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A reduced theory for thin-film micromagnetics

by

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Abstract

Micromagnetics is a nonlocal, nonconvex variational problem. Its minimizer represents the ground state magnetization pattern of a ferromagnetic body under a specified external field. This paper identifies a physically relevant thin film limit, and shows that the limiting behavior is described by a certain "reduced" variational problem. Our main result is the Γ–convergence of suitably scaled 3D micromagnetic problems to a 2D reduced problem; this implies, in particular, convergence of minimizers for any value of the external field. The reduced problem is degenerate but convex; as a result it determines some (but not all) features of the ground state magnetization pattern in the associated thin film limit.

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Contents

1 Introduction

The micromagnetic variational principle is a nonconvex variational problem whose local minima represent the stable magnetization patterns of a ferromagnetic body. Applied in various ways, it captures the remarkable multiscale complexity of magnetic materials [16].

One widely–explored theme is the analysis of global minimizers, i.e. ground state magnetization patterns. The motivation is not that a ferromagnet easily reaches its ground state, but rather that the most robust features of the ground state may be shared by all physically accessible local minima [12].

It is natural to address the problem by considering its limiting behavior in various asymptotic regimes. Restricting our attention to films, i.e. thin cylinder–shaped bodies, such regimes are defined by special relations between

$$\begin{aligned} t &= \text{the thickness of the film,} \\ \ell &= \text{the length scale of the cross–section, and} \\ d &= \text{a characteristic length scale of the magnetic material} \end{aligned}$$

(see Section 2 for a careful definition of d). Several regimes are well–understood, including:

(a) The *large body limit*, in which $d/\ell \to 0$ while t/ℓ stays fixed. The asymptotic variational problem for this case is nonlocal but convex [8] (see also [13, 19, 21, 24, 25]).

(b) The *small aspect ratio limit*, in which $t/\ell \to 0$ while d/ℓ stays fixed. The asymptotic variational problem for this case predicts a uniform magnetization when the external field is constant [15].

The present paper considers a different limit, which mixes the properties of (a) and (b): Our films are large in the sense that $d^2/\ell t \to 0$, but thin in the sense that $t/\ell \to 0$. The resulting asymptotic variational problem is convex as in (a), but two–dimensional as in (b).

Our regime is of interest because it is readily accessible experimentally but quite inaccessible numerically. Indeed, our regime is appropriate for the analysis of permalloy films tens of microns in diameter — whose magnetizations under moderate external fields are certainly not constant [16]. Numerical simulation, by contrast, is generally restricted to submicron–size films, whose behavior is very different. The proper mathematical framework for the analysis of asymptotic variational problems is the notion of Γ–convergence [6, 7]. Our main result is thus the Γ–convergence of appropriately scaled 3D micromagnetic problems in our thin film limit to a 2D reduced problem. It follows (using basic properties of Γ–convergence) that for any external field, an asymptotically–energy–minimizing sequence of magnetization patterns converges in the thin film limit to a minimizer of the reduced problem. This implies convergence of minimizers, and much more. Let $e^{(\nu)}$ be the energy of the ground state at thickness $t^{(\nu)}$, cross–section length scale $\ell^{(\nu)}$, and magnetic length scale $d^{(\nu)}$; and suppose $\nu \uparrow \infty$ corresponds to our regime. Then Γ–convergence implies the convergence of all magnetization patterns whose energies $\tilde{e}^{(\nu)}$ satisfy $\lim_{\nu\uparrow\infty} \tilde{e}^{(\nu)} = \lim_{\nu\uparrow\infty} e^{(\nu)}$. Under our scaling, the wall energies will be higher–order terms. Therefore our asymptotic problem describes not just the limits of ground states, but also the limits of patterns that differ from the ground state mainly by having different wall patterns.

Besides Γ–convergence, we also prove a weak regularity result for the asymptotic variational problem. It is a constrained, convex optimization, whose physical interpretation lies mainly in the associated Euler–Lagrange equation. Our regularity result, though probably far from optimal, permits rigorous discussion of the Euler–Lagrange equation. It is crucial for understanding which features of the magnetization are uniquely determined by the reduced problem. Not surprisingly, these features are also the experimentally robust ones.

Mathematically speaking, our main results are Γ–convergence and regularity. But physically speaking, even the *choice* of problem represents a significant contribution. We have, in essence, identified an asymptotic thin film regime appropriate for materials like permalloy, and we delimit the region of parameter space where it applies. The physical consequences of our model are discussed further in [11, 12]; those papers also include a numerical treatment of the reduced problem and direct comparison to experimental data.

