Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 73 2001

PATHOLOGIES IN ALEKSANDROV SPACES OF CURVATURE BOUNDED ABOVE

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Abstract. We construct in the paper two examples of Aleksandrov spaces A with curvature bounded above, which possess a pathological properties. In the first we give a $CAT(-1)$-space A, which is homeomorphic to $R^n, n \geq 5$, while its hyperbolic boundary in Gromov sense is not topological manifold. This construction is much simpler than in corresponding example of Davies-Januszkiewicz. In the second A has curvature ≤ 0 and entropy dimension (around some point) strongly more than (equal) topological and Hausdorff dimensions.

1. Introduction and main results.

In this paper we prove the part of results on A.D.Aleksandrov spaces of the curvature $\leq K$ (see [1],[3],[5], [6]) announced earlier in the paper [7]. Let us give necessary definitions and notations.

The distance between two points x, y of a metric space M is denoted by xy. *(Locally) inner* (or *(locally) interior* or *(locally) length*) *metric space* is a metric space in which (locally) any two points x, y can be joined by a path with the length arbitrary close to xy. A path joining a points x, y in M is called *shortest arc* or *segment* (with the ends x, y) if its length is equal to xy. *(Locally) geodesic space* is a metric space in which (locally) any two points can be joined by shortest arc.

A point y lies between a points x, z if $xz = xy + yz$ and $y \neq x, y \neq z$; the notation is (xyz). An open (closed) ball in M of the radius r and the center x is denoted by $U(x, r)$ (respectively $B(x, r)$); S_K denotes complete simply connected three-dimensional Riemannian manifold of constant sectional curvature K.

We define *excess* of ordered triple (a, b, c) in M to be the number $\varepsilon(b; a, c) := ab + bc - ac$.

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[0]MR Subject Classification 2000: 53C23, 57Mxx, 52Bxx,

[0]Keywords:Aleksandrov space with curvature bounded above, hyperbolic boundary, topological dimension, Hausdorff dimension, entropy dimension

[0]This paper was written while the author enjoyed the hospitality of the Max Planck Institute für Mathematik in Leipzig.

Date: September 22, 2001.

For a point p of a metric space M, we will denote by $\Omega_p(M)$ the space of all directions to M at point p and by $\omega_p(M)$ the subspace of directions to M at the point p, defined by shortest arcs starting at p (see[1], [3] or [6]). The distance between two directions is defined to be the upper angle between corresponding curves (respectively, shortest arcs, see [1] or [3]). The corresponding 0-cones $M_p := C_0\Omega_p(M)$ and $m_p := C_0\omega_p(M)$ (see [3]) as well as their Hausdorff completions we will call *the tangent spaces to M at point p.*

We construct some examples of a spaces with curvature $\leq K$ possessing pathological properties.Using some constructions from [8], the author proved earlier in [9] that any n-dimensional sphere $S^n, n \geq 5$, admits geodesic inner metric d of curvature ≤ 1 (even $CAT(1)$-metric, see [6]) such that at some points $x \in S^n$, the space of direction $\Omega_x(S^n, d)$ is not a manifold and hence is not homeomorphic to S^{n-1}. This example gives the negative answer to the question of A.D.Aleksandrov and the author whether at every point of a n-dimensional manifold with curvature bounded above the space of directions is homeomorphic to S^{n-1} (see [2]). Let's remark that on the other hand, the tangent space $T_x(S^n, d)$ at any point $x \in S^n$ for constructed metric d is homeomorphic to R^n. As a modification of this construction we prove the following theorem.

