Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

A generalization of Rellich's theorem and regularity of varifolds minimizing curvature

by

Roger Moser

A generalization of Rellich's theorem and regularity of varifolds minimizing curvature

Roger Moser
MPI for Mathematics in the Sciences
D-04103 Leipzig (Germany)

September 24, 2001

Abstract

We prove a generalization of Rellich's theorem for weakly differentiable functions on curvature varifolds (see definitions below) and apply it to prove regularity of minimizers of curvature integrals under certain assumptions.

1 Introduction

Let M be a C^2-submanifold of dimension n of the Euclidean space $\mathbb{R}^N$, and let $\mathbf{B}$ be the second fundamental form of M. One may assign to M the number

$$\int_M F(\mathbf{B}) \, d\mathcal{H}^n$$

for some function F, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. This defines a functional on the set of such submanifolds. Further one may ask whether there are minimizing submanifolds or critical points of this functional and what their properties are.

For the existence problem, a reasonably satisfactory answer has been given by Hutchinson [8] (and by Mantegazza [9] in the presence of a boundary). Namely, under reasonable assumptions, a minimizer of the functional exists in the class of curvature varifolds (or curvature varifolds with boundary) defined in those papers. It is no surprise that the space where the functional is defined has to be enlarged, especially as the direct method in the calculus of variations is used to find these minima. This however raises other questions, in particular concerning the regularity of the solutions of the problem.

A very important tool for attacking such questions for variational problems in general is the theory of Sobolev spaces. In this case however we do not have it at our disposition, since the underlying structures are no longer manifolds but varifolds, which furthermore may vary. On the other hand, the second fundamental form does still behave like a derivative in many ways, indeed it may be expressed in terms of approximate derivatives, as was shown in [9]. Moreover, various results for Sobolev spaces apply also in the varifold setting, like e. g. the Sobolev inequality (see [10]).

The aim of this paper is to give a generalization of Rellich's theorem for the situation of functions on (varying) curvature varifolds, and to apply it to the regularity problem mentioned above. We now give an outline of the content. In Sect. 2, we repeat some of the definitions and results of Allard [1] (on which Hutchinson's paper is based) and of Hutchinson [8] and Mantegazza [9]. More definitions and results of [8] are looked at in Sect. 3, along with a few generalizations and new results. In Sect. 4, we give a definition of a notion of differentiability with respect to a varifold. We prove some results for functions with that property, including in particular a generalization of Rellich's theorem. We also show that the Sobolev inequality of Michael-Simon [10] holds for that situation. We turn our attention back to functionals as discussed above in Sect. 5. We obtain a regularity result for minimizers under certain assumptions. However these assumptions are not easily verified in general, in fact we can only prove them under very strong conditions. We will prove that they hold true in a special situation in Sect. 6, namely for varifolds given as graphs of Lipschitz functions. Of course this assumption is in general not justified a priori. The results of Sect. 5 and 6 are therefore to be considered either as preliminary results of as a demonstration of how the previous results might be applied.

We would like to point out one special example of a functional as above. The Willmore functional for a compact surface M in $\mathbb{R}^N$ is defined by

$$\mathcal{F}(M) = \frac{1}{4} \int_M |\mathbf{H}|^2 \, d\mathcal{H}^2,$$

where H is the mean curvature vector of M. This can be written

$$\mathcal{F}(M) = \frac{1}{4} \int_M |\mathbf{B}|^2 \, d\mathcal{H}^2 + 2\pi(1 - g)$$

by the Gauss-Bonnet formula, where g is the genus of the surface. If g is fixed, we therefore have a functional of the form as described above. Existence and regularity of minimizing surfaces of the Willmore functional have been studied by Simon [12]. That paper actually provides tools to fill the gaps in our reasoning to prove regularity for minimizers of the Willmore functional, but it also renders our methods redundant in that case, for regularity is already proved there.

Other papers worth mentioning are [2], where similar problems are studied in the context of currents rather than varifolds, and [4, 5, 6], which provide the basic ideas for some of the concepts presented here.

2 Varifolds and curvature varifolds

We first repeat some of the definitions, ideas, and results of Allard [1], of Hutchinson [8], and of Mantegazza [9]. Standard reference for unexplained notation is [11] (see also [3]). We always work in an open set $\Omega \subset \mathbb{R}^N$ as ambient space.

By $G(N, n)$ we denote the Grassmann manifold consisting of all n-dimensional subspaces of $\mathbb{R}^N$. If S is a subset of R^N, we write $G_n(S) = S \times G(N, n)$. We may identify any $P \in G(N, n)$ with the orthogonal projection onto P and therefore with a matrix in $\mathbb{R}^{N \times N}$. We will usually denote this matrix with the same symbol, thus we may write $P = (P_{ij})$.

In this paper we will often consider Radon measures on $G_n(\Omega)$. A special role will be played by the following particular kind of Radon measures. Let $M \subset \Omega$ be a countably n-rectifiable (subsequently simply called n-rectifiable) and $\mathcal{H}^n$-measurable set, and $\theta : M \to [0, \infty)$ locally integrable with respect to $\mathcal{H}^n \llcorner M$. Then for $\mathcal{H}^n$-almost every $x \in M$, there exists a unique approximate tangent plane $T_x M$ to M at x. Moreover the measure V on $G_n(\Omega)$ characterized by

$$\int \phi \, dV = \int_M \theta(x)\phi(x, T_x M) \, d\mathcal{H}^n(x)$$

is a Radon measure. We write $V = \mathbf{v}(M, \theta)$.

Definition 2.1 *A Radon measure V on $G_n(\Omega)$ is called an n-varifold in Ω. If $V = \mathbf{v}(M, \theta)$ as above, then it is called a rectifiable n-varifold. If moreover θ takes integer values, then V is an integral varifold.*

If V is an n-varifold in Ω, then we denote by $\mu_V = \pi_\# V$ the induced measure by the canonical projection $\pi : G_n(\Omega) \to \Omega$.

By varifold convergence we mean convergence in the measure sense, i. e. weak* convergence in the dual space of $C_0^0(G_n(\Omega))$, the space of continuous functions on $G_n(\Omega)$ with compact support. We write $V_k \to V$ if V_k converges to V in the varifold sense.

Varifolds, in particular integral varifolds, are supposed to be generalizations of submanifolds. When studying curvature integrals, one needs a generalization of the second fundamental form, or at least the mean curvature vector, as well. The latter can be defined as follows (cf. [1]).

Let $V = \mathbf{v}(M, \theta)$ be an integral varifold in Ω, and define its first variation

$$\delta V(X) = \int \operatorname{div}_M X \, d\mu_V, \quad X \in C_0^1(\Omega, \mathbb{R}^N),$$

where $\operatorname{div}_M X$ is the tangential divergence of the vector field X with respect to M,

$$\operatorname{div}_M X = (T_x M)_{ij} \frac{\partial X^i}{\partial x^j}.$$

Here and throughout the paper we sum over repeated latin indices from 1 to N. (The first variation can also be defined for general varifolds, but we are only interested in this case.) This is a linear functional on $C_0^1(\Omega, \mathbb{R}^N)$. If it has a continuous extension to $C_0^0(\Omega, \mathbb{R}^N)$, then it can be represented by a vector valued Radon measure, which we can split into an absolutely continuous part and a singular part with respect to μ_V, using the theorem of Radon-Nikodym. We obtain the representation

$$\int \operatorname{div}_M X \, d\mu_V = -\int X \cdot \mathbf{H} \, d\mu_V - \int X \cdot \bar{\nu} \, d\sigma_V, \tag{1}$$

where $\mathbf{H} \in L^1_{\mathrm{loc}}(\mu_V, \mathbb{R}^N)$, σ_V is a Radon measure on Ω, which is singular with respect to μ_V, and $\bar{\nu} \in L^1_{\mathrm{loc}}(\sigma_V, \mathbb{S}^{N-1})$. Considering the integration by parts formula on C^2-manifolds, it is natural to call $\mathbf{H}, \bar{\nu}, \sigma_V$ respectively the generalized mean curvature vector, the generalized inner normal and the generalized boundary of V.

The notion of a second fundamental form was generalized to varifolds by Hutchinson [8]. His definition was further generalized by Mantegazza [9] to include varifolds with boundary.

Definition 2.2 *Let V be an n-varifold in Ω. We say that V is a curvature varifold with boundary, if there exist functions $A_{ijk} \in L^1_{\mathrm{loc}}(V)$, called the generalized curvature of V, and an $\mathbb{R}^N$-valued Radon measure ∂V on $G_n(\Omega)$, such that*

$$\int \left(P_{ij} \frac{\partial \phi}{\partial x^j}(x,P) + \frac{\partial \phi}{\partial P_{jk}}(x,P) A_{ijk}(x,P) + \phi(x,P) A_{jij}(x,P) \right) dV(x,P) \quad (2)$$

$$+ \int \phi(x,P) \, d(\partial V_i)(x,P) = 0$$

for all $\phi \in C^1_0(\Omega \times \mathbb{R}^{N \times N})$ and for $i = 1, \ldots, N$. If $\partial V = 0$, then V is simply called a curvature varifold. A rectifiable curvature varifold is a rectifiable varifold which is also a curvature varifold, etc.

Note that if we choose a ϕ which is independent of P, then (2) simply reads

$$\int P_{ij} \frac{\partial \phi}{\partial x^j} \, dV = - \int \phi A_{jij} \, dV - \int \phi \, d(\partial V_i).$$

If $V = \mathbf{v}(M, \theta)$ is a rectifiable varifold, then any function f on $G_n(\Omega)$ agrees V-almost everywhere with the function $g(x,P) = f(x, T_x M)$, which does not depend on P. We may therefore identify functions on $G_n(\Omega)$ with functions on Ω. Writing $\mathbf{H} = (\mathbf{H}_i)$ for $\mathbf{H}_i = A_{jij}$ and $\partial V = \bar{\nu} \sigma_V$, where σ_V is a Radon measure on Ω and $\bar{\nu} \in L^1_{\mathrm{loc}}(\sigma_V, \mathbb{S}^{N-1})$, we recover (1).

According to [8, 9], the functions A_{ijk} and the measure ∂V are uniquely determined by (2). Moreover, they have the following properties:

(i) $A_{ijk} = A_{ikj}$,

(ii) $A_{ijj} = 0$,

(iii) $A_{ijk} = P_{jr} A_{irk} + P_{rk} A_{ijr}$.

If M is a C^2-submanifold of Ω with second fundamental form $\mathbf{B}(x) : T_x M \times T_x M \to (T_x M)^\perp$, then they satisfy:

(iv) $\mathbf{B}^k_{ij} = P_{rj} A_{ikr}$,

(v) $A_{ijk} = \mathbf{B}^k_{ij} + \mathbf{B}^j_{ik}$,

where the $\mathbf{B}^k_{ij}$ are given by extending $\mathbf{B}(x)$ to $\mathbb{R}^N \times \mathbb{R}^N$ by composition with the orthogonal projection onto $T_x M$ and setting $\mathbf{B}^k_{ij} = e_k \cdot \mathbf{B}(e_i, e_j)$ for the standard unit vectors $e_1, \ldots, e_N$ in $\mathbb{R}^N$. Again we identify functions on Ω with functions on $G_n(\Omega)$. All these properties are proved in [8].

