}}$-invariant measures on $\overline{\mathcal{A}}$ and the measures on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$. Hence, there is a unique normalized $\overline{\mathcal{G}}$-invariant Borel measure $\mu_{\overline{\mathcal{A}},\mathrm{YM}}$ on $\overline{\mathcal{A}}$, whose image measure on $\overline{\mathcal{A}/\mathcal{G}}$ equals μ_{YM}. Sometimes we will write instead of $\mu_{\overline{\mathcal{A}},\mathrm{YM}}$ simply μ_{YM}. We get

Corollary 5.1 Under the assumptions of Proposition 4.11 we have

$$\int_{\overline{\mathcal{A}}} T_{\vec{\phi},\phi} \circ \pi_{\mathcal{F}} \, \mathrm{d}\mu_{\overline{\mathcal{A}},\mathrm{YM}} = \sqrt{\dim \phi} \prod_{I=1}^{n(\Gamma)} \Bigl(\sqrt{\dim \phi_I} \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_I} |G_I|} \Bigr) \tag{7}$$

and

$$\int_{\overline{\mathcal{A}}} \chi_\phi \circ \pi_\beta \, \mathrm{d}\mu_{\overline{\mathcal{A}},\mathrm{YM}} = \dim \phi \; \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_\phi |G|}. \tag{8}$$

The theory of measures on projective limits [83] allows to map μ_{YM} consistently to the lattice gauge theories.

Definition 5.1 Let Γ be some (again connected) graph and $\boldsymbol{\alpha}$ be a weak fundamental system for Γ. Then we denote by

- $\mu_{0,\boldsymbol{\alpha}} := \pi_{\boldsymbol{\alpha}*}\mu_0$ the image measure of μ_0 on $\overline{\mathcal{A}}_{\boldsymbol{\alpha}} \cong \mathbf{G}^{\#\boldsymbol{\alpha}} \equiv \mathbf{G}^{n(\Gamma)}$ and
- $\mu_{\mathrm{YM},\boldsymbol{\alpha}} := \pi_{\boldsymbol{\alpha}*}\mu_{\overline{\mathcal{A}},\mathrm{YM}}$ the image measure of $\mu_{\overline{\mathcal{A}},\mathrm{YM}}$ on $\mathbf{G}^{n(\Gamma)}$.

Since $\boldsymbol{\alpha}$ is a weak fundamental system, we have $\mu_{0,\boldsymbol{\alpha}} = \mu_{\mathrm{Haar}}^{\#\boldsymbol{\alpha}}$. The continuity of $\pi_{\boldsymbol{\alpha}}$ implies

Lemma 5.2 For every graph Γ and every weak fundamental system (hence, in particular every moderately independent flag world) $\boldsymbol{\alpha}$ of Γ, the measure $\mu_{\mathrm{YM},\boldsymbol{\alpha}}$ on $\mathbf{G}^{\#\boldsymbol{\alpha}}$ is Ad-invariant, regular and Borel.

We know that μ_{YM} and μ_0 can be reconstructed from the corresponding self-consistent families $(\mu_{\mathrm{YM},\boldsymbol{\alpha}})_{\boldsymbol{\alpha}}$ and $(\mu_{0,\boldsymbol{\alpha}})_{\boldsymbol{\alpha}}$, respectively.[5] To study the relation between μ_{YM} and μ_0 below, we first investigate the relations between $\mu_{\mathrm{YM},\boldsymbol{\alpha}}$ and $\mu_{0,\boldsymbol{\alpha}}$. The easiest case is that of a single flag β.[6] If $\mu_{\mathrm{YM},\beta}$ were absolutely continuous w.r.t. $\mu_{0,\beta}$, i.e. $\mu_{\mathrm{YM},\beta} = \chi_\beta \odot \mu_{0,\beta}$ for some appropriate $\chi_\beta : \mathbf{G} \longrightarrow \mathbb{C}$, we would have

$$(\chi_{\vec{\mathbf{n}}}, \chi_\beta)_{\mathrm{Haar}} = \int_{\mathbf{G}} \overline{\chi_{\vec{\mathbf{n}}}} \, \chi_\beta \, \mathrm{d}\mu_{0,\beta} = \int_{\mathbf{G}} \overline{\chi_{\vec{\mathbf{n}}}} \, \mathrm{d}\mu_{\mathrm{YM},\beta} = d_{\vec{\mathbf{n}}} \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}} |G|},$$

i.e., integrability assumed, $\chi_\beta = \sum_{\vec{\mathbf{n}}} d_{\vec{\mathbf{n}}} \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G|} \chi_{\vec{\mathbf{n}}}$. Indeed the existence of such a χ_β has been proven in [75]. Here we even show that χ_β is continuous and grows at most polynomially for vanishing $|G_\beta|$.

Proposition 5.3 For every flag β there is an Ad-invariant continuous[7] function $\chi_\beta : \mathbf{G} \longrightarrow \mathbb{C}$ with $\mu_{\mathrm{YM},\beta} = \chi_\beta \odot \mu_{0,\beta}$. Moreover, we have:

[5] This is true indeed, because every arbitrary (not necessarily connected and simple) graph can be refined to a connected and simple graph.

[6] In order to avoid confusion of a flag f with functions f, we denote flags (as general closed paths) by α or β and write $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ instead of $\mathcal{F}$ for flag worlds analogously.

[7] One can even prove $\chi_\beta \in C^\infty(\mathbf{G})$.

1. The Fourier series
$$\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} d_{\vec{\mathbf{n}}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\, \chi_{\vec{\mathbf{n}}} \tag{9}$$
converges absolutely and uniformly to χ_β.
2. There are constants const_ν that depend only on ν (and $\mathbf{G}$), but not on $|G_\beta|$, such that
$$\|\chi_\beta\|_\infty \equiv \sup_{g\in\mathbf{G}} |\chi_\beta(g)| = \chi_\beta(e_{\mathbf{G}}) \leq 1 + \sum_{\nu=1}^{\dim\mathbf{G}} \mathrm{const}_\nu |G_\beta|^{-\frac{\nu}{2}}.$$

The following proof requires some estimates that are contained for reasons of readability in Appendix A.

Proof We set
$$\chi_\beta := \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} d_{\vec{\mathbf{n}}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\, \chi_{\vec{\mathbf{n}}}.$$

- For all $\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})$, $\vec{\mathbf{n}}\neq\vec{0}$, we have by Proposition 3.7 and 3.8
$$\begin{aligned}\|d_{\vec{\mathbf{n}}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\, \chi_{\vec{\mathbf{n}}}\|_\infty &= d_{\vec{\mathbf{n}}}^2\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|} \qquad (\text{by } \sup_{g\in\mathbf{G}}|\chi_{\vec{\mathbf{n}}}(g)| = \chi_{\vec{\mathbf{n}}}(e_{\mathbf{G}}) = d_{\vec{\mathbf{n}}})\\ &\leq \mathrm{const}_{\mathbf{G}}^2\, \|\vec{\mathbf{n}}\|^{\dim\mathbf{G}_{\mathrm{ss}}-l}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_-|G_\beta|\|\vec{\mathbf{n}}\|^2}.\end{aligned}$$
Define $f(r) := \mathrm{const}_{\mathbf{G}}^2\, (r+\sqrt{k+l})^{\dim\mathbf{G}_{\mathrm{ss}}-l}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_-|G_\beta| r^2}$. Remember that k and l are determined by $\mathbf{G} = (\mathbf{G}_{\mathrm{ss}}\times U(1)^k)/\mathbf{N}$ (see the beginning of Section 3). Here l is the dimension of a maximal torus in $\mathbf{G}_{\mathrm{ss}}$.
Let $\vec{x}\in\mathbb{R}^{k+l}_{\geq 0}$, $\vec{x}\neq\vec{0}$, be arbitrary. Moreover, let
$$\mathrm{W}^-_{\vec{\mathbf{n}},k+l} := \{\vec{x}\in\mathbb{R}^l\times\mathbb{R}^k \mid n_i - 1 < x_i \leq n_i\ \forall i\}$$
denote the semi-open cube with edge length 1 in $\mathbb{R}^l\times\mathbb{R}^k$, that is determined by the corners $\vec{n}-\vec{1}$ and $\vec{n}$. Now, choose some $\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{N}^k$ with $\vec{x}\in\mathrm{W}^-_{\vec{\mathbf{n}},k+l}$. Then $\|\vec{\mathbf{n}}\|\geq\|\vec{x}\|$ and $\|\vec{x}\|+\sqrt{k+l}\geq\|\vec{\mathbf{n}}\|-\|\vec{\mathbf{n}}-\vec{x}\|+\sqrt{k+l}\geq\|\vec{\mathbf{n}}\|$, hence $f(\|\vec{x}\|)\geq\mathrm{const}_{\mathbf{G}}^2\, \|\vec{\mathbf{n}}\|^{\dim\mathbf{G}_{\mathrm{ss}}-l}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_-|G_\beta|\|\vec{\mathbf{n}}\|^2}$.
- Corollary A.2 yields
$$\begin{aligned}&\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} \sup_{g\in\mathbf{G}} |d_{\vec{\mathbf{n}}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\, \chi_{\vec{\mathbf{n}}}|\\ \leq\ & 1 + \sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} \sup_{g\in\mathbf{G}} |d_{\vec{\mathbf{n}}}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\, \chi_{\vec{\mathbf{n}}}|\\ \leq\ & 1 + \\ & \underbrace{\int_0^\infty \mathrm{const}_{\mathbf{G}}^2 2^k \sum_{\nu=1}^{k+l}\binom{k+l}{\nu}\frac{\pi^{\frac{\nu}{2}}}{2^{\nu-1}\Gamma(\frac{\nu}{2})}\, r^{\nu-1}\,(r+\sqrt{k+l})^{\dim\mathbf{G}_{\mathrm{ss}}-l}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_-|G_\beta|\, r^2}}_{=:\ \text{polynom}\ \sum_{\nu=0}^{\dim\mathbf{G}-1} p_\nu r^\nu}\mathrm{d}r\\ =\ & 1 + \sum_{\nu=0}^{\dim\mathbf{G}-1} \mathrm{const}_\nu\, |G_\beta|^{-\frac{\nu+1}{2}}.\end{aligned}$$
Here we have used $\int_0^\infty r^{\nu-1}\,\mathrm{e}^{-a^2r^2}\,\mathrm{d}r = \frac{1}{2}\Gamma(\frac{\nu}{2})a^{-\nu}$. [84]
- Hence, by Lemma 3.10 the Fourier series (9) is absolutely and uniformly convergent.
- Moreover, obviously, $\sup_{g\in\mathbf{G}}|\chi_\beta(g)| = \chi_\beta(e_{\mathbf{G}})$.

We are now left with the proof of $\mu_{\mathrm{YM},\beta} = \chi_\beta \odot \mu_{0,\beta}$.
- Since $\mu_{\mathrm{YM},\beta}$ and $\chi_\beta\odot\mu_{0,\beta}$ are in each case Ad-invariant regular Borel measures, we are to prove only $\int_{\mathbf{G}} f\,\mathrm{d}\mu_{\mathrm{YM},\beta} = \int_{\mathbf{G}} f\chi_\beta\,\mathrm{d}\mu_{0,\beta}$ for all Ad-invariant $f\in C(\mathbf{G})$.

Since $\mathrm{span}_{\mathbb{C}}\{\chi_{\vec{\mathbf{n}}} \mid \vec{\mathbf{n}} \in \mathcal{D}(\mathbf{G})\}$ is dense in $C_{\mathrm{Ad}}(\mathbf{G})$ by the Peter-Weyl theorem, this follows from

$$\begin{aligned}
\int_{\mathbf{G}} \chi_{\vec{\mathbf{n}}} \chi_\beta \, \mathrm{d}\mu_{0,\beta} &= (\chi_\beta, \chi_{\vec{\mathbf{n}}})_{\mathrm{Haar}} && (\chi_\beta \text{ real (see below)}) \\
&= d_{\vec{\mathbf{n}}} \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}} |G_\beta|} && (\text{Definition of } \chi_\beta) \\
&= \int_{\overline{\mathcal{A}}} \chi_{\vec{\mathbf{n}}} \circ \pi_\beta \, \mathrm{d}\mu_{\overline{\mathcal{A}},\mathrm{YM}} && (\text{Corollary 5.1}) \\
&= \int_{\mathbf{G}} \chi_{\vec{\mathbf{n}}} \, \mathrm{d}\mu_{\mathrm{YM},\beta} && (\text{Definition of } \mu_{\mathrm{YM},\beta})
\end{aligned}$$

for all $\vec{\mathbf{n}} \in \mathcal{D}(\mathbf{G})$.