Theorem 1.1. *Every n-dimensional ariphmetic space $R^n, n \geq 5$, admits (compatible with the topology) complete inner metric d of the curvature ≤ -1 (even $CAT(-1)$-metric) such that the hyperbolic Gromov boundary* [17] *$hb(R^n, d)$ is not a topological manifold hence is not homeomorphic to S^{n-1} (but it is a triangulable $(n-1)$-dimensional space with the homologies of S^{n-1}).*

For this aim we take arbitrary triangulated $(n-2)$-dimensional nonsimply connected closed manifold N with homologies of S^{n-2}. Then supply N by an inner metric δ of 1-region (see [8], [3] or [6]). The required space M is $C_{-1}C_1(N, \delta)$, where C_k denotes the construction of the cone with curvature k over given space [3]. It follows from [8] that (M, d) is $CAT(-1)$-space. The space M is homeomorphic to R^n by R.D.Edwards theorem on double suspension [15]. We will prove that $hb(R^n, d)$ is homeomorphic to the space of directions to M at the vertex of cone C_{-1}, i.e. to $C_1(N, \delta)$ what is not a manifold (see [15]).

Remark 1.2. The result from theorem 1.1 have been proved previously in the paper [16] for $n = 5$ and stated for all $n \geq 5$ with the help of more complicated techniques of hyperbolization of polyhedra, based on Gromov's idea. M.Davies and T.Januskiewicz hyperbolize the double suspension of N above and then take the universal cover of resulting space. We metrize directly the double suspension of N without one point, i.e the cone over suspension of N. With connection to this subject, note also that the statement on the page 266 in [6] that one can use a theorem of Rolfsen in [18] to show that hyperbolic boundary of a complete $CAT(0)$ 3-manifold M is homeomorphic to a 2-sphere and $\overline{M}$ is a 3-ball, requires an additional arguments.

Remark 1.3. In the same manner one could prove

Theorem 1.4. *Every compact simply connected* 2*-dimensional manifold* Σ^3 *is homeomorphic to a space of directions at some point* x *in a* $CAT(1)$*-space, homeomorphic to* S^4*, or to Gromov hyperbolic boundary of a* $CAT(-1)$*- space, homeomorphic to* R^4*.*

To prove this theorem one could endow any triangulation T of Σ^3 by inner metric d of 1-region from the papers [8] or [3]; all simplices of the first barycentric subdivision of T gives all-right spherical complex relative to metric d in the sense of [14] or [6]. Then one takes $C_1(\Sigma^3, d)$ for the first statement and $C_{-1}(\Sigma^3, d)$ for the second statement of the theorem. One could use also in the proof of this theorem the recent known topological result that the suspension $S(\Sigma^3)$ is homeomorphic to S^4. The author knew this result in the conversation with R.J.Daverman at 1994 but doesn't know the exact reference to its proof. Let us note also that the same construction from [8] was used in the proof of the main result in [4] which states that the interior of any contractible compact n-manifold ($n \geq 5$) with boundary admits $CAT(0)$-metric.

Remark 1.5. It follows from the previous remark that if Σ^3 is not necessarily homeomorphic to S^3 (i.e. the famous Poincare conjecture is false) then the answer to the question in [2], mentioned above, is also negative in dimension 4. In other words, the positive answer to this question would imply the positive answer to Poincare conjecture.

In the paper [10] by Burago, Gromov and Perel'man it was proved that for locally compact A.D.Aleksandrov spaces with inner metric of curvature bounded from below in the case of finite Hausdorff dimension: 1)any sufficiently small neighborhood U has equal topological, Hausdorff and entropy (there called rough) dimensions $dim_T U, dim_H U$, and $dim_E U$; 2)the dimensions of sufficiently small neighborhoods of any two different points coincide.

Both these statements can fail for spaces of curvature bounded above.

Theorem 1.6. *There is compact space* M *with inner metric of curvature* ≤ 0 *such that*

(1) *at some point* $x \in M$*, for any its neighborhood* U*,*

$$2 = dim_T U = dim_H U < dim_E U = \beta,$$

where β can be any real number more than 2 and even $+\infty$;

(2) *on the other hand, there are other points, where all these dimensions are equal to 2.*

With this aim we can take a subspace

$$M := \{(\alpha, t), \alpha \in R, 0 \leq t \leq f(\alpha)\} \tag{1.1}$$

in C_0R (see [3]) with the induced inner metric. Here R is real line with usual metric; for a finite number β or $+\infty$ one needs take the function

$$f(\alpha) := (1 + |\alpha|)^{-\frac{1}{\beta}}, 2 < \beta. \tag{1.2}$$

or

$$f(\alpha) := (1 + |\alpha|)^{-\frac{1}{1+\sqrt{\log(1+|\alpha|)}}}. \tag{1.3}$$

Remark 1.7. Probably, this is the first known example of this kind even in the context of inner metric spaces. Bakhtin's example in [11] with nonequal Hausdorff and entropy dimensions is countable.