The preceding discussion of our accomplishments is of course somewhat vague. To make it precise, we begin in Section 2 by briefly reviewing the variational problem of micromagnetics. Then in Section 3 we give the full statement of our Γ–convergence result, and a more complete discussion of its mathematical context. Section 4 continues to lay necessary groundwork, explaining our choice of scaling by considering the energies of basic structures such as Néel walls and vortices. Our discussion emphasizes the fact that these structures are related to critical Sobolev embeddings. Sections 5, 6, and 7 form the mathematical heart of the paper. They are virtually independent of one another: Section 5 gives our regularity result for the reduced problem; Section 7 gives the "upper bound" half of the Γ–convergence argument; Section 6 gives the "lower bound" half of the Γ–convergence argument, and proves that certain features of the magnetization are uniquely determined by the reduced problem.

2 The model

The micromagnetic model states that the experimentally observed ground state for the magnetization m and for the magnetostatic potential U of the stray field is the minimizer of a variational problem.

2.1 The admissibility set

The open set $\Omega \subset \mathbb{R}^3$ denotes the ferromagnetic sample. The set of admissible vector fields/potentials $(m\colon \Omega \to \mathbb{R}^3, U\colon \mathbb{R}^3 \to \mathbb{R})$ is constrained by a unity spontaneous magnetization

$$|m|^2 \;=\; 1 \;\text{ in }\; \Omega, \tag{1}$$

and the static Maxwell equations, which we formulate variationally

$$\int_{\mathbb{R}^3} \nabla U \cdot \nabla \zeta \, dx \;=\; \int_\Omega m \cdot \nabla \zeta \, dx \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^3). \tag{2}$$

For the classical version of (2), see Subsection 4.1.

2.2 The functional

The micromagnetic energy is given by

$$\begin{aligned} E \;&:=\; d^2 \int_\Omega |\nabla m|^2 \, dx \;+\; Q \int_\Omega m_2^2 + m_3^2 \, dx \\ &\quad + \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \;-\; 2 \int_\Omega H_{ext} \cdot m \, dx. \end{aligned} \tag{3}$$

Let us now explain and comment on these four terms.

- The first term is the so–called exchange energy, it penalizes spatial variations of m through the Dirichlet integral of m.
- The second term is the anisotropy energy: Crystalline anisotropy favors certain magnetization axes. Here we assume that the material is uni-axial, i. e. it favors a single axis, which we label as the first coordinate axis m_1.
- $H_{str} = -\nabla U \colon \mathbb{R}^3 \to \mathbb{R}^3$ is the so–called stray field. The third term is the energy of the stray field, which we call the magnetostatic energy.
- $H_{ext} \colon \mathbb{R}^3 \to \mathbb{R}^3$ denotes the external field. The corresponding term in (3) favors alignment of the magnetization with the external field; we call it the external field energy.

2.3 Nonconvexity and nonlocality

The functional (3) is the sum of a quadratic and a linear part. It is the constraint (1) which makes the variational problem a nonconvex one. The magnetostatic energy in (3) makes it a nonlocal variational problem in m, since the energy density depends in a nonlocal way on the order parameter m, namely through the equation (2) which determines the potential U.

In view of (1) and the Dirichlet integral in (3), one might think of our variational problem as a perturbation of the harmonic mapping problem, since the remaining terms in (3) are order zero terms and thus form a compact perturbation. As we shall see, due to the multiscale nature of the problem, this perspective is misleading in the regime we will consider.

2.4 Units and material parameters

This model is already partially non–dimensionalized: The magnetization m, the fields H_{ext} and $H_{str} = -\nabla U$, and the energy density are dimensionless. So is the "quality factor" Q, a material parameter measuring the relative strength of anisotropy with respect to magnetostatic energy.

However, length is still dimensional. As can be inferred from (3), there are two length scales, which are material parameters:

$$d \quad \text{and} \quad d/Q^{\frac{1}{2}}.$$

d measures the relative strength of exchange energy with respect to the magnetostatic energy, $d/Q^{\frac{1}{2}}$ measures the relative strength of exchange energy with respect to anisotropy.

2.5 Geometry

We now specify the geometry we are interested in.

- We consider Ω a cylindrical domain of thickness t with cross section Ω'

  $$\Omega \;=\; \Omega' \times (0,t).$$

  Here and in the sequel, the prime denotes the projection on the first two components (the "in–plane" components) of a three–dimensional object.

- The anisotropy favors the first axis, which is in–plane.

- The external field is assumed to be in–plane and constant in x_3 (the "thickness direction"), that is,

  $$H_{ext} \;=\; (H'_{ext}, 0) \quad \text{and} \quad H_{ext} \;=\; H_{ext}(x').$$

2.6 Length scales

Let ℓ denote the diameter of the cross–section Ω'. We have four length scales: The two intrinsic scales (i. e. only depending on the material) and two extrinsic scales (i. e. only depending on the sample geometry)

$$\begin{array}{rl} \text{intrinsic scales:} & d \;\text{ and }\; d/Q^{\frac{1}{2}} \\ \text{extrinsic scales:} & t \;\text{ and }\; \ell \end{array} \qquad (4)$$

This multiscale nature of the variational problem, together with its nonconvexity and nonlocality, leads to a rich behavior and pattern formation on intermediate scales.