In general we take (iv) as a definition for the generalized second fundamental form $\mathbf{B} = (\mathbf{B}^k_{ij})$. The property (v) then follows as in [8]. Moreover, the following holds true.

Lemma 2.1 *The generalized second fundamental form $\mathbf{B} = (\mathbf{B}^k_{ij})$ of a curvature varifold satisfies $P_{rk} \mathbf{B}^k_{ij} = 0$ almost everywhere with respect to V.*

Proof. From (iv) and (v) we conclude that

$$\mathbf{B}^k_{ij} = P_{rj} A_{ikr} = P_{rj}(\mathbf{B}^r_{ik} + \mathbf{B}^k_{ir}).$$

Thus we see that

$$P_{rk}\mathbf{B}^k_{ij} = P_{rs}P_{sk}\mathbf{B}^k_{ij} = P_{rs}(\mathbf{B}^j_{is} - P_{sk}\mathbf{B}^j_{ik}) = P_{rs}\mathbf{B}^j_{is} - P_{rk}\mathbf{B}^j_{ik} = 0,$$

using the fact that the P_{ij} belong to orthogonal projections. $\square$

3 Measure-function pairs

Another useful notion from [8] is that of measure-function pairs. We first recall the definition. In the following, E is a subset of some Euclidean space, and $1 \le p < \infty$.

Definition 3.1 *A measure-function pair over E with values in $\mathbb{R}^s$ is a pair (μ, f), where μ is a Radon measure on E and $f \in L^1_{\mathrm{loc}}(\mu, \mathbb{R}^s)$.*

Example. Let V be a curvature n-varifold in Ω with generalized second fundamental form $\mathbf{B}$, then $(V, \mathbf{B})$ is a measure-function pair over $G_n(\Omega)$ with values in $\mathbb{R}^{N \times N \times N}$.

Definition 3.2 *Let $\{(\mu_k, f_k)\}$ be a sequence of measure-function pairs over E with values in $\mathbb{R}^s$ such that $f_k \in L^p(\mu_k, \mathbb{R}^s)$. We say that (μ_k, f_k) converges weakly in L^p to a measure-function pair (μ, f), if*

- $\mu_k \to \mu$ *as $k \to \infty$ in the sense of measures,*

- $\mu_k \llcorner f_k \to \mu \llcorner f$ *as $k \to \infty$ in the sense of vector-valued measures, and*

- *the norms $\|f_k\|_{L^p(\mu_k)}$ are uniformly bounded.*

In this case, we write $(\mu_k, f_k) \overset{L^p}{\rightharpoonup} (\mu, f)$. We say that the convergence is strong in L^p, and write $(\mu_k, f_k) \overset{L^p}{\to} (\mu, f)$, if

- $\lim_{k \to \infty} \int \phi(x, f_k(x)) \, d\mu_k(x) = \int \phi(x, f(x)) \, d\mu(x)$ *for all $\phi \in C^0_0(E \times \mathbb{R}^s)$, and*

- $\lim_{j \to \infty} \int_{S_{kj}} |f_k|^p \, d\mu_k = 0$ *uniformly in k, where*

$$S_{kj} = \{x \in E : \ |x| \ge j \ \text{or} \ |f_k(x)| \ge j\}.$$

When working with measure-function pairs, it is often convenient to consider the graph measures associated to them.

Definition 3.3 *Let (μ, f) be a measure-function pair over E with values in $\mathbb{R}^s$. The graph measure $[\mu, f]$ on $E \times \mathbb{R}^s$ is defined by*

$$[\mu, f] = G_\# \mu,$$

where $G : E \to E \times \mathbb{R}^s$ is given by $G(x) = (x, f(x))$.

5

Definition 3.4 *Let Γ be a Radon measure on $E \times \mathbb{R}^s$ and $\pi : E \times \mathbb{R}^s \to E$ the projection. Then $\|\Gamma\| = \pi_\# \Gamma$ is called the weight measure of Γ. If $\|\Gamma\|$ is a Radon measure, then the fiber measure $\Gamma^{(x)}$ can be defined for $\|\Gamma\|$-almost every $x \in E$ by*

$$\int \phi \, d\Gamma^{(x)} = \lim_{r \searrow 0} \frac{1}{\|\Gamma\|(B_r(x))} \int_{B_r(x) \times \mathbb{R}^s} \phi(y) \, d\Gamma(x, y).$$

The following results are due to Hutchinson [8].

Proposition 3.1 *Let $1 < p < \infty$. Suppose that (μ_k, f_k), $k \in \mathbb{N}$, and (μ, f) are measure-function pairs over E with values in $\mathbb{R}^s$.*

(i) *If $(\mu_k, f_k) \overset{L^p}{\rightharpoonup} (\mu, f)$, then there exist a subsequence $\{k'\} \subset \mathbb{N}$ and a Radon measure Γ on $E \times \mathbb{R}^s$ such that $[\mu_{k'}, f_{k'}] \to \Gamma$. Moreover, $\|\Gamma\| = \mu$ and*

$$f(x) = \int y \, d\Gamma^{(x)}(y)$$

μ-almost everywhere.

(ii) *the following statements are equivalent:*

- $(\mu_k, f_k) \overset{L^p}{\to} (\mu, f)$
- $[\mu_k, f_k] \to [\mu, f]$ *and*

$$\int_{S_j} |y|^p \, d[\mu_k, f_k](x, y) \to 0$$

as $j \to \infty$ uniformly in k, where

$$S_j = \left\{ (x, y) \in E \times \mathbb{R}^k : |x| \geq j \text{ or } |y| \geq j \right\}.$$

Theorem 3.1 *Let $1 < p < \infty$. Suppose that (μ_k, f_k), $k \in \mathbb{N}$, are measure-function pairs over E.*

(i) *If $\mu_k(K)$ and $\|f_k\|_{L^p(\mu_k)}$ are uniformly bounded for any compact $K \subset E$, then there exists a subsequence which converges weakly in L^p.*

(ii) *Suppose $(\mu_k, f_k) \overset{L^p}{\rightharpoonup} (\mu, f)$. Then*

$$\|f\|_{L^p(\mu)} \leq \liminf_{k \to \infty} \|f_k\|_{L^p(\mu_k)}.$$

(iii) *If equality holds, then the convergence is strong in L^p.*

This is one example which shows that weak and strong convergence in L^p for measure-function pairs behave in some aspects like weak and strong convergence in the Banach spaces $L^p(\mu)$ for a fixed measure μ, although we have not even a vector space in this case. The next result provides another example.

Proposition 3.2 *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that μ_k and μ are Radon measures on E and that $f_k \in L^p(\mu_k, \mathbb{R}^s)$, $f \in L^p(\mu, \mathbb{R}^s)$, $g_k \in L^q(\mu_k, \mathbb{R}^s)$, and $g \in L^q(\mu, \mathbb{R}^s)$. Suppose further that $(\mu_k, f_k) \overset{L^p}{\to} (\mu, f)$ and $(\mu_k, g_k) \overset{L^q}{\rightharpoonup} (\mu, g)$. Then $(\mu_k, f_k \cdot g_k) \overset{L^1}{\rightharpoonup} (\mu, f \cdot g)$.*

Proof. Consider the Radon measures $\tilde{\mu}_k = [\mu_k, f_k]$ and the functions $\tilde{g}_k(x, y) = g_k(x)$ on $E \times \mathbb{R}^s$. We have

$$\int |\tilde{g}_k|^q \, d\tilde{\mu}_k = \int |g_k|^q \, d\mu_k,$$

which is uniformly bounded, hence $(\tilde{\mu}_k, \tilde{g}_k)$ are measure-function pairs over $E \times \mathbb{R}^s$ which satisfy the conditions of Theorem 3.1.(i). Thus for a subsequence (which we denote the same as the whole sequence) there exists a weak limit $(\tilde{\mu}, \tilde{g})$ in L^q. By Proposition 3.1, $\tilde{\mu} = [\mu, f]$.

Now choose $\phi \in C_0^0(E)$, and let $\psi_j \in C_0^0(\mathbb{R}^s)$ satisfy $0 \le \psi_j \le 1$ and $\psi_j(y) = 1$ for $|y| \le j$. Then we have

$$
\begin{aligned}
\lim_{k \to \infty} \lim_{j \to \infty} \int \phi(x) \psi_j(y) \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) &= \lim_{k \to \infty} \int \phi(x) \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) \\
&= \lim_{k \to \infty} \int \phi(x) g_k(x) \, d\mu_k(x) \\
&= \int \phi g \, d\mu,
\end{aligned}
$$

using Lebesgue's convergence theorem, and

$$
\begin{aligned}
\lim_{j \to \infty} \lim_{k \to \infty} \int \phi(x) \psi_j(y) \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) &= \lim_{j \to \infty} \int \phi(x) \psi_j(y) \tilde{g}(x, y) \, d\tilde{\mu}(x, y) \\
&= \int \phi(x) \tilde{g}(x, f(x)) \, d\mu(x).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left| \int \phi(x) \psi_j(y) \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) \right. &\left. - \int \phi(x) \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) \right| \\
&= \left| \int \phi(\psi_j \circ f_k - 1) g_k \, d\mu_k \right| \\
&\le \int_{\{x \in E : \, |f_k(x)| \ge j\}} |\phi \, g_k| \, d\mu_k \\
&\le \left(\sup_E |\phi| \right) \mu_k (\operatorname{supp} \phi \cap \{x \in E : \, |f_k(x)| \ge j\})^{\frac{1}{p}} \|g_k\|_{L^q(\mu_k)}.
\end{aligned}
$$

The right hand side converges to 0 uniformly in k as $j \to \infty$. It follows that $g(x) = \tilde{g}(x, f(x))$ for μ-almost every $x \in E$.

Similarly we compute

$$\lim_{k \to \infty} \lim_{j \to \infty} \int \phi(x) \psi_j(y) y \cdot \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) = \lim_{k \to \infty} \int \phi \, f_k \cdot g_k \, d\mu_k$$

and

$$
\begin{aligned}
\lim_{j \to \infty} \lim_{k \to \infty} \int \phi(x) \psi_j(y) y \cdot \tilde{g}_k(x, y) \, d\tilde{\mu}_k(x, y) &= \int \phi(x) f(x) \cdot \tilde{g}(x, f(x)) \, d\mu(x) \\
&= \int \phi \, f \cdot g \, d\mu.
\end{aligned}
$$

7

That the convergence for $j \to 0$ is uniform in k is proved like before, except that we now have on the right-hand side of the corresponding estimate the expression

$$\left(\sup_E |\phi| \right) \left(\int_{\{x \in E \, : \, |f_k(x)| \geq j\}} |f_k|^p \, d\mu_k \right)^{\frac{1}{p}} \|g_k\|_{L^q(\mu_k)}.$$

According to Proposition 3.1, this also converges uniformly to 0, and hence

$$\lim_{k \to \infty} \int \phi \, f_k \cdot g_k \, d\mu_k = \int \phi \, f \cdot g \, d\mu.$$

The proof is thus finished. □

We now extend the definition of measure-function pairs to include multiple-valued functions.