Remark 1.8. Gromov told to the author at ICM'94 that similar examples can be constructed for R-trees (see [19]). In this case we will have topological and Hausdorff dimensions equal to one. Then we can use the direct multiplication with the real line to get an example of topological dimension two. This is true, but we think that some other features of constructed spaces deserve an attention.

2. Proofs

In this section we prove the theorems from introduction providing pathological examples of Aleksandrov spaces with curvature bounded above.

Proof. of theorem 1.1. The construction of metric d was given in introduction. So we need only prove the following proposition because of discussion in the introduction. □

Proposition 2.1. *The space $M := C_{-1}S$, where S is any $CAT(1)$-space, has the hyperbolic boundary homeomorphic to S.*

Proof. Since (M, d) is $CAT(-1)$-space by [3], it is hyperbolic in the sense of the paper [17], so the hyperbolic boundary $hb(M)$ is defined (see [17]).The hyperbolic boundary $hb(M)$ is defined in [17] as follows. First M.Gromov defines the "scalar product" for points $x, y \in M$ with respect to some fixed basis point p:

$$(x.y) := \frac{1}{2}(px + py - xy) = \frac{1}{2}\varepsilon(p; x, y).$$

Then a sequence $x_k \in M$ is called *convergent at infinity*, if $(x_k.x_l) \to \infty$ when $k, l \to \infty$. The latter notion is independent on the choice of a point p (see [17]). Since (M, d) is hyperbolic, the equality

$$\lim \inf_{k,l\to+\infty} (x_k.y_l) = \infty$$

is an equivalence relation on the set of a sequences, converging to ∞. The hyperbolic boundary $hb(M)$ is by definition the set of all equivalence classes induced by this equivalence relation. If a sequence x_k is contained in a class $\alpha \in hb(M)$, we write $x_k \to \alpha$ for $k \to \infty$. M.Gromov defines the natural topology on $M \cup hb(M)$ such that M is dense in $M \cup hb(M)$. More exactly, it's defined as follows. For a sequences $\tilde{x} = (x_k)$ and $\tilde{y} = (y_k)$ define

$$(\tilde{x}.\tilde{y}) := \lim \inf_{k\to+\infty} (x_k.y_k).$$

For a points $x, y \in M \cup hb(M)$ set

$$(x.y) := \inf(\tilde{x}.\tilde{y}),$$

with $\tilde{x} = (x_k)$ converging to x and $\tilde{y} = (y_k)$ converging to y. In particular, $(x.x) = \infty$, if $x \in hb(M)$; $(x.y) < \infty$, if $x, y \in hb(M), x \neq y$. The initial topology on M coincides with induced one from $M \cup hb(M)$, while a base of neighborhoods for a point $x \in hb(M)$ consists of a sets

$$N_{x,u} := \{y \in M \cup hb(M) : (x.y) > u\},$$

where u is some nonnegative real number (see [13]).

In our case any point $x \in M$ has a form (a, t), where $a \in S$, t is nonnegative number and all the points of the form $(a, 0)$ are identified with the vertex $p \in M$ of cone C_{-1}.

Calculate at first $(x.y)$ for $x = (a, t), y = (b, s) \in M$ and $\gamma := ab$. If $\gamma \geq \pi$, then $(x.y) = 0$. If $\gamma = 0$, then $(x.y) = \min(t, s)$.