- We still show that χ_β is real: Since the character of the dual representaton equals the complex conjugate character of the original representation, we have
$$d_{\phi^*} \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi^*} |G_\beta|} = \int_{\mathbf{G}} \chi_{\phi^*} \, \mathrm{d}\mu_{\mathrm{YM},\beta} = \overline{\int_{\mathbf{G}} \chi_\phi \, \mathrm{d}\mu_{\mathrm{YM},\beta}} = \overline{d_\phi \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_\phi |G_\beta|}},$$
i.e. the imaginary parts in the Fourier series of χ_β cancel each other.

Remark The just proven continuity statement can also be gained by means of the so-called heat-kernel. One can show ([68], see also [42]), that the heat-kernel $K : \mathbf{G} \times (0, \infty) \longrightarrow \mathbb{R}$ for the diffusion operator $\partial_t - \Delta$ is a C^∞-function and fulfills the equation
$$K(g,t) = \sum_{\vec{\mathbf{n}}} d_{\vec{\mathbf{n}}} \, \mathrm{e}^{-c_{\vec{\mathbf{n}}} t} \chi_{\vec{\mathbf{n}}}(g).$$
Hence we can identify χ_β with the heat-kernel at time $t = \frac{1}{2}\mathrm{g}^2 |G_\beta|$. This way we could have used the asymptotics $\mathrm{tr}\, \mathrm{e}^{-t\Delta} \approx t^{-\frac{1}{2}\dim \mathbf{G}} p(t)$ with some power series $p(t)$ [71] for the proof of the second statement.
However, since we are interested in more general assertions on χ_β, we decided despite those general results for the direct proof above.

Even when calculating the expectation values of the Yang-Mills measure one sees that the integrations w.r.t. non-overlapping flag factorize. This is confirmed by

Proposition 5.4 For every graph Γ and every moderately independent flag world $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_n\}$ of Γ we have
$$\mu_{\mathrm{YM},\boldsymbol{\beta}} = \chi_{\boldsymbol{\beta}} \odot \mu_{0,\boldsymbol{\beta}} \quad \text{with} \quad \chi_{\boldsymbol{\beta}}(\vec{g}) := \chi_{\beta_1}(g_1) \cdots \chi_{\beta_n}(g_n).$$
In particular, $\chi_{\boldsymbol{\beta}} : \mathbf{G}^n \longrightarrow \mathbb{C}$ is an Ad-invariant and continuous (even C^∞) function.

Proof Together with the single χ_{β_ν}, also $\chi_{\boldsymbol{\beta}}$ is continuous and Ad-invariant. By Proposition 5.3, $\chi_{\beta_\nu} = \sum_{\phi_\nu \in \mathcal{D}(\mathbf{G})} \dim \phi_\nu \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_\nu} |G_{\beta_\nu}|} \chi_{\phi_\nu}$ is absolutely and uniformly convergent for every flag β_ν. Hence, all the subsequent rearrangements are allowed:

$$\begin{aligned}
\chi_{\boldsymbol{\beta}}(\vec{g}) &= \prod_{\nu=1}^n \chi_{\beta_\nu}(g_\nu) \\
&= \sum_{\vec{\phi} \in \mathcal{D}(\mathbf{G})^n} \prod_{\nu=1}^n \dim \phi_\nu \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_\nu} |G_{\beta_\nu}|} \, \chi_{\phi_\nu}(g_\nu) \\
&= \sum_{\vec{\phi} \in \mathcal{D}(\mathbf{G})^n} \sum_{\phi \in \vec{\phi}} \sum_{\vec{\imath},\vec{\jmath}} \mathcal{C}^{\vec{\jmath}\vec{\imath}}_{\vec{\phi},\phi} \prod_{\nu=1}^n \dim \phi_\nu \, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_\nu} |G_{\beta_\nu}|} \phi_\nu^{i_\nu j_\nu}(g_\nu) \\
&\qquad \left(\textstyle\prod_\nu \chi_{\phi_\nu}(g_\nu) = \sum_{\vec{\imath},\vec{\jmath}} \mathbf{1}^{\vec{\jmath}\vec{\imath}}_{\vec{\phi}} \prod_\nu \phi_\nu^{i_\nu j_\nu}(g_\nu) \text{ and } \mathbf{1}_{\vec{\phi}} = \sum_{\phi \in \vec{\phi}} \mathcal{C}_{\vec{\phi},\phi}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\vec\phi \in \mathcal{D}(\mathbf{G})^n} \sum_{\phi \in \vec\phi} \Bigl(\sqrt{\dim\phi} \prod_{\nu=1}^n \bigl(\sqrt{\dim\phi_\nu}\, \mathrm{e}^{-\frac{1}{2}g^2 c_{\phi_\nu}|G_{\beta_\nu}|}\bigr)\Bigr) \times \\
&\qquad\qquad \Bigl(\frac{1}{\sqrt{\dim\phi}} \sum_{\vec\imath,\vec\jmath} \mathcal{C}^{\vec\jmath\vec\imath}_{\vec\phi,\phi} \prod_{\nu=1}^n \sqrt{\dim\phi_\nu}\, \phi_\nu^{i_\nu j_\nu}(g_\nu)\Bigr) \\
&= \sum_{\vec\phi \in \mathcal{D}(\mathbf{G})^n} \sum_{\phi \in \vec\phi} \Bigl(\int_{\overline{\mathcal{A}}} T_{\vec\phi,\phi} \circ \pi_{\boldsymbol\beta}\, \mathrm{d}\mu_{\overline{\mathcal{A}},\mathrm{YM}}\Bigr) T_{\vec\phi,\phi} \qquad \text{(Corollary 5.1)} \\
&= \sum_{\vec\phi \in \mathcal{D}(\mathbf{G})^n} \sum_{\phi \in \vec\phi} \Bigl(\int_{\mathbf{G}^n} T_{\vec\phi,\phi}\, \mathrm{d}\mu_{\mathrm{YM},\boldsymbol\beta}\Bigr) T_{\vec\phi,\phi} \qquad \text{(Definition of } \mu_{\mathrm{YM},\boldsymbol\beta}\text{).}
\end{aligned}
$$

Here, in part, we used $\vec\phi$ and $\bigotimes_\nu \phi_\nu$ synonymously. So, e.g., $\phi \in \vec\phi$ just denotes an irreducible representation ϕ contained in $\bigotimes_\nu \phi_\nu$.
Consequently,

$$
\begin{aligned}
\int_{\mathbf{G}^n} T_{\vec\phi,\phi}\, \chi_{\boldsymbol\beta}\, \mathrm{d}\mu_{0,\boldsymbol\beta} &= (\chi_{\boldsymbol\beta}, T_{\vec\phi,\phi})_{\mathrm{Haar},n} \qquad (\mu_{0,\boldsymbol\beta} = \mu^n_{\mathrm{Haar}} \text{ and } \chi_{\boldsymbol\beta} \text{ real}) \\
&= \sum_{\vec\phi',\phi'} \Bigl(\int_{\mathbf{G}^n} T_{\vec\phi',\phi'}\, \mathrm{d}\mu_{\mathrm{YM},\boldsymbol\beta}\Bigr) (T_{\vec\phi',\phi'}, T_{\vec\phi,\phi})_{\mathrm{Haar},n} \\
&= \int_{\mathbf{G}^n} T_{\vec\phi,\phi}\, \mathrm{d}\mu_{\mathrm{YM},\boldsymbol\beta} \\
&\qquad \text{(Orthonormalization of the } n\text{-characters (Theorem 3.14))}
\end{aligned}
$$

for all n-characters $T_{\vec\phi,\phi}$. But, because these by the Peter-Weyl theorem span a dense subspace of $C_{\mathrm{Ad}}(\mathbf{G}^n)$, we have

$$
\int_{\mathbf{G}^n} f\, \chi_{\boldsymbol\beta}\, \mathrm{d}\mu_{0,\boldsymbol\beta} = \int_{\mathbf{G}^n} f\, \mathrm{d}\mu_{\mathrm{YM},\boldsymbol\beta}
$$

for all $f \in C_{\mathrm{Ad}}(\mathbf{G}^n)$ as in the proposition above, hence $\mu_{\mathrm{YM},\boldsymbol\beta} = \chi_{\boldsymbol\beta} \odot \mu_{0,\boldsymbol\beta}$. **qed**

Corollary 5.5 Let Γ be a graph and $\boldsymbol\beta = \{\beta_1, \ldots, \beta_n\}$ be a moderately independent flag world in Γ.
Then we have $\mu_{\mathrm{YM},\boldsymbol\beta}(U_1 \times \cdots \times U_n) = \mu_{\mathrm{YM},\beta_1}(U_1) \cdots \mu_{\mathrm{YM},\beta_n}(U_n)$ for all measurable and Ad-invariant $U_i \subseteq \mathbf{G}$, $i = 1, \ldots, n$.

Proof

$$
\begin{aligned}
& \mu_{\mathrm{YM},\boldsymbol\beta}(U_1 \times \cdots \times U_n) \\
= {} & \textstyle\int_{\mathbf{G}^n} 1_{U_1 \times \cdots \times U_n}\, \chi_{\boldsymbol\beta}\, \mathrm{d}\mu_{0,\boldsymbol\beta} \\
= {} & \textstyle\int_{\mathbf{G}^n} 1_{U_1}(g_1) \cdots 1_{U_n}(g_n)\, \chi_{\beta_1}(g_1) \cdots \chi_{\beta_n}(g_n)\, \mathrm{d}\mu_{0,\beta_1}(g_1) \cdots \mathrm{d}\mu_{0,\beta_n}(g_n) \\
= {} & \bigl(\textstyle\int_{\mathbf{G}} 1_{U_1}(g_1)\chi_{\beta_1}(g_1)\, \mathrm{d}\mu_{0,\beta_1}(g_1)\bigr) \cdots \bigl(\textstyle\int_{\mathbf{G}} 1_{U_n}(g_n)\chi_{\beta_n}(g_n)\, \mathrm{d}\mu_{0,\beta_n}(g_n)\bigr) \\
= {} & \mu_{\mathrm{YM},\beta_1}(U_1) \cdots \mu_{\mathrm{YM},\beta_n}(U_n)
\end{aligned}
$$

qed

Finally we show that the integration of (not necessarily continuous) cylindrical functions always can be reduced to the analysis of absolutely convergent Fourier series.

Proposition 5.6 Let $f \in L^2(\mathbf{G})$ be an Ad-invariant function. Then we have for all flags $\beta \in \mathcal{HG}$

$$
\int_{\mathbf{G}} f\, \mathrm{d}\mu_{\mathrm{YM},\beta} = \sum_{\vec{\mathbf{n}} \in \mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} \mathrm{e}^{-\frac{1}{2}g^2 c_{\vec{\mathbf{n}}}|G_\beta|},
$$

whereas the rhs always converges absolutely.