Definition 3.5 *If S is a set, then we write S^∞ for the set of all sequences in S. Let μ be a Radon measure on E and $\theta : E \to \mathbb{N}$ a μ-measurable function. If $f = (f_{(1)}, f_{(2)}, \ldots) : E \to (\mathbb{R}^s)^\infty$ is a multiple-valued function such that $(\mu \llcorner \{\theta \leq l\}, f_{(l)})$ is a measure-function pair for all $l \in \mathbb{N}$, then we call (μ, f) a multiple-valued measure-function pair over E with values in $\mathbb{R}^s$ and with multiplicity θ. We define the graph measure $[\mu, f]_\theta$ on $E \times \mathbb{R}^s$ by*

$$\int \phi(x, y) \, d[\mu, f]_\theta(x, y) = \int \theta(x)^{-1} \sum_{l=1}^{\theta(x)} \phi(x, f_{(l)}(x)) \, d\mu(x).$$

We use the notation

$$\int f \, d\mu = \int \theta(x)^{-1} \sum_{l=1}^{\theta(x)} f_{(l)}(x) \, d\mu(x).$$

when it is clear what the multiplicity function is (which will usually be the case). In this case we also drop the subscript θ for the graph measure and write $[\mu, f] = [\mu, f]_\theta$.

If $1 \leq p < \infty$, then we write (using this notation)

$$\|f\|_{L^p(\mu)} = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Let (μ_k, f_k), $k \in \mathbb{N}$, and (μ, f) be multiple-valued measure-function pairs with multiplicities θ_k and θ, respectively. We say that (μ_k, f_k) converges weakly in L^p to (μ, f), if μ_k converges to μ in the sense of measures, $\lim_{k \to \infty} \int \phi f_k \, d\mu_k = \int \phi f \, d\mu$ for all $\phi \in C_0^0(E)$, and $\|f_k\|_{L^p(\mu_k)}$ is uniformly bounded. The convergence is strong in L^p, if $[\mu_k, f_k] \to [\mu, f]$ and

$$\int_{S_j} |y|^p \, d[\mu_k, f_k](x, y) \to 0$$

as $j \to \infty$ uniformly in k, where S_j is defined as in Proposition 3.1.

Remarks.

- Given a (single-valued) measure-function pair over E and a multiplicity function $\theta : \Omega \to \mathbb{N}$ which is measurable with respect to μ, we can (and often will) regard f as a multiple-valued measure-function pair by considering the sequence $(f, f, \ldots)$.

- Let (μ, f) and (μ, g) be multiple-valued measure-function pairs over E with values in $\mathbb{R}$ and with the same multiplicity θ. Then the multiple-valued measure function pairs $(\mu, f + g)$ and (μ, fg) can be defined by component-wise addition and multiplication (the latter provided that the product is in $L^1_{\mathrm{loc}}(\mu)$). The same procedure works for other operations, e. g. the scalar product.

We have the following version of Proposition 3.2 for multiple-valued measure-function pairs.

Proposition 3.3 *Let $1 < p < \infty$ and $1 < q < \infty$. Suppose that (μ_k, f_k) are multiple-valued measure-function pairs over E with values in $\mathbb{R}^s$ and multiplicity θ_k, such that $(\mu_k, f_k) \overset{L^p}{\to} (\mu, f)$ for a multiple-valued measure-function pair (μ, f) with multiplicity θ. Suppose further that (μ_k, g_k) are multiple-valued measure-function pairs over E with values in $\mathbb{R}^t$ and multiplicity θ_k, such that $\|g_k\|_{L^q(\mu_k)}$ is uniformly bounded.*

(i) There exist a subsequence $\{k'\} \subset \mathbb{N}$ and a multiple-valued function $g : E \to (\mathbb{R}^t)^\infty$ such that (μ, g) is a multiple-valued measure-function pair over E with multiplicity θ and

$$\int \phi(x, f(x)) \cdot g(x) \, d\mu(x) = \lim_{k' \to \infty} \int \phi(x, f_{k'}(x)) \cdot g_{k'}(x) \, d\mu_{k'}(x) \qquad (3)$$

for all $\phi \in C^0_0(E \times \mathbb{R}^s, \mathbb{R}^t)$. Moreover, $(\mu_{k'}, g_{k'}) \overset{L^q}{\rightharpoonup} (\mu, g)$ and

$$\|g\|_{L^q(\mu)} \leq \liminf_{k' \to \infty} \|g_{k'}\|_{L^q(\mu_{k'})}.$$

(ii) If (μ_k, g_k) are single-valued measure-function pairs and if $(\mu_k, g_k) \overset{L^q}{\to} (\mu, g)$ for a measure-function pair (μ, g), then

$$\int \psi(x, f(x), g(x)) \, d\mu(x) = \lim_{k \to \infty} \int \psi(x, f_k(x), g_k(x)) \, d\mu_k(x)$$

for all $\psi \in C^0_0(E \times \mathbb{R}^s \times \mathbb{R}^t)$.

Proof. Define $\tilde{\mu}_k = [\mu_k, f_k]$ and $\tilde{\mu} = [\mu, f]$. Further define on $E \times \mathbb{R}^s$ the functions

$$\tilde{g}_k(x, y) = \begin{cases} (g_k)_{(l)}(x), & \text{if } y = (f_k)_{(l)}(x), \\ \text{anything}, & \text{else.} \end{cases}$$

This is well-defined $\tilde{\mu}_k$-almost everywhere only if for μ_k-almost every $x \in E$ we have for all $l_1, l_2 \in \{1, \ldots \theta(x)\}$ either $(f_k)_{(l_1)}(x) \neq (f_k)_{(l_2)}(x)$ or $(g_k)_{(l_1)}(x) = (g_k)_{(l_2)}(x)$. However we may assume without loss of generality that this is true.

9

We have
$$\int |\tilde{g}_k|^q \, d\tilde{\mu}_k = \int |g_k|^q \, d\mu_k,$$
hence by Theorem 3.1 there exists a function $\tilde{g} : E \times \mathbb{R}^s \to \mathbb{R}^t$ such that $(\tilde{\mu}_{k'}, \tilde{g}_{k'}) \overset{L^q}{\rightharpoonup} (\tilde{\mu}, \tilde{g})$ for a subsequence.

To prove (i), set $g(x) = \tilde{g}(x, f(x))$. Then (3) and the inequality follow immediately. For the weak convergence, use estimates like in the proof of Proposition 3.2.

Under the conditions of (ii), define $\tilde{h}(x, y) = g(x)$. It is easy to see that for $\phi \in C_0^0(E \times \mathbb{R}^s, \mathbb{R}^t)$, the functions

$$\eta_k(x) = \theta_k(x)^{-1} \sum_{l=1}^{\theta_k(x)} \phi(x, (f_k)_{(l)}(x)) \quad \text{and} \quad \eta(x) = \theta(x)^{-1} \sum_{l=1}^{\theta(x)} \phi(x, f_{(l)}(x))$$

satisfy $(\mu_k, \eta_k) \overset{L^r}{\rightharpoonup} (\mu, \eta)$ for any $r < \infty$. According to Proposition 3.2, this implies (3) for the given function g. Hence $\tilde{g} = \tilde{h}$. We conclude that $\|\tilde{g}\|_{L^q(\tilde{\mu})} = \lim_{k \to \infty} \|\tilde{g}_k\|_{L^q(\tilde{\mu}_k)}$, and the convergence of $(\tilde{\mu}_k, \tilde{g}_k)$ to $(\tilde{\mu}, \tilde{g})$ is strong in L^q by Theorem 3.1. This proves (ii). $\qquad\square$

4 Weak derivatives

Using formula (2), we can integrate by parts C^1-functions on a curvature varifold. We want to generalize the notion of differentiability of a function based on such an integration by parts formula. We will consider multiple-valued functions however.

Definition 4.1 *Let $V = \mathbf{v}(M, \theta)$ be an integral curvature n-varifold in Ω with generalized curvature $A = (A_{ijk})$. A multiple-valued function $f : \Omega \to (\mathbb{R}^s)^\infty$ is called weakly differentiable with respect to V, if (μ_V, f) is a multiple-valued measure-function pair over Ω with values in $\mathbb{R}^s$ and multiplicity θ, satisfying*

$$|A(x, P)|\theta(x)^{-1} \sum_{l=1}^{\theta(x)} |f_{(l)}(x)| \in L_{\mathrm{loc}}^1(V),$$

and if there exists a multiple-valued function $g : \Omega \to (\mathbb{R}^{N \times s})^\infty$ such that also (μ_V, g) is a multiple-valued measure-function pair with multiplicity θ, and such that

$$\int \Big(P_{ij} \frac{\partial \phi}{\partial x^j}(x, f(x), P) + \frac{\partial \phi}{\partial y^\alpha}(x, f(x), P) g_{i\alpha}(x) \tag{4}$$

$$+ \frac{\partial \phi}{\partial P_{jk}}(x, f(x), P) A_{ijk}(x, P) + \phi(x, f(x), P) A_{jij}(x, P) \Big) \, dV(x, P) = 0$$

for all $\phi = \phi(x, y, P) \in C_0^1(\Omega \times \mathbb{R}^s \times \mathbb{R}^{N \times N})$ and for $i = 1, \dots, N$. (Here the notation of Definition 3.5 is used, regarding the integrand as a multiple-valued function on $G_n(\Omega)$. We sum over alpha from 1 to s.) We call g the weak gradient of f with respect to V and write $g = \nabla^V f$.

Example. The function $P(x) = T_x M$ is weakly differentiable with respect to V with

$$\nabla_i^V P_{jk}(x) = A_{ijk}(x, T_x M).$$

Proposition 4.1 *Let $V = \mathbf{v}(M, \theta)$ be an integral curvature n-varifold in Ω and $f : \Omega \to (\mathbb{R}^s)^\infty$ a multiple-valued function that is weakly differentiable with respect to V. If $\psi : \Omega \times \mathbb{R}^s \times \mathbb{R}^{N \times N} \to \mathbb{R}^t$ is a Lipschitz map, then $g(x) = \psi(x, f(x), P(x))$, where P is defined as above, is weakly differentiable with respect to V and satisfies*

$$\nabla_i^V g(x) = P_{ij}(x) \frac{\partial \psi}{\partial x^j}(x, f(x), P(x)) + \frac{\partial \psi}{\partial y^\alpha}(x, f(x), P(x)) \nabla_i^V f^\alpha(x)$$

$$+ \frac{\partial \psi}{\partial P_{jk}}(x, f(x), P(x)) A_{ijk}(x, P(x)).$$

Proof. If ψ is smooth and has the property $|\psi(x, y, P)| \geq |y|$ for $y \notin K$, where K is a compact subset of $\mathbb{R}^s$, then this follows simply by inserting $\psi(x, f(x), P(x))$ into (4). Otherwise approximate ψ by such maps and apply Lebesgue's convergence theorem to compute the limits. $\square$

Remarks.