If $0 < \gamma < \pi$, then by cosine theorem of hyperbolic geometry we have

$$\cosh xy = \cosh t \cosh s - \cos\gamma \sinh t \sinh s =$$
$$\cosh t \cosh s(1 - \cos\gamma \tanh t \tanh s),$$

i.e.

$$\frac{e^{xy}}{2}(1 + e^{-2xy}) =$$
$$\frac{e^t}{2}\frac{e^s}{2}(1 - \cos\gamma \tanh t \tanh s)(1 + e^{-2t})(1 + e^{-2s})$$

and

$$\frac{e^{t+s-xy}}{2} = \frac{1 + e^{-2xy}}{(1 + e^{-2t})(1 + e^{-2s})(1 - \cos\gamma \tanh t \tanh s)}.$$

Thus

$$(x.y) = \frac{1}{2}\log\frac{2(1+e^{-2xy})}{(1+e^{-2t})(1+e^{-2s})(1-\cos\gamma\tanh t\tanh s)}.$$

If $\gamma \to \gamma_\infty, 0 < \gamma_\infty \leq \pi; s, t \to \infty$, we get in the limit

$$(x_\infty.y_\infty) = \frac{1}{2}\log\frac{2}{1-\cos\gamma_\infty}, \tag{2.1}$$

because $xy \to \infty$. In general case we get an upper bound

$$(x.y) \leq \frac{1}{2}\log\frac{4}{1-\cos\gamma\tanh t\tanh s}.$$

If

$$px_k, py_k \to \infty,$$

then

$$\lim\inf_{k,l\to+\infty}(x_k.y_l) = \frac{1}{2}\log\frac{2}{1-\cos\gamma_\infty},$$

where $\gamma_\infty = \lim\sup a_k b_l$.

Thus $\lim\inf(x_k.y_l) = \infty$ if and only if $\lim\sup a_k b_l = 0$, i.e. a_k, b_l converge to one and the same limit point $a = b$ in S. In particular, this is true for (x_k) which is equivalent to (y_l). Thus we get the bijection between $hb(M)$ and S. Under this it follows from the formula 2.1 that

$$\gamma_\infty = \arccos(1 - 2e^{-2(x_\infty.y_\infty)}).$$

Hence for $x_\infty \in hb(M)$,

$$N_{x_\infty,u} \cap hb(M) = \{y_\infty \in hb(M) : a(x_\infty)b(y_\infty) < \arccos(1-2e^{-2u})\}.$$

So, the topology in $hb(M)$ coincides with the topology in S. The proposition is proved. □

Remark 2.2. When M is a finitely compact ([12]) (in other terminology, proper) $CAT(0)$-space, there exist the other possibilities to define $hb(M)$ and the same topology in $M \cup hb(M)$ (see [6]).

Informally the entropy dimension $dim_E(M,d)$ of a metric compact (M,d) is defined as follows. Let $c_{r_0}M$ be the number of elements in a minimal r_0-net in (M,d) (i.e. with a minimal number of elements) for a real number $r_0 > 0$. Then $dim_E(M,d)$ is defined as a nonnegative real number β such that $c_{r_0}M$ has the same order as $(r_0)^{-\beta}$, when $r_0 \to 0$. This (in general, nonexact) definition will work quite well in our later considerations. For us it will be more convenient to use in this definition instead of $c_{r_0}M$ the maximal number $N_{r_0}M$ of points in a subset $W \subset M$ such that every distance between different points in W is no less than $2r_0$. Note that for inner metric space M, the number $N_{r_0}M$ is equal to a greatest number of pairwise disjoint open balls of radiuse r_0 in M. We need to show that the numbers $c_{r_0}M$ and $N_{r_0}M$ have equal orders when $r_0 \to 0$.

Lemma 2.3. *For any metric compact* (M, d) *and any real number* $r_0 > 0$, *one has the inequality*

$$N_{r_0}M \leq c_{r_0}M \leq N_{\frac{r_0}{2}}M.$$

Proof. Indeed, let V be a r_0-net in M and W be a set mentioned above.Then any element w of W is contained in r_0-neighborhood of some choosen element $g(w)$ of V. Then $g(w_1) \neq g(w_2)$, if $w_1 \neq w_2$. In opposite case we would have $d(w_1, w_2) < 2r_0$ by triangle inequality, which is impossible by definition of W. It follows from here the first inequality of the lemma.

