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On Artin's braid group and polyconvexity in the calculus of variations

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $F : \Omega \times \mathbb{R}^{2\times 2}_+ \to \mathbb{R}$ be a Carathèodory integrand such that $F(x, \cdot)$ is polyconvex for $\mathcal{L}^2$- a.e. $x \in \Omega$. Moreover assume that F is bounded from below and satisfies the condition $F(x, \xi) \to \infty$ as $\det \xi \to 0^+$ for $\mathcal{L}^2$- a.e. $x \in \Omega$. In this article we study the effect of domain topology on the existence and multiplicity of strong local minimizers of the functional

$$\mathbb{F}[u] := \int_\Omega F(x, \nabla u(x))\, dx,$$

where the map u lies in the Sobolev space $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$ with $p \geq 2$ and satisfies the pointwise condition $\det \nabla u(x) > 0$ for $\mathcal{L}^2$-a.e. $x \in \Omega$. We settle the question by establishing that $\mathbb{F}[\cdot]$ admits a set of strong local minimizers on $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$ that can be indexed by the group $\mathbb{P}_n \oplus \mathbb{Z}^n$, the direct sum of Artin's pure braid group on n strings and n copies of the infinite cyclic group. The dependence on the domain topology is through the number of holes n in Ω and the different mechanisms that give rise to such local minimizers are fully exploited by this particular representation.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain (open connected set) with a Lipschitz boundary $\partial\Omega$ and let $F : \Omega \times \mathbb{R}^{2\times 2}_+ \to \mathbb{R}$ be a Carathèodory integrand such that $F(x, \cdot)$ is polyconvex for $\mathcal{L}^2$- a.e. $x \in \Omega$. Moreover assume that F is bounded from below and satisfies the condition $F(x, \xi) \to \infty$ as $\det \xi \to 0^+$ for $\mathcal{L}^2$- a.e. $x \in \Omega$. Here $\mathbb{R}^{2\times 2}_+ = \{\xi \in \mathbb{R}^{2\times 2} : \det \xi > 0\}$.

In this article we address the question of existence and multiplicity of *local minimizers* of the functional

$$\mathbb{F}[u] = \int_\Omega F(x, \nabla u(x))\, dx$$

where the map u lies in the Sobolev space $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2) = W^{1,p}_0(\Omega, \mathbb{R}^2) + {\rm id}$ with $p \geq 2$ and satisfies the condition $\det \nabla u(x) > 0$ for $\mathcal{L}^2$-a.e. $x \in \Omega$.

It is often convenient to extend the integrand F to the entire space $\mathbb{R}^{2\times 2}$ by setting $F(x,\xi) = \infty$ for $\mathcal{L}^2$-a.e. $x \in \Omega$ and each ξ outside the set $\mathbb{R}^{2\times 2}_+$. In this way any map $u \in W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$ with $\mathbb{F}[u] < \infty$ will verify the condition $\det \nabla u(x) > 0$ $\mathcal{L}^2$-a.e. $x \in \Omega$. In the sequel we shall speak of F in this extended sense.

As the term local minimizer has different meanings in different contexts, let us proceed by clarifying the terminology. Assume that $1 \le q \le \infty$ and let $\bar{u} \in W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$ satisfy $\mathbb{F}[\bar{u}] < \infty$. Then $\bar{u}$ is referred to as an L^q (respectively $W^{1,q}$) local minimizer of $\mathbb{F}$ if and only if there exists $\delta > 0$ such that for all $u \in W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$ it holds that $\mathbb{F}[\bar{u}] \le \mathbb{F}[u]$ provided that $||u - \bar{u}||_{L^q} \le \delta$ (respectively $||u - \bar{u}||_{W^{1,q}} \le \delta$). We also adopt a classical terminology from the calculus of variations and refer to a $W^{1,q}$ local minimizer with $1 \le q < \infty$ or an L^q local minimizer with $1 \le q \le \infty$ as a *strong* local minimizer.

The central aim in this article is to study the interaction between domain topology and multiplicity of strong local minimizers of $\mathbb{F}[\cdot]$ in $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$. Such questions can quickly attract one's attention when one observes that for star-shaped domains Ω, when F does not depend on the spatial variable x and is additionally strictly $W^{1,p}$-quasiconvex at $\xi = I$ in the sense of J. Ball & F. Murat [2], any $W^{1,p}$ local minimizer of $\mathbb{F}[\cdot]$ is bound to coincide with the identity map (the argument being similar to that in [14] pp. 2). On the other hand for any annular domain in $\mathbb{R}^2$, a heuristic argument of F. John [7] followed by the work of K. Post & J. Sivalogonathan [12], imply that $\mathbb{F}[\cdot]$ admits at least countably many strong local minimizers in the space $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$, a sharp contrast to the former uniqueness result. (As will become clear later, the heuristic argument of F. John corresponds to what topologists often call a *Dehn*-twist and quite surprisingly such arguments were known to the latter community as early as the thirties (cf. e.g. M. Dehn [6])).

Motivated by this observation, we focus on the case of Ω being an arbitrary bounded Lipschitz domain in $\mathbb{R}^2$ and address the question of how the topology of Ω relates to the number of strong local minimizers of $\mathbb{F}[\cdot]$ in the space $W^{1,p}_{\rm id}(\Omega, \mathbb{R}^2)$. In particular can one obtain multiplicity bounds for such minimizers in terms of the homotopy or homology groups of Ω, a question that was raised in the general multi-dimensional setting in [14].

The answer to the above questions in the two dimensional case studied here is affirmative and depends only on the number of holes in Ω. For clarification, we refer to the domain Ω as having n holes (with n being a non negative integer) provided that $\partial\Omega$ consists of $n+1$ connected components. That for bounded Lipschitz domains this number is finite is immediate (cf. Section 2). It is quite interesting that this connection can be simply stated in terms of the number of holes, as opposed to the higher dimensional case where one has to account for other phenomenon as well (cf. [15]). The link to the homotopy and homology groups of Ω is now natural as Ω is an aspherical space and thus the non-trivial groups in the homotopy and homology sequences correspond respectively to the fundamental group $\pi_1(\Omega)$ being the free group of rank n with its Abelianization $H_1(\Omega, \mathbb{Z})$ being isomorphic to $\mathbb{Z}^n$ (clearly the connectedness of Ω implies that

$H_0(\Omega, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$). With this brief introduction we can now arrive at the main result of this article:

Main Theorem. *Let $F : \Omega \times \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be polyconvex as described and suitably coercive. Then the functional $\mathbb{F}[\cdot]$ admits a set of L^1 local minimizers indexed by the group $\mathbb{P}_n \oplus \mathbb{Z}^n$.*

The group $\mathbb{P}_n \oplus \mathbb{Z}^n$ here is a direct sum of the *pure* braid group $\mathbb{P}_n$ on n string introduced by E. Artin and n copies of the infinite cyclic group. In addition to a multiplicity result, as will be clear later, this presentation also describes the different mechanisms that are involved in giving rise to such local minimizers. We refer the reader to Section 3 for a detailed discussion on the structure of this group. Finally we note that the restriction to the boundary values of identity is merely for convenience and the above theorem holds true for any boundary data corresponding to the restriction to $\partial\Omega$ of an orientation preserving homeomorphism of the closure of the domain Ω into $\mathbb{R}^2$.

NOTATION AND TERMINOLOGY. For $a \in \mathbb{R}^2$ and $r > 0$, we denote the open (closed) disk with center a and radius r by $D(a,r)$ (respectively $D[a,r]$) and put $D = D(0,1)$. The circle $c(a,r)$ refers to the boundary of the disk $D(a,r)$. For $0 < r < R$ we denote the open (closed) annulus with center a and inner and outer radii r and R by $A(a,r,R)$ (respectively $A[a,r,R]$) and put $A(r,R) = A(0,r,R)$. For any $E \subset \mathbb{R}^2$ we denote by $\operatorname{int} E$ its interior and by $\operatorname{cls} E$ its closure. The space of real 2×2 matrices is denoted by $\mathbb{R}^{2\times 2}$ and the open subset containing those with positive determinant by $\mathbb{R}^{2\times 2}_+$. Group isomorphism is denoted by $\cong$. The Greek letters ϕ, ψ are used to denote self-maps and in particular self-homeomorphisms of a domain whereas τ often denotes a homeomorphism (diffeomorphism) between two different domains. Finally homotopy between maps is denoted by $\simeq$.