- It is easy to verify that $\nabla^V f$ is uniquely determined by V and f.

- If ψ is a Lipschitz function of $G_n(\Omega)$, then it defines a weakly differentiable function with respect to V by Proposition 4.1. It is therefore convenient sometimes to think of functions on $G_n(\Omega)$ rather than Ω.

- The reason why we consider multiple-valued functions is that we want to prove a generalization of Rellich's theorem for pairs of integral curvature varifolds and weakly differentiable functions. However the following example shows that this is not possible for single-valued functions. Consider the varifolds $V_k = \mathbf{v}(\mathbb{R} \times \{0, \frac{1}{k}\}, 1)$ in $\mathbb{R}^2$ and the function

$$f(x, y) = \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0. \end{cases}$$

 Clearly f is differentiable with respect to V_k for any reasonable definition of differentiability with $\nabla^{V_k} f = 0$. On the other hand, $V_k \to \mathbf{v}(\mathbb{R} \times \{0\}, 2)$ in the varifold sense, but the measure-function pairs (μ_{V_k}, f) do not converge in the strong sense unless we allow the multiple-valued function $(0, 1, \dots)$ as the limit.

- Weak differentiability with respect to integral curvature varifolds with boundary can be defined similarly. For simplicity we do not consider that case however.

Definition 4.2 *Let $1 \leq p < \infty$ and $V = \mathbf{v}(M, \theta)$ be an integral curvature n-varifold in Ω. We denote by $W^{1,p}(V, \mathbb{R}^s)$ (or $W^{1,p}(V)$ if $s = 1$) the set of all multiple-valued functions $f : \Omega \to (\mathbb{R}^s)^\infty$ that are weakly differentiable with respect to V and satisfy*

$$\|f\|_{W^{1,p}(V)} := \|\nabla^V f\|_{L^p(\mu_V)} + \||A| f\|_{L^p(\mu_V)} < \infty,$$

where A is the generalized curvature of V.

The Sobolev inequality of Michael–Simon [10] holds also for this situation.

Theorem 4.1 *Suppose that $V = \mathbf{v}(M, \theta)$ is an integral curvature n-varifold in Ω and $1 \leq p < \infty$. Let $f \in W^{1,p}(V, \mathbb{R}^s)$. Suppose further that $\operatorname{supp} f \Subset \Omega$. Then*

$$\|f\|_{L^q(\mu_V)} \leq C\|f\|_{W^{1,p}(V)}$$

for $q = \frac{np}{n-p}$ and $C = \frac{4^{n+1}p(n-1)N}{\omega_n^{1/n}(n-p)}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. Assume first that $p = 1$, $s = 1$, and $f \geq 0$ almost everywhere with respect to μ. From (4) we derive that

$$0 = \int \left(P_{ij}\frac{\partial \phi}{\partial x^j}f + \phi \nabla_i^V f + \phi f \mathbf{H}_i \right) d\mu_V$$

for all $\phi \in C_0^\infty(\Omega)$, where $\mathbf{H} = (\mathbf{H}_i)$ is the generalized mean curvature vector of V. Moreover, a corresponding formula holds for $\psi \circ f$ for all Lipschitz functions $\psi : \mathbb{R} \to \mathbb{R}$. The proof of Theorem 2.1 in [10] now carries over to this situation almost word for word and provides the inequality

$$\left(\int f^{\frac{n}{n-1}} d\mu_V \right)^{\frac{n-1}{n}} \leq \frac{4^{n+1}}{\omega_n^{1/n}} \int (|\nabla^V f| + f|\mathbf{H}|) d\mu_V.$$

In general, insert $|f|^{\frac{p(n-1)}{n-p}}$ instead of f into this estimate and obtain

$$\left(\int |f|^{\frac{np}{n-p}} d\mu_V \right)^{\frac{n-1}{n}}$$

$$\leq \frac{4^{n+1}}{\omega_n^{1/n}} \int \left(\frac{p(n-1)}{n-p}|f|^{\frac{n(p-1)}{n-p}}|\nabla^V f| + |f|^{\frac{p(n-1)}{n-p}}|\mathbf{H}| \right) d\mu_V$$

$$\leq \frac{4^{n+1}p(n-1)}{\omega_n^{1/n}(n-p)} \left(\int |f|^{\frac{np}{n-p}} d\mu_V \right)^{1-\frac{1}{p}} \left(\int (|\nabla^V f|^p + |f|^p|\mathbf{H}|^p) d\mu_V \right)^{\frac{1}{p}}.$$

The claim follows immediately. $\qquad\square$

To prove our version of Rellich's theorem, we need some preparations first. The basic idea for the next lemma is due to Mantegazza [9] (cf. also [4, 5, 6]).

Lemma 4.1 *Let $V = \mathbf{v}(M, \theta)$ be an integral curvature n-varifold in Ω and $f : \Omega \to [0, \infty)^\infty$ weakly differentiable with respect to V. Consider the set $M' = M \times [-1, \infty)$ and the multiplicity function*

$$\theta'(x, y) = |\{l \in \{1, \ldots, \theta(x)\}: f_{(l)}(x) \geq y\}|, \quad x \in \Omega, \; y \geq -1.$$

Set $V' = \mathbf{v}(M', \theta')$. Then V' is an integral curvature $(n + 1)$-varifold with boundary in $\Omega \times \mathbb{R}$. Its generalized curvature is

$$A'_{ijk}(x, y, Q) = \begin{cases} A_{ijk}(x, P), & \text{if } 1 \leq i, j, k \leq N \text{ and } Q = P \times \mathbb{R}, \\ 0 & , \quad \text{else,} \end{cases}$$

where $A = (A_{ijk})$ is the generalized curvature of V. Moreover, we have

$$|\partial V'|(G_{n+1}(\Omega \times \mathbb{R})) = \left\| \sqrt{|\nabla^V f|^2 + 1} + 1 \right\|_{L^1(\mu_V)},$$

and for all $\phi \in C_0^0(\Omega \times \mathbb{R})$,

$$\int \phi(x,y) \, d\|\partial V'_{N+1}(x,y,P)\| = \int \phi \, d[\mu_V, f] - \int \phi(x,-1) \, d\mu_V(x). \qquad (5)$$

Proof. Obviously we have $T_{(x,y)}M' = T_x M \times \mathbb{R}$ for $\mathcal{H}^{n+1}$-almost every $(x,y) \in M'$. Hence

$$\int \phi(x,y,Q) \, dV'(x,y,Q) = \int \int_{-1}^{f(x)} \phi(x,y,P \times \mathbb{R}) \, dy \, dV(x,P)$$

for all $\phi \in C_0^0(G_{n+1}(\Omega \times \mathbb{R}))$. Thus we compute

$$\int \left(Q_{ij} \frac{\partial \phi}{\partial x^j}(x,y,Q) + \frac{\partial \phi}{\partial Q_{jk}}(x,y,Q) A'_{ijk}(x,y,Q) \right.$$
$$\left. + \phi(x,y,Q) A'_{jij}(x,y,Q) \right) dV'(x,y,Q)$$

$$= \int \left(P_{ij} \int_{-1}^{f(x)} \frac{\partial \phi}{\partial x^j}(x,y,P \times \mathbb{R}) \, dy + \int_{-1}^{f(x)} \frac{\partial \phi}{\partial P_{jk}}(x,y,P \times \mathbb{R}) \, dy \, A_{ijk}(x,P) \right.$$
$$\left. + \int_{-1}^{f(x)} \phi(x,y,P \times \mathbb{R}) \, dy \, A_{jij}(x,P) \right) dV(x,P)$$

$$= -\int \phi(x,f(x),P \times \mathbb{R}) \nabla_i^V f(x) \, dV(x,P)$$

for $i = 1, \ldots, N$, owing to (4). As usual we sum all indices from 1 to N, even in the first line. Note however that the corresponding terms for $j = N + 1$ or $k = N + 1$ vanish.

Furthermore we have

$$\int Q_{N+1,N+1} \frac{\partial \phi}{\partial y}(x,y,Q) \, dV'(x,y,Q)$$

$$= \int \int_{-1}^{f(x)} \frac{\partial \phi}{\partial y}(x,y,P \times \mathbb{R}) \, dy \, dV(x,P)$$

$$= \int (\phi(x,f(x),P \times \mathbb{R}) - \phi(x,-1,P \times \mathbb{R})) \, dV(x,P).$$

All claims now follow easily from these formulae. $\qquad \square$

Now we are ready to proof the main result of this section.

Theorem 4.2 *Let $1 < p < n$ and $\frac{np}{np-n+p} < q < \infty$. Suppose that Ω is bounded, and that $V_k = \mathbf{v}(M_k, \theta_k)$ are integral curvature n-varifolds in Ω and $f_k \in W^{1,p}(V_k)$, such that we have the uniform bounds*

$$\int (|A_k|^q + 1) \, dV_k \leq C_0,$$

where A_k is the generalized curvature of V_k, and

$$\|f_k\|_{W^{1,p}(V_k)} \le C_1.$$

Suppose further that $\operatorname{supp} f_k \Subset \Omega$ for each k. Then for any $r \in (1, \frac{np}{n-p})$, there exist a subsequence $\{k'\} \subset \mathbb{N}$, an integral curvature n-varifold $V = \mathbf{v}(M,\theta)$ in Ω with generalized curvature A, and a multiple-valued function $f : \Omega \to \mathbb{R}^\infty$ such that

$$(\mu_{V_{k'}}, f_{k'}) \xrightarrow{L^r} (\mu_V, f) \quad and \quad (V_{k'}, A_{k'}) \xrightarrow{L^q} (V, A).$$

If $(V_{k'}, A_{k'}) \xrightarrow{L^q} (V, A)$, then $f \in W^{1,p}(V)$ and

$$(\mu_{V_{k'}}, \nabla^{V_{k'}} f_{k'}) \xrightarrow{L^p} (\mu_V, \nabla^V f).$$

Proof. We may assume that the f_k take only non-negative values. Otherwise we consider the functions $\max\{f_k + 1, 0\}$ and $\min\{f_k - 1, 0\}$ instead. If the result holds for both of these sequences, then it holds also for f_k.

Since we have a uniform bound on $\|f_k\|_{L^{(np)/(n-p)}(\mu_k)}$ by Theorem 4.1, it is easy to show that

$$\int_{S_j} |y|^r \, d[\mu_k, f_k](x, y) \to 0$$

uniformly as $j \to \infty$, where

$$S_j = \{(x, y) \in \Omega \times \mathbb{R} : \ |y| \ge j\}.$$

Let V_k' be the $(n+1)$-varifolds associated to V_k and f_k by Lemma 4.1. Then the quantities

$$\mu_{V_k'}(\Omega \times \mathbb{R}) + \int |A_k'|^t \, dV_k' + |\partial V_k'|(G_{n+1}(\Omega \times \mathbb{R}))$$

are uniformly bounded for some $t > 1$. Hence by Theorem 6.1 in [9], we may pick a subsequence (which we denote the same as the original sequence), such that $V_k' \to V'$ for an integral curvature $(n+1)$-varifold V' with boundary in $\Omega \times \mathbb{R}$. We may also assume that $V_k \to V$, where $V = \mathbf{v}(M, \theta)$ is an integral curvature varifold in Ω, according to Theorem 5.3.2 in [8], and that $(V_k, A_k) \xrightarrow{L^q} (V, A)$ for the generalized curvature A of V.