On the other hand, if W above is taken for number $\frac{r_0}{2}$, then it must be a r_0-net in M because of maximality, so we get the second inequality. □

The cone $P := C_0R$, where R is real line with usual inner metric, supplies an example of (noncompact) $CAT(0)$-space whose direction space $\Omega_O P$ at vertex O has diameter π and is locally isometric to R. So $\Omega_O P$ is not inner metric space. Moreover an angle of "shortest cone of directions" joining two given directions (see [1]) maybe arbitrary large. The desired (for theorem 1.6) space $M \subset P$ with induced (inner) metric was described in introduction by formulas (1.1) and (1.2). First we need to find when M, defined by formula (1.1) for a positive continuous even function $f(\alpha)$ with condition $f(\alpha) \to 0$, when $\alpha \to \infty$, will be a (metrically) convex subspace of P.

For this consider an Euclidean triangle ΔOAB with angle $\angle AOB = \alpha, 0 < \alpha < \pi$, and sides $OA = r_1 > 0, OB = r_2 > 0$; C be a point on side $[AB]$ such that $\angle AOC = \alpha_1, \angle BOC = \alpha_2$, where $\alpha_1 + \alpha_2 = \alpha$. Then direct calculation shows that $OC = r$ is given by formulas

$$r = \frac{r_1 r_2 \sin\alpha}{r_1 \sin\alpha_1 + r_2 \sin\alpha_2}, \frac{1}{r} = \frac{1}{r_1}\frac{\sin\alpha_2}{\sin\alpha} + \frac{1}{r_2}\frac{\sin\alpha_1}{\sin\alpha}. \tag{2.2}$$

Now M, defined by formula (1.1), is convex if and only if M together with a points $(\alpha_1, r_1), (\alpha_2, r_2)$, where $0 < \alpha_2 - \alpha_1 < \pi$, contains the all shortest arc in P, joining these two points. As a corollary of formulas (1.1) and (2.2), M is metrically convex if and only if the function $\phi(\alpha) := \frac{1}{f(\alpha)}$ satisfies the inequality

$$\phi(t_1\alpha_1 + t_2\alpha_2) \leq \phi(\alpha_1)\frac{\sin t_1(\alpha_2 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} + \phi(\alpha_2)\frac{\sin t_2(\alpha_2 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)}, \tag{2.3}$$

(where $t_1 + t_2 = 1, 0 \leq t_1 \leq 1$ and α_k have other sense than in (2.2)), i.e. function ϕ is "sinus-convex". The used term is connected with the fact that after removing "sin" in the formula (2.3) we get the convexity condition for the function ϕ. It is geometrically evident that the function $g(x) := \frac{\sin x}{x}$ is strongly decreasing on interval $0 < x < \pi$. It follows from here that

$$\frac{\sin t_k(\alpha_2 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} \geq t_k; k = 1, 2.$$

Then every (positive continuous) convex function is sinus-convex. The opposite statement is also true.Its proof is more difficult even uses only standard tools from real analysis.The main steps of the proof are that a sinus-convex function is "infinitesimally convex" and then locally and globally convex.We omit this proof as well as the proof of the last statement in the proposition 2.4 below. As a corollary of (2.3) and discussion above we get the following proposition.