2 Circular domains in $\mathbb{R}^2$

A bounded domain $\Omega \subset \mathbb{R}^2$ is referred to as circular if its boundary consist of a finite number of pairwise disjoint circles. If for some non negative integer n, the boundary $\partial\Omega$ consist of $n+1$ such circles we often refer to Ω as an n-circular domain. A simple dilatation and translation in the plane shows that any circular domain is diffeomorphic to one whose outer boundary is the unit circle. The following observation will be frequently used in the sequel.

Proposition 2.1. *Assume that Ω_1 and Ω_2 are two circular domains with n_1 and n_2 holes each. Then $\operatorname{cls}\Omega_1$ and $\operatorname{cls}\Omega_2$ are diffeomorphic if and only if $n_1 = n_2$.*

Proof. We establish only the "if" part as the other one is clear. For this let n denote the common value of n_1 and n_2 and assume that $\{a^1, ..., a^n\}$ and $\{b^1, ..., b^n\}$ are the centers of the inner boundary components of Ω_1 and Ω_2 with the outer boundary of each being the unit circle.

We first observe that, up to diffeomorphism, the radii of the inner boundaries can be assumed small and arbitrary, in particular equal. For this take $0 < s < R \le 1$ and $0 < t < R \le 1$ and let $\tau_1 : A[s,R] \to A[t,R]$ be the map defined in the polar coordinates (r,θ) by

$$\tau_1 : (r,\theta) \mapsto (\varrho(r),\theta), \tag{2.1}$$

where $\varrho \in \mathrm{C}^\infty[s,1]$ verifies $\varrho(s) = t$, $\varrho(r) = r$ for $r \in [R,1]$ and $\varrho' > 0$ on $[s,1]$. That τ_1 is a diffeomorphism is clear and simple calculations show that $\det \nabla\tau_1 = \varrho\varrho'/r$. The assertion now follows by applying a suitable translate of τ_1 at each boundary circle $c(a,r)$ (respectively $c(b,r)$) by setting $s = r$ and t to be the desired radius and selecting R so that the circle $c(a,R)$ (respectively $c(b,R)$) does not intersect any of the other boundary components.

We now claim that for any pair of points $a,b \in D$ and any $r > 0$ sufficiently small there is a smooth map that pulls the disk $D(a,r)$ to $D(b,r)$ along the line segment joining a to b in D. More specifically that we can associate to such data an orientation preserving diffeomorphism $\tau_2 : \operatorname{cls} D\backslash D(a,r) \to \operatorname{cls} D\backslash D(b,r)$ that coincides with the identity map outside a strip parallel to the line segment joining a to b.

Indeed assume that the coordinate axis are translated and rotated such that b lies at the origin and a lies on the vertical axis at a distant ℓ above b. Let τ_2 be the map defined by

$$\tau_2 : (x_1,x_2) \mapsto (x_1, x_2 - \ell\zeta(x_1)\eta(x_2)), \tag{2.2}$$

where $\zeta \in \mathrm{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp}\zeta \subset [-2r,2r]$ such that $\zeta = 1$ on $[-r,r]$ with $\zeta' \ge 0$ in $(-2r,-r)$ and $\zeta' \le 0$ in $(r,2r)$, and $\eta \in \mathrm{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp}\eta \subset [-2r,\ell+2r]$ such that $\eta = 1$ on $[\ell-r,\ell+r]$ with $0 \le \eta' < 1/\ell$ in $[-2r,\ell-r]$ and $\eta' \le 0$ in $(\ell+r,\ell+2r)$. (The existence of such η follows easily by mollifying the piecewise affine function that is obtained by slightly perturbing the function (again piecewise affine) having the same values as η at the end points of the corresponding sub-intervals). Thus clearly $\det \nabla\tau_2 = 1 - \ell\zeta\eta > 0$ in $\operatorname{cls} D$.

Applying the diffeomorphism τ_2 as many times as necessary it is possible to show that when r is sufficiently small, one can even pull the disk $D(a,r)$ to $D(b,r)$ along any *polygonal* line joining a to b in D, having no self-intersection and with the resulting diffeomorphism coinciding with the identity map outside a strip parallel to the polygonal line.

Using this idea it is now simple to complete the proof. We assume that for each $1 \le i \le n$, the points a^i and b^i are distinct (otherwise we remove them from the sets and proceed with the remaining ones). We connect the points in $\{a^1,...,a^n\}$ to the ones in $\{b^1,...,b^n\}$ with the same index by disjoint polygonal lines having no self-intersections (that this is always possible follows form the fact that the set obtained by removing a finite union of pairwise disjoint, simple Jordan arcs from the unit disk is connected (cf. e.g. [5] or [10])). We let $10r > 0$ denote the minimum distance between any two such lines or between one and the boundary of D. Applying the diffeomorphism τ_1 we shrink the inner disks to ones with all having equal radius r. Then use the diffeomorphism τ_2

to move these new disks inside D, each along the corresponding polygonal line and finally scale back to get the boundary circles of Ω_2. This gives the desired diffeomorphism, indeed orientation preserving. The proof is thus complete. $\square$

We now make a simple observation regarding arbitrary bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$. Indeed we claim that for such Ω there exists a finite collection of closed Jordan curves (infact Lipschitz) $\{\gamma_0, \gamma_1, ..., \gamma_n\}$ in the Euclidean plane such that $\Omega = \Omega_0 \setminus \cup_{i=1}^{n} \operatorname{cls} \Omega_i$, where for $0 \le i \le n$ we denote by Ω_i the inside of γ_i. To justify this assertion we note that from Ω being Lipschitz, it follows that $\partial\Omega$ has only finitely many connected components (as otherwise one could take a sequence of points $x_j \to x_\infty$ with each x_j lying on a distinct boundary component and reach a contradiction by visualising that Ω does not lie on one side of its boundary at x_∞). As each connected component of the boundary is locally a Lipschitz graph and no self-intersection can occur, the argument can be completed using standard devices from plane topology (cf. e.g. [5] and [10]). For convenience and for future reference, in the above representation of $\partial\Omega$ we refer to γ_0 as the outer boundary of Ω and often write $\partial^{\mathrm{o}}\Omega$.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists a non negative integer n and for every n-circular domain Ω_1 a homeomorphism $\tau : \operatorname{cls}\Omega \to \operatorname{cls}\Omega_1$ such that $\tau(\partial^{\mathrm{o}}\Omega) = \partial^{\mathrm{o}}\Omega_1$. In addition τ can be chosen to be a diffeomorphism when restricted to Ω.*

Proof. It follows from the discussion prior to the proposition that for some non negative integer n the boundary $\partial\Omega$ consists of $n+1$ connected components each homeomorphic to the unit circle. There are now several ways to establish the required homeomorphism. We choose a version of the Riemann mapping theorem that asserts that Ω is conformally equivalent to an n-circular domain Ω_2 (cf. [5] pp. 106). Moreover according to a variant of an extension theorem of Carathéodory (cf. [5] pp. 82) this conformal map can be extended to a homeomorphism $\tau : \operatorname{cls}\Omega \to \operatorname{cls}\Omega_2$. The proof can now be concluded by appealing to the previous proposition. $\square$

Remark 2.1. Note that under sufficient regularity of the boundary $\partial\Omega$, one can extend the homeomorphism τ in Proposition 2.2, to a homeomorphism between $\operatorname{cls}\Omega$ and $\operatorname{cls}\Omega_1$ with any desired degree of smoothness (cf. e.g. [11]). The lack of continuity up to the boundary for the derivatives of τ in the general case considered here does not cause any inconvenience to us (cf. Section 4).

3 The mapping class group of Ω

We denote by $\Lambda^{\mathrm{o}}_{\mathrm{id}}(\Omega)$ the group consisting of those self-homeomorphisms of $\operatorname{cls}\Omega$ that are identity on the outer boundary $\partial^{\mathrm{o}}\Omega$. The subgroup consisting of the homeomorphisms that are identity on the entire boundary is denoted by $\Lambda_{\mathrm{id}}(\Omega)$.