Note that for any $\phi \in C_0^0(\Omega \times (-1, \infty))$ with $\phi \ge 0$, and for $h < 0$, we have

$$\int \phi(x, y + h) \, d\mu_{V_k'}(x, y) \le \int \phi \, d\mu_{V_k'}.$$

Therefore $\mu_{V'}$ must have the same property. It follows that V' is of the form $V' = \mathbf{v}(M', \theta')$, where $M' = \tilde{M} \times [-1, \infty)$ for an n-rectifiable and $\mathcal{H}^n$-measurable set $\tilde{M} \subset \Omega$, and $\theta'(x, y)$ is decreasing in y. Hence there is a multiple-valued function $f : \tilde{M} \to \mathbb{R}^\infty$ such that

$$\theta'(x, y) = |\{l \in \mathbb{N} : \ f_{(l)}(x) \ge y\}|$$

for $\mathcal{H}^{n+1}$-almost every $(x, y) \in M'$. From (5) we see that $\tilde{M} = M$ and that $[\mu_{V_k}, f_k] \to [\mu_V, f]$.

Now assume that $(V_{k'}, A_{k'}) \overset{L^q}{\to} (V, A)$. Choose $\phi \in C_0^1(\Omega \times \mathbb{R} \times \mathbb{R}^{N \times N})$. By Proposition 3.3, we have $(\mu_{V_k}, (f_k(x), T_x M_k)) \overset{L^r}{\to} (\mu_V, (f(x), T_x M))$. Hence

$$\int P_{ij} \frac{\partial \phi}{\partial x^j}(x, f(x), P) \, dV(x, P) = \lim_{k \to \infty} \int P_{ij} \frac{\partial \phi}{\partial x^j}(x, f_k(x), P) \, dV_k(x, P).$$

Moreover we see that for any $\psi \in C_0^0(\Omega \times \mathbb{R} \times \mathbb{R}^{N \times N})$ and for any $t < \infty$, the functions $\tilde{f}_k(x) = \theta_k(x)^{-1} \sum_{l=1}^{\theta_k(x)} \psi(x, (f_k)_{(l)}(x), T_x M_k)$ satisfy $(\mu_{V_k}, \tilde{f}_k) \overset{L^t}{\to} (\mu_V, \tilde{f})$, where $\tilde{f}(x) = \theta(x)^{-1} \sum_{l=1}^{\theta(x)} \psi(x, f_{(l)}(x), T_x M)$. Thus

$$\int \frac{\partial \phi}{\partial P_{jm}}(x, f(x), P) A_{ijm}(x, P) \, dV(x, P)$$
$$= \lim_{k \to \infty} \int \frac{\partial \phi}{\partial P_{jm}}(x, f_k(x), P)(A_k)_{ijm}(x, P) \, dV_k(x, P),$$
$$\int \phi(x, f(x), P,) A_{jij}(x, P) \, dV(x, P)$$
$$= \lim_{k \to \infty} \int \phi(x, f_k(x), P)(A_k)_{jij}(x, P) \, dV_k(x, P)$$

by Proposition 3.2. Owing to Proposition 3.3, there exists a multiple-valued function $g : \Omega \to (\mathbb{R}^N)^\infty$ such that $(\mu_{V_k}, \nabla^{V_k} f_k) \overset{L^p}{\to} (\mu_V, g)$ and

$$\int \frac{\partial \phi}{\partial y}(x, f(x), P) g_i(x) \, dV(x, P) = \lim_{k \to \infty} \int \frac{\partial \phi}{\partial y}(x, f_k(x), P) \nabla_i^{V_k} f_k(x) \, dV_k(x, P).$$

Therefore f and g satisfy (4). Moreover,

$$\|g\|_{L^p(\mu_V)} + \| \, |A| \, |f| \, \|_{L^p(\mu_V)} \leq \liminf_{k \to \infty} \|f\|_{W^{1,p}(V)} < \infty,$$

and we conclude that $f \in W^{1,p}(V)$ and $\nabla^V f = g$. $\qquad \square$

5 Minimizers of curvature integrals

In the following we sketch a possible application of the results obtained so far.

For simplicity we restrict the class of varifolds that we work with. Namely, we assume now that $N = n + 1$, i. e. that we have co-dimension 1. Moreover, we want the varifolds to possess a weakly differentiable normal vector.

Definition 5.1 *Let $V = \mathbf{v}(M, \theta)$ be an integral curvature n-varifold in Ω. If there exists a function $\nu : \Omega \to (\mathbb{S}^n)^\infty$ which is weakly differentiable with respect to V, such that $\nu_{(l)}(x) \perp T_x M$ for μ_V-almost every $x \in \Omega$ and $l = 1, \ldots, \theta(x)$, then such a ν is called a differentiable normal vector of V.*

Proposition 5.1 *Let V be an integral curvature n-varifold in Ω with generalized second fundamental form $\mathbf{B} = (\mathbf{B}_{ij}^k)$ and differentiable normal vector ν. Then*

$$\mathbf{B}_{ij}^k = -\nu^k \nabla_i^V \nu^j$$

μ_V-almost everywhere. In particular, $|\mathbf{B}| = |\nabla^V \nu|$.

Proof. We have μ_V-almost everywhere

$$0 = \nabla_i^V(P_{jk}\nu^k) = A_{ijk}\nu^k + P_{jk}\nabla_i^V\nu^k \tag{6}$$

and

$$0 = \nabla_i^V|\nu|^2 = 2\nu^j\nabla_i^V\nu^j.$$

From the second identity we conclude that

$$\nabla_i^V\nu^j = P_{jk}\nabla_i^V\nu^k.$$

Hence

$$\mathbf{B}_{ij}^k\nu^k = P_{rj}A_{ikr}\nu^k = -P_{rj}\nabla_i^V\nu^r = -\nabla_i^V\nu^j.$$

Here the properties (i) and (iv) of Sect. 2 and (6) have been used. Since $\mathbf{B}_{ij}^k = \nu^k\nu^r\mathbf{B}_{ij}^r$ by Lemma 2.1, the proof is finished. $\qquad\square$

Definition 5.2 *Let V_k, $k \in \mathbb{N}$, and V be integral curvature n-varifolds in Ω with generalized second fundamental form $\mathbf{B}_k$ and $\mathbf{B}$, respectively. Let $1 < p < \infty$. We say that V_k converges to V weakly (strongly) in the $W^{2,p}$-sense, and write $V_k \overset{W^{2,p}}{\rightharpoonup} V$ ($V_k \overset{W^{2,p}}{\rightarrow} V$), if*

$$(V_k, \mathbf{B}_k) \overset{L^p}{\rightharpoonup} (V, \mathbf{B}) \quad ((V_k, \mathbf{B}_k) \overset{L^p}{\rightarrow} (V, \mathbf{B})).$$

For any $L > 0$, let $\mathcal{W}_L^{2,p}(\Omega)$ denote the set of all weak limits in the $W^{2,p}$-sense of sequences $V_k = \mathbf{v}(M_k, 1)$, where M_k are oriented C^2-submanifolds of Ω satisfying

$$\int_{M_k}(|\mathbf{B}_k|^p + 1)\,d\mathcal{H}^n \leq L,$$

for their second fundamental forms $\mathbf{B}_k$. Moreover, define

$$\mathcal{W}^{2,p}(\Omega) = \bigcup_{L>0}\mathcal{W}_L^{2,p}(\Omega).$$

Lemma 5.1 *The sets $\mathcal{W}_L^{2,p}(\Omega)$ are sequentially compact with respect to weak convergence in the $W^{2,p}$-sense.*

Proof. Let $V_k \in \mathcal{W}_L^{2,p}(\Omega)$ be the weak limits in the $W^{2,p}$-sense of $V_{km} = \mathbf{v}(M_{km}, 1)$, where M_{km} have the properties required in the definition above. According to Theorem 5.3.2 in [8], there exist a subsequence (also denoted by V_k) and an integral curvature n-varifold V in Ω such that $V_k \overset{W^{2,p}}{\rightharpoonup} V$. We have to prove that $V \in \mathcal{W}_L^{2,p}(\Omega)$.

Since $C_0^0(G_n(\Omega)$ is separable, we can easily construct a subsequence of the V_{km} that converges to V in the varifold sense. Again by Theorem 5.3.2 in [8], this convergence also holds in the weak $W^{2,p}$-sense. Hence $V \in \mathcal{W}_L^{2,p}(\Omega)$. $\qquad\square$

Lemma 5.2 *Any $V \in \mathcal{W}^{2,p}(\Omega)$ has a differentiable normal vector.*

Proof. The condition that V has a differentiable normal vector is equivalent to requiring that V is an oriented integral varifold (cf. [8]) which satisfies (2) in the sense of oriented varifolds. The claim therefore follows from the arguments of [8]. $\qquad\square$

16

Lemma 5.3 *For $V \in \mathcal{W}^{2,p}(\Omega)$ with differentiable normal vector ν we have*

$$\frac{\partial}{\partial x^i}(\mu_V \llcorner \nu^j) - \frac{\partial}{\partial x^j}(\mu_V \llcorner \nu^i) = 0, \quad 1 \leq i, j \leq n+1, \tag{7}$$

in the distribution sense.

Proof. If V corresponds to a C^2-submanifold M of Ω, then this follows from the fact that $\mu_V \llcorner \nu$ represents locally the distributional derivative of the characteristic function of a set having M as part of its boundary. The set of varifolds in $\mathcal{W}^{2,p}(\Omega)$ satisfying (7) is clearly closed with respect to weak convergence in the $W^{2,p}$-sense, and the claim follows. $\quad\square$

Now consider the functional

$$\mathcal{F}_p(V) = \int |\mathbf{B}|^p \, dV$$

for $V \in \mathcal{W}^{2,p}(\Omega)$, where $\mathbf{B}$ denotes the generalized second fundamental form of V.