3 The main result

The goal of this paper is to rigorously derive a reduced theory for the case of a film, that is, a cylinder of small aspect ratio

$$t \;\ll\; \ell. \qquad (5)$$

Depending on the further relation among the scales (4), there are many possible reduced theories. We aim at recovering a reduced theory which reproduces the following gross features of experimental observations

- m does not depend on the thickness direction x_3,
- m has no out–of–plane component m_3,
- m is divergence–free in the absence of an external field.

At the same time, we seek a

- reduced theory with single length scale.

We aim at *identifying* the parameter regime of validity of the reduced theory: Under which asymptotic assumptions on the various parameters (the length scales (4), the strength of the external field, ...) is the reduced problem a good approximation to the full problem? Further comments concerning our choice of regime, and its relation to prior work, will be given in Subsection 3.2; but first we now give careful statements of our main results.

3.1 Rigorous statements

In order to formulate the rigorous statement, we non–dimensionalize length by measuring it in units of the diameter ℓ of the cross section Ω'

$$\omega' \;=\; \frac{1}{\ell}\,\Omega'.$$

Definition 1 *Let $\omega' \subset I\!R^2$ be open, bounded with smooth boundary. Let $h'_{ext} \in L^1(I\!R^2)^2$ and $q \geq 0$.*

i) By the reduced variational problem we understand the following: The set of admissible $(m'\colon \omega' \to I\!R^2, u\colon I\!R^3 \to I\!R)$ is given by all

$$|m'|^2 \;\leq\; 1 \;\; in \;\; \omega' \tag{6}$$

and

$$\int_{\mathbf{R}^3} \nabla u \cdot \nabla \zeta \, dx \;=\; \int_{\omega'} m' \cdot \nabla' \zeta \, dx' \quad \text{for all } \zeta \in C_0^\infty(I\!R^3), \tag{7}$$

for a classical version of (7), see Subsection 4.1. The functional e' is given by

$$e' \;=\; q \int_{\omega'} m_2^2 \, dx' + \int_{\mathbf{R}^3} |\nabla u|^2 \, dx - 2 \int_{\omega'} h'_{ext} \cdot m' \, dx', \tag{8}$$

ii) Let $d, t > 0$ and set

$$\omega := \omega' \times (0,t), \quad Q = t\,q, \quad H_{ext} = t \begin{pmatrix} h'_{ext} \\ 0 \end{pmatrix}. \tag{9}$$

By the full variational problem we understand the following: The set of admissible $(m\colon \omega \to I\!R^3, U\colon I\!R^3 \to I\!R)$ is given by

$$|m|^2 = 1 \ \ in\ \omega, \tag{10}$$

and

$$\int_{\mathbf{R}^3} \nabla U \cdot \nabla \zeta \, dx = \int_\omega m \cdot \nabla \zeta \, dx \quad for\ all\ \zeta \in C_0^\infty(I\!R^3). \tag{11}$$

The functional e is given by

$$\begin{aligned} t^2\, e &= d^2 \int_\omega |\nabla m|^2 \, dx + Q \int_\omega m_2^2 + m_3^2 \, dx \\ &\quad + \int_{\mathbf{R}^3} |\nabla U|^2 \, dx - 2 \int_\omega H_{ext} \cdot m \, dx. \end{aligned} \tag{12}$$

iii) We introduce the following notion of convergence: Consider a sequence $\{t^{(\nu)}\}_{\nu\uparrow\infty}$ of positive numbers converging to zero. An admissible sequence $\{(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ for the full problem converges to an admissible (m', u) for the reduced problem if

$$\begin{aligned} \frac{1}{t^{(\nu)}} \int_0^{t^{(\nu)}} m^{(\nu)}(\cdot, x_3)\, dx_3 &\stackrel{w}{\rightharpoonup} \begin{pmatrix} m' \\ 0 \end{pmatrix} \quad in\ L^2(\omega')^3, \\ \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} &\stackrel{w}{\rightharpoonup} \nabla u \quad in\ L^2(I\!R^3)^3. \end{aligned}$$

The nonlocal term of our reduced problem is $\int_{\mathbf{R}^3} |\nabla u|^2 \, dx$ with u defined by (7). We shall explain its origin and meaning in Section 4. But we remark here that it amounts to the squared $H^{-\frac{1}{2}}$ Sobolev norm of $\nabla' \cdot m'$:

$$\int_{\mathbf{R}^3} |\nabla u|^2 \, dx = \frac{1}{2} \|\nabla' \cdot m'\|^2_{H^{-\frac{1}{2}}(\mathbf{R}^2)} = \frac{1}{2} \|(\nabla')^{-\frac{1}{2}} (\nabla' \cdot m')\|^2_{L^2(\mathbf{R}^2)}.$$

The main result of this paper is a rigorous connection between the full three–dimensional model and the reduced two–dimensional model.