Proof By Corollary 3.4, $(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}\chi_{\vec{\mathbf{n}}}$ converges to f in $L^2(\mu_{\mathrm{Haar}}) \equiv L^2(\mathbf{G})$. Since the function χ_β with $\mu_{\mathrm{YM},\beta} = \chi_\beta \odot \mu_{0,\beta}$ from Proposition 5.3 is continuous, hence bounded, we have $L^2(\mu_{\mathrm{Haar}}) \subseteq L^2(\mu_{\mathrm{YM},\beta})$, i.e., $(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}\chi_{\vec{\mathbf{n}}}$ converges in $L^2(\mu_{\mathrm{YM},\beta})$ to f as well. Hence,

$$\begin{aligned} \textstyle\int_{\mathbf{G}} f \,\mathrm{d}\mu_{\mathrm{YM},\beta} &\equiv (1, f)_{\mathrm{YM},\beta} \\ &= \textstyle\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} (1, \chi_{\vec{\mathbf{n}}})_{\mathrm{YM},\beta} \\ &= \textstyle\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}} |G_\beta|}. \end{aligned}$$

By $|(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}| \leq \|\chi_{\vec{\mathbf{n}}}\|_{\mathrm{Haar}} \|f\|_{\mathrm{Haar}} = \|f\|_{\mathrm{Haar}}$, hence

$$\bigl|(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}\bigr| \leq \|f\|_{\mathrm{Haar}} \mathrm{const}_{\mathbf{G}} \|\vec{\mathbf{n}}\|^{\dim \mathbf{G}_{\mathrm{ss}} - l} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_- |G_\beta| \|\vec{\mathbf{n}}\|^2},$$

and by Corollary A.3 the series converges even absolutely:

$$g(r) := r^{\dim \mathbf{G}_{\mathrm{ss}} - l} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_- |G_\beta| r^2}$$

decreases for sufficiently large r monotonically, and $g(r) r^{k+l-1}$ is integrable on $[0, \infty]$.

qed

Completely analogously we get

Proposition 5.7 Let Γ be a graph and $f \in L^2(\mathbf{G}^{n(\Gamma)})$ be an Ad-invariant function. Then we have for all moderately independent flag worlds $\boldsymbol{\beta}$ in Γ

$$\int_{\mathbf{G}^n} f \,\mathrm{d}\mu_{\mathrm{YM},\boldsymbol{\beta}} = \sum_{\vec{\phi},\phi} (T_{\vec{\phi},\phi}, f)_{\mathrm{Haar},n(\Gamma)} \sqrt{\dim \phi} \prod_{I=1}^{n(\Gamma)} \Bigl(\sqrt{\dim \phi_I}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_I} |G_I|} \Bigr).$$

Remark The definition of the expectation values of μ_{YM} started with the Wilson action, hence with a quantity directly gained from the standard Yang-Mills action $\frac{1}{2}(F, F)$. This, however, has the disadvantage that the existence of the continuum limit in Proposition 4.9 had to be proven laboriously.

There is another possibility to process: One can put the calculated expectation values directly into the definition of the regularization of S_{YM}. One simply defines for all graphs Γ and moderately independent flag worlds $\mathcal{F}$ in Γ

$$S_{\mathrm{YM},\Gamma} := -\ln\Bigl(\sum_{\vec{\phi},\phi} \sqrt{\dim \phi} \prod_{I=1}^{n(\Gamma)} \Bigl(\sqrt{\dim \phi_I}\, \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\phi_I} |G_I|} \Bigr) \overline{T}_{(\mathcal{F},\vec{\phi},\phi)} \Bigr) \qquad (10)$$

and defines then

$$\mathbf{E}(f_\Gamma \circ \pi_\Gamma) := \int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{e}^{-S_{\mathrm{YM},\Gamma}} f_\Gamma \circ \pi_\Gamma \,\mathrm{d}\mu_0.$$

This way one avoids the problem of the limit and gets the "correct" expectation values directly. This variant has been used, e.g., by Aroca and Kubyshin [2]. (However, there only some flags has been considered what is insufficient for the determination of the full measure.) Typically the action (10) is also called heat-kernel action or Villain action [52]. Of course, this method has the disadvantage that the relation to the standard Yang-Mills theory is not as close as in the case of the Wilson approximation; the problem is only shifted.

6 Support of the Yang-Mills Measure

In this section we are going to prove that the Yang-Mills measure is purely singular w.r.t. the Ashtekar-Lewandowski measure. Moreover, we present an explicit (however, of course, non-unique) decomposition of $\overline{\mathcal{A}/\mathcal{G}}$ into disjoint subsets that support the one and the other measure, respectively. Finally, we investigate the impact of smooth connections and of the Gribov problem.

6.1 General Remarks on Singularity Proofs

To prove the singularity of a (finite Borel) measure μ w.r.t. to the Ashtekar-Lewandowski measure μ_0 we need to show that there is no $L^1(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$-measureable function f fulfilling $\mu = f \odot \mu_0$ (or $\mathrm{d}\mu = f\,\mathrm{d}\mu_0$). However, there is a very simple criterion for that:

Proposition 6.1 Let μ be some (normalized) regular Borel measure on $\overline{\mathcal{A}/\mathcal{G}}$. Then μ is singular w.r.t. to μ_0 if there are uncountably many non-zero spin-network expectation values.

Obviously, there is no $L^2(\mu_0)$-function f with $\mu = f \odot \mu_0$. Namely, if this were not the case, then this f could be expanded into a spin-network series [20] with uncountably many non-vanishing "Fourier" coefficients which is impossible in a Hilbert space. The idea for the proof in the L^1-case is due to Jerzy Lewandowski:

Proof Suppose μ would be absolutely continuous w.r.t. μ_0. Then there would be an $L^1(\mu_0)$-function with $\mu = f \odot \mu_0$. Since the cylindrical functions form a dense subalgebra of the continuous functions in $\overline{\mathcal{A}/\mathcal{G}}$, they form a dense subspace in the Banach space $L_1(\mu_0)$. Hence, there is a sequence (f_n) of (w.l.o.g. real) cylindrical functions with $f_n \to f$ in L^1. Obviously, then $Tf_n \to Tf$ in L^1, i.e.

$$\overline{(T,f_n)_{\mu_0}} \equiv \int_{\overline{\mathcal{A}/\mathcal{G}}} Tf_n\,\mathrm{d}\mu_0 \to \int_{\overline{\mathcal{A}/\mathcal{G}}} Tf\,\mathrm{d}\mu_0 = \int_{\overline{\mathcal{A}/\mathcal{G}}} T\mathrm{d}\mu \equiv \langle T\rangle \tag{11}$$

for every continuous function T on $\overline{\mathcal{A}/\mathcal{G}}$. Since every cylindrical function can be written as a sum of countably many spin-networks, there are at most countably many spin-network states T with non-vanishing $(T,f_n)_{\mu_0}$ for some n. But, by assumption there are uncountably many spin-network states T having non-vanishing μ-expectation value $\langle T\rangle$. We get a contradiction to (11). **qed**

Now, since Wilson loops can be regarded as special spin networks, the Yang-Mills measure fits to the condition of the proposition above. Consequently, μ_{YM} is not absolutely continuous w.r.t. to μ_0. Note, more general, that measures having a continuous symmetry typically fulfill the condition above.

It remains now the proof of the pure singularity. This means that there is a μ_0-zero subset having the full μ-measure 1. General arguments are provided, i.e., by the notion of ergodicity. More precisely, let H be some (not necessarily continuous) transformation group on the measure space X. Then every two H-ergodic and H-(quasi-)invariant measures are equivalent or purely singular to each other [83]. In the Yang-Mills case we know that the measure is invariant under the action of area and analyticity preserving automorphisms. It is well-known that μ_0 is not only invariant under these transformations as well, but even ergodic. Therefore, if we were able to prove the corresponding ergodicity of μ_{YM}, we would get the pure singularity. However, up to now, we do not know whether the ergodocity is given (although we guess it is). Moreover, it would not immediately yield the partition of $X = \overline{\mathcal{A}/\mathcal{G}}$ into the supports of μ_{YM} and μ_0 that will be given below.

Finally, we note that using ergodicity the inequivalence of the Fock measures and the Ashtekar-Lewandowski measure has been proven [11, 80].

6.2 Idea

The proof of the inequivalence of μ_{YM} and μ_0 is based on the following two facts:

1. If the flag β shrinks, the measure $\mu_{\mathrm{YM},\beta}$ concentrates around $e_{\mathbf{G}}$.
2. The measure w.r.t. non-overlapping flags is the product of the measures for the single flags.

More precisely, for the first item we know that the Radon-Nikodym derivative χ_β of the Yang-Mills measure equals $\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} d_{\vec{\mathbf{n}}} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|} \chi_{\vec{\mathbf{n}}}$ for a single flag. Obviously, the Fourier coefficients of this expansion fall the slowlier, the smaller $|G_\beta|$ is. The slow falling of a Fourier series typically corresponds to a strong concentration of the original function in a point. (Remember the extremal cases δ-"function" with $\delta(\varphi) = \sum_{n\in\mathbb{Z}} 1 \cdot \mathrm{e}^{\mathrm{i}n\varphi}$ and 1-function with $1(\varphi) = \sum_{n\in\mathbb{Z}} \delta_{n0} \cdot \mathrm{e}^{\mathrm{i}n\varphi}$.) This way it should be possible to find a neighbourhood U of $e_{\mathbf{G}}$ with Haar measure ε whose (pushed-forward) Yang-Mills measure $\mu_{\mathrm{YM},\beta}(U) \approx \chi_\beta(e_{\mathbf{G}})\, \mu_{\mathrm{Haar}}(U)$ goes to 1 for shrinking flags. Using the second item above and an increasing number of shrinking flags we let on the one hand the Haar measure of $U^n \subseteq \mathbf{G}^n$ go to 0 with ε^n and let on the other hand the corresponding Yang-Mills measure go to a non-vanishing value. For instance, we can always choose some flag β_n with $\mu_{\mathrm{YM},\beta_n}(U) = \mathrm{e}^{-2^{-n}}$. Namely, here $\prod \mathrm{e}^{-2^{-n}} = \mathrm{e}^{-1} > 0$. Hence, the Yang-Mills measure is definitely not absolutely continuous w.r.t. μ_0. This argument can be used even for the proof of the full singularity. After a refinement of the indicated estimates, we will show that the Yang-Mills measure for U^n can adopt for appropriate flags β_n every value different from 1. Hence we reach the full Yang-Mills measure 1.

We will assume thoughout this section that $\mathbf{G}$ is nontrivial. Otherwise, $\mathcal{A}/\mathcal{G} = \overline{\mathcal{A}/\mathcal{G}}$, i.e. every generalized connection would be regular, and thus $\mu_{\mathrm{YM}}(\mathcal{A}/\mathcal{G}) = \mu_{\mathrm{YM}}(\overline{\mathcal{A}/\mathcal{G}}) = 1$. Additionally, $\mu_{\mathrm{YM}} = \mu_0$, i.e. μ_{YM} would be absolutely continuous w.r.t. μ_0.

6.3 Pure Singularity of μ_{YM} w.r.t. μ_0

Let us choose some fixed $\varepsilon \in (0,1)$.

Lemma 6.2 There is a constant $c \geq 0$ depending only on ε and $\mathbf{G}$ and an open Ad-invariant subset $U \subseteq \mathbf{G}$, such that we have for all flags $\beta \in \mathcal{HG}$
- $\mu_{0,\beta}(U) \leq \varepsilon$ and
- $\mu_{\mathrm{YM},\beta}(U) \geq 1 - c|G_\beta|$.

Proof Let us choose for U some open Ad-invariant neighbourhood of the identity $e_{\mathbf{G}}$ with $\mu_{\mathrm{Haar}}(U) \leq \varepsilon$. Obviously, $\mu_{0,\beta}(U) \equiv \mu_{\mathrm{Haar}}(U) \leq \varepsilon$; so we are left with the proof of $\mu_{\mathrm{YM},\beta}(U) \geq 1 - c|G_\beta|$.
- Let $f : \mathbf{G} \longrightarrow [0,1]$ be some Ad-invariant C^∞-function with $\operatorname{supp} f \subseteq U$ and $f(e_{\mathbf{G}}) = 1$.
- By Corollary 3.13 the Fourier series $f = \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} \chi_{\vec{\mathbf{n}}}$ converges absolutely and uniformly. In particular,
$$f(e_{\mathbf{G}}) = \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} = 1 \qquad (12)$$
converges absolutely.
- Hence we get
$$\begin{aligned}
\mu_{\mathrm{YM},\beta}(U) &\geq \int_{\mathbf{G}} f \,\mathrm{d}\mu_{\mathrm{YM},\beta} \\
&= \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|} \\
&= \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}} \\
&\quad - \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} (\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}} d_{\vec{\mathbf{n}}}\, \frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|\, \frac{1 - \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}}{\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}
\end{aligned}$$

$$= \quad 1 - |G_\beta| \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} \underbrace{(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}\, d_{\vec{\mathbf{n}}}\, \frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}\, \frac{1 - \mathrm{e}^{-\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}}{\frac{1}{2}\mathrm{g}^2 c_{\vec{\mathbf{n}}}|G_\beta|}}_{=:\, \xi_{\vec{\mathbf{n}},|G_\beta|}}.$$

In the first step we used $f \leq 1_U$ and in the second the integration formula from Proposition 5.6. For the third step we used the absolute convergence of both series ($\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G})} |(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}| d_{\vec{\mathbf{n}}}$ is a convergent majorant) and in the fourth Equation (12).