If $\mathcal{E} \neq \emptyset$ is a sequentially closed subset of $\mathcal{W}^{2,p}(\Omega)$ such that there exist numbers $C_1, C_2 > 0$ satisfying

$$\mu_V(\Omega) \leq C_2$$

for any $V \in \mathcal{E}$ with $\mathcal{F}_p(V) \leq C_1$, and if $C_1 > \inf_{\mathcal{E}} \mathcal{F}_p$, then we may apply Theorem 6.1 in [8] and find a $V^* \in \mathcal{E}$ with the property

$$\mathcal{F}_p(V^*) = \inf_{\mathcal{E}} \mathcal{F}_p.$$

Example. Suppose $\Omega \Subset \mathbb{R}^{n+1}$ and $1 < p < n$. Let V_0 be an integral curvature n-varifold in $\mathbb{R}^{n+1}$ with compact support, and set $W = V_0 \llcorner G_n(\mathbb{R}^{n+1} \setminus \Omega)$. Let $\mathcal{E}$ be the set of all $V \in \mathcal{W}^{2,p}(\Omega)$ such that $V + W$ is an integral curvature varifold in $\mathbb{R}^{n+1}$. As in [8], we find that

$$\mu_V(\Omega)^{\frac{n-1}{n}} \leq \mu_{V+W}(\mathbb{R}^{n+1})^{\frac{n-1}{n}} \leq c \left(\int |\mathbf{B}|^p \, d(V+W) \right)^{\frac{n-1}{n-p}} \tag{8}$$

for a constant $c = c(n, p)$, where $\mathbf{B}$ is the generalized second fundamental form of $V + W$. Therefore if $\mathcal{E}$ contains an element such that the right hand side of (8) is finite, then there is a minimizer of $\mathcal{F}_p$ in $\mathcal{E}$.

Next we want to compute the Euler-Lagrange equation for such a minimizing varifold of $\mathcal{F}$, or at least the leading term of it.

Choose a C^2-diffeomorphism $\Phi : \Omega \to \Omega$ such that $\mathrm{supp}\,(\Phi - id) \Subset \Omega$ with inverse map $\Psi = \Phi^{-1}$. Then for any integral n-varifold $V = \mathbf{v}(M, \theta)$ in Ω, the varifold $\Phi_\# V = \mathbf{v}(\Phi(M), \theta \circ \Psi)$ is also an integral n-varifold in Ω. As a measure in $G_n(\Omega)$, the transformation reads

$$\int \phi \, d(\Phi_\# V) = \int \phi(\Phi(x), D\Phi(x)P)|\Lambda_n D\Phi(x)| \, dV(x, P).$$

Here $\Lambda_n D\Phi(x)$ is the map $\Lambda_n \mathbb{R}^{n+1} \to \Lambda_n \mathbb{R}^{n+1}$ induced by $D\Phi(x)$. The notation $D\Phi(x)P$ refers to P as an n-dimensional subspace of $\mathbb{R}^{n+1}$, not the corresponding projection.

If V is generated by an orientable C^2-submanifold M with second fundamental form $\mathbf{B}$, then we can compute the second fundamental form of $\Phi(M)$ as follows. Let $G^0(n+1, n)$ be the manifold of all oriented n-subspaces of $\mathbb{R}^{n+1}$ and $T : G^0(n+1, n) \to \mathbb{S}^n$ one of the two smooth maps assigning to every element of $G^0(n+1, n)$ one of its normal vectors. Define $S : \mathbb{S}^n \times \mathrm{Gl}(n+1, \mathbb{R}) \to \mathbb{S}^n$ by

$$S(\nu, Z) = T(ZT^{-1}(\nu)).$$

Furthermore let $Q(\nu, Z)$ be the orthogonal projection along $S(\nu, Z)$. If ν is a differentiable normal vector of M, then $\hat{\nu}$ defined by

$$\hat{\nu}(\Phi(x)) = S(\nu(x), D\Phi(x))$$

is a differentiable normal vector for $\hat{M} = \Phi(M)$. Hence the second fundamental form of $\hat{M}$ has the components

$$
\begin{aligned}
\hat{\mathbf{B}}^k_{ij}(\Phi(x)) =& \\
S^k(\nu(x), &D\Phi(x))Q_{ia}(\nu(x), D\Phi(x)) \\
\cdot \left(\frac{\partial S^j}{\partial \nu^b}(\nu(x), D\Phi(x))\nu^c(x)\mathbf{B}^c_{bd}(x) - \frac{\partial S^j}{\partial Z_{rs}}(\nu(x), D\Phi(x))\frac{\partial^2 \Phi^r(x)}{\partial x^s \partial x^d} \right) & \frac{\partial \Psi^d(\Phi(x))}{\partial x^a}.
\end{aligned}
\tag{9}
$$

In particular

$$\int_{\hat{M}} |\hat{\mathbf{B}}|^p \, d\mathcal{H}^n \le C \int_M (|\mathbf{B}|^p + 1) \, d\mathcal{H}^n$$

for a constant $C = C(n, p, \|\Phi\|_{C^2})$. We see immediately that $\Phi_\#$ maps $\mathcal{W}^{2,p}(\Omega)$ onto itself. By approximation with C^2-submanifolds we also see that formula (9) holds in general for varifolds in $\mathcal{W}^{2,p}(\Omega)$.

Suppose now that we have a family of C^2-diffeomorphisms $\Phi_t(x) = x + tX(x)$ for $X \in C^2_0(\Omega, \mathbb{R}^{n+1})$ and $-\delta < t < \delta$. If $V \in \mathcal{W}^{2,p}(\Omega)$ is a minimizer of $\mathcal{F}_p$ in some set that is mapped onto itself by such diffeomorphisms, then we compute from the condition

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_p((\Phi_t)_\# V) = 0,$$

that

$$\int |\mathbf{B}|^{p-2} \mathbf{B}^k_{ij} \left(\nu^k P_{ia} \frac{\partial S^j}{\partial Z_{rs}}(\nu, \mathrm{id})\frac{\partial^2 X^r}{\partial x^s \partial x^a} + \alpha_{ijk}(\nu, DX, \mathbf{B}) \right) d\mu_V = 0 \tag{10}$$

for functions α_{ijk} depending smoothly on ν and linearly on DX and $\mathbf{B}$. (We are not going to compute these functions.) Note that we have the representation

$$S^j(\nu, Z) = \frac{\nu^i (Z^{-1})_{ij}}{\sqrt{\sum_k (\nu^i (Z^{-1})_{ik})^2}}$$

for S and can thus compute

$$\frac{\partial S^j}{\partial Z_{rs}}(\nu, \mathrm{id})Z_{rs} = -Z_{rj}\nu^r + Z_{rs}\nu^r \nu^s \nu^j.$$

18

Let now $p = 2$. We want to use the results from the previous section to generalize a well-known method for certain variational methods to our situation and prove a regularity result for minimizers of $\mathcal{F}_2$. We need however some additional assumptions, namely generalizations of the Poincaré inequality and of a reverse Poincaré inequality (Caccioppoli inequality) for the normal vectors of minimizing solutions of (10). The former is a well-known result for the usual Sobolev spaces, but it is not clear if it holds for pairs of curvature varifolds and weakly differentiable functions in general. The latter can be proved for a variety of variational problems, but it is more difficult for the case that we consider here. We are not able to prove either of these inequalities in general. For $n = 2$ and if the varifold we consider is the strong limit in the $W^{2,2}$-sense of smooth manifolds, then both can be proved using the graphical decomposition lemma from [12], but in that case there are easier methods to prove regularity than those we will use. We will prove the Poincaré and the reverse Poincaré inequality for the normal vectors of varifolds corresponding to Lipschitz graphs however.

Notation. Let $B_r(x_0)$ denote the ball in $\mathbb{R}^{n+1}$ with center x_0 and radius r.

Definition 5.3 *Let $\epsilon_0 > 0$ and $C_0 > \omega_n$ be given, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. For $V \in \mathcal{W}^{2,2}(\Omega)$ and $B_r(x_0) \subset \Omega$, let $\mathcal{C}(V, B_r(x_0), \mathbb{R}^s)$ be the set of all multiple-valued functions $f_0 : B_r(x_0) \to (\mathbb{R}^s)^\infty$ such that $\tilde{f}(x, P) = f(x)$ agrees $V \llcorner G_n(B_r(x_0))$-almost everywhere with a locally constant function on $\operatorname{supp} V$. Let $f : \Omega \to (\mathbb{R}^s)^\infty$ be weakly differentiable with respect to V. We say that (V, f) satisfies the (ϵ_0, C_0)-Poincaré inequality with constant C, if for any ball $B_r(x_0) \subset \Omega$ satisfying $\mu_V(B_r(x_0)) \le C_0 r^n$ and*

$$\int_{B_r(x_0)} |\mathbf{B}|^2 \, d\mu_V \le \epsilon_0^2 r^{n-2},$$

where $\mathbf{B}$ is the generalized second fundamental form of V, the inequality

$$\inf_{f_0 \in \mathcal{C}(V, B_r(x_0), \mathbb{R}^s)} \|f - f_0\|_{L^2(\mu_V \llcorner B_{r/2}(x_0))}$$

$$\le Cr(\|\nabla^V f\|_{L^2(\mu_V \llcorner B_r(x_0))} + \| \, |\mathbf{B}| \, |f| \, \|_{L^2(\mu_V \llcorner B_r(x_0))})$$

holds. We say that the reverse (ϵ_0, C_0)-Poincaré inequality is satisfied with constant C, if under the conditions above, we always have

$$\|\nabla^V f\|_{L^2(\mu_V \llcorner B_{r/2}(x_0))} \le Cr^{-1} \inf_{f_0 \in \mathcal{C}(V, B_r(x_0), \mathbb{R}^s)} \|f - f_0\|_{L^2(\mu_V \llcorner B_r(x_0))}.$$

Lemma 5.4 *Let $V \in \mathcal{W}^{2,2}(\Omega)$ with generalized second fundamental form $\mathbf{B}$ and differentiable normal vector ν be a solution of (10) for $p = 2$. Suppose there exist constants $\epsilon_0, C_1 > 0$ and $C_0 \in (\omega_n, 2\omega_n)$, such that (V, ν) satisfies the (ϵ_0, C_0)-Poincaré inequality and the reverse (ϵ_0, C_0)-Poincaré inequality, both with constant C_1. Then there exist numbers $\epsilon > 0$ and $\tau \in (0, 1)$, both depending only on n, ϵ_0, C_0, and C_1, such that for any ball $B_r(x_0) \subset \Omega$, the conditions*

$$\mu_V(B_r(x_0)) \le C_0 r^n, \quad \mu_V(B_{2\tau r}(x_0)) \le C_0(2\tau r)^n,$$

and

$$\int_{B_r(x_0)} |\mathbf{B}|^2 \, d\mu_V \le \epsilon^2 r^{n-2}$$

imply

$$(\tau r)^{2-n} \int_{B_{\tau r}(x_0)} |\mathbf{B}|^2 \, d\mu_V \leq \frac{1}{2} r^{2-n} \int_{B_r(x_0)} |\mathbf{B}|^2 \, d\mu_V.$$

Proof. It suffices to consider the case $B_r(x_0) = B_1(0)$, for we may rescale everything otherwise. We argue by contradiction. If the lemma is false, then for any fixed $\tau \in (0, \frac{1}{4})$, there exist integral curvature n-varifolds $V_k = \mathbf{v}(M_k, \theta_k)$ in $B_1(0) \subset \mathbb{R}^{n+1}$ with second fundamental forms $\mathbf{B}_k$ and differentiable normal vectors ν_k, which are solutions of (10) and which satisfy the (ϵ_0, C_0)-Poincaré inequality and the reverse (ϵ_0, C_0)-Poincaré inequality with constant C_1 and

$$\mu_{V_k}(B_1(0)) \leq C_0, \quad \mu_{V_k}(B_{2\tau}(0)) \leq C_0 (2\tau)^n,$$
$$\int_{B_1(0)} |\mathbf{B}_k|^2 \, d\mu_{V_k} =: \epsilon_k^2 \to 0 \quad (k \to \infty),$$

but

$$\tau^{2-n} \int_{B_\tau(0)} |\mathbf{B}_k|^2 \, d\mu_{V_k} > \frac{1}{2} \epsilon_k^2. \tag{11}$$

Choose $\nu_{0k} \in \mathcal{C}(V_k, B_1(0), \mathbb{R}^{n+1})$ such that

$$\|\nu_k - \nu_{0k}\|_{L^2(\mu_{V_k} \llcorner B_{1/2}(0))} \leq 2 \inf_{f_0 \in \mathcal{C}(V_k, B_1(0), \mathbb{R}^{n+1})} \|\nu_k - f_0\|_{L^2(\mu_{V_k} \llcorner B_{1/2}(0))}.$$

We may assume that $\epsilon_k \leq \epsilon_0$ for all k and thus

$$\|\nu_k - \nu_{0k}\|_{L^2(\mu_{V_k} \llcorner B_{1/2}(0))} \leq 2 C_1 \epsilon_k$$

by the Poincaré inequality.