Theorem 1 *Let* $\{t^{(\nu)}\}_{\nu\uparrow\infty}$ *and* $\{d^{(\nu)}\}_{\nu\uparrow\infty}$ *be sequences of positive numbers such that*

$$t^{(\nu)} \;\to\; 0 \quad \textit{and} \quad (d^{(\nu)})^2\,\frac{\log\frac{1}{t^{(\nu)}}}{t^{(\nu)}} \;\to\; 0. \tag{13}$$

Then the reduced variational problem (Definition 1 i)) is the Γ*–limit of the full variational problem (Definition 1 ii)) under the convergence stated in Definition 1 iii). This means*

i) Lower semicontinuity. Let $\{(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ *be an admissible sequence for the full problem such that* $\{e(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ *is bounded. Then, there exists an* (m', u) *admissible for the reduced problem, and a subsequence such that we have convergence as in Definition 1 iii). For any of these possible limits* (m', u) *we have*

$$e'(m', u) \;\leq\; \liminf_{\nu\uparrow\infty}\, e(m^{(\nu)}, U^{(\nu)}).$$

ii) Construction. Let (m', u) *be admissible for the reduced variational problem. Then there exists an admissible sequence* $\{(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ *for the full problem, which converges to* (m', u) *in the sense of Definition 1 iii) and which satisfies*

$$e'(m', u) \;\geq\; \limsup_{\nu\uparrow\infty}\, e(m^{(\nu)}, U^{(\nu)}).$$

We notice that the reduced variational problem is *convex*. It is even strictly convex in the potential u and the m_2–component of the magnetization (provided, the reduced quality factor q is positive). Consequently, u and $q\,m_2$ are *unique*. On the other hand, if $q = 0$, the reduced variational problem is highly degenerate in m': for constant external field $h'_{ext} = const$, e' depends on m' only through its in–plane divergence $\nabla'\!\cdot m'$. Nevertheless, there is some more hidden strict convexity, since the pointwise constraint (6) on $m'(x')$ is strictly convex, which we shall elucidate now. For sufficiently small external fields, however, the constraint (6) is not active and the Euler–Lagrange equation for the reduced variational problem is given by

$$q\begin{pmatrix} 0 \\ m_2 \end{pmatrix} + \nabla' u + h'_{ext} \;=\; 0 \quad \text{on } \omega'. \tag{14}$$

Think of $q = 0$ for a moment: In this case, (14) states that the stray field $h_{str} = -\nabla u$ compensates the external field within the sample (in our thin–film framework). This is analogous to electrostatics, where charges on $\partial\omega$

arrange themselves to shield the interior ω of the conductor from an external field. Two reduced models have been proposed in the physics literature, both are based on (14), which is heuristically derived by this electrostatic analogy. The first model [26] considers the case of no external field (next to $q = 0$), which in view of (14) predicts that the stray field and thus $\nabla' \cdot m'$ vanishes ("flux closure"). Also the second model [5], which allows for an external field, determines $\nabla' u$ and thus $\nabla' \cdot m'$ from (14).

But the constraint (6) on the magnetization limits the "charges", unlike the case in electrostatics — we will explain the concept of magnetic charges in the Subsection 4.1. Hence stronger external fields "penetrate" into the sample:

$$h'_{pen} \;:=\; q \begin{pmatrix} 0 \\ m_2 \end{pmatrix} + \nabla' u + h'_{ext} \;\neq\; 0.$$

Therefore, our model extends the one in [5]. As we shall state in Corollary 1, m' is unique on the penetrated region

$$\omega'_{pen} \;:=\; \{\, x' \in \omega' \,|\, h'_{pen}(x') \neq 0 \,\}, \tag{15}$$

which itself is unique, since it only depends on the uniquely determined u and $q\, m_2$. In order to properly define ω'_{pen}, we need some regularity of u, which is stated in the next proposition.