Two homeomorphisms $\phi, \psi \in \Lambda^{\mathrm{o}}_{\mathrm{id}}(\Omega)$ with $\phi|_{\partial\Omega} = \psi|_{\partial\Omega}$ are referred to as homotopic (or often homotopic in $\Lambda^{\mathrm{o}}_{\mathrm{id}}(\Omega)$) if and only if there exists a continuous map $h : [0,1] \times \operatorname{cls}\Omega \to \operatorname{cls}\Omega$ with $h(t,x) = \phi(x)$ for every $x \in \partial\Omega$ and $t \in [0,1]$

such that $h(0,\cdot) = \phi(\cdot)$ and $h(1,\cdot) = \psi(\cdot)$. It is thus clear that if $\phi, \psi \in \Lambda_{\rm id}(\Omega)$ are homotopic, then for each $t \in [0,1]$ the homeomorphism $h(t,\cdot) \in \Lambda_{\rm id}(\Omega)$. In this case we refer to the two homeomorphisms as being homotopic in $\Lambda_{\rm id}(\Omega)$.

Of special importance in the sequel is the space $\mathrm{C}_{\rm id}(\Omega)$ consisting of those continuous maps from $\mathrm{cls}\,\Omega$ to itself that are identity on the entire boundary. It follows from standard arguments based on degree theory that indeed any such map is onto. Similar to that of homeomorphisms we say that two continuous maps $\phi, \psi \in \mathrm{C}_{\rm id}(\Omega)$ are homotopic (or often homotopic in $\mathrm{C}_{\rm id}(\Omega)$) if and only if there exists a continuous map $h : [0,1] \times \mathrm{cls}\,\Omega \to \mathrm{cls}\,\Omega$ with $h(t,x) = x$ for every $x \in \partial\Omega$ and $t \in [0,1]$ such that $h(0,\cdot) = \phi(\cdot)$ and $h(1,\cdot) = \psi(\cdot)$. We denote by $\mathcal{C}_\Omega$ the *set* consisting of the homotopy classes of maps in $\mathrm{C}_{\rm id}(\Omega)$ or equivalently $\pi_0(\mathrm{C}_{\rm id}(\Omega), \mathrm{id})$.

For notational convenience in the sequel the product $\phi\,\psi$ of two homeomorphisms ϕ and $\psi \in \Lambda^{\rm o}_{\rm id}(\Omega)$ (respectively two continuous maps ϕ and $\psi \in \mathrm{C}_{\rm id}(\Omega)$) refers to the homeomorphism $\psi \circ \phi \in \Lambda^{\rm o}_{\rm id}(\Omega)$ (respectively continuous map $\psi \circ \phi \in \mathrm{C}_{\rm id}(\Omega)$). This convention will be particularly helpful in the proof of the main result.

The *mapping class group* of Ω is the quotient group of $\Lambda_{\rm id}(\Omega)$ subject to the equivalence relation of homotopy of maps or equivalently $\pi_0(\Lambda_{\rm id}(\Omega), \mathrm{id})$. We denote this group by $\mathcal{M}_\Omega$. Our aim in this section is to give a convenient characterisation of this group in terms of certain topological invariants of the domain Ω. That this is indeed possible is the content of the following proposition.

Proposition 3.1. *The mapping class group $\mathcal{M}_\Omega$ is isomorphic to the group $\mathbb{P}_n \oplus \mathbb{Z}^n$ where n denotes the number of holes in Ω.*

Before focusing on the proof of this proposition we develop some necessary tools that are required for the subsequent arguments. In doing so we make extensive use of Artin's braid groups on n strings. We refer the interested reader to the monograph by J. Birman [3] and the original paper of E. Artin [1] for further reference.

The *full* braid group on n strings (with $n \geq 1$) is denoted throughout by $\mathbb{B}_n$ and the subgroup consisting of the pure braids is denoted by $\mathbb{P}_n$. For notational consistency we set $\mathbb{B}_0$ and $\mathbb{P}_0$ to be the trivial group. According to the representation theorem of E. Artin (cf. [1] or [3] pp. 18) when $n \geq 2$, the group $\mathbb{B}_n$ is isomorphic to the abstract group generated by the $n-1$ elements $\sigma_1, ..., \sigma_{n-1}$ and subject to the defining relations

$$\begin{cases} \sigma_i\,\sigma_j = \sigma_j\,\sigma_i & \text{when } |i-j| \geq 2, \\ \sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} & \text{when } 1 \leq i \leq n-2. \end{cases}$$

The trivial braid is represented by ε_n. We also recall a result of W. Chow (cf. [4]) to the effect that for $n \geq 2$, the center of the group $\mathbb{B}_n$ is infinite cyclic and generated by the braid $\sigma_c = (\sigma_1 \ldots \sigma_{n-1})^n$. The braid σ_c has a simple and interesting geometric interpretation corresponding to a simultaneous twist of the n strings.

We denote by F_n the free (non-Abelian) group of rank n (and when necessary refer to its generators as $q_1, ..., q_n$). It is easy to see that this group is isomorphic to the fundamental group of any bounded Lipschitz domain with n holes. Moreover any homeomorphism $\phi \in \Lambda_{\mathrm{id}}(\Omega)$ (respectively any continuous map $\phi \in \mathrm{C}_{\mathrm{id}}(\Omega)$) induces in a natural way an automorphism (respectively an endomorphism) on this latter group. A representation theorem of E. Artin (cf. e.g. [3] pp. 30) characterises the braid group $\mathbb{B}_n$ as being isomorphic to a suitable subgroup of the group of automorphisms of F_n. This observation would be of particular relevance in the sequel.

It is technically more convenient to carry out much of the subsequent analysis on circular domains first and then pass on to the general case of Lipschitz domains by suitable homeomorphisms. To this end we define for each non negative integer n an n-circular domain $D_n \subset \mathbb{R}^2$ obtained by removing n smaller disks from the interior of the unit disk D, in such a way that the centers of these disks $\{a^1, ..., a^n\}$ lie on the horizontal axis, equi-distant from one another and the boundary and that the radii of these disks are all $r_n = 1/(4n+4)$.

Our initial goal in this section is to associate to each domain D_n a family of homeomorphisms in $\Lambda^{\mathrm{o}}{}_{\mathrm{id}}(D_n)$ that are of fundamental importance in the study of the mapping class group of D_n. First we introduce two basic homeomorphisms ϕ and ψ on two fixed domains Ω_1 and Ω_2. Then we use these particular homeomorphisms as building blocks to construct the stated family of maps.

The twist homeomorphisms: Let $\Omega_1 \subset \mathbb{R}^2$ denote the annulus $A(1/2, 1)$ and choose a monotone function $\varrho_1 \in \mathrm{C}^\infty(\mathbb{R})$ such that $0 \le \varrho_1 \le 1$ with $\varrho_1(r) = 0$ for $r \le 1/2$ and $\varrho_1(r) = 1$ for $r \ge 1$. Set $\phi \in \mathrm{C}^\infty(\mathrm{cls}\,\Omega_1, \mathbb{R}^2)$ to be the map defined in the polar coordinates (r, θ) via

$$\phi : (r, \theta) \mapsto (r, \theta - 2\pi\varrho_1(r)). \tag{3.1}$$

Simple calculations show that $\det \nabla\phi = 1$ everywhere in Ω_1 and that $\phi|_{\partial\Omega_1} = \mathrm{id}$. It is therefore clear that $\phi \in \Lambda_{\mathrm{id}}(\Omega_1)$. Indeed ϕ is a map that keeps the inner boundary of Ω_1 fixed while rotates the outer boundary by 2π in the clockwise direction.

Assume now that $\Omega \subset \mathbb{R}^2$ is given and let γ be a closed Jordan curve in Ω. Let $\tau : \mathrm{cls}\,\Omega_1 \to \mathrm{cls}\,U_\gamma$ be an orientation preserving homeomorphism where U_γ is a neighbourhood of γ in Ω. Then a Dehn-twist along γ refers to a homeomorphism $\phi^\gamma \in \Lambda_{\mathrm{id}}(\Omega)$ that extends the map $\tau \circ \phi \circ \tau^{-1}$ from U_γ to $\mathrm{cls}\,\Omega$ by identity.

Dehn-twists are crucial instruments in the study of self-homeomorphisms of surfaces (cf. M. Dehn [6]). In the context of planar domains, we recall that when Ω is circular, any homeomorphism $\phi \in \Lambda_{\mathrm{id}}(\Omega)$ is homotopic in $\Lambda_{\mathrm{id}}(\Omega)$ to a product of Dehn-twists along closed Jordan curves in Ω (cf. W. Lickorish [8] pp. 537). A further observation in this regard is that isotopic curves in Ω give rise to homotopic twists in $\Lambda_{\mathrm{id}}(\Omega)$. We often use this fact to replace a twist, up to homotopy, with another one by taking a suitable isotopy of the curve corresponding to the initial twist.