Choose $\zeta \in C_0^\infty(B_{1/2}(0))$ with $\zeta \equiv 1$ in $B_{1/4}(0)$ and $|\nabla \zeta| \leq 8$. Define

$$f_k = \frac{1}{\epsilon_k} \zeta (\nu_k - \nu_{0k}).$$

These functions are weakly differentiable with respect to V_k with

$$\nabla_i^{V_k} f_k(x) = \frac{1}{\epsilon_k} \left((T_x M_k)_{ij} \frac{\partial \zeta(x)}{\partial x^j} (\nu_k(x) - \nu_{0k}(x)) + \zeta(x) \nabla_i^{V_k} \nu_k(x) \right).$$

Thus

$$\|f_k\|_{W^{1,2}(V_k)} \leq 16 C_1 + 5.$$

By Theorem 4.2, we may assume that $(\mu_{V_k}, f_k) \overset{L^2}{\to} (\mu_V, f)$ and $(\mu_{V_k}, \nabla^{V_k} f_k) \overset{L^2}{\to} (\mu_V, \nabla^V f)$ for an integral curvature varifold $V = \mathbf{v}(M, \theta)$ with generalized curvature $A = 0$ and a function $f : B_1(0) \to (\mathbb{R}^{n+1})^\infty$ which is weakly differentiable with respect to V. The results of [7] imply that V is a union of hyper-planes. Moreover, we have $\mu_V(B_1(0)) \leq C_0 < 2\omega_n$, and we can therefore choose τ so small that at most one of these hyper-planes intersects $B_{2\tau}(0)$.

Let $\phi \in C_0^\infty(B_{1/4}(0))$. We compute

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x^i} \nu_{0k}^j \, d\mu_{V_k} &= \int \nu_k \cdot \nabla \phi \, \nu_k^i \nu_{0k}^j \, d\mu_{V_k} - \int \phi \nu_{0k}^j \mathbf{H}_{ki} \, d\mu_{V_k} \\
&= \int \nu_k \cdot \nabla \phi \, \nu_k^i (\nu_{0k}^j - \nu_k^j) \, d\mu_{V_k} + \int \nu_k \cdot \nabla \phi \, \nu_k^i \nu_k^j \, d\mu_{V_k} \\
&\quad - \int \phi (\nu_{0k}^j - \nu_k^j) \mathbf{H}_{ki} \, d\mu_{V_k} - \int \phi \nu_k^i \nu_k^j |\mathbf{H}_k| \, d\mu_{V_k},
\end{aligned}$$

where $\mathbf{H}_k = (\mathbf{H}_{ki})$ is the generalized mean curvature vector of V_k. Thus by Lemma 5.3,

$$\int \left(\frac{\partial \phi}{\partial x^i} f_k^j - \frac{\partial \phi}{\partial x^j} f_k^i \right) d\mu_{V_k}$$

$$= \frac{1}{\epsilon_k} \left(\int \nu_k \cdot \nabla\phi \left(\nu_k^i (\nu_{0k}^j - \nu_k^j) - \nu_k^j (\nu_{0k}^i - \nu_k^i) \right) d\mu_{V_k} \right. \tag{12}$$

$$\left. - \int \phi ((\nu_{0k}^j - \nu_k^j) \mathbf{H}_{ki} - (\nu_{0k}^i - \nu_k^i) \mathbf{H}_{kj}) d\mu_{V_k} \right).$$

We estimate

$$\left| \frac{1}{\epsilon_k} \int \phi ((\nu_{0k}^j - \nu_k^j) \mathbf{H}_{ki} \, d\mu_{V_k} \right| \leq \left(\sup_{B_{1/4}(0)} |\phi| \right) \|f_k\|_{L^2(\mu_{V_k})} \|\mathbf{B}_k\|_{L^2(\mu_{V_k})} \to 0$$

as $k \to \infty$, and the same with i and j interchanged. Assume now that $\operatorname{supp}\phi$ intersects only one of the hyper-planes generating V, and that $\nu \cdot \nabla\phi = 0$, where ν is the differentiable normal vector of V. Then

$$\int |\nu_k \cdot \nabla\phi|^2 \, d\mu_{V_k} = \sum_{i=1}^{n+1} \int \left| (\delta_{ij} - P_{ij}) \frac{\partial \phi}{\partial x^j} \right|^2 dV_k \to 0,$$

and therefore the whole right-hand side of (12) converges to 0. Assuming that the hyper-plane in question is $P_1 = x_1 + (\mathbb{R}^n \times \{0\})$ for some $x_1 \in B_1(0)$, we see immediately that f^{n+1} is locally constant and $\frac{\partial f^j}{\partial x^i} - \frac{\partial f^i}{\partial x^j} = 0$ in the distribution sense for i, j ranging from 1 to n, on $M_1 = (P_1 \cap B_{1/4}(0)) \backslash \overline{(M \backslash P_1)}$. Knowing however that the derivative of f exists in the weak sense, we conclude that in fact the function $f_1 := f|_{\overline{M}_1}$ satisfies

- f_1^{n+1} is constant,

- $(f_1^1, \ldots f_1^n) = (\frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^n})$ for some function $u : \overline{M}_1 \to \mathbb{R}$.

From (10) we derive the partial differential equation $\Delta^2 u = 0$, where Δ denotes the Laplace operator on $\overline{M}_1$, using the fact that the $\alpha_{ijk}(\nu, DX, \mathbf{B})$ are linear in $\mathbf{B}$ and observing the remark following (10). Since

$$\|\Delta u\|_{L^2(\overline{M}_1)} \leq n \liminf_{k \to \infty} \|\nabla^{V_k} f_k\|_{L^2(\mu_{V_k} \llcorner B_{1/4}(0))} \leq (16C_1 + 5)n$$

and thus

$$\|\Delta u\|_{L^\infty(\overline{M}_1 \cap B_{1/8}(0))} \leq 8^n (16C_1 + 5)n$$

by the mean value theorem, there is a constant $C = C(n, C_1)$ such that

$$\inf_{\alpha \in \mathbb{R}^{n+1}} \|f - \alpha\|_{L^2(\mu_V \llcorner B_{2\tau}(0))} \leq C \tau^{\frac{n+2}{2}}.$$

Hence

$$\inf_{\alpha \in \mathbb{R}^{n+1}} \|\nu_k - \nu_{0k} - \alpha\|_{L^2(\mu_{V_k} \llcorner B_{2\tau}(0))} \leq 2C\tau^{\frac{n+2}{2}} \epsilon_k$$

for all sufficiently large k. Now the reverse Poincaré inequality implies

$$\|\nabla^{V_k} \nu_k\|_{L^2(\mu_{V_k} \llcorner B_\tau(0))} \leq CC_1 \tau^{\frac{n}{2}} \epsilon_k.$$

Choose $\tau \leq (2CC_1)^{-1}$, then we have a contradiction to (11). The lemma is therefore proved. $\qquad\square$

We want to apply Lemma 5.4 inductively in order to obtain a decay of the L^2-norm of $\mathbf{B}$ of the form

$$\int_{B_r(x_0)} |\mathbf{B}|^2 \, d\mu_V \leq r^{n-2+\alpha} \epsilon^2$$

for some $\alpha > 0$. But for this we need to control the area contained in the balls $B_r(x_0)$. This is done by the following lemma, which is an adaption of a result of Hutchinson [7].

Lemma 5.5 *Let $1 < p < \infty$. Suppose V is a curvature varifold in Ω with generalized curvature A satisfying*

$$\int_{G_n(B_\rho(x_0))} |A|^p \, d\mu_V \leq \rho^{n-p+\alpha} \epsilon^p$$

for all $\rho \in [r_1, r_2]$, where $B_{r_1}(x_0) \subset B_{r_2}(x_0) \subset \Omega$ and $\alpha > 0$. Let $\psi \in C^1(\mathbb{R}^{N \times N})$ satisfy $0 \leq \psi \leq 1$ and $|\nabla \psi| \leq \lambda \psi$ for a constant $\lambda > 0$. Then

$$\left(r_1^{-n} \int_{G_n(B_{r_1}(x_0))} \psi \, dV \right)^{\frac{1}{p}} \leq \left(r_2^{-n} \int_{G_n(B_{r_2}(x_0))} \psi \, dV \right)^{\frac{1}{p}} + (1+\lambda)\frac{p}{\alpha}(r_2^{\alpha/p} - r_1^{\alpha/p})\epsilon.$$

Proof. We proceed as in [7]. Choose $\rho \in [r_1, r_2]$. For $i \in \{1, \ldots, n+1\}$ let $\phi_i(x, P) = \gamma(r)x^i\psi(P)$, where $r = |x|$ and $\gamma \in C^1(\mathbb{R})$ with $\gamma(t) = 0$ for $t \geq \rho$, $\gamma(t) = 1$ for $t \leq \frac{\rho}{2}$, and $\gamma' \leq 0$. Inserting ϕ_i as test function in (2) and summing over i yields

$$0 = \int \left(P_{ij}\gamma'(r)x^j x^i r^{-1}\psi + P_{ij}\gamma(r)\delta_{ij}\psi + A_{ijk}\gamma(r)x^i \frac{\partial \psi}{\partial P_{jk}} + A_{jij}\gamma(r)x^i\psi \right) dV$$

$$= \int \left(\gamma'(r)|Px|^2 r^{-1}\psi + n\gamma(r)\psi + A_{ijk}\gamma(r)x^i \frac{\partial \psi}{\partial P_{jk}} + A_{jij}\gamma(r)x^i\psi \right) dV.$$