Proposition 2.4.
1. *Every positive continuous real function is sinus-convex if and only if it is convex.*
2. *The space M, defined by formula (1.1), is metrically convex if and only if the function $\phi(\alpha) := \frac{1}{f(\alpha)}$ is convex.*
3. *If $f(\alpha)$ is any positive continuous even real function such that $f(\alpha) \to 0$ when $\alpha \to \infty$, then M, defined by formula (1.1) und equipped with induced inner metric from P, is compact.*

It is well known that for a general compact metric space M we have the inequalities $dim_T M \leq dim_H M \leq dim_E M$. One can prove that in the case of metrically convex subspace $M \subset P$ from proposition 2.4 the extreme dimensions above coincide, hence all these dimensions are equal. So we need consider noncovex $M \subset P$ equipped with induced inner metric. But then arose the problem to find a connection of induced inner metric d in M with old metric ρ and to prove that (M, d) is $CAT(0)$-space. We suppose that every space (M, d) with conditions, as in the last statement of proposition 2.4, is $CAT(0)$-space. More generally, it is very similar that we can suggest the following conjecture.

Conjecture 1. *Let M be a simply connected closed subset of a $CAT(0)$-space P of topological dimension two, which admits (finite) inner metric d, induced from P. Then (M, d) is also $CAT(0)$-space.*

One can check that for the function $f(\alpha)$ in formula (1.2) or (1.3), the function $\phi(\alpha) := \frac{1}{f(\alpha)}$ has positive first and negative second derivatives for positive α. Hence by proposition 2.4, the formulas (1.1) and (1.2) or (1.3) define a subspace M which is not metrically convex in P, so we need the following lemma.

Lemma 2.5. *A subspace $M \subset P$, defined by formula (1.1) for the function (1.2) or (1.3) and endowed with inner metric d induced from P, is a compact $CAT(0)$-space.*

Proof. One can easily see that every subset of a form

$$M_{[\alpha_1,\alpha_2]} := \{(\alpha, t) \in M : \alpha_1 \leq \alpha \leq \alpha_2\},$$

where $0 < \alpha_2 - \alpha_1 < \pi$, will be a metrically convex subset of (M, d) and $(M_{[\alpha_1,\alpha_2]}, d)$ is isometric to a closed region D in Euclidean plane, bounded by a curvilinear Euclidean triangle ΔOAB with a segments $[OA], [OB]$ and a concave arc AB and endowed with induced inner metric δ. Evidently, the arc AB is a geodesic in (D, δ), which is a shortest arc if and only if its (always finite!) length is less than $OA + OB = f(\alpha_1) + f(\alpha_2)$. The last condition is garanteed if $\alpha_2 - \alpha_1$ is small enough (depending on α_1 or α_2), so we can assume that this condition is fulfilled. Then ΔOAB is a real triangle in (D, δ) as well as the corresponding bounding curve of $M_{[\alpha_1,\alpha_2]}$. Now one can prove easily (directly or) with the help of Aleksandrov lemma (lemma 2.16 in [6]) and any version of limit in the corollary 3.10, [6], that (D, δ) (hence $M_{[\alpha_1,\alpha_2]}$) is $CAT(0)$-space. Since M can be obtained by gluing of subspaces of the form $M_{[\alpha_1,\alpha_2]}$ along shortest arcs in linear order, it is $CAT(0)$-space by Reshetnyak theorem (Basic Gluing Theorem 11.1 in [6]). □

Lemma 2.6.
If $f(\alpha)$ is any positive continuous even real function such that $f(\alpha)$ is decreasing, when $\alpha > 0$, then on (M, d), defined by formula (1.1) and equipped with induced inner metric d from (P, ρ), we have the inequalities

$$\rho \leq d \leq 2\rho.$$

So, the metrics d and ρ on M are bilipshitz homeomorphic.

Proof. The first inequality is evident.

Note that under mentioned conditions,

$$d((\alpha_1, r_1), (\alpha_2, r_2)) = \rho((\alpha_1, r_1), (\alpha_2, r_2)),$$

if $r_1 = r_2$, or $\alpha_1 - \alpha_2$, or $|\alpha_1 = \alpha_2| \geq \pi$. Let suppose that $r_1 < r_2$ and $0 < |\alpha_1 - \alpha_2| < \pi$. Then by triangle inequality,

$$\begin{gathered} a := \rho((\alpha_1, r_1), (\alpha_2, r_2)) \leq d((\alpha_1, r_1), (\alpha_2, r_2)) \leq \\ d((\alpha_1, r_1), (\alpha_2, r_1)) + d((\alpha_2, r_1), (\alpha_2, r_2)) = \\ \rho((\alpha_1, r_1), (\alpha_2, r_1)) + \rho((\alpha_2, r_1), (\alpha_2, r_2)) = \\ 2r_1 \sin \frac{|\alpha_1 - \alpha_2|}{2} + |r_1 - r_2| := b + c. \end{gathered}$$