In the case of Ω being the circular domain D_n introduced above and with $n \ge 1$, we consider the collection of circles $\{c_1, ..., c_n\}$ where $c_i = c(a^i, 3r_n/2)$ for $1 \le i \le n$. Then every such circle bounds an inner component of the boundary and by taking as neighbourhoods $U_{c_i} = A(a^i, r_n, 2r_n)$ with the obvious choice of $\tau(\cdot) = a^i + \cdot/(2n+2)$, we get a collection of Dehn-twists along the latter circles. For future reference we denote this set by $\Phi_n = \{\phi_1, ..., \phi_n\}$.

The shift homeomorphisms: Put $a^+ = (1/2, 0)$, $a^- = (-1/2, 0)$ and let $\Omega_2 \subset \mathbb{R}^2$ denote the 2-circular domain obtained by removing the closed disks $D[a^\pm, 1/8]$ from the unit disk D.

Choose $\varrho_2 \in \mathrm{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} \varrho_2 \subset [1/4, 3/4]$ such that $\varrho_2 = 1$ on $[3/8, 5/8]$ with $\varrho_2' \geq 0$ in $(1/4, 3/8)$ and $\varrho_2' \leq 0$ in $(5/8, 3/4)$. Let $\psi \in \mathrm{C}^\infty(\operatorname{cls}\Omega_2, \mathbb{R}^2)$ be the map defined in the polar coordinates (r, θ) via

$$\psi : (r, \theta) \mapsto (r, \theta + \pi \varrho_2(r)). \tag{3.2}$$

Simple calculations again show that $\det \nabla \psi = 1$ everywhere in Ω_2 and that $\psi|_{\partial D} = \mathrm{id}$. Thus $\psi \in {\Lambda^o}_{\mathrm{id}}(\Omega_2)$. The homeomorphism ψ corresponds to a half rotation in a suitable inner strip of Ω_2 that results in the interchange of the two inner boundary circles.

We now construct for any $n \geq 2$, a family of homeomorphisms $\Psi_n = \{\psi_1, ..., \psi_{n-1}\}$ in $\Lambda^o_{\mathrm{id}}(D_n)$ that are naturally associated to the simple braids in $\Sigma_n = \{\sigma_1, ..., \sigma_{n-1}\}$. For this put $\rho = 2/(n+1)$ and for $1 \leq i \leq n-1$ let α^i denote the midpoint of the interval (a^i, a^{i+1}). Then $\psi_i \in \mathrm{C}^\infty(\operatorname{cls} D_n, \mathbb{R}^2)$ refers to the homeomorphism that extends $\alpha^i + \rho\psi((\cdot - \alpha^i)/\rho)$ from $\operatorname{cls} D_n \cap D(\alpha^i, \rho)$ to $\operatorname{cls} D_n$ by identity.

Following the terminology for simple braids, in the sequel we refer to any of the homeomorphisms in Ψ_n as simple shifts and similarly to any of the homeomorphisms in Φ_n as simple twists. It is easy to see that the simple twists commute with one another and the simple shifts (indeed they lie in the center of the group $\Lambda^o_{\mathrm{id}}(D_n)$).

Lemma 3.1. *Assume that ϕ^i and ψ^i for $i = 1, 2$ are arbitrary homeomorphisms in $\Lambda^o_{\mathrm{id}}(\Omega)$ with $\psi^1 \simeq \psi^2$ in $\Lambda^o_{\mathrm{id}}(\Omega)$. Then*
(i) $\phi^1\,\psi^1\,\phi^2 \simeq \phi^1\,\psi^2\,\phi^2$ in $\Lambda^o_{\mathrm{id}}(\Omega)$,
(ii) if in addition $\phi^1\,\psi^1\,\phi^2 \in \Lambda_{\mathrm{id}}(\Omega)$, then $\phi^1\,\psi^1\,\phi^2 \simeq \phi^1\,\psi^2\,\phi^2$ in $\Lambda_{\mathrm{id}}(\Omega)$, and
(iii) if $\phi^1\,\phi^2 \simeq \mathrm{id}$ in $\Lambda_{\mathrm{id}}(\Omega)$ and $\psi^1 \simeq \mathrm{id}$ in $\Lambda_{\mathrm{id}}(\Omega)$, then $\phi^1\,\psi^1\,\phi^2 \simeq \mathrm{id}$ in $\Lambda_{\mathrm{id}}(\Omega)$.

Proof. The cases (i) and (ii) are easy. For the case (iii) we first use (ii) to deduce that $\phi^1\,\psi^1\,\phi^2 \simeq \phi^1\,\phi^2$ and then conclude. $\square$

Lemma 3.2. *The simple shifts in Ψ_n satisfy the relations*
(i) $\psi_i\,\psi_j = \psi_j\,\psi_i$ for $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$,
(ii) $\psi_i\,\psi_{i+1}\,\psi_i \simeq \psi_{i+1}\,\psi_i\,\psi_{i+1}$ for $1 \leq i \leq n-2$, and
(iii) for any $2 \leq k \leq n-1$,

$$\prod_{i=1}^{k} \psi_i \prod_{j=1}^{m} \psi_{k-j} \simeq \prod_{j=1}^{m} \psi_{k-j+1} \prod_{i=1}^{k} \psi_i, \tag{3.3}$$

where $1 \leq m \leq k-1$.

Proof. The cases (i) and (ii) follow from the definition of the simple shifts and by direct verification (notice the analogy with the defining relations for the braid group $\mathbb{B}_n$). To establish (iii) we fix k within the given range and argue by induction on m. Indeed, when $m = 1$

$$\prod_{i=1}^{k} \psi_i \, \psi_{k-1} \simeq \prod_{i=1}^{k-1} \psi_i \, (\psi_k \, \psi_{k-1} \, \psi_k) \simeq \psi_k \prod_{i=1}^{k} \psi_i, \tag{3.4}$$

where in the last two steps we have used (i) and (ii) above followed by a repeated application of Lemma 3.1. Assume now that the assertion holds true for $1 \leq m \leq k-2$. Then we can write

$$\begin{aligned}
\prod_{i=1}^{k} \psi_i \prod_{j=1}^{m+1} \psi_{k-j} \quad &\simeq \quad \prod_{i=1}^{k} \psi_i \prod_{j=1}^{m} \psi_{k-j} \; \psi_{k-m-1} \simeq \prod_{j=1}^{m} \psi_{k-j+1} \prod_{i=1}^{k} \psi_i \; \psi_{k-m-1} \\
&\simeq \quad \prod_{j=1}^{m} \psi_{k-j+1} \prod_{i=1}^{k-m-1} \psi_i \; \psi_{k-m} \; \psi_{k-m-1} \prod_{i=k-m+1}^{k} \psi_i \\
&\simeq \quad \prod_{j=1}^{m} \psi_{k-j+1} \; \psi_{k-m} \prod_{i=1}^{k-m-1} \psi_i \; \psi_{k-m} \prod_{i=k-m+1}^{k} \psi_i \\
&\simeq \quad \prod_{j=1}^{m+1} \psi_{k-j+1} \prod_{i=1}^{k} \psi_i,
\end{aligned} \tag{3.5}$$

where again we have used (i) and (ii) above and Lemma 3.1. This establishes the assertion for $m+1$. The proof is thus complete. □

The significance of the next lemma is that it allows us to transfer information from braids in $\mathbb{B}_n$ to homeomorphisms in $\Lambda^{\mathrm{o}}_{\mathrm{id}}(D_n)$ that are accurate up to a homotopy.