Proposition 1 *Assume in addition that ω' is simply connected and that h'_{ext} has all partial derivatives up to second order in $L^1(I\!R^2)$, i. e. $h'_{ext} \in H^{2,1}(I\!R^2)^2$. Then any minimizer (m', u) of the reduced problem satisfies*

$$\nabla' u \;\in\; L^4_{loc}(\omega'). \tag{16}$$

As a consequence of the Γ–convergence stated in Theorem 1 and the above mentioned partial strict convexity, we will obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem 1 and Proposition 1 we have*

i) Let (m', u) and $(\tilde{m}', \tilde{u})$ be two minimizers of the reduced variational problem. Then we have

$$|m'|^2 \;=\; 1 \quad \text{a. e. on } \omega'_{pen}, \tag{17}$$

$$q\, m_2 \;=\; q\, \tilde{m}_2 \quad \text{and} \quad \nabla u \;=\; \nabla \tilde{u} \quad \text{a. e. on } I\!R^3, \tag{18}$$

$$m' \;=\; \tilde{m}' \quad \text{a. e. on } \omega'_{pen}, \tag{19}$$

where ω'_{pen} is defined in (15).

ii) Let the sequences $\{t^{(\nu)}\}_{\nu\uparrow\infty}$ *and* $\{d^{(\nu)}\}_{\nu\uparrow\infty}$ *be as in Theorem 1. Let* $(m^{(\nu)}, U^{(\nu)})$ *be admissible for the full problem with energy close to the minimal energy* $\inf e^{(\nu)}$ *in the sense of*

$$\limsup_{\nu\uparrow\infty} e(m^{(\nu)}, U^{(\nu)}) \;\le\; \limsup_{\nu\uparrow\infty} \inf e^{(\nu)}. \tag{20}$$

Then we have for any minimizer (m', u) *of the reduced variational problem*

$$\frac{1}{t^{(\nu)}} \int_{\omega'_{pen}\times(0,t^{(\nu)})} \left| m^{(\nu)} - \begin{pmatrix} m' \\ 0 \end{pmatrix} \right|^2 dx \;\to\; 0, \tag{21}$$

$$\frac{q}{t^{(\nu)}} \int_{\omega} (m_2^{(\nu)} - m_2)^2 \, dx + \int_{\boldsymbol{R}^3} \left| \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} - \nabla u \right|^2 dx \;\to\; 0, \tag{22}$$

$$e(m^{(\nu)}, U^{(\nu)}) - e'(m', u) \;\to\; 0. \tag{23}$$

Informal statements of Theorem 1 and Corollary 1 were given in our papers [11, 12], which mainly addressed the physical meaning of these results. We note, however, that the version of Theorem 1 presented here is slightly stronger than the one stated in [11, 12]. Indeed, those papers effectively imposed a lower bound as well as an upper bound on d. Here, by contrast, condition (13) imposes only an upper bound.

3.2 Mathematical context

Our thin film regime is very different from the one considered in [15]. That paper studied the case $t/\ell \to 0$, $d/\ell \sim 1$, obtaining an asymptotic problem in which

(a) the magnetostatic term survives only as an anisotropy term favoring $m_3 = 0$, while

(b) the exchange term survives intact, favoring $\nabla m = 0$.

This limit is very rigid, because it is dominated by the exchange energy. The asymptotic variational problem is insensitive to the shape of the cross–section, because the part of the magnetostatic term favoring $\nabla' \cdot m' = 0$ has been lost. The energy–minimizing magnetization is constant whenever the external field is constant.

Our regime has $t/\ell \to 0$ and $\frac{d^2}{\ell t}\log(\ell/t) \to 0$. The resulting theory is much less rigid, because it is dominated by the magnetostatic term rather than the exchange energy. Indeed, in our limit

(a') the magnetostatic energy imposes $m_3 = 0$ as a constraint, and introduces a nonlocal term favoring $\nabla' \cdot m' = 0$ in ω' and $m' \cdot \nu' = 0$ at $\partial\omega'$;

(b') the exchange term disappears, and the nonconvex 3D constraint $|m|^2 = 1$ gets relaxed to the convex 2D constraint $|m'|^2 \leq 1$.

A key advantage of our regime is its broad applicability: it requires *only* that a film be large enough (in the sense that $\ell t \log^{-1}(\ell/t) \gg d^2$) and that the aspect ratio be small enough. The corresponding disadvantage is the degeneracy of the reduced problem. Such degeneracy — and loss of the constraint $|m|^2 = 1$ — is directly linked to the disappearance of the exchange energy. This effect is familiar from work on the large–body limit [8]. One might ask whether there isn't a limit in which *both* the magnetostatically–induced shape effects and the exchange energy survive at principal order. The answer appears to be no. To explain why, let us consider the 3D micromagnetic energy restricted to magnetizations m that are independent of x_3. We may ignore the anisotropy and external field terms — they can always be made to interact with the surviving terms by scaling Q and H_{ext} appropriately. As usual we use the scaled spatial variables in which the cross–section of the film is $\omega' = \Omega'/\ell$, with diameter 1. With no further simplification, the exchange energy becomes $d^2 t \int_{\omega'} |\nabla' m|^2$. The magnetostatic energy, being nonlocal, is more complicated, but we can approximate it by

$$\frac{1}{2}\, t^2\, \ell\, \|(\nabla')^{-\frac{1}{2}}\, (\nabla' \cdot m')\, \|^2_{L^2(\boldsymbol{R}^2)} + \ell^2\, t \int_{\omega'} m_3^2\, dx' \;.$$