- For the study of the convergence of the ξ-series we need the following estimates ($\vec{\mathbf{n}} \neq \vec{0}$): First, $\frac{1-\mathrm{e}^{-x}}{x} \leq 1$ for all $x \in \mathbb{R}_+$, and second $|(\chi_{\vec{\mathbf{n}}}, f)_{\mathrm{Haar}}| \leq \mathrm{const}_{s,f}\, c_{\vec{\mathbf{n}}}^{-s}$ for all $s \in \mathbb{N}_+$ by $f \in C^\infty(\mathbf{G})$ and by Proposition 3.9. Hence, for all $s \in \mathbb{N}$
$$\left|\xi_{\vec{\mathbf{n}},|G_\beta|}\right| \leq \frac{1}{2}\mathrm{g}^2 \mathrm{const}_{s,f}\, c_{\vec{\mathbf{n}}}^{1-s} d_{\vec{\mathbf{n}}}.$$
In particular, for $2s = \frac{1}{2}(\dim \mathbf{G} + k + l) + 3$ the relation $\frac{1}{2}(\dim \mathbf{G}_{\mathrm{ss}} - l) + 2(1-s) = -(k+l+1)$ is fulfilled. By the convergence criterion in Lemma 3.11 we have
$$\sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} \xi_{\vec{\mathbf{n}},|G_\beta|} \leq \frac{1}{2}\mathrm{g}^2 \mathrm{const}_{s,f} \sum_{\vec{\mathbf{n}}\in\mathcal{D}(\mathbf{G}),\vec{\mathbf{n}}\neq\vec{0}} c_{\vec{\mathbf{n}}}^{1-s} d_{\vec{\mathbf{n}}} =: c < \infty,$$
independent of $|G_\beta|$. Finally, we get
$$\mu_{\mathrm{YM},\beta}(U) \geq 1 - c|G_\beta|.$$

qed

Lemma 6.3 Let $F \in (0,1)$ and let $(\beta_i)_{i\in\mathbb{N}_+} \subseteq \mathcal{HG}$ be a sequence of flags, such that

1. $|G_{\beta_i}| \leq \frac{F}{2^i c}$ for all $i \in \mathbb{N}_+$, where c is the constant of Lemma 6.2, and
2. for every $\Lambda \in \mathbb{N}_+$ the set $\boldsymbol{\beta}_\Lambda := \{\beta_1, \ldots, \beta_\Lambda\}$ is a moderately independent flag world in the graph Γ_Λ spanned by $\boldsymbol{\beta}_\Lambda$.[8]

Then there are open Ad-invariant $V_\Lambda \subseteq \mathbf{G}^\Lambda$, such that we have for all $\Lambda \in \mathbb{N}_+$

- $\mu_{0,\boldsymbol{\beta}_\Lambda}(V_\Lambda) \leq \varepsilon^\Lambda$,
- $\mu_{\mathrm{YM},\boldsymbol{\beta}_\Lambda}(V_\Lambda) \geq 1 - F$ and
- $\pi_{\boldsymbol{\beta}_{\Lambda+1}}^{-1}(V_{\Lambda+1}) \subseteq \pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)$.

Proof Choose the Ad-invariant $U \subseteq \mathbf{G}$ of the preceding lemma and define $V_\Lambda := U^\Lambda \subseteq \mathbf{G}^\Lambda$. Obviously, V_Λ is always open and Ad-invariant. Moreover, we have for all $\Lambda \in \mathbb{N}_+$:

- $\mu_{0,\boldsymbol{\beta}_\Lambda}(V_\Lambda) = \mu_{\mathrm{Haar}}^\Lambda(U^\Lambda) = (\mu_{\mathrm{Haar}}(U))^\Lambda \leq \varepsilon^\Lambda$.
- By Corollary 5.5 we have
$$\begin{aligned}\mu_{\mathrm{YM},\boldsymbol{\beta}_\Lambda}(V_\Lambda) &= \mu_{\mathrm{YM},\boldsymbol{\beta}_\Lambda}(U^\Lambda) = \textstyle\prod_{i=1}^\Lambda \mu_{\mathrm{YM},\beta_i}(U) \\ &\geq \textstyle\prod_{i=1}^\Lambda (1 - c|G_{\beta_i}|) \geq \prod_{i=1}^\Lambda (1 - \frac{F}{2^i}) \geq 1 - F.\end{aligned}$$
Here we used $c|G_{\beta_i}| \leq \frac{F}{2^i} < \frac{1}{2}$ and (in the last step) the relation $\prod_{i=1}^\Lambda (1 - \frac{F}{2^i}) \geq 1 - F$ valid for all $F \in (0,1)$.[9]
- By
$$\begin{aligned}\overline{A} \in \pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda) &\iff \big(h_{\beta_1}(\overline{A}), \ldots, h_{\beta_\Lambda}(\overline{A})\big) = \pi_{\boldsymbol{\beta}_\Lambda}(\overline{A}) \in V_\Lambda = U^\Lambda \\ &\iff h_{\beta_i}(\overline{A}) \in U \qquad \forall i = 1, \ldots, \Lambda\end{aligned}$$
we have $\pi_{\boldsymbol{\beta}_{\Lambda+1}}^{-1}(V_{\Lambda+1}) \subseteq \pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)$.

qed

[8] Descriptively, the flags β_i are just non-overlapping.

[9] Using the absolute convergence of the Taylor series of $\ln(1-F)$ we have for $0 < F < 1$:
$$\begin{aligned}\ln \textstyle\prod_{i=1}^\Lambda (1 - \frac{F}{2^i}) &= \textstyle\sum_{i=1}^\Lambda \ln(1 - \frac{F}{2^i}) = -\sum_{i=1}^\Lambda \sum_{j=1}^\infty \frac{1}{j}\left(\frac{F}{2^i}\right)^j = -\sum_{j=1}^\infty \sum_{i=1}^\Lambda \frac{1}{j}\left(\frac{F}{2^i}\right)^j \\ &= -\textstyle\sum_{j=1}^\infty \frac{F^j}{j} \underbrace{\left(\sum_{i=1}^\Lambda \frac{1}{2^{ij}}\right)}_{<1} \geq -\sum_{j=1}^\infty \frac{F^j}{j} = \ln(1-F).\end{aligned}$$

Lemma 6.4 For every $F \in (0,1)$ there is a measurable $W_F \subseteq \overline{\mathcal{A}}/\overline{\mathcal{G}}$ with
$$\mu_0(W_F) = 0 \text{ and } \mu_{\mathrm{YM}}(W_F) \geq 1 - F.$$

Proof Obviously there is a sequence $(\beta_i)_{i\in\mathbb{N}_+} \subseteq \mathcal{HG}$ of flags with the properties listed in Lemma 6.3. Let again $\boldsymbol{\beta}_\Lambda := \{\beta_1, \ldots, \beta_\Lambda\}$. According to the lemma above choose for every $\Lambda \in \mathbb{N}_+$ some Ad-invariant subset $V_\Lambda \subseteq \mathbf{G}^\Lambda$ and define $W_F := \bigcap_{\Lambda\in\mathbb{N}_+} \pi(\pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda))$. By the Ad-invariance of V_Λ we have $\pi^{-1}(W_F) = \bigcap_{\Lambda\in\mathbb{N}_+} \pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)$. Hence, by $\pi_{\boldsymbol{\beta}_{\Lambda+1}}^{-1}(V_{\Lambda+1}) \subseteq \pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)$ and by the theorem on monotone convergence of measures
1. $\mu_0(W_F) = \lim_{\Lambda\to\infty} \mu_0(\pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)) = \lim_{\Lambda\to\infty} \mu_{0,\boldsymbol{\beta}_\Lambda}(V_\Lambda) = \lim_{\Lambda\to\infty} \varepsilon^\Lambda = 0$ and
2. $\mu_{\mathrm{YM}}(W_F) = \lim_{\Lambda\to\infty} \mu_{\overline{\mathcal{A}},\mathrm{YM}}(\pi_{\boldsymbol{\beta}_\Lambda}^{-1}(V_\Lambda)) = \lim_{\Lambda\to\infty} \mu_{\mathrm{YM},\boldsymbol{\beta}_\Lambda}(V_\Lambda) \geq 1-F$. **qed**

Consequently, μ_{YM} is not absolutely continuous w.r.t. μ_0. The pure singularity comes from

Theorem 6.5 There is a measurable $W \subseteq \overline{\mathcal{A}}/\overline{\mathcal{G}}$ with
$$\mu_0(W) = 0 \text{ and } \mu_{\mathrm{YM}}(W) = 1.$$

Proof Define $W := \bigcup_{n\in\mathbb{N},n>1} W_{\frac{1}{n}}$, where $W_{\frac{1}{n}}$ is defined as in Lemma 6.4. Then we have
1. $0 \leq \mu_0(W) \leq \sum_{n\in\mathbb{N},n>1} \mu_0(W_{\frac{1}{n}}) = 0$ and
2. $1 \geq \mu_{\mathrm{YM}}(W) \geq \sup_{n\in\mathbb{N},n>1} \mu_{\mathrm{YM}}(W_{\frac{1}{n}}) = \sup_{n\in\mathbb{N},n>1}\{1 - \frac{1}{n}\} = 1$. **qed**

6.4 μ_{YM}-Almost Global Triviality of the Generic Stratum

In [33] the impact of the Gribov problem on the kinematical level, i.e. w.r.t. μ_0, has been investigated. This is strongly related to the existence of so-called almost global trivializations of the generic stratum[10]. This means that there is a covering of the generic stratum consisting of gauge-invariant sets having full measure 1. This way it has been shown that there are sections in the fibering $\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ being continuous almost everywhere – the Gribov problem is concentrated on a zero subset. In this subsection we will see that this is true dynamically as well, i.e. for the Yang-Mills measure.

Theorem 6.6 The generic stratum of $\overline{\mathcal{A}}$ is $\mu_{\overline{\mathcal{A}},\mathrm{YM}}$-almost globally trivial.

Proof We simply show that the μ_0-almost global trivialization from [33] is also a $\mu_{\overline{\mathcal{A}},\mathrm{YM}}$-almost global trivialization. We recall that there every element of the covering of the generic stratum was the preimage $\pi_{\boldsymbol{\alpha}}^{-1}(V)$ of some Ad-invariant set $V \subseteq \mathbf{G}^{\#\boldsymbol{\alpha}}$ with Haar measure 1 where $\boldsymbol{\alpha}$ can be chosen to be a moderately independent flag world.
Now we have $\mu_{\overline{\mathcal{A}},\mathrm{YM}}(\pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{G}^n \setminus V)) = \mu_{\mathrm{YM},\boldsymbol{\alpha}}(\mathbf{G}^n \setminus V) = 0$ with $n := \#\boldsymbol{\alpha}$, hence $\mu_{\overline{\mathcal{A}},\mathrm{YM}}(\pi_{\boldsymbol{\alpha}}^{-1}(V)) = 1$, for all Ad-invariant $V \subseteq \mathbf{G}^n$ with $\mu_{\mathrm{Haar}}^n(\mathbf{G}^n \setminus V) = 0$, since $\mu_{\mathrm{YM},\boldsymbol{\alpha}}$ is absolutely continuous w.r.t. $\mu_{0,\boldsymbol{\alpha}} = \mu_{\mathrm{Haar}}^n$. **qed**

We get immediately

Theorem 6.7 We have $\mu_{\mathrm{YM}}(\overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}) = 1$.

[10]The generic stratum contains exactly those connections that have minimal stabilizer w.r.t. the action of gauge transforms [38, 33]. It has been proven that a stabilizer is minimal iff it contains precisely the constant center-valued gauge transforms. Moreover, the generic stratum has induced Haar measure 1.

6.5 Smooth Connections

Finally we are going to show that smooth connections are not only contained in a μ_0-zero subset [51], but also in a μ_{YM}-zero subset.

Theorem 6.8 $\mathcal{A}/\mathcal{G}$ is contained in subset of μ_{YM}-measure 0.

The idea of the proof extends that of the proof (see [51]) in the μ_0-case.