There exists an Euclidean triangle Δ with the sides a, b, c. The angle α between the sides b, c in Δ is evaluated as follows:

$$\alpha = \pi - \frac{\pi - |\alpha_1 - \alpha_2|}{2} = \frac{\pi}{2} + \frac{|\alpha_1 - \alpha_2|}{2} \geq \frac{\pi}{2}.$$

Thus

$$a \geq \sqrt{b^2 + c^2}.$$

If $\delta := \max(b, c)$, then

$$a \geq \sqrt{b^2 + c^2} \geq \delta \geq \frac{b + c}{2},$$

i.e.

$$d((\alpha_1, r_1), (\alpha_2, r_2)) \leq b + c \leq 2a = 2\rho((\alpha_1, r_1), (\alpha_2, r_2)),$$

as required. $\square$

Proof. of theorem 1.6. Onemore consider a space M as in the last statement of proposition 2.4 and assume also that $f(\alpha)$ is decreasing function when α is positive. By lemma 2.6, we can use the metric ρ on M instead of metric d for calculation of $dim_E(M, d)$. We use this possibility below.

Denote by M_{r_0}, where $0 < r_0 < f(0)$, compact subset in M, defined by formula

$$M_{r_0} := \{(\alpha, t) \in M : r_0 \leq f(\alpha)\}.$$

Suppose that $f(\alpha_0) = r_0, \alpha_0 \geq 0$, and denote $\psi(r_0) := \alpha_0 = f^{-1}(r_0)$. Then well-known formula for area in polar coordinates gives the expression

$$\sigma(M_{r_0}) = \int_0^{\alpha_0} f^2(\alpha)\, d\alpha \tag{2.4}$$

for metric ρ in P.

Evaluate below the number $N_{r_0}M$ from lemma 2.3.Denote by m the maximal nonnegative integer number such that $2mr_0 \leq f(0)$. Evidently, the distance between the points $[\alpha_1, 2kr_0]$ and $[\alpha_2, 2kr_0]$ in M will be no less than $2r_0$, if $|\alpha_1 - \alpha_2| \geq 2\arcsin(\frac{1}{2k})$. The maximal number of points in M, satisfying pairwise this condition under fixed k, will be no less than

$$\frac{2\psi(2kr_0)}{2\arcsin\frac{1}{2k}} = \frac{\psi(2kr_0)}{\arcsin\frac{1}{2k}}.$$

Thus by taking all $k = 0, 1, ..., m$, we get at least

$$\sum_{k=1}^{m} \frac{\psi(2kr_0)}{\arcsin\frac{1}{2k}} := n_{r_0}M$$

points in M with pairwise distances between them no less than $2r_0$. Under this for any $k \geq 1$, by the inequality mentioned above

$$\frac{\sin \arcsin \frac{1}{2k}}{\arcsin \frac{1}{2k}} \geq \frac{\sin \arcsin \frac{1}{2}}{\arcsin \frac{1}{2}},$$

i.e.

$$\arcsin \frac{1}{2k} \leq \frac{\frac{1}{2k} \arcsin \frac{1}{2}}{\frac{1}{2}} = \frac{\pi}{6k} < \frac{2}{3k}.$$

Thus

$$N_{r_0}M \geq n_{r_0}M \geq \frac{3}{4}\sum_{k=1}^{m} 2k\psi(2kr_0) \geq \frac{3}{4}2\psi(2r_0). \tag{2.5}$$

For the function (1.2) (respectively, (1.3)), we get

$$\psi(r) = f^{-1}(r) = r^{-\beta} - 1, (\psi(r) = f^{-1}(r) = r^{-(1+\sqrt{\log(1+|\alpha|)})} - 1).$$

It follows from the last equalities in both cases and formula (2.5) that β (respectively $+\infty$) will be the entropy dimension of M, if an upper bound of $N_{r_0}M$ in the first case has the same order relative to $2r_0$ as the last term in (2.5), i.e. as $\psi(2r_0) = (2r_0)^{-\beta} - 1$ when $r_0 \to 0$.Of course, in the second case we need no upper bound for $N_{r_0}M$.