Hence

$$\int (n\gamma(r) + r\gamma'(r))\psi \, dV \leq (1+\lambda)\int |A|\gamma(r)r\psi \, dV.$$

Now let $\eta \in C^1(\mathbb{R})$ such that $\eta(t) = 1$ for $t \leq \frac{1}{2}$, $\eta(t) = 0$ for $t \geq 1$, and $\eta' \leq 0$. Set $\gamma(r) = \eta(r/\rho)$. Then

$$\frac{d}{d\rho} \left(\rho^{-n} \int \eta\left(\frac{r}{\rho}\right) \psi \, dV \right)$$

$$= -\rho^{-n-1} \int (n\gamma(r) + r\gamma'(r))\psi \, dV$$

$$\geq -(1+\lambda)\rho^{-n-1} \int |A|\gamma(r)r\psi \, dV$$

$$\geq -(1+\lambda)\rho^{-n} \left(\int_{G_n(B_\rho(x_0))} |A|^p \, dV \right)^{\frac{1}{p}} \left(\int \eta\left(\frac{r}{\rho}\right) \psi \, dV \right)^{1-\frac{1}{p}}$$

$$\geq -(1+\lambda)\rho^{\alpha/p-1}\epsilon \left(\rho^{-n} \int \eta\left(\frac{r}{\rho}\right) \psi \, dV \right)^{1-\frac{1}{p}}.$$

Therefore

$$\frac{d}{d\rho}\left[\left(\rho^{-n}\int \eta\left(\frac{r}{\rho}\right)\psi\,dV\right)^{\frac{1}{p}}\right] \geq -(1+\lambda)\epsilon\rho^{\alpha/p-1}.$$

Approximating the characteristic function of the interval $(-\infty, 1]$ by η and integrating over $[r_1, r_2]$ finally proves the lemma. $\qquad\square$

Proposition 5.2 *Let $V \in \mathcal{W}^{2,2}(\Omega)$ be a solution of (10) for $p = 2$. Let $\mathbf{B}$ be the generalized second fundamental form and ν a differentiable normal vector of V. Suppose there exist constants $\epsilon_0, C_1 > 0$ and $C_0 \in (\omega_n, 2\omega_n)$ such that (V, ν) satisfies for $p = 2$ the (ϵ_0, C_0)-Poincaré inequality and the reverse (ϵ_0, C_0)-Poincaré inequality, both with constant C_1. Then for any $c_0 < C_0$, there exist numbers $\epsilon, \alpha, C > 0$, depending only on n, ϵ_0, c_0, C_0, and C_1, such that for any ball $B_r(x_0) \subset \Omega$ satisfying $\mu_V(B_r(x_0)) \leq c_0 r^n$ and*

$$\int_{B_r(x_0)} |\mathbf{B}|^2 \, d\mu_V \leq r^{n-2}\epsilon^2, \tag{13}$$

we have

$$\int_{B_\rho(x_0)} |\mathbf{B}|^2 \, d\mu_V \leq C r^{n-2+\alpha}\epsilon^2. \tag{14}$$

Proof. This follows by applying Lemma 5.4 inductively. Note that Lemma 5.5 with $\psi = 1$ provides the right bounds for the area at each step, provided that ϵ is chosen sufficiently small. $\qquad\square$

Theorem 5.1 *Under the conditions of Proposition 5.2, the number ϵ can be chosen such that there exists a number $\delta > 0$ with the property that $\operatorname{supp}\mu_V \cap B_\delta(x_0)$ decomposes into a finite collection of smooth manifolds, provided that $\mu_V(B_r(x_0)) \leq c_0 r^n$ and (13) hold.*

Proof. We may assume that the conditions of Proposition 5.2 hold with x_0 replaced by any $x \in B_\delta(x_0)$, if δ is small enough. Therefore we obtain (14) around any such point. Then $C^{1,\beta}$-regularity can be proved with the same arguments as in [7], replacing the monotonicity formula 3.1 in that paper by Lemma 5.5 and making the obvious adaptions. Higher regularity follows as in Lemma 3.2 and the following arguments of [12]. (The proofs in [7] and in [12] are rather involved, which is why we do not repeat the arguments here.) $\qquad\square$

6 Lipschitz graphs

The purpose of this section is to show that the Poincaré and the reverse Poincaré inequality can be proved in a special situation.

We assume that $\Omega = \Omega' \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and that $V = \mathbf{v}(M, 1) \in \mathcal{W}^{2,2}$ for the graph

$$M = \{(x, u(x)) : \ x \in \Omega'\},$$

where $u : \Omega' \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant bounded by $L > 0$. Suppose that V is a minimizer of $\mathcal{F}_2$ in the following sense: For any ball $B_r(x_0) \Subset \Omega'$ and for any Lipschitz function $\tilde{u} : B_r(x_0) \to \mathbb{R}$ such that the "combined" graph

$$\tilde{M} = (M \setminus (B_r(x_0) \times \mathbb{R})) \cup \{(x, \tilde{u}(x)) : \ x \in B_r(x_0)\}$$

defines an integral curvature varifold $\tilde{V} = \mathbf{v}(\tilde{M}, 1) \in \mathcal{W}^{2,2}(\Omega)$, we have $\mathcal{F}_2(V) \leq \mathcal{F}_2(\tilde{V})$.

Lemma 6.1 *For any $\epsilon_0 > 0$ and any $C_0 > \omega_n$, and for any $f \in W^{1,2}(V)$, the pair (V, f) satisfies the (ϵ_0, C_0)-Poincaré inequality for a constant depending only on n and L.*

Proof. This follows easily from the fact that M is a Lipschitz manifold. $\qquad\square$

Lemma 6.2 *Under the conditions above, the pair (V, ν) satisfies the reverse (ϵ_0, C_0)-Poincaré inequality for any $\epsilon_0 > 0$ and any $C_0 > \omega_n$ for a constant depending only on n and L, where ν is the normal vector of V given by*

$$\nu^i = \frac{\frac{\partial u}{\partial x^i}}{\sqrt{1 + |\nabla u|^2}}, \quad i = 1, \dots, n, \qquad \nu^{n+1} = \frac{1}{\sqrt{1 + |\nabla u|^2}}.$$

Proof. We prove that for any $\lambda > 0$, the inequality

$$\int_{B_{r/2}(x_0)} |\nabla^2 u|^2 \, dx \leq \lambda \int_{B_r(x_0)} |\nabla^2 u|^2 \, dx + C_1 r^{-1} \int_{B_r(x_0)} |\nabla u - \sigma|^2 \, dx \quad (15)$$

holds for any ball $B_r(x_0) \subset \Omega'$ and for a constant $C_1 = C_1(n, L, \lambda)$, where

$$\sigma = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} \nabla u \, dx.$$

Then by a standard argument (see e. g. Lemma 2.8.2 in [13]) we can get rid of the first term on the right-hand side, provided that we choose λ sufficiently small, and the reverse Poincaré inequality follows.

Fix $B_r(x_0) \subset \Omega'$ and $\lambda \in (0, \frac{1}{4}]$, and choose a radius $\rho \in [\frac{r}{2}, r - \lambda r]$ such that

$$\int_{B_{\rho(1+\lambda)}(x_0) \setminus B_\rho(x_0)} |\nabla^2 u|^2 \, dx \leq C\lambda \int_{B_r(x_0)} |\nabla^2 u|^2 \, dx. \quad (16)$$

Here and subsequently C denotes indiscriminately several constants depending only on n and L. Choose $\zeta \in C_0^\infty(B_{\rho(1+\lambda)}(x_0))$ satisfying $\zeta \equiv 1$ in $B_\rho(x_0)$, $0 \leq \zeta \leq 1$, and $|\nabla \zeta|^2 + |\nabla^2 \zeta| \leq C(\lambda\rho)^{-2}$. We may assume that

$$\int_{B_r(x_0)} u \, dx = 0,$$

so that

$$\int_{B_r(x_0)} |u(x) - a \cdot x|^2 \, dx \leq C r^2 \int_{B_r(x_0)} |\nabla u - a|^2 \, dx \quad (17)$$

for all $a \in \mathbb{R}^n$ by the (ordinary) Poincaré inequality. Otherwise we add a constant to u. Set

$$v(x) = \zeta(x)\sigma \cdot x + (1 - \zeta(x))u(x), \quad x \in \Omega'.$$

We have

$$|\nabla v(x)| = |\zeta(x)\sigma + (1 - \zeta(x))\nabla u(x) + \nabla \zeta(x)(\sigma \cdot x - u(x))| \leq C(\lambda\rho)^{-1}$$

24

and

$$|\nabla^2 v(x)| \le (1 - \zeta(x))|\nabla^2 u(x)| + 2|\nabla\zeta(x)||\nabla u(x) - \sigma| + |\nabla^2\zeta(x)||u(x) - \sigma \cdot x|.$$

Hence

$$\int_{B_{\rho(1+\lambda)}(x_0)} |\nabla^2 v|^2 \, dx \le C\lambda \int_{B_r(x_0)} |\nabla^2 u|^2 \, dx + Cr^{-2}\lambda^{-4} \int_{B_r(x_0)} |\nabla u - \sigma|^2 \, dx$$

by (16) and (17). The minimality of $\mathcal{F}_2(V)$ implies the same inequality with $\int_{B_{\rho(1+\lambda)}(x_0)} |\nabla^2 u|^2 \, dx$ on the left-hand side (and with a different constant, still depending only on n and L however). Replacing λ by $C^{-1}\lambda$, we find that (15) holds true. $\qquad\square$

Combining this with Theorem 5.1, we find that we can prove C^∞-regularity for minimizers of $\mathcal{F}_2$ under certain conditions if we already have Lipschitz regularity.

References

[1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[2] G. Anzellotti, R. Serapioni, and I. Tamanini, *Curvatures, functionals, currents*, Indiana Univ. Math. J. **39** (1990), 617–669.

[3] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

[4] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents and variational problems for mappings into spheres*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1989), 393–485.

[5] ———, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **106** (1989), 97–159.

[6] ———, *Partial regularity of Cartesian currents which minimize certain variational integrals*, Partial differential equations and the calculus of variations, Vol. II, Birkhäuser, Boston, 1989, pp. 563–587.

[7] J. E. Hutchinson, $C^{1,\alpha}$ *multiple function regularity and tangent cone behaviour for varifolds with second fundamental form in L^p*, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Amer. Math. Soc., Providence, 1986, pp. 281–306.

[8] ———, *Second fundamental form for varifolds and the existence of surfaces minimising curvature*, Indiana Univ. Math. J. **35** (1986), 45–71.

[9] C. Mantegazza, *Curvature varifolds with boundary*, J. Differential Geom. **43** (1996), 807–843.

[10] J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of R^n*, Comm. Pure Appl. Math. **26** (1973), 361–379.

[11] L. Simon, *Lectures on geometric measure theory*, Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[12] _____, *Existence of surfaces minimizing the Willmore functional*, Comm. Anal. Geom. **1** (1993), 281–326.

[13] _____, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Math. ETH Zürich, Birkhäuser, Basel, 1996.