Proof

- Consider $\mathbf{G}$ again as a subset of some $U(N) \subseteq \mathrm{Gl}_{\mathbb{C}}(N) \subseteq \mathbb{C}^{N\times N}$, hence $\mathfrak{g} \subseteq \mathfrak{gl}_{\mathbb{C}}(N) = \mathbb{C}^{N\times N}$. Choose some Ad$\mathbf{G}$-invariant norm $\|\cdot\|_\bullet$ on $\mathbb{C}^{N\times N}$ and define $B_\varepsilon(e_{\mathbf{G}}) := \{g \in \mathbf{G} \mid \|g - e_{\mathbf{G}}\|_\bullet < \varepsilon\}$ for all $\varepsilon \in \mathbb{R}_+$. Obviously, $B_\varepsilon(e_{\mathbf{G}})$ is always an Ad-invariant set.
  Next we choose some bounded domain $U \subseteq \mathbb{R}^2$ given the Euclidian metric. We assume that the image of every $\alpha \in \mathcal{HG}$ used in this proof is contained in U.
- Now we define for all α and all real $r \in \mathbb{R}_+$ the (by the Ad-invariance of $B_\varepsilon(e_{\mathbf{G}})$) $\overline{\mathcal{G}}$-invariant set
  $$U_{\alpha,r} := \pi_\alpha^{-1}(B_{r|G_\alpha|}(e_{\mathbf{G}})) \subseteq \overline{\mathcal{A}}.$$
  We have
  $$\begin{aligned} \mu_{\overline{\mathcal{A}},\mathrm{YM}}(U_{\alpha,r}) &= \mu_{\mathrm{YM},\alpha}(B_{r|G_\alpha|}(e_{\mathbf{G}})) \\ &\leq \|\chi_\alpha\|_\infty\, \mu_{0,\alpha}(B_{r|G_\alpha|}(e_{\mathbf{G}})) \qquad (\mu_{\mathrm{YM},\alpha} = \chi_\alpha \odot \mu_{0,\alpha}) \\ &\leq \Bigl(\sum_{\nu=0}^{\dim \mathbf{G}} \mathrm{const}_\nu |G_\alpha|^{-\frac{1}{2}\nu}\Bigr)\, c(r|G_\alpha|)^{\dim \mathbf{G}} \\ &\qquad\qquad \text{(Proposition 5.3 and Lemma C.1)}, \end{aligned}$$
  where the last term is a polynom in $\sqrt{|G_\alpha|}$ whose lowest order equals $-\dim \mathbf{G} + 2\dim \mathbf{G} = \dim \mathbf{G} \geq 1$. ($\mathbf{G}$ has been assumed nontrivial and connected.) Hence, $\mu_{\mathrm{YM}}(U_{\alpha,r})$ goes to 0 for $|G_\alpha| \downarrow 0$.
- Now let $(\alpha_i)_{i\in\mathbb{N}}$ be some sequence of circles in U with $|G_{\alpha_i}| \downarrow 0$, where each two circles have m as unique common point. We define
  $$U_r := \bigcap_{i\in\mathbb{N}} U_{\alpha_i,r}.$$
  Obviously, $\mu_{\mathrm{YM}}(U_r) \leq \inf_i\{\mu_{\mathrm{YM}}(U_{\alpha_i,r})\} = 0$.
- On the other hand, for every $A \in \mathcal{A}$ there is a $c_A \in \mathbb{R}_+$ with $A \in U_{\alpha,c_A} \equiv \pi_\alpha^{-1}(B_{c_A|G_\alpha|}(e_{\mathbf{G}}))$ for all circles α (see Corollary B.2). Hence $A \in U_{c_A}$. Consequently, $U := \bigcup_{r\in\mathbb{N}_+} U_r$ is obviously a μ_{YM}-zero subset containing $\mathcal{A}$. Since U is $\overline{\mathcal{G}}$-invariant as well, $U/\overline{\mathcal{G}}$ is again a μ_{YM}-subset containing now $\mathcal{A}/\mathcal{G}$. **qed**

7 Generalization

Originally, we expected μ_{YM} to be absolutely continuous w.r.t. μ_0. This presumption has been induced by the observation that although S_{YM} cannot be extended from $\mathcal{A}/\mathcal{G}$ to $\overline{\mathcal{A}/\mathcal{G}}$, i.e. a direct definition via $\mathrm{d}\mu_{\mathrm{YM}} := \mathrm{e}^{-S_{\mathrm{YM}}}\mathrm{d}\mu_0$ is impossible, after exchanging limit and integral the expectation values are indeed completely well-defined. Moreover, it has been doubtful whether S_{YM} on $\mathcal{A}/\mathcal{G}$ can serve as a starting point for the definition of such an action on $\overline{\mathcal{A}/\mathcal{G}}$ because $\mathcal{A}/\mathcal{G}$ is contained in a μ_0-zero subset. However, as we have seen in the last section, the Yang-Mills measure and the Ashtekar-Lewandowski measure *are* inequivalent. This has the following fundamental consequence:

The interaction measure μ_{YM} cannot be constructed from μ_0 using the action method.

This means, there is no measurable function $\overline{S_{\mathrm{YM}}}$ on $\overline{\mathcal{A}/\mathcal{G}}$, such that $\mu_{\mathrm{YM}} = \mathrm{e}^{-\overline{S_{\mathrm{YM}}}} \odot \mu_0$.

This immediately raises the question, whether the Ashtekar-Lewandowski measure were simply the wrong starting point for the construction of the Yang-Mills measure. This argumentation is indeed entitled because in the naive limit of an infinite coupling ($\mathrm{g} \to \infty$) both measures are identical or – in other words – the Ashtekar-Lewandowski measure μ_0 is simply the Yang-Mills measure for infinite coupling. Physically it is obvious that both cases of finite and infinite coupling have to be essentially different. To underpin this measure-theoretically the corresponding measures are to be inequivalent. But, why should one take the measure of the rigid theory as a kinematical measure? Typically, one starts with the free theory anyway, hence with vanishing coupling ($\mathrm{g} = 0$). In this case, however, one sees that μ_{YM} is simply the Dirac measure in $e_{\mathbf{G}}$ which obviously is singular w.r.t. to the Yang-Mills measure of finite coupling as well. Not only that is why we consider μ_0 as a kinematically destined measure. On the one hand, μ_0 personifies (in complete contrast to a point measure) by its $\overline{\mathcal{G}}$-invariance and its even larger invariance on the graph level the principle of equal a-priori probability: Only the dynamics should tell us which configurations are favoured by the system. On the other hand, via μ_0 one can easily construct all continuum measures that are at the lattice level absolutely continuous w.r.t. the Haar measure. This is true as we already noticed even if the continuum measure is purely singular. Could not the continuum limit be the real deeper reason for the singularity of the interaction measure?

In order to study this question (on a very preliminary stage, of course) we review the proofs for the singularity in the Yang-Mills case and try to understand their basic ideas. First we have proven that the lattice measures for single flags are absolutely continuous (Proposition 5.3). For this it was crucial that the Fourier series $\chi_\beta := \sum_{\vec{\mathbf{n}}} \langle \chi_{\vec{\mathbf{n}}} \rangle_\beta \, \chi_{\vec{\mathbf{n}}}$ converges because then χ_β can be considered as a well-defined density function of $\mu_{\mathrm{YM},\beta}$ w.r.t. $\mu_{0,\beta} \equiv \mu_{\mathrm{Haar}}$. The convergence itself followed from the fast, here even exponential falling of $\langle \chi_{\vec{\mathbf{n}}} \rangle_\beta$ for increasing $\|\vec{\mathbf{n}}\|$. The absolute continuity of arbitrary lattice measures now came from the fact that certain (here precisely the non-overlapping) flags are independent random variables (Proposition 5.4). For the singularity of the continuum measure the for decreasing β increasing concentration of the density function χ_β around the identity of $\mathbf{G}$ is responsible (Lemma 6.2). Altogether we see that the presence of absolutely continuous lattice measures and purely singular continuum measures depends less on the concrete model under consideration, but more on three general properties of the expectation values. In the following we will deduce these three properties from three (physically relatively plausible) criteria [34]. For this we assume that we are given some physical theory that can be described within the Ashtekar approach using some (possibly unknown) measure μ and that provides us with some appropriate expectation values. Here neither the compact structure group $\mathbf{G}$ is fixed, nor the dimension of M is restricted to 2. Now we are going to explain the three mentioned properties.

7.1 Principle 1: Universality of the Coupling Constant

We are aiming at the following statement: If the theory considered has a (in a certain sense) universal coupling constant that by itself describes the coupling strength between the elementary (matter) particles of that theory, then $\langle \mathrm{tr}\ \phi(h_\beta) \rangle$ is determined completely by $\langle \mathrm{tr}\ h_\beta \rangle$ and the representation ϕ. Here $\langle f \rangle$ always denotes the physical expectation value of a function f.[11]

Let us consider the simplest case of a Yang-Mills theory with structure group $U(1)$. The elementary matter particles are the single-charged particles; the coupling constant be $\mathrm{g} = e$. Classically, the interaction, i.e. the potential between a particle and its antiparticle, is obviously proportional to g^2. Now we call the coupling constant to be *universal* if it yields

[11] In the following, we always assume $\langle \mathrm{tr}\ h_\beta \rangle \geq 0$.

immediately the (classical) interaction between *arbitrarily* charged particles: In particular, for composed particles with charges n and $-n$, resp., it is proportional $(ng)^2$. In general, one assumes that also the Wilson-loop expectation values $\langle h_\beta \rangle$ describe the potential between two oppositely charged static particles [81, 63]. Namely, if β is a rectangular loop running in space between $\vec{x}$ and $\vec{y}$ and in time between 0 and Δt, then the potential between the elementary particles resting in $\vec{x}$ and $\vec{y}$, resp., is given by

$$V_1(\vec{x} - \vec{y}) = - \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \ln \langle h_\beta \rangle.$$

A Wilson loop so just carries the interaction between an elementary particle-antiparticle pair; consequently, n loops should yield the interaction between a pair of an n-times charged particle and its antiparticle. On the other hand, (by the assumed universality of the coupling constant) the corresponding potential V_n is to be $n^2 V_1$. Hence, we have

$$n^2 V_1(\vec{x} - \vec{y}) = V_n(\vec{x} - \vec{y}) = - \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \ln \langle h_\beta^n \rangle.$$

Translating these two equations to the level of Wilson-loop expectation values, we get (at least in the limit $\Delta t \to \infty$)

$$\langle h_\beta^n \rangle = \langle h_\beta \rangle^{n^2}. \tag{13}$$

Indeed, the Wilson-loop expectation values of the $U(1)$ theory for $d = 2$ dimensions in the Ashtekar framework fulfill equation (13) – and namely not only for loops being large w.r.t. the time, but for all loops. Hence, it is by no means unrealistic to identify the validity of (13) for *all* loops with the existence of a universal coupling constant.

Let us now turn to gauge theories having general compact structure group $\mathbf{G}$. Using the following translation table

	$U(1)$	$\longmapsto$	$\mathbf{G}$
irreducible representation	n	$\longmapsto$	ϕ
dimension	1	$\longmapsto$	d_ϕ
normalized character	g^n	$\longmapsto$	$\frac{1}{d_\phi}\chi_\phi(g)$
Casimir eigenvalue	n^2	$\longmapsto$	c_ϕ

,

Equation (13) becomes

$$\frac{\langle \chi_\phi(h_\beta) \rangle}{d_\phi} = \Bigl(\frac{\langle \chi_{\phi_1}(h_\beta) \rangle}{d_{\phi_1}} \Bigr)^{\frac{c_\phi}{c_1}}, \tag{14}$$

where ϕ_1 denotes some nontrivial representation of $\mathbf{G}$, e.g., the standard one of $\mathbf{G} \subseteq U(N)$ on $\mathbb{C}^N$. Therefore, we will call a theory *having a universal coupling constant* iff Equation (14) is fulfilled for all nontrivial irreducible representations ϕ and all "non-selfoverlapping" loops β.

From the physical point of view such an assumption has a very interesting consequence: If a theory describes confinement (in the sense of an area law) between the elementary particles, *all other* charged particle-antiparticle pairs are confined as well. In the case of QCD this just explains why only quark-product particles consisting exclusively of baryons and mesons are freely observable; they are simply those particles whose total color charge $\sqrt{c_\phi}$ equals zero, i.e. whose quark product state transforms according the trivial $SU(3)$ representation. We remark that this discussion is not new because already about twenty years ago Yang-Mills theories with non-elementary charges have been considered (cf., e.g., [63]) and it has been shown that there occurs an area law as well. However, there one started with the action $\frac{1}{2}(\phi(F), \phi(F))$ specially taylored to those charges, such that a comparison between differently charged particles is not possible within *one* model – in contrast to our description.