Lemma 3.3. *Assume that the product of a finite collection of simple braids or their inverses coincide with the trivial braid in $\mathbb{B}_n$. Then the product of the corresponding simple shifts or their inverses is homotopic in $\Lambda_{\mathrm{id}}(D_n)$ to the identity map.*

Proof. We first recall that any product of simple braids or their inverses is equal to the trivial braid ε_n only through the defining relations of the group $\mathbb{B}_n$ indicated earlier. Consider now the following simple cases:
(i) $\sigma_i \, \sigma_i^{-1} = \varepsilon_n$ or $\sigma_i^{-1} \, \sigma_i = \varepsilon_n$
for $1 \leq i \leq n-1$
(ii) $\sigma_i \, \sigma_j \, \sigma_i^{-1} \, \sigma_j^{-1} = \varepsilon_n$ or $\sigma_i^{-1} \, \sigma_j^{-1} \, \sigma_i \, \sigma_j = \varepsilon_n$
for $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$,
(iii) $\sigma_i \, \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}^{-1} \, \sigma_i^{-1} \, \sigma_{i+1}^{-1} = \varepsilon_n$ or $\sigma_{i+1}^{-1} \, \sigma_i^{-1} \, \sigma_{i+1}^{-1} \, \sigma_i \, \sigma_{i+1} \, \sigma_i = \varepsilon_n$
for $1 \leq i \leq n-2$.
In view of (i) and (ii) in Lemma 3.2, it is immediate that replacing any occurance

of a simple braids by the corresponding simple shift in the left-hand sides of all the above cases results a homeomorphism in $\Lambda_{\rm id}(D_n)$ homotopic to the identity map (indeed in the cases (i) and (ii) the resulting homeomorphism is the identity map itself).

We argue by induction on the length m of the word in the expression for ε_n. Assume that the assertion holds true for any word with length not exceeding $m \geq 1$ and take a word of length $m+1$. As for any product of simple braids or their inverses of length $m+1$ and equal to the trivial braid ε_n, it must be that one of the three possibilities listed above occur in the product, by removing the corresponding word (which is of length either 2, 4 or 6) one arrives at a new product again equal to the trivial braid but of length not exceeding m. Using the basis of the induction, replacing the braids with shifts in this new word results in a homeomorphism, homotopic to the identity map. As for the product that was temporarily removed the same result holds true, it follows from (iii) in Lemma 3.1 that putting the corresponding shifts back in their original place also results in a homeomorphism that is homotopic to the identity map. Hence the conclusion holds for words of length $m+1$. The proof is thus complete. □

As indicated earlier, the collection of Dehn-twists along closed Jordan curves in any circular domain Ω provides a convenient way of representing arbitrary homeomorphisms in $\Lambda_{\rm id}(\Omega)$ up to homotopy. In the study of the mapping class group of D_n that constitutes part of our subsequent analysis, it is more useful to go a step further and instead present the homeomorphisms in $\Lambda_{\rm id}(D_n)$ via the simple shifts and the simple twists (together with their inverses). As a starting point one could address the question of whether one can write a Dehn-twist along a closed Jordan curve in D_n in the desired form.

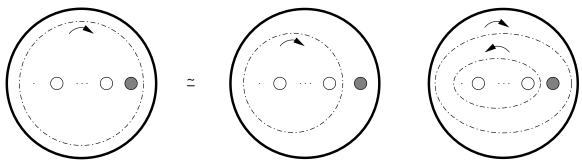

Figure 1: Here two homotopic maps in $\Lambda_{\rm id}(D_{k+1})$ are illustrated. The one on the left corresponds to a Dehn-twist along the dotted circle that bounds all the $k+1$ inner boundary components. The one on the right is a product of a Dehn-twist along the first k boundary components and a homeomorphism that corresponds to pulling the $(k+1)$-th boundary components in the counter-clockwise direction once around the outer boundary. The shaded disk in each case corresponds to the $(k+1)$-th boundary component and arrows in the clockwise direction represent a positive twist.

For this let γ be an arbitrary closed Jordan curve in D_n and let ϕ^γ denote the Dehn-twist along γ. Let $c_{i_1}, ..., c_{i_k}$ for $1 \leq k \leq n$ denote a sub-collection of the circles introduced earlier that lie inside γ (here we allow ourself to take any suitable Jordan curve isotopic to γ as by doing so we do not leave the homotopy class to which ϕ^γ belongs). We now consider two cases.

Case 1. The circles $c_{i_1}, ..., c_{i_k}$ bounded by the curve γ are consecutive. For simplicity we assume that after re-labeling these are $c_1, ..., c_k$. It is clear that when $k = 1$ the homeomorphism ϕ^γ is homotopic to the simple twist ϕ_1 and hence the assertion follows. Thus in the sequel we assume that $k \geq 2$. We claim that under these assumptions

$$\phi^\gamma \simeq \left(\prod_{i=1}^{k} \phi_i\right)^{2-k} \left(\prod_{j=1}^{k-1} \psi_j\right)^{k}, \tag{3.6}$$

with $\phi_i \in \Phi_n$ for $1 \leq i \leq k$ and $\psi_j \in \Psi_n$ for $1 \leq j \leq k-1$. (Note that the second brackets correspond to the braid $\sigma_c = (\sigma_1 \dots \sigma_{k-1})^k$ that generates the center of $\mathbb{B}_k$.) To establish the claim we argue by induction on k. Indeed when $k = 2$ the homeomorphism ϕ^γ is easily seen to be homotopic to ψ_1^2 and thus (3.6) holds true. Assume now that the assertion holds for some integer $k \geq 2$. Refering to Figure 1 it is clear that the twist ϕ^γ is homotopic to a product of the form $\phi^{\gamma_1} \psi$ where γ_1 is a curve bounding the boundary components $\{c_1, ..., c_k\}$ and ψ is a homeomorphism that pulls c_{k+1} once along the boundary ∂D in the counter-clockwise direction. It is easy to check that (e.g. by multiplying the simple shifts involved in a step by step fashion and then adjusting the twists along each boundary component)

$$\psi \simeq (\phi_1^{-1} \dots \phi_k^{-1} \, \phi_{k+1}^{1-k}) \prod_{i=1}^{k} \psi_{k-i+1} \prod_{j=1}^{k} \psi_j. \tag{3.7}$$

Thus recalling that the simple twists commute with one another and the simple shifts, we can write

$$\begin{aligned} \phi^\gamma &\simeq \phi^{\gamma_1} \psi \\ &\simeq \left(\prod_{i=1}^{k} \phi_i\right)^{2-k} \left(\prod_{j=1}^{k-1} \psi_j\right)^{k} (\phi_1^{-1} \dots \phi_k^{-1} \, \phi_{k+1}^{1-k}) \prod_{i=1}^{k} \psi_{k-i+1} \prod_{j=1}^{k} \psi_j \\ &\simeq \left(\prod_{i=1}^{k+1} \phi_i\right)^{1-k} \left(\prod_{j=1}^{k-1} \psi_j\right)^{k} \prod_{i=1}^{k} \psi_{k-i+1} \prod_{j=1}^{k} \psi_j. \end{aligned} \tag{3.8}$$

However ignoring the twists temporarily, the last three sequence of products can be written as

$$\begin{aligned}\left(\prod_{j=1}^{k-1}\psi_j\right)^k\prod_{i=1}^{k}\psi_{k-i+1}\prod_{j=1}^{k}\psi_j &\simeq \left(\prod_{j=1}^{k-1}\psi_j\right)^m\prod_{i=1}^{k}\psi_i\prod_{j=1}^{m}\psi_{k-j}\left(\prod_{j=1}^{k}\psi_j\right)^{k-m}\\ &\simeq \left(\prod_{j=1}^{k-1}\psi_j\right)^m\prod_{i=1}^{m}\psi_{k-i+1}\prod_{j=1}^{k}\psi_j\left(\prod_{j=1}^{k}\psi_j\right)^{k-m}\\ &\simeq \left(\prod_{j=1}^{k}\psi_j\right)^{k+1} \qquad (3.9)\end{aligned}$$

with $m = k-1, ..., 1$. Substituting (3.9) into (3.8) gives the assertion for $k+1$.

Case 2. The circles $c_{i_1}, ..., c_{i_k}$ bounded by the curve γ are not consecutive. Then we can reduce the problem to the previous case using homeomorphisms $\psi \in \Lambda^{\mathrm{o}}_{\mathrm{id}}(D_n)$ in the form of a finite product of simple shifts or their inverses such that $\phi^\gamma \simeq \psi\,\phi^{\gamma_1}\,\psi^{-1}$ with $\phi^{\gamma_1} \in \Lambda_{\mathrm{id}}(D_n)$ being a Dehn-twist along a Jordan curve γ_1 in D_n. The main idea being that by simultaneously pre and post multiplying a twist by a simple shift or its inverse we arrive at a new twist that includes or respectively excludes a boundary component associated to the latter shift. Proceeding step by step we reduce to case 1 and easily conclude.