(This approximation is valid if the typical length scale over which m varies is large compared to the film thickness t: see Section 4.1. With the possible exceptions of Néel walls in thick films, this approximation is valid in our regime). With this approximation, the sum of exchange plus magnetostatic energy scales as

$$\ell^2\, t\, \left\{ \frac{d^2}{\ell^2} \int_{\omega'} |\nabla' m|^2\, dx' + \frac{t}{2\,\ell} \|(\nabla')^{-\frac{1}{2}}\, (\nabla' \cdot m')\, \|^2_{L^2(\boldsymbol{R}^2)} + \int_{\omega'} m_3^2\, dx' \right\}.$$

Our regime has $\frac{d^2}{\ell^2} \ll \frac{t}{\ell} \ll 1$ so the exchange term is negligible and the m_3^2 term becomes a constraint. To capture both exchange and magnetostatic shape effects at principal order in a thin film limit it seems necessary to consider the regime

$$\frac{d^2}{\ell^2} \sim \frac{t}{\ell} \ll 1.$$

But an asymptotic theory for this regime would apparently impose $m_3 = 0$ and $|m'|^2 = 1$ in ω', $m' \cdot \nu = 0$ at $\partial\omega'$, and $\int_{\omega'} |\nabla' m|^2 < \infty$. There is no such function m.

We conclude, at least heuristically, that the degeneracy of our reduced problem cannot be fixed by considering a different scaling. Rather, the exchange energy (and hence the wall energy) must remain a higher–order term — a singular perturbation that breaks the degeneracy of our reduced problem. It would be interesting to understand the effect of this perturbation, by somehow identifying the energy–minimizing wall structures and locations. In fact, different wall structures are expected for different ranges of thicknesses within the parameter regime for which our theory is valid. Thus, we anticipate that several distinct higher–order corrections can emerge, each specific to a wall type and to a restricted range of film thicknesses.

It would be interesting to identify these higher–order terms and to understand their effect. This appears to be a very difficult problem, for which the appropriate mathematical tools are not yet available. There has, however, been considerable progress on several closely related problems [1], [2], [20], [22]. A related, but more qualitative and simpler question is whether the constraint $|m|^2 = 1$ is preserved in the limit (which is to be expected only for $q = 0$). This question can be rephrased as an issue of compactness; it has been settled affirmatively for several singularly perturbed variational problems mimicking micromagnetism [3], [10], [17], [23], [18].

Our focus is energy minimization, but the dynamics and switching of soft thin–film ferromagnets is also an important topic. The paper [14] proposes a 2D reduction of the Landau–Lifshitz–Gilbert equations of micromagnetic dynamics, and demonstrates its validity through asymptotic analysis and direct comparison with 3D numerical simulations.

Our films are small in just one dimension, but it is also of interest to consider bodies that are small in every dimension. The paper [9] studies this *small* *particle limit* and the corresponding local minima using Γ–convergence, and [4] takes a different approach using an implicit function theorem.

4 Interpretation, heuristics and optimality

Our main result, which we stated in the previous section, can be paraphrased as follows: Suppose that (5) holds (which is encoded in the first limit of (13)) and that the sample is not too small in the sense of

$$d^2 \ll \ell\, t \log^{-1} \frac{\ell}{t}, \tag{24}$$

which is encoded in the second limit of (13). Suppose further that the anisotropy and the external field are sufficiently weak in the sense of

$$Q = O(\frac{t}{\ell}) \quad \text{and} \quad H_{ext} = O(\frac{t}{\ell}), \tag{25}$$

conditions which are encoded in (9). Then (23) states that to leading order, the minimal energy behaves as

$$\min_{(m,U) \text{ satisfies } (1,2)} E(m) \approx \ell\, t^2 \min_{(m',u) \text{ satisfies } (6,7)} e'(m'). \tag{26}$$

Notice the relaxation (6) of the nonconvex constraint (1). This relaxation is quite intuitive once one understands why one may drop the exchange energy term in (3) (observe that there is no gradient term in (8)). The main insight of our result is that (24) is what is needed for the exchange term to become negligible to leading order in the scaling of the minimal energy (26).

Over the next subsections, we will give a heuristic argument why (24) is natural. At the same time, this will also motivate (25). The proof of Theorem 1 is inspired by these arguments. In this heuristic part, we restrict ourselves to the regime of so–called "thin" films, i. e.

$$t \ll d. \tag{27}$$

This means that in view of (5), (24) and (25), t is the smallest of the four length scales in (4). Our result also holds in a regime when (27) is violated (so–called thick films), but the proof of Theorem 1 is motivated by (27).