The measure-theoretical implication of a universal coupling constant is now summarized in the following

Proposition 7.1 Let us given a theory with a universal coupling constant. Then we have:

	expectation value		measure
equality	$0 = \langle\chi_1\rangle_\beta$	$\Longrightarrow$	$\mu_\beta = \mu_{\text{Haar}}$
absolute continuity	$0 < \langle\chi_1\rangle_\beta < d_1$	$\Longrightarrow$	$\mu_\beta = \chi_\beta \odot \mu_{\text{Haar}}$
singularity	$\langle\chi_1\rangle_\beta = d_1$	$\Longrightarrow$	$\mu_\beta = \delta_{e_{\mathbf{G}}}$.

χ_β is again some smooth function.

Here, $\langle\cdot\rangle_\beta$ denotes the expectation value w.r.t. the image measure $\mu_\beta \equiv \pi_{\beta *}\mu$. For brevity, we write χ_1 instead of χ_{ϕ_1} etc.

Proof The cases $\langle\chi_1\rangle_\beta$ equals 0 or d_1 are clear.
Let now $0 < \langle\chi_1\rangle_\beta < d_1$. Define $\chi_\beta := \sum_{\vec{\mathbf{n}}}\langle\chi_{\vec{\mathbf{n}}}\rangle_\beta\, \chi_{\vec{\mathbf{n}}}$ and $b := -\frac{1}{c_1}\ln(\frac{1}{d_1}\langle\chi_1\rangle_\beta) > 0$. The absolute convergence of χ_β now follows as in Proposition 5.3 from

$$|\chi_\beta(g)| \leq \sum_{\vec{\mathbf{n}}}\langle\chi_{\vec{\mathbf{n}}}\rangle_\beta \|\chi_{\vec{\mathbf{n}}}\|_\infty = \sum_{\vec{\mathbf{n}}} d_{\vec{\mathbf{n}}}\Big(\frac{\langle\chi_1\rangle_\beta}{d_1}\Big)^{\frac{c_{\vec{\mathbf{n}}}}{c_1}} d_{\vec{\mathbf{n}}} = \sum_{\vec{\mathbf{n}}} \mathrm{e}^{-b\, c_{\vec{\mathbf{n}}}}\, d_{\vec{\mathbf{n}}}^2$$

for all $g \in \mathbf{G}$ and the standard estimates for $c_{\vec{\mathbf{n}}}$ and $d_{\vec{\mathbf{n}}}$. As above χ_β is even smooth. The relation $\mu_\beta = \chi_\beta \odot \mu_{\text{Haar}}$ comes again as in Proposition 5.3. **qed**

Finally, we note that just the universality of the coupling constant might be a desirable property of unified theories.

7.2 Principle 2: Independence Principle

It is well-known that non-overlapping loops yield independent random variables in the two-dimensional Yang-Mills theory. This means, for all finite sets $\beta_1, \ldots, \beta_n$ of such loops we have

$$\langle\chi_{\phi_1}(h_{\beta_1}) \,\cdots\, \chi_{\phi_n}(h_{\beta_n})\rangle = \langle\chi_{\phi_1}(h_{\beta_1})\rangle \,\cdots\, \langle\chi_{\phi_n}(h_{\beta_n})\rangle \tag{15}$$

for all representations $\phi_1, \ldots, \phi_n$ of the structure group $\mathbf{G}$ or – more precisely in terms of loop-networks –

$$\langle T_{\vec{\phi},\phi}\rangle_{\boldsymbol{\beta}} = \sqrt{d_\phi}\prod_\nu \frac{\langle\chi_{\phi_\nu}\rangle_{\beta_\nu}}{\sqrt{d_{\phi_\nu}}} \tag{16}$$

for all $\vec{\phi}$ and ϕ. However, to demand Equation (16) being satisfied for general theories is too restrictive physically because then every quantum state will be ultralocal and the Hamiltonian vanishes [62]. Actually we do not need such a general statement for all non-overlapping loops. What we rather need is a sufficiently large number of "small" loops fulfilling the relations above. Of course, non-overlapping loops remain natural candidates for this although their precise definition is worth discussing – in particular from dimension 3 on. As a minimal version one could view a set of loops as non-overlapping if there is a surface in the space-time such that these loops form a set of non-overlapping loops. However, this condition seems to be too restrictive. Perhaps one could resort to the knot theory instead; maybe there are physically interesting measures where Equation (16) is fulfilled for all sets of loops that have Gauss winding number 0.

Stopping this discussion here, we now just define a set of loops to be measure-theoretically independent iff it fulfills Equation (16) for all $\vec{\phi}$ and ϕ. We have analogously to Proposition 5.4

Proposition 7.2 The lattice measure $\mu_{\boldsymbol{\beta}}$ for a measure-theoretically independent weak fundamental system $\boldsymbol{\beta}$ is absolutely continuous if all single lattice measures μ_{β_i} are absolutely continuous.
The density function of $\mu_{\boldsymbol{\beta}}$ w.r.t. $\mu_{\text{Haar}}^{\#\boldsymbol{\beta}}$ then equals $\chi_{\boldsymbol{\beta}} = \chi_{\beta_1}\cdots\chi_{\beta_n}$.

Finally we declare a theory to obey the *independence principle* if there is an infinite number of loops of decreasing geometrical size (see below) that are independent both graph-theoretically and measure-theoretically.

7.3 Principle 3: Geometrical Regularity

After we have discussed two principles on the level of a fixed lattice, we are now going to discuss the continuum limit. If a theory is to have a continuum limit, then the holonomy along a loop should go to the identity when shrinking the loop to a point. In other words, since a measure in general encodes the distribution of certain objects, this suggests that the smaller the loop – the more the corresponding lattice measure should concentrate around the identity [82]. One could even demand that the lattice measure goes to the δ-distribution. Hence, it should be clear that the continuum limit naturally leads to singular measures.

In order to retrace this effect also quantitatively, we transfer it to the level of expectation values. First it is obvious that $\langle\chi_\phi\rangle_\beta$ should go to the dimension d_ϕ of the representation ϕ, if the (non-selfoverlapping) loop β becomes small. In the case of the two-dimensional Yang-Mills theory, one can even prove that $d_\phi - \langle\chi_\phi\rangle_\beta < \text{const}|G_\beta|$ holds, i.e. the expectation values are Hölder continuous w.r.t. the area $|G_\beta|$ enclosed by the loop β. Therefore we will call a theory *geometrically regular* iff there is a non-negative real function $\sigma(\beta)$ such that first

$$\frac{d_\phi - \langle\chi_\phi\rangle_\beta}{\sigma(\beta)} \tag{17}$$

is bounded as a function of β and second σ goes to 0 for shrinking β. For technical reasons we assume here that ϕ is the representation having smallest non-zero Casimir eigenvalue. Examples of conceivable functions $\sigma(\beta)$ are the area $|G_\beta|$ enclosed by β or the length $L(\beta)$ of β. Now we have

Proposition 7.3 In a theory with universal coupling constant, independence principle and geometrical regularity the continuum measure μ is always purely singular w.r.t. to the Ashtekar-Lewandowski measure μ_0.

We note that the geometrical regularity implies immediately the convergence of the density function χ_β to the δ-distribution in $e_{\mathbf{G}}$ for $\sigma(\beta) \to 0$.

Proof We can assume $\langle\chi_\phi\rangle_\beta \neq d_\phi$ for all these independent β. Otherwise even the lattice measure would be singular and the continuum measure all the more. The possibility $\langle\chi_\phi\rangle_\beta = 0$ is excluded by Equation (17).
The proof now follows mostly the proofs in Subsection 6.3, such that we present here the modifications only. First all special expectation values are to be substituted by $\langle\chi_{\vec{\mathbf{n}}}\rangle_\beta$ and then $|G_\beta|$ by the more general geometrical function $\sigma(\beta)$. Moreover, we have to observe that from $c_{\vec{\mathbf{n}}} \geq c_\phi$ for all $\vec{\mathbf{n}} \neq \vec{0}$ always

$$\frac{1 - \frac{1}{d_{\vec{\mathbf{n}}}}\langle\chi_{\vec{\mathbf{n}}}\rangle_\beta}{c_{\vec{\mathbf{n}}}\sigma(\beta)} = c_\phi \frac{1 - (\frac{1}{d_\phi}\langle\chi_\phi\rangle_\beta)^{\frac{c_{\vec{\mathbf{n}}}}{c_\phi}}}{\frac{c_{\vec{\mathbf{n}}}}{c_\phi}\sigma(\beta)} \leq c_\phi \frac{1 - \frac{1}{d_\phi}\langle\chi_\phi\rangle_\beta}{\sigma(\beta)}$$

follows. Hence the first term is uniformly bounded w.r.t. $\vec{\mathbf{n}}$ as a function of β. This suffices to transfer Lemma 6.2. Lemma 6.3 follows from the independence principle, and Lemma 6.4 is obvious. The present proof now follows from that of Theorem 6.5. **qed**

We remark that not only the singularity statement itself is true, but also the construction of the partition of $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ into disjoint supports of μ_0 and μ can be reused.

7.4 Examples

The first example for a purely singular measure has been already studied in the sections before – the Yang-Mills measure for $\mathbb{R}^2$. There are also striking hints that the same results can be gained for the other Yang-Mills theories on two-dimensional spaces as well. Namely, Sengupta [66] could prove on the classical level that in certain graphs (e.g. simple and small graphs that only contain homotopically trivial loops) the lattice measures are given by heat-kernel measures as in the $\mathbb{R}^2$-case. It can be expected that these results can be transferred to the Ashtekar approach as for $\mathbb{R}^2$ because holonomies outside a graph have been unimportant for the continuum limit in $\mathbb{R}^2$. In contrast to this, calculations of Aroca and Kubyshin [2] indicate for compact space-time that the area of the complement of a graph influences the expectation values by its finiteness. Hence, the universality of the coupling constant is given only approximatively. However, the interpretation of our principles has to be handled with care for compact space-times anyway: A limit $\Delta t \to \infty$ is hard to define. Nevertheless, in general one can expect purely singular continuum measures, hence a failure of the action method for $d = 2$.

To get a larger class of theories with purely singular continuum measure observe that the geometrical regularity is given for every theory with an area law $\langle \mathrm{tr}\ \phi(h_\beta)\rangle = d_\phi\ \mathrm{e}^{-\mathrm{const}|G_\beta|}$ or a length law $\langle \mathrm{tr}\ \phi(h_\beta)\rangle = d_\phi\ \mathrm{e}^{-\mathrm{const} L(\beta)}$. The former one is regarded as an indicator for confinement, and the latter one for deconfinement. Since among our three criteria just the geometrical regularity is the most important one for the singularity of the continuum measure, one could expect for both classes of theories that the action method fails. However, we have to mention that both the deconfinement and the confinement criterion need the corresponding laws for loops that are *large* in the time direction, but we actually need loops of *small* size to prove the singularity of the measure (at least in two dimensions). Both requirements can be matched together only in the area-law case: Here one can still generate loops with small area by choosing very narrow loops that are large w.r.t. the time which is impossible in the length-law case. Therefore, up to now, we can only claim that the appearance of an area law is a convincing indicator for a purely singular continuum measure.

However, if we are looking only for a failure of the action method (i.e. only for singular, not for purely singular measures), Proposition 6.1 implies that probably almost no theory can be gained using the action method on the continuum level.

8 Concluding Remarks

In the present article we have shown that a theory having a universal coupling constant and obeying an independence principle has absolutely continuous lattice measures, but a singular continuum measure if the theory is even geometrically regular. That is why neither non-generic connections nor the Gribov problem play any rôle in such a theory – provided one only looks at phenomena that can be discussed using the physical measure. However, it comes to a significant concentration of the continuum measure in a neighbourhood of the singular strata, since for small β the concentration of the density function χ_β increases in a neighbourhood of the "most" singular element $e_{\mathbf{G}} \in \mathbf{G}$.[12] This (qualitative) observation strengthens the conjecture of Emmrich and Römer [29] that singularities typically lead to concentrations of the wave functions. Maybe that this way the singular strata indeed get some influence although it cannot be described measure-theoretically.