By triangle inequality in M, there is at most one point (α, r) in the set $W \subset M$ above with $r < r_0$. Thus all maybe but one points in W are contained in M_{r_0} and have a form $(\alpha, r), r \geq r_0$. So we can suppose that all points in W have the last two properties. It follows from lemma (2.5) that the set M_{r_0} is canonically closed subset in M (i.e.it is the closure of open subset in M), bounded by four geodesics in (M, d) (!), namely shortest arcs $[OA], [OC]$ and geodesic arcs AB, BC, where $A = [-\alpha_0, r_0], B = [0, 1], A = [\alpha_0, r_0]$. Since $f(0) = 1, f'(0) = -\frac{1}{\beta}$, it's clear that the angle between the arcs BA, BC at the point B is equal to $\gamma_0 := 2 \arctan \beta$. Then $\gamma_0 > \frac{\pi}{2}$ because $\beta > 2$. Since $f'(\alpha) < 0$, when $\alpha > 0$, the inner angles of M_{r_0} at points A, C are also more than right angle. At the end the plane inner angle of M_{r_0} at vertex O is equal to $2\alpha_0$. Since we are interested in the case $r_0 \to 0$, when $\alpha_0 \to \infty$, we also can suppose that the last angle is more than right angle. Using the mentioned properties of sets W and M_{r_0} and the fact that arcs AB, BC are concave (geodesics) in M_{r_0}, one can easily see that the intersection of every open ball in P of radius r_0 and center at a point in W has an area more than $\frac{\pi(r_0)^2}{4}$ (relative to metric in P), if r_0 is small enough. Evidently, all these balls are mutually disjoint, while area

$$\sigma(M_{r_0}) = \frac{\beta}{\beta - 2}[(1 + \alpha_0)^{\frac{\beta-2}{\beta}} - 1] = \frac{\beta}{\beta - 2}((r_0)^{2-\beta} - 1)$$

as a corollary of formula (2.4), applied to the function (1.2).

Using the above considerations, we get for $N_{r_0}M$ the following upper bound

$$N_{r_0}M \leq \frac{\sigma(M_{r_0})}{\frac{\pi(r_0)^2}{4}} \leq \frac{4\beta}{\pi(\beta-2)}(r_0)^{-\beta},$$

i.e. the quantity of desired order.

On the other hand, the space P can be represented as the countable union of sets

$$P_{[k\pi,(k+1)\pi]} := \{(\alpha,t) \in P : k\pi \leq \alpha \leq (k+1)\pi\},$$

where k is any integer number. Every such set is isometric to Euclidean semiplane, hence has Hausdorff dimension 2. Then the space P also has Hausdorff dimension 2 by known property of Hausdorff dimension.Hence the same is true for $M \subset P$. Evidently, M has the topological dimension two.The theorem is proved. □

Remark 2.7. In the paper [7] instead of the function (1.3) it was erroneously taken the function $f(\alpha) = (1+|\alpha|)^{-\frac{1}{1+|\alpha|}}$. The space M, defined by (1.1) and this function, is noncompact.

Question 1. *Does there exist a compact inner metric space M (maybe, with curvature bounded above) with inequality $dim_T M < dim_H M$?*

Question 2. *In constructed example M possess the property that every but one point in M has a neighborhood where all three dimensions coincide. Is it possible to construct an example of (inner metric) space M with curvature bounded above such that every neighborhood of any point in a dense subset $V \subset M$ has different entropy and Hausdorff dimensions?*

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