Also observe that (5) and (24) imply the following ordering of three of these length scales

$$t \ll d \ll \ell. \tag{28}$$

As a first consequence of (27), it is safe to assume that m does not depend on the thickness variable x_3

$$m(x) = m(x'). \tag{29}$$

4.1 The magnetostatic energy

The magnetostatic part of the energy (3) is crucial — it makes our variational problem qualitatively different from the harmonic mapping problem. Therefore, a few remarks are in order.

We start by observing that the classical formulation of (2) is

$$\left.\begin{array}{rcl} \nabla^2 U & = & \left\{\begin{array}{ccc} \nabla\cdot m & \text{in} & \Omega \\ 0 & \text{in} & I\!R^3-\Omega \end{array}\right\} \\ {[U]} & = & 0 \quad \text{and} \quad \left[\frac{\partial U}{\partial \nu}\right] = \nu\cdot m \quad \text{on } \partial\Omega \end{array}\right\}, \tag{30}$$

where $[\cdot]$ denotes the jump of quantity $\cdot$ across the boundary $\partial\Omega$ with normal ν. In view of (30), it is clear that there are two sources of stray field $H_{str} = -\nabla U$. By the electrostatic analogy, they are called magnetic volume charges and magnetic surface charges:

$$\begin{array}{lccc} \text{volume charge density:} & \nabla\cdot m & \text{in} & \Omega, \\ \text{surface charge density:} & \nu\cdot m & \text{on} & \partial\Omega. \end{array} \tag{31}$$

Under the assumption (29), the volume charge density is the in–plane divergence $\nabla'\cdot m'$, whereas the out–of–plane component m_3 is a surface charge. A simple calculation reveals that the magnetostatic energy splits into two parts, which penalize the in–plane divergence and the out–of–plane component separately:

$$\int_{\mathbf{R}^3} |\nabla U|^2\, dx$$
$$= t\int_{\mathbf{R}^2} f(\frac{t}{2}\,|\xi'|)\left|\frac{\xi'}{|\xi'|}\cdot\mathcal{F}(m')\right|^2 d\xi' + t\int_{\mathbf{R}^2} g(\frac{t}{2}\,|\xi'|)\,|\mathcal{F}(m_3)|^2\, d\xi', \tag{32}$$

where $\mathcal{F}(m')(\xi')$ and $\mathcal{F}(m_3)(\xi')$ denote the Fourier transforms of $m'(x')$ resp. $m_3(x')$, when extended trivially on $I\!R^2$. Here the Fourier multipliers are given by

$$g(z) \;=\; \frac{\sinh(z)}{|z|\,\exp(z)} \quad \text{and} \quad f(z) \;=\; 1 - g(z)$$

and thus display a cross–over at wavenumbers of $O(\frac{1}{t})$ resp. length scales of $O(t)$. Since (27) implies that the intrinsic length scales $d, d/Q^{\frac{1}{2}}$ are much larger than t, we expect that (32) is approximated as follows

$$\begin{aligned}\int_{\boldsymbol{R}^3} |\nabla U|^2\,dx \;&\approx\; \frac{t^2}{2}\int_{\boldsymbol{R}^2} \frac{1}{|\xi'|}\,|\xi'\cdot\mathcal{F}(m')|^2\,d\xi' + t\int_{\boldsymbol{R}^2} |\mathcal{F}(m_3)|^2\,d\xi' \\ &=\; t^2\int_{\boldsymbol{R}^3} |\nabla u|^2\,dx + t\int_{\Omega'} m_3^2\,dx',\end{aligned} \tag{33}$$

where u is defined as in (7) (with the normalized ω' replaced by Ω').

Notice that u is the single layer potential of the volume charge density $\nabla'\cdot m'$. Classically, (7) turns into

$$\left.\begin{array}{rcl} \nabla^2 u &=& 0 \quad \text{in } I\!R^3 - (\Omega'\times\{0\}) \\ {[u]} \;=\; 0 \quad \text{and} & \left[\frac{\partial u}{\partial x_3}\right] = \nabla'\cdot m' & \text{on } \Omega'\times\{0\} \end{array}\right\}, \tag{34}$$

where $[\cdot]$ denotes the jump of a quantity $\cdot$ across the plane $I\!R^2\times\{0\}$. This characterization of u implicitly contains the assumption that the normal component of m' does not jump across a possible discontinuity line of m', in particular

$$m'\cdot\nu' \;=\; 0 \quad \text{on } \partial\Omega', \tag{35}$$

where ν' denotes the normal to the boundary $\partial\Omega'$ of the cross section Ω', more on this in Subsection 4.4.