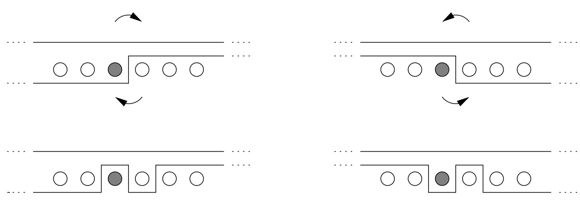

Figure 2: A collection of Dehn-twists along polygonal lines is illustrated. First row: ϕ^a and ϕ^b, second row: ϕ^c and ϕ^d. The domain $\Omega = D_n$ and the shaded disk corresponds to the i-th boundary component. It is easy to see that here $\phi^a \simeq \psi_i^{-1}\phi^c\psi_i$, and similarly $\phi^b \simeq \psi_i\phi^d\psi_i^{-1}$.

Lemma 3.4. *Any homeomorphism in* $\Lambda_{\mathrm{id}}(D_n)$ (*respectively any continuous map in* $\mathrm{C}_{\mathrm{id}}(D_n)$) *is homotopic in* $\Lambda_{\mathrm{id}}(D_n)$ (*respectively in* $\mathrm{C}_{\mathrm{id}}(D_n)$) *to a product of simple twists or their inverses and simple shifts or their inverses.*

Proof. We focus only on the case where the continuous map $\phi \in \mathrm{C}_{\mathrm{id}}(D_n)$. The case $\phi \in \Lambda_{\mathrm{id}}(D_n)$ is similar (or one can argue as above). First observe that ϕ is homotopic in $\mathrm{C}_{\mathrm{id}}(D_n)$ to a map that is identity on a strip near the boundary and which does not map any point in D_n outside this strip into the strip itself. Indeed it is enough to extend ϕ by identity to a neighbourhood of $\mathrm{cls}D_n$ and proceed with a construction similar to that in the first part of the proof of Proposition 2.1 (along each boundary component) to get for any small $\delta > 0$

and $t \in [0,1]$ an orientation preserving diffeomorphism $\tau_t : \mathrm{cls}D_n \to K_{\delta t}$, where $K_{\delta t} = \{x \in \mathbb{R}^2 : \mathrm{dist}_2(x, D_n) \le \delta t\}$. The homotopy $(t,x) \mapsto \tau_t^{-1} \circ \phi \circ \tau_t(x)$ would then lead to the conclusion. For convenience we continue to denote the map corresponding to $t = 1$ by ϕ. Note that we can make this latter map smooth while maintaining the other properties via a standard mollification.

Consider now the pointed space $(\mathrm{cls}D_n, x_0)$ where $x_0 = (-1,0) \in \partial D$. Under the above assumptions, it is possible to check that the map ϕ induces an endomorphism on $\pi_1(\mathrm{cls}\, D_n, x_0)$ that verifies the hypotheses of the representation theorem of E. Artin (cf. [3] pp. 30). (Notice that since ϕ coincides with the identity map in a strip near ∂D_n, it does not change the orientation of the closed curves representing the standard generators of $\pi_1(\mathrm{cls}D_n, x_0)$). Therefore this endomorphism is indeed an automorphism of F_n and consequently can be associated to a braid $\sigma \in \mathbb{B}_n$ (indeed in $\mathbb{P}_n$). Rewriting σ as a product of simple braids or their inverses and letting $\bar{\psi} \in \Lambda_{\mathrm{id}}(D_n)$ to be the corresponding product of simple shifts or their inverses, it follows that $\bar{\phi} = \phi\bar{\psi}^{-1} \in \mathrm{C}_{\mathrm{id}}(D_n)$ induces the identity automorphism on F_n. Hence it remains to show that any map $\bar{\phi}$ satisfying such assumptions should be homotopic in $\mathrm{C}_{\mathrm{id}}(D_n)$ to a product of simple twists or their inverses.

We argue by induction on the number of holes n. When $n = 0$, it is evident that any map in $\mathrm{C}_{\mathrm{id}}(D)$ is homotopic in $\mathrm{C}_{\mathrm{id}}(D)$ to the identity map. Assume now that the assertion holds for some n and pick a representative of the generator q_1 of $\pi_1(\mathrm{cls}D_n, x_0)$. By considering the winding number of continuous curves or otherwise, it is easy to see that for this curve to be mapped into a curve that is again a representative of q_1, it must be that the segment (x_0, x_m) should wind around the first inner boundary component and do so at most finitely many times (finiteness follows from continuity of $\bar{\phi}$). By multiplying $\bar{\phi}$ by ϕ_1^k for some $k \in \mathbb{Z}$ we then arrive up to a homotopy in $\mathrm{C}_{\mathrm{id}}(D_n)$ at a map that coincides with the identity in a strip near the segment (x_0, x_m). Cutting the domain D_{n+1} along this segment leaves us with a domain with n holes, homeomorphic to D_n and a map that coincides with the identity on the boundary. We use the basis of the induction and conclude. □

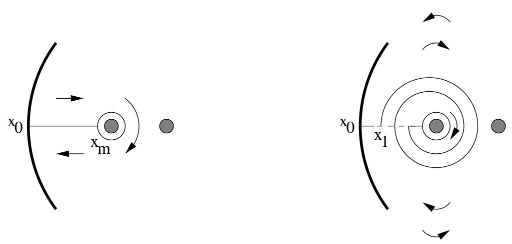

Figure 3: A representative of the generator q_1 and its image under $\bar{\phi}$ is illustrated. It is clear that the image is again a representative of q_1. The inner and outer arrows on the right show the direction in which the image curve is traversed before and after it winds around the first inner boundary component of D_n.

Proof of Proposition 3.1. For convenience we present this in two steps.
Step 1. Here we consider the case where $\Omega \subset \mathbb{R}^2$ is the n-circular domain D_n. We define a group epimorphism $H : \Lambda^{\text{o}}_{\text{id}}(D_n) \to \mathbb{B}_n$ in the following way: pick $\phi \in \Lambda^{\text{o}}_{\text{id}}(D_n)$ and extend it to an element of $\Lambda_{\text{id}}(D)$. (If $\phi \in \Lambda_{\text{id}}(D_n)$ then simply extend it by setting $\phi = \text{id}$ in the inner disks, otherwise extend it by assigning to each inner disk a homeomorphism that takes it to the interior of the disk to which the boundary of the initial disk is mapped to. That such a homeomorphism exists is an easy consequence of Schoenflies theorem). Let h denote the homotopy obtained by the application of the so-called Alexander's lemma, that is

$$h(t,x) = \begin{cases} x & \text{when } t \leq |x| \leq 1, \\ t\phi(\frac{x}{t}) & \text{when } 0 \leq |x| < t. \end{cases}$$

Thus $h(t,\cdot) \in \Lambda_{\text{id}}(D)$ for $0 \leq t \leq 1$ with $h(0,x) = x$ and $h(1,x) = \phi(x)$ for every $x \in \text{cls}D$. By restricting h to the centers a^i of the interior disks we obtain a representative of an element σ of the full braid group $\mathbb{B}_n$. We put $H(\phi) = \sigma$ and note that $H(\psi_i) = \sigma_i$ for every $1 \leq i \leq n-1$ (this clarifies the connection between the simple shifts and the simple braids).

$$\begin{array}{ccccccc}
\langle \Delta_n \rangle / \simeq & \longrightarrow & \mathcal{M}_\Omega & \longrightarrow & \Lambda^{\text{o}}_{\text{id}}(\Omega)/\simeq & \longrightarrow & \text{Aut}(\text{F}_\text{n}) \\
{\scriptstyle G}\big\downarrow & & {\scriptstyle \text{onto}}\big\downarrow & & \big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow{\scriptstyle \text{identity}} \\
0 & \longrightarrow & \mathbb{P}_n & \longrightarrow & \mathbb{B}_n & \longrightarrow & \text{Aut}(\text{F}_\text{n})
\end{array}$$

That H is a group homomorphism is clear. We proceed by showing that it is surjective. The cases $n = 0, 1$ are immediate as in both cases $\mathbb{B}_n$ is the trivial group, so we shall assume that $n \geq 2$. As the collection of the simple braids Σ_n is a system of generators for $\mathbb{B}_n$, each braid $\sigma \in \mathbb{B}_n$ can be written as a product, unique up to the defining relation of $\mathbb{B}_n$

$$\sigma = \prod_{j=1}^{k} \sigma_{i_j}^{e_{i_j}}, \qquad e_{i_j} \in \{\pm 1\}. \tag{3.10}$$

To associate a homeomorphism in $\Lambda^{\text{o}}_{\text{id}}(D_n)$ to the braid σ, it is enough to take the product

$$\phi = \prod_{j=1}^{k} \psi_{i_j}^{e_{i_j}}, \tag{3.11}$$

with e_{i_j} as in (3.10). It is clear that $\phi \in \Lambda^{\text{o}}_{\text{id}}(\Omega)$ and according to Lemma 3.3, independent, up to homotopy of the particular representation of σ. Also it is interesting to observe that the group epimorphism H when restricted to $\Lambda_{\text{id}}(D_n)$ gives a group epimorphism (again denoted by H for brevity) $H : \Lambda_{\text{id}}(D_n) \to \mathbb{P}_n$. As homotopic maps in $\Lambda_{\text{id}}(D_n)$ have identical images under H we are led to a quotient group epimorphism $G : \mathcal{M}_{D_n} \to \mathbb{P}_n$. Thus to finish the proof in step 1 it is enough to identify the kernel of G.