However, from our point of view much more important is the realization that the singular-

[12] At the first glance, this seems to be a contradiction to $\chi_\beta \longrightarrow \delta_{e_{\mathbf{G}}}$. But, this is not correct because such a limiting process runs over different lattices and does therefore not yield a comparably convergent process after lifting to the level of $\overline{\mathcal{A}}/\overline{\mathcal{G}}$.

ity of the full interaction measure μ can be regarded as a typical property of the continuum. Hence, in particular, regular continuum limit and action method exclude each other: Assuming regularity, the definition of the interaction measure via $\mu := \mathrm{e}^{-S} \odot \mu_0$ is *impossible.* If one uses the action method, one can at most "approximate" it by lattice measures constructed this way. For all that it is mostly tried to get μ via the action method on the continuum level. Maybe that just this sticking to the action method is a deeper reason for the problems with the continuum limit or quantizations occuring permanently up to now. The desired absolute continuity seems to be a deceptfully simple tool, since it hides important physical phenomena. But, the singularity of a measure *per se* is completely harmless. There is no singularity in the dual picture, i.e. for the expectation values. Moreover, strictly speaking, the measure is no physically relevant quantity; only expectation values are detectable. So far it is to be evaluate absolutely positive that the interaction measure μ has not been used in our principles, but rather some of its expectation values. It has been completely sufficient to know that μ does *exist* at all for extracting properties of μ from our physical principles in a mathematically rigorous way. Thus, a measure is only the mathematical arena where anything happens. To know it might be superfluous from the physical point of view; however, one must be able to rely on it.

Acknowledgements

The author thanks Gerd Rudolph, Eberhard Zeidler and, in particular, Jerzy Lewandowski for very fruitful discussions. The author is supported by the Reimar-Lüst-Stipendium of the Max-Planck-Gesellschaft.

Appendix

A Estimates for the Fourier Analysis

In this appendix we give some criteria needed for the convergence proofs of series over $\mathbb{N}^l$. Before doing this we introduce some notation. The set $\mathrm{W}^-_{\vec{n},l} := \{\vec{x} \in \mathbb{R}^l \mid n_i - 1 < x_i \leq n_i \,\forall i\}$ describes a half-open cube with edge length 1 in $\mathbb{R}^l$ that is determined by the two corners $\vec{n} - \vec{1}$ and $\vec{n}$. Analogously we define $\mathrm{W}_{\vec{n},l} := \{\vec{x} \in \mathbb{R}^l \mid n_i - \frac{1}{2} \leq x_i < n_i + \frac{1}{2} \,\forall i\}$.

Proposition A.1 Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be some function, $l \in \mathbb{N}_+$ and $\nu_{\vec{n}} := \#\{i \mid n_i \neq 0\}$.
If there exists some $\rho \in \mathbb{R}_{\geq 0}$ and some function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_{\geq 0}$ with

- $\int_\rho^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r < \infty$ for all $\nu \in \mathbb{N}$, $1 \leq \nu \leq l$, and
- $|f(\|\vec{n}\|)| \leq g(\|\vec{x}\|)$ for all $\vec{n} \in \mathbb{N}^l$ with $\|\vec{n}\| \geq \rho + 1$ and for all $\vec{x} \in \mathrm{W}^-_{\vec{n},l} \cap \mathbb{R}^l_{\geq 0}$,

then $\sum_{\vec{n} \in \mathbb{N}^l \setminus \{\vec{0}\}} f(\|\vec{n}\|)$ converges absolutely and we have

$$\sum_{\substack{\vec{n} \in \mathbb{N}^l \\ \vec{n} \neq \vec{0} \\ \|\vec{n}\| \geq \rho + \sqrt{\nu_{\vec{n}}}}} |f(\|\vec{n}\|)| \leq \sum_{\nu=1}^{l} \binom{l}{\nu} \frac{\pi^{\frac{\nu}{2}}}{2^{\nu-1}\Gamma(\frac{\nu}{2})} \int_\rho^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r.$$

Here, Γ is the Gamma-function.

Proof We only consider functions with $f \geq 0$. (Here, convergence equals absolute convergence.)

- We divide the sum over all $\vec{n} \in \mathbb{N}^l \setminus \{\vec{0}\}$ into $2^l - 1$ partial sums. For this, $I \subseteq \{1, \ldots, l\}$ be some non-empty subset. We define the partial sum S_I belonging to I by $S_I := \sum_{\vec{n} \in \mathbb{N}^l, n_i \neq 0 \Longleftrightarrow i \in I} f(\|\vec{n}\|)$. Since $\mathbb{N}^l \setminus \{\vec{0}\}$ is the disjoint union of all

$\{\vec{n} \in \mathbb{N}^l \mid n_i \neq 0 \Longleftrightarrow i \in I\}$ with running I, it suffices to show the (absolute) convergence of S_I for every I.

- Obviously,
$$S_I \;=\; \sum_{\vec{n}\in\mathbb{N}^l, n_i\neq 0 \Longleftrightarrow i\in I} f(\|\vec{n}\|) \;=\; \sum_{\vec{n}'\in\mathbb{N}^{\#I}, n'_i\neq 0\ \forall i} f(\|\vec{n}'\|).$$
Hence we are left to prove the convergence of $S_\nu := \sum_{\vec{n}\in\mathbb{N}^\nu_+} f(\|\vec{n}\|)$ for all $1 \leq \nu \leq l$. We set
$$S_{I,\rho} := \sum_{\substack{\vec{n}\in\mathbb{N}^l \\ n_i\neq 0 \Longleftrightarrow i\in I \\ \|\vec{n}\|\geq\rho+\sqrt{\nu_{\vec{n}}}}} f(\|\vec{n}\|) \qquad \text{and} \qquad S_{\nu,\rho} := \sum_{\substack{\vec{n}\in\mathbb{N}^\nu_+ \\ \|\vec{n}\|\geq\rho+\sqrt{\nu}}} f(\|\vec{n}\|).$$
We have $S_I = S_{\#I}$ and $S_{I,\rho} = S_{\#I,\rho}$.
- It is clear that the union
$$\bigcup_{\vec{n}\in\mathbb{N}^\nu_+} \mathrm{W}^-_{\vec{n},\nu} = \mathbb{R}^\nu_+$$
is disjoint, and we have $\bigcup_{\vec{n}\in\mathbb{N}^\nu_+, \|\vec{n}\|<\rho+\sqrt{\nu}} \mathrm{W}^-_{\vec{n},\nu} \supseteq B^\nu_\rho \cap \mathbb{R}^\nu_+$: Let $\vec{x} \in B^\nu_\rho \cap \mathbb{R}^\nu_+$, and let $\vec{n}$ be that element in $\mathbb{N}^\nu_+$ with $\vec{x} \in \mathrm{W}^-_{\vec{n},\nu}$; then $\|\vec{n}\| \leq \|\vec{n}-\vec{x}\| + \|\vec{x}\| < \sqrt{\nu}+\rho$. Here, B^ν_ρ is the ball around the origin $\vec{0}$ with radius ρ. Now we have
$$\begin{aligned}
S_{\nu,\rho} &= \sum_{\vec{n}\in\mathbb{N}^\nu_+, \|\vec{n}\|\geq\rho+\sqrt{\nu}} f(\|\vec{n}\|) \\
&\leq \sum_{\vec{n}\in\mathbb{N}^\nu_+, \|\vec{n}\|\geq\rho+\sqrt{\nu}} \int_{\mathrm{W}^-_{\vec{n},\nu}} g(\|\vec{x}\|)\, \mathrm{d}^\nu\vec{x} \\
&\qquad (f(\|\vec{n}\|) \leq g(\|\vec{x}\|) \text{ for all } \vec{n}\in\mathbb{N}^\nu_+,\ \|\vec{n}\|\geq\rho+\sqrt{\nu}, \text{ and all } \vec{x}\in\mathrm{W}^-_{\vec{n},\nu}) \\
&\leq \int_{\mathbb{R}^\nu_+\setminus B^\nu_\rho} g(\|\vec{x}\|)\, \mathrm{d}^\nu\vec{x} \qquad (\text{by } \textstyle\bigcup_{\|\vec{n}\|\geq\rho+\sqrt{\nu}} \mathrm{W}^-_{\vec{n},\nu} \subseteq \mathbb{R}^\nu_+ \setminus B^\nu_\rho) \\
&= \frac{1}{2^\nu} \int_{\mathbb{R}^\nu\setminus B^\nu_\rho} g(\|\vec{x}\|)\, \mathrm{d}^\nu\vec{x} \\
&= \frac{1}{2^\nu} \operatorname{vol}(\partial B^\nu_1) \int_\rho^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r \\
&< \infty
\end{aligned}$$
by assumption. Since the set of all $\vec{n} \in \mathbb{N}^\nu_+$ with $\|\vec{n}\| < \rho + \sqrt{\nu}$ is finite, also $S_\nu \equiv \sum_{\vec{n}\in\mathbb{N}^\nu_+, \|\vec{n}\|<\rho+\sqrt{\nu}} f(\|\vec{n}\|) + S_{\nu,\rho}$ is finite.
- Therefore $\sum_{\vec{n}\in\mathbb{N}^l\setminus\{\vec{0}\}} f(\|\vec{n}\|)$ converges (absolutely).
- Moreover, we have
$$\begin{aligned}
\sum_{\vec{n}\in\mathbb{N}^l, \|\vec{n}\|\geq\rho+\sqrt{\nu_{\vec{n}}}} f(\|\vec{n}\|) &= \sum_{I\subseteq\{1,\ldots,l\}, I\neq\varnothing} S_{I,\rho} = \sum_{\nu=1}^{l} \binom{l}{\nu} S_{\nu,\rho} \\
&\leq \sum_{\nu=1}^{l} \binom{l}{\nu} \frac{1}{2^\nu} \operatorname{vol}(\partial B^\nu_1) \int_\rho^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r.
\end{aligned}$$
- The assertion follows from $\operatorname{vol}(\partial B^\nu_1) = \frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}$.

The case of an arbitrary function f is now clear. **qed**

Corollary A.2 Let $k, l \in \mathbb{N}$, $k + l \neq 0$, and let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be some function. If there exists a function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_{\geq 0}$ with
- $\int_0^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r < \infty$ for all $\nu \in \mathbb{N}$, $1 \leq \nu \leq k + l$, and
- $|f(\|\mathbf{\vec{n}}\|)| \leq g(\|\vec{x}\|)$ for all $\mathbf{\vec{n}} \in \mathbb{N}^l \times \mathbb{N}^k$, $\mathbf{\vec{n}} \neq \vec{0}$, and all $x \in \mathrm{W}^-_{\mathbf{\vec{n}},k+l} \cap \mathbb{R}^{l+k}_{\geq 0}$,

then $\sum_{\mathbf{\vec{n}}\in\mathbb{N}^l\times\mathbb{Z}^k, \mathbf{\vec{n}}\neq\vec{0}} f(\|\mathbf{\vec{n}}\|)$ converges absolutely and we have in this case

$$\sum_{\substack{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k\\ \vec{\mathbf{n}}\neq\vec{0}}} |f(\|\vec{\mathbf{n}}\|)| \leq 2^k \sum_{\nu=1}^{k+l} \binom{k+l}{\nu} \frac{\pi^{\frac{\nu}{2}}}{2^{\nu-1}\Gamma(\frac{\nu}{2})} \int_0^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r.$$

Proof For all $\vec{\mathbf{n}} \in \mathbb{N}^l \times \mathbb{N}^k$ we have obviously $\|\vec{\mathbf{n}}\| \geq \sqrt{\nu_{\vec{\mathbf{n}}}}$. Hence by Proposition A.1 the series $\sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{N}^k,\vec{\mathbf{n}}\neq\vec{0}} |f(\|\vec{\mathbf{n}}\|)|$ converges absolutely with

$$\sum_{\substack{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{N}^k\\ \vec{\mathbf{n}}\neq\vec{0}}} |f(\|\vec{\mathbf{n}}\|)| \leq \sum_{\nu=1}^{k+l} \binom{k+l}{\nu} \frac{\pi^{\frac{\nu}{2}}}{2^{\nu-1}\Gamma(\frac{\nu}{2})} \int_0^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r.$$

Consequently, $\sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} |f(\|\vec{\mathbf{n}}\|)| \leq 2^k \sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{N}^k,\vec{\mathbf{n}}\neq\vec{0}} |f(\|\vec{\mathbf{n}}\|)|$ converges absolutely as well. **qed**

If one is not interested in a concrete estimate, but only in a convergence statement, one can get help from

Corollary A.3 Let k, l and f be as above.
If there is a $\rho \in \mathbb{R}_{\geq 0}$ and a function $g : \mathbb{R}_{\geq\rho} \longrightarrow \mathbb{R}$, such that

- $\int_\rho^\infty g(r)\, r^{k+l-1}\, \mathrm{d}r < \infty$,
- g is monotonically decreasing on $[\rho,\infty)$ and
- $|f(r)| \leq g(r)$ on $[\rho,\infty)$,

then $\sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} f(\|\vec{\mathbf{n}}\|)$ is absolutely convergent.