Let $\Delta_n := \{[\phi_1], ..., [\phi_n]\}$, that is the collection of equivalence classes of simple twists in Φ_n. It is easy to see that for each $1 \le i \le n$ the equivalence class $[\phi_i]$ lies in the kernel of G. We claim that $\mathrm{Ker}(G) = \langle \Delta_n \rangle$, that is the subgroup generated by Δ_n. To this end let $[\phi] \in \mathcal{M}_{D_n}$ be an arbitrary equivalence class with representative ϕ. According to Lemma 3.4,

$$[\phi] = \prod_{j=1}^{m-1} [\phi_{i_j}]^{d_j} \prod_{j=m}^{k} [\psi_{i_j}]^{e_{i_j}} \tag{3.12}$$

for a collection of simple twists and simple shifts with $d_j \in \mathbb{Z}$ for $1 \le j \le m-1$ and $e_{i_j} \in \{\pm 1\}$ for $m \le j \le k$. Applying G to both sides of (3.12) and demanding $G([\phi]) = \varepsilon_n$ it follows that

$$G([\phi]) = \prod_{j=m}^{k} \sigma_{i_j}^{e_{i_j}} = \varepsilon_n. \tag{3.13}$$

However in view of Lemma 3.3, this implies that the corresponding product of simple shifts should be homotopic in $\Lambda_{\mathrm{id}}(D_n)$ to the identity map. Hence

$$[\phi] = \prod_{j=1}^{m-1} [\phi_{i_j}]^{d_j} \tag{3.14}$$

and so $\mathrm{Ker}(G) = \langle \Delta_n \rangle$. As this latter group is isomorphic to $\mathbb{Z}^n$ we deduce that $\mathrm{Ker}(G) \cong \mathbb{Z}^n$. This completes the proof for the case where $\Omega = D_n$.

Step 2. We now show that the general case can be reduced to that of the previous step. Indeed let Ω be an arbitrary bounded Lipschitz domain with n holes and let D_n be the n-circular domain of the previous step. According to Proposition 2.2, $\mathrm{cls}\,\Omega$ and $\mathrm{cls}D_n$ are homeomorphic and thus $\mathcal{M}_\Omega \cong \mathcal{M}_{D_n}$. The conclusion now follows form that of step 1. The proof is thus complete. □

Proposition 3.2. *The homotopy classes of maps in the space* $\mathrm{C}_{\mathrm{id}}(\Omega)$ *can be indexed by the group* $\mathbb{P}_n \oplus \mathbb{Z}^n$.

Proof. As by now clear we first establish the claim for the case where $\Omega = D_n$ and then conclude with the aid of Proposition 2.2. To this end it is enough to prove that firstly every homotopy class of maps in $\mathrm{C}_{\mathrm{id}}(D_n)$ contains a homeomorphism of $\Lambda_{\mathrm{id}}(D_n)$ and secondly that two homeomorphisms ϕ^1 and $\phi^2 \in \Lambda_{\mathrm{id}}(D_n)$ are homotopic in $\Lambda_{\mathrm{id}}(D_n)$ if they are homotopic in $\mathrm{C}_{\mathrm{id}}\ (D_n)$. The first assertion was proved in Lemma 3.4, so we proceed with the second one. We assume without loss of generality that ϕ^2 is the identity map. According to Lemma 3.4 we can write $\phi^1 \simeq \phi\psi$ in $\Lambda_{\mathrm{id}}(D_n)$ where ϕ and ψ are each products of simple twists or their inverses and simple shifts or their inverses respectively. If ψ is not homotopic to the identity map in $\Lambda_{\mathrm{id}}(D_n)$, then it corresponds to a braid $\sigma \ne \varepsilon_n$ (via the group epimorphism constructed in step 1 in the proof of Proposition 3.1). Appealing again to the representation theorem of E. Artin the braid σ is then associated to an automorphism of F_n different from the identity automorphisms. Observing that the following diagram commutes

$$\begin{tikzcd} \Lambda^{\circ}_{\mathrm{id}}(D_n) \arrow[d, "*"'] & \langle \Psi_n \rangle \arrow[l] \arrow[d, "H"] \\ \mathrm{Aut}(\mathrm{F_n}) & \mathbb{B}_n \arrow[l] \end{tikzcd}$$

we arrive at a contradiction as ϕ^1 and ψ induce the same automorphism on F_n and by assumption ϕ^1 is homotopic to the identity map in $\mathrm{C}_{\mathrm{id}}(D_n)$. Hence to complete the proof we need to establish that the product of the simple twists, that is ϕ is homotopic to the identity map in $\Lambda_{\mathrm{id}}(D_n)$. But this is easy as ϕ is homotopic to the identity map in $\mathrm{C}_{\mathrm{id}}(D_n)$ and therefore can not contain any twist without its inverse (as otherwise the base point preserving map associated to ϕ from the pointed space $(\mathbb{S}^1, x_0)$ to itself and the constant map of $(\mathbb{S}^1, x_0)$ associated to the identity will be homotopic in $(\mathbb{S}^1, x_0)$ while having different Brouwer-Hopf degrees). Hence ϕ^1 is homotopic to the identity map in $\Lambda_{\mathrm{id}}(D_n)$.

We have therefore established that the homotopy classes of maps in $\mathrm{C}_{\mathrm{id}}(D_n)$ are in a one-to-one correspondence with the homotopy classes of maps in $\Lambda_{\mathrm{id}}(D_n)$. Thus the conclusion follows by an application of Proposition 3.1. $\square$

4 Proof of the main result

The purpose of this section is to finalise the proof of the main theorem stated earlier in the introduction (cf. Theorem 4.1 below).

Recall that $\Omega \subset \mathbb{R}^2$ is assumed to be a bounded Lipschitz domain and that the integrand $F : \Omega \times \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}$ is bounded from below and for $\mathcal{L}^2$- a.e. $x \in \Omega$ polyconvex in ξ. More specifically we assume that F satisfies:

(H1) there exists $\Phi : \Omega \times \mathbb{R}^5 \to \overline{\mathbb{R}}$ such that
(i) $\Phi(\cdot, J)$ is $\mathcal{L}^2$- measurable for every $J \in \mathbb{R}^5$,
(ii) $\Phi(x, \cdot)$ is continuous and convex on $\mathbb{R}^5$ for $\mathcal{L}^2$- a.e. $x \in \Omega$,
(iii) for $\mathcal{L}^2$- a.e. $x \in \Omega$, $\Phi(x, \xi, \det \xi) = \infty$ if and only if $\xi \notin \mathbb{R}^{2\times 2}_{+}$,
(iv) $F(x, \xi) = \Phi(x, \xi, \det \xi)$ for $\mathcal{L}^2$-a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{2\times 2}$,

(H2) there exists a locally bounded function $G : \mathbb{R}^{2\times 2} \to \mathbb{R}$ such that $F(x, \xi) \leq G(\xi)$ for $\mathcal{L}^2$- a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{2\times 2}_{+}$, and finally that

(H3) there exist $f \in L^1(\Omega)$ and constants $c > 0$, $p \geq 2$ such that

$$F(x, \xi) \geq f(x) + c|\xi|^p$$

for $\mathcal{L}^2$- a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{2\times 2}_{+}$.

It is an immediate consequence of *(iii)* and *(iv)* in (H1) that any map $u \in W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ with $\mathbb{F}[u] < \infty$ verifies the condition $\det \nabla u(x) > 0$ $\mathcal{L}^2$- a.e. in Ω. In view of a result of S. Vodopyanov & V. Gol'dshtein [16] any such u can be represented by a continuous map in $\mathrm{C}(\mathrm{cls}\, \Omega, \mathbb{R}^2)$ with monotone coordinate functions. The boundary values of u would then imply that u can be represented by a map in the space $\mathrm{C}_{\mathrm{id}}(\Omega)$.

We now fix the domain Ω and let n denote the number of holes in Ω. Then according to Propositions 3.1 and 3.2, the mapping class group $\mathcal{M}_{\Omega}$ and the set

$\mathcal{C}_\Omega$ both can be indexed by the group $\mathbb{P}_n \oplus \mathbb{Z}^n$. For each λ in this latter group, we let $\mathrm{C}_\lambda(\Omega)$ and $\Lambda_\lambda(\Omega)$ denote the corresponding homotopy classes of maps in $\mathrm{C}_{\mathrm{id}}(\Omega)$ and $\Lambda_{\mathrm{id}}(\Omega)$ respectively and put

$$W_\lambda(\Omega) := \{u \in W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2) : \mathbb{F}[u] < \infty, u \in \mathrm{C}_\lambda(\Omega)\}.$$

Notice that we have taken advantage of the earlier comment that any Sobolev map $u \in W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ with $\mathbb{F}[u] < \infty$ can be represented by a map in $\mathrm{C}_{\mathrm{id}}(\Omega)$.

It is clear from the above definition that every class $W_\lambda(\Omega)$ corresponds to a component $\mathrm{C}_\lambda(\Omega)$ and hence $\Lambda_\lambda(\Omega)$. We now intend to show that this correspondence goes in the reverse direction as well. Indeed fix $\Lambda_{\lambda'}(\Omega) \subset \Lambda_{\mathrm{id}}(\Omega)$ and let $\Lambda_{\lambda'}(D_n) \subset \Lambda_{\mathrm{id}}(D_n)$ be the component associated to the latter via the homeomorphism $\tau : \mathrm{cls}\,\Omega \to \mathrm{cls} D_n$ of Proposition 2.2. In view of the discussion preceeding Lemma 3.4, $\Lambda_{\lambda'}(D_n)$ admits a representative, say $\tilde{\phi}$, that can be written as a composition of finitely many Dehn-twists along closed Jordan curves in D_n (notice that we can take these curves to be smooth by isotopy). Since each such twist corresponds to a smooth map with a positive determinant (cf. (3.1) and the discussion subsequent to it) and as the latter curves are away from the boundary of D_n, appealing to Proposition 2.2, it follows that composing with τ and τ^{-1} respectively also gives a smooth representative ϕ of the component $\Lambda_{\lambda'}(\Omega)$ with a positive determinant. (This can be easily checked by setting $\phi(x) = \tau^{-1} \circ \tilde{\phi} \circ \tau(x)$ for $x \in \mathrm{cls}\,\Omega$ and using the chain rule for the derivatives and the product formula for the determinants). Thus $\det \nabla\phi(x) > 0$ for $x \in \mathrm{cls}\Omega$ and hence the latter provides us with a representative of a Sobolev map $u \in W_{\lambda'}(\Omega)$ that in view of (H2) verifies $\mathbb{F}[u] < \infty$.

Theorem 4.1. *Let the integrand F satisfy $(H1)$, $(H2)$ and $(H3)$. Then the functional $\mathbb{F}[\cdot]$ admits a set of L^1 local minimizers that can be indexed by the group $\mathbb{P}_n \oplus \mathbb{Z}^n$.*

Proof. For each $\lambda \in \mathbb{P}_n \oplus \mathbb{Z}^n$ let $\bar{u}_\lambda \in W_\lambda(\Omega)$ be defined via

$$\mathbb{F}[u_\lambda] := \inf_{W_\lambda(\Omega)} \mathbb{F}[\cdot].$$

The existence of a minimizer to the above problem follows from the direct method of the calculus of variations. Indeed let $\{u^{(j)}\} \subset W_\lambda(\Omega)$ be a minimizing sequence for $\mathbb{F}[\cdot]$. It follows from the coercivity of F that $\{u^{(j)}\}$ is bounded in $W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ and that $\det \nabla u^{(j)} > 0$ for each j and $\mathcal{L}^2$-a.e. $x \in \Omega$. As $p \geq 2$ this implies that there exists $\bar{u}_\lambda \in W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ such that by passing to a subsequence, $u^{(j)} \rightharpoonup \bar{u}_\lambda$ in $W^{1,p}$. Moreover since $\mathbb{F}[\cdot]$ is sequentially lower semicontinuous with respect to $W^{1,p}$-weak convergence we deduce that

$$\mathbb{F}[\bar{u}_\lambda] \leq \liminf_{j\to\infty} \mathbb{F}[u^{(j)}] < \infty.$$

Thus $\bar{u}_\lambda$ is a minimizer of $\mathbb{F}[\cdot]$ on $W_\lambda(\Omega)$ once we show that $\bar{u}_\lambda \in W_\lambda(\Omega)$. To this end we claim that the sequence $\{u^{(j)}\}$ can be represented by a relatively compact sequence in $\mathrm{C}(\mathrm{cls}\,\Omega, \mathbb{R}^2)$. When $p > 2$ this is an immediate consequence of the Rellich-Kondrachov's compactness theorem. In the borderline case $p = 2$ we appeal first to the result of Vodopyanov & Gol'dshtein mentioned earlier to infer that the sequence $\{u^{(j)}\}$ can be represented by a bounded sequence in $\mathrm{C}(\mathrm{cls}\,\Omega, \mathbb{R}^2)$ and secondly to a theorem of C. Morrey (cf. [9] pp. 110) to deduce that the corresponding sequence is equicontinuous. The assertion then follows from the Arzela-Ascoli theorem. Therefore in both cases, by passing to a further subsequence if necessary we have (for the corresponding representatives) that $u^{(j)} \to \bar{u}_\lambda$ in $\mathrm{C}(\mathrm{cls}\,\Omega, \mathbb{R}^2)$. As by the assumption $u^{(j)}$ has a representative in $\mathrm{C}_\lambda(\Omega)$, the latter convergence implies that the same holds true for $\bar{u}_\lambda$. Thus $\bar{u}_\lambda \in W_\lambda(\Omega)$.

It therefore remains to justify the assertion that such a $\bar{u}_\lambda$ is an L^1 local minimizer of $\mathbb{F}[\cdot]$. Indeed if this were not the case there would exist a sequence $\{u^{(j)}\}$ in $W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ such that $u^{(j)} \to \bar{u}_\lambda$ in L^1 and

$$\mathbb{F}[u^{(j)}] < \mathbb{F}[\bar{u}_\lambda] < \infty.$$

It then follows from the coercivity assumption on the integrand F that the sequence $\{u^{(j)}\}$ is bounded in $W^{1,p}_{\mathrm{id}}(\Omega, \mathbb{R}^2)$ and therefore by passing to a subsequence $u^{(j)} \rightharpoonup \bar{u}_\lambda$ in $W^{1,p}$. Applying an argument similar to the above, we deduce by passing to a further subsequence (again for the corresponding representatives) that $u^{(j)} \to \bar{u}_\lambda$ in $\mathrm{C}(\mathrm{cls}\,\Omega, \mathbb{R}^2)$ and therefore for sufficiently large j it must be that $u^{(j)}$ has a representative in $\mathrm{C}_\lambda(\Omega)$. This however is a contradiction to $\bar{u}_\lambda$ being an absolute minimizer of $\mathbb{F}[\cdot]$ over the class $W_\lambda(\Omega)$. The proof is thus complete. □

Remark 4.1. The restriction on the Sobolev maps to coincide with the identity on the boundary is to a large extent for convenience. Indeed without much extra effort it is possible to show that the same result holds true for any boundary data that is the restriction to $\partial\Omega$ of an orientation preserving homeomorphism of $\mathrm{cls}\,\Omega$ into $\mathbb{R}^2$. We refer the interested reader to [15] for further results in this direction.

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