Proof Set $\rho' := \rho + \sqrt{k+l}$.
By $\rho' \geq 1$ we have first $g(r)r^{\nu-1} \leq g(r)r^{k+l-1}$ for all $r \geq \rho'$ and $\nu \leq k+l$. Hence, also $\int_{\rho'}^\infty g(r)\, r^{\nu-1}\, \mathrm{d}r \leq \int_{\rho'}^\infty g(r)\, r^{k+l-1}\, \mathrm{d}r < \infty$ for all $\nu \leq k+l$.
Second we have $\|\vec{x}\| \geq \|\vec{\mathbf{n}}\| - \|\vec{\mathbf{n}}-\vec{x}\| \geq \rho'+1-\sqrt{k+l} = \rho+1$ for all $\|\vec{\mathbf{n}}\| \geq \rho'+1$ and $\vec{x} \in \mathrm{W}^-_{\vec{\mathbf{n}},k+l} \cap \mathbb{R}^{k+l}_{\geq 0}$. For monotonicity reasons we have $|f(\|\vec{\mathbf{n}}\|)| \leq g(\|\vec{\mathbf{n}}\|) \leq g(\|\vec{x}\|)$ for all $\vec{\mathbf{n}} \in \mathbb{N}^l \times \mathbb{N}^k$ with $\|\vec{\mathbf{n}}\| \geq \rho'+1$ and all $\vec{x} \in \mathrm{W}^-_{\vec{\mathbf{n}},k+l} \cap \mathbb{R}^{k+l}_{\geq 0}$.
Proposition A.1 yields the assumption for the sum over $\mathbb{N}^l \times \mathbb{N}^k$. For the sum over $\mathbb{N}^l \times \mathbb{Z}^k$ one argues as in Corollary A.2. **qed**

Corollary A.4 Let $k, l \in \mathbb{N}$ with $k+l \neq 0$ and $\rho \in \mathbb{R}_+$ be arbitrary, and let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be some function with $|f(x)| \leq \mathrm{const}\frac{1}{x^{k+l+1}}$ for all $x \geq \rho$.
Then $\sum_{\vec{\mathbf{n}}\in\mathbb{N}^l\times\mathbb{Z}^k,\vec{\mathbf{n}}\neq\vec{0}} f(\|\vec{\mathbf{n}}\|)$ converges absolutely.

Proof The function $g(x) := \mathrm{const}\frac{1}{x^{k+l+1}}$ is obviously monotonically decreasing on the whole $\mathbb{R}_+$. Moreover, we have $\int_\rho^\infty g(r)\, r^{k+l-1}\, \mathrm{d}r = \mathrm{const} \int_\rho^\infty r^{-2}\, \mathrm{d}r < \infty$. Corollary A.3 gives the assertion. **qed**

B Small Holonomies

In this appendix we study the behaviour of holonomies for small loops not restricting ourselves to two-dimensional manifolds. $\mathbf{G}$ is always considered as a subset of some $U(N) \subseteq \mathrm{Gl}_{\mathbb{C}}(N)$; hence, $\mathfrak{g} \subseteq \mathfrak{gl}_{\mathbb{C}}(N) = \mathbb{C}^{N\times N}$. Additionally, $\|\cdot\|_\bullet$ is some algebra norm on $\mathfrak{gl}_{\mathbb{C}}(N)$. To compute holonomies locally we choose a local chart $U \subseteq M$ with the chart mapping $\kappa : U \longrightarrow \kappa(U)$ and the coordinate functions x^μ. We arrange for every positive c

Definition B.1 Let $\alpha : [0,T] \longrightarrow U$ be some path in U and $x := \kappa \circ \alpha$ its image on $\kappa(U)$. We call α $(\mu\nu, c)$-round (or shortly c-**round**) iff

- im α is contained completely in the surface spanned by the coordinates x^μ and x^ν,
- α is a closed Jordan curve in that surface,
- $\sum_i |\dot{x}^i(t)|^2 = 1$ for all $t \in [0,T]$ and
- $T^2 \leq c|G_\alpha|_U$.

Here, G_α is the domain in the $\mu\nu$-surface enclosed by α. $G_{\alpha,U} := \int_{\kappa(G_\alpha)} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is its "oriented" and $|G_\alpha|_U := |\int_{\kappa(G_\alpha)} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu|$ its "absolute" area. ($|G_\alpha|_U$ is just the Euclidian area of G_α.)

We have chosen the term "round" because of the last condition $T^2 \leq c|G_\alpha|_U$. Due to the isoperimetrical inequality we have always $T^2 \geq 4\pi|G_\alpha|_U$ with equality precisely for the circle. Additionally, α has to become similar to a circle if we let c decrease. Nevertheless, for $c > 4\pi$ there is a huge number of paths α fulfilling the conditions above.

Now we have

Proposition B.1 Let the image of U in $\mathbb{R}^{\dim M}$ be convex and let $|x^\mu|$ for all μ be bounded on U by some $C \in \mathbb{R}_+$. Moreover, let $c \geq 4\pi$ be arbitrary, but fixed. Then for all $A \in \mathcal{A}$ there is a constant $\mathrm{const}_A \in \mathbb{R}$ (depending only on A, U, c and the algebra norm $\|\cdot\|_\bullet$), such that

$$\|h_A(\alpha) - (\mathbf{1} - F_{\mu\nu}(m_0)G_{\alpha,U})\|_\bullet \leq \mathrm{const}_A(|G_\alpha|_U)^{\frac{3}{2}}$$

for all μ, ν and all $(\mu\nu, c)$-round α in U with base point $m_0 \in U$.
Here, $F_{\mu\nu} := \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$ is the curvature for A.

The proof is not very difficult, but quite technical, and is therefore dropped here. It can be found in [39].

Corollary B.2 For all $A \in \mathcal{A}$ there is a constant $c_A \in \mathbb{R}$ depending on A, such that $\|h_A(\alpha) - \mathbf{1}\|_\bullet \leq c_A|G_\alpha|_U$ for all c-round paths $\alpha \in \mathcal{HG}$ in U.

Proof By Proposition B.1 there is a constant $\mathrm{const}_A \in \mathbb{R}$ for every $A \in \mathcal{A}$, such that

$$\|h_A(\alpha) - (\mathbf{1} - F_{\mu\nu}(m)G_{\alpha,U})\|_\bullet \leq \mathrm{const}_A(|G_\alpha|_U)^{\frac{3}{2}},$$

provided $\alpha \in \mathcal{HG}$ is a c-round path contained in the coordinate surface $(x^\mu, x^\nu) \subseteq U$. Since $|G_\alpha|_U$ is bounded, there is in each case some $c_A \in \mathbb{R}$ with $\|h_A(\alpha) - \mathbf{1}\|_\bullet \leq c_A|G_\alpha|_U$ for all round $\alpha \in \mathcal{HG}$. **qed**

C Haar Measure Estimate

We estimate the Haar measure of all $g \in \mathbf{G}$ whose distance to $e_\mathbf{G}$ is smaller than ε. Again we consider $\mathbf{G}$ as a subset of some $U(N) \subseteq \mathrm{Gl}_\mathbb{C}(N)$, hence $\mathfrak{g} \subseteq \mathfrak{gl}_\mathbb{C}(N) = \mathbb{C}^{N\times N}$, and choose some algebra norm $\|\cdot\|_\bullet$ on $\mathfrak{gl}_\mathbb{C}(N)$. Correspondingly we set

- $B_\varepsilon(e_\mathbf{G}) := \{g \in \mathbf{G} \mid \|g - e_\mathbf{G}\|_\bullet < \varepsilon\}$,
- $B_\varepsilon(\mathbf{1}) := \{g \in \mathrm{Gl}_\mathbb{C}(N) \mid \|g - \mathbf{1}\|_\bullet < \varepsilon\}$ and
- $B_\varepsilon(\mathbf{0}) := \{X \in \mathfrak{gl}_\mathbb{C}(N) \mid \|X\|_\bullet < \varepsilon\}$

for $\varepsilon \in \mathbb{R}_+$. Note that $e_\mathbf{G} = \mathbf{1}$, but $e_\mathbf{G}$ is used for $\mathbf{G}$ and $\mathbf{1}$ for $\mathrm{Gl}_\mathbb{C}(N)$.

Lemma C.1 There is a constant c with $\mu_{\mathrm{Haar}}(B_\varepsilon(e_\mathbf{G})) \leq c\,\varepsilon^{\dim \mathbf{G}}$ for all $\varepsilon > 0$.

Proof
- We consider the log-function [43]

$$\ln g = \sum_{n=1}^{\infty} (-1)^{k+1} \frac{(g-\mathbf{1})^k}{k}.$$

For $g \in B_1(\mathbf{1})$ this series converges absolutely and fulfills $\exp(\ln g) = g$. Additionally, we have $\ln(\exp X) = X$ for all $X \in B_{\ln 2}(\mathbf{0})$. Hence, for $g \in B_{\frac{1}{2}}(\mathbf{1})$ we get

$$\|\ln g\|_\bullet \leq \|g-\mathbf{1}\|_\bullet \sum_{n=1}^{\infty} \frac{\|g-\mathbf{1}\|_\bullet^{k-1}}{k} \leq (2\ln 2)\|g-\mathbf{1}\|_\bullet,$$

i.e. $B_\varepsilon(\mathbf{1}) = \exp(\ln(B_\varepsilon(\mathbf{1}))) \subseteq \exp(B_{(2\ln 2)\varepsilon}(\mathbf{0}))$ for all $\varepsilon < \frac{1}{2}$.

- $\exp : \mathfrak{g} \longrightarrow \mathbf{G}$ is a local diffeomorphism. Hence, there is an $\varepsilon_0 > 0$ (w.l.o.g. $\varepsilon_0 < 1$), such that $\exp^{-1} : B_{\varepsilon_0}(e_\mathbf{G}) \longrightarrow \exp^{-1}(B_{\varepsilon_0}(e_\mathbf{G}))$ is a diffeomorphism, hence a chart mapping.
  Since the Haar measure on $\mathbf{G}$ is a Lebesgue measure , there is a nowhere vanishing C^∞-function f with $(\exp^{-1})_*\mu_{\mathrm{Haar}} = f \odot \lambda^{\dim\mathfrak{g}}$ on $\exp^{-1}(B_{\varepsilon_0}(e_\mathbf{G}))$. Here, $\lambda^{\dim\mathfrak{g}}$ is the Lebesgue measure on $\mathfrak{g}$.
  By the compactness of $\exp^{-1}(\overline{B}_{\frac{1}{2}\varepsilon_0}(e_\mathbf{G}))$, $|f|$ has a maximum $f_0 < \infty$ there; hence
  $$\begin{aligned}\mu_{\mathrm{Haar}}(B_\varepsilon(e_\mathbf{G})) &\leq \mu_{\mathrm{Haar}}(\exp(B_{(2\ln 2)\varepsilon}(e_\mathbf{G}))) \equiv (\exp^{-1})_*\mu_{\mathrm{Haar}}(B_{(2\ln 2)\varepsilon}(e_\mathbf{G})) \\ &\leq f_0 \mathrm{vol}\,(B^{\dim\mathfrak{g}})(2\ln 2)^{\dim\mathfrak{g}}\varepsilon^{\dim\mathfrak{g}} =: \widetilde{c}\,\varepsilon^{\dim\mathfrak{g}}\end{aligned}$$
  for all $\varepsilon \leq \frac{1}{2}\varepsilon_0$. Here, $\mathrm{vol}\,(B^n)$ is the volume of the unit ball of the n-dimensional Euclidian space.
- Setting $c := \max\{\widetilde{c}, (\frac{2}{\varepsilon_0})^{\dim\mathfrak{g}}\}$ we get
  1. $\mu_{\mathrm{Haar}}(B_\varepsilon(e_\mathbf{G})) \leq \widetilde{c}\,\varepsilon^{\dim\mathfrak{g}} \leq c\,\varepsilon^{\dim\mathfrak{g}}$ for $\varepsilon \leq \frac{1}{2}\varepsilon_0$ and
  2. $\mu_{\mathrm{Haar}}(B_\varepsilon(e_\mathbf{G})) \leq 1 \leq (\frac{2\varepsilon}{\varepsilon_0})^{\dim\mathfrak{g}} \leq c\,\varepsilon^{\dim\mathfrak{g}}$ for $\varepsilon \geq \frac{1}{2}\varepsilon_0$ by the normalization of μ_{Haar}. **qed**

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