4.2 A dimensional argument

At this stage, we can already give a simple dimensional argument which shows that (24) is optimal up to the logarithmic term. Indeed, here is the (formal) scaling of the four terms in (3):

$$\begin{aligned} d^2\int_\Omega |\nabla m|^2\,dx \;&\overset{(29)}{=}\; d^2\,t\int_{\Omega'} |\nabla' m|^2\,dx' \\ &=\; O(d^2\,t),\end{aligned}$$

$$
\begin{aligned}
Q \int_\Omega m_2^2 + m_3^2 \, dx &\overset{(29)}{=} Q\, t \int_{\Omega'} m_2^2 + m_3^2 \, dx' \\
&= O(Q\, \ell^2\, t), \\
\int_{\boldsymbol{R}^3} |\nabla U|^2 \, dx &\overset{(33)}{\approx} t^2 \int_{\boldsymbol{R}^3} |\nabla u|^2 \, dx + t \int_{\Omega'} m_3^2 \, dx' \\
&= O(\ell\, t^2) + O(\ell^2\, t) \\
\int_\Omega H_{ext} \cdot m \, dx &\overset{(29)}{=} t \int_{\Omega'} H_{ext} \cdot m \, dx' \\
&= O(|H_{ext}|\, \ell^2\, t).
\end{aligned} \tag{36}
$$

As a first conclusion, we observe that in our regime (5), the larger of the two magnetostatic terms is the cost of the out–of–plane component m_3. Therefore we expect $m_3 \approx 0$. Together with (29), this shows that m should essentially be a two–dimensional vector field as assumed in the reduced variational problem.

We also observe that (25) is chosen so that the cost of the volume charge generated by the in–plane divergence $\nabla' \cdot m'$ is of the same order as the anisotropy energy and the external field energy, namely $O(\ell\, t^2)$, which is the predicted scaling of the minimal energy (26). We finally notice that the condition

$$
d^2 \ll \ell\, t \tag{37}
$$

seems indeed to ensure that the exchange energy is of higher order.

We now will show, in the next several subsections, that the logarithmic factor in (24) is also natural. In order to achieve the relaxation (6), we will have to construct sequences which oscillate on a scale smaller than ℓ without affecting the leading order $O(\ell\, t^2)$–terms of the energy. Experiments show that in general, the way magnetization oscillates on a scale smaller than the sample size is by forming regions in which the magnetization varies smoothly, separated by transition layers, which have a typical width much smaller than the typical size of these regions. The regions are called domains, the transition layers are called walls. Under the assumption (29), the domains are two–dimensional objects and the walls are essentially one–dimensional objects. In the following, we will look at a typical type of wall in thin films (27).

4.3 Néel walls

The Néel wall is the favored wall type when $t \ll d$, i. e. (27). The reason is simple: A Néel wall avoids surface charges (which are penalized like $\ell^2\, t$,

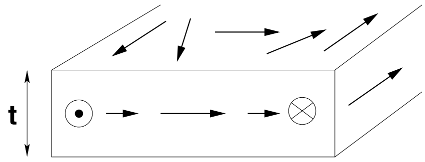

Figure 1

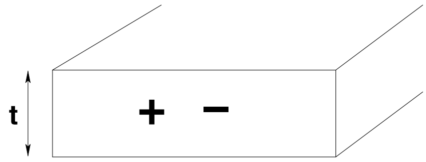

Figure 2

see (36)) at the expense of volume charges (which are only penalized at the lower level $\ell\, t^2$). A Néel wall achieves this by an entirely in–plane rotation of the magnetization. Figure 1 and 2 show a sketch of the magnetization and the volume charge within a Néel wall.

The prototype of a Néel wall separating two domains of diameter $O(\ell)$ has thus the following form

$$m \;=\; \begin{pmatrix} m' \\ 0 \end{pmatrix}, \quad m \;=\; m(x_1), \quad \left\{ \begin{array}{ccc} m_2(x_1 = 0) & = & -1 \\ m_2(x_1 = \ell) & = & 1 \end{array} \right\}. \tag{38}$$

This prototype connects the magnetizations $\binom{0}{-1}$ and $\binom{0}{1}$ by a one–dimensional profile over the distance ℓ, and is thus called a 180^o Néel wall. As we shall see, the assumption that the rotation takes place over the *finite* length ℓ is important. Since $m_3 = 0$, there are no surface charges. On the other hand, the constraint $|m'|^2 = |m|^2 = 1$ together with the boundary conditions in (38) implies that $m_1 = \pm 1$ somewhere in $(0, \ell)$. Hence $\nabla' \cdot m' = \frac{\partial m_1}{\partial x_1} \neq 0$ and thus there are volume charges.

Our goal now is to infer the scaling of the specific energy (i. e. the energy per length in the tangential x_2–direction) of such a prototypical Néel wall. This means that we wish to deduce the scaling of the minimum of

$$E_{Neel} \;:=\; d^2\, t \int_0^\ell |\frac{dm'}{dx_1}|^2\, dx_1 + t^2 \int_{-\infty}^{\infty} |\xi_1|\, |\mathcal{F}(m_1)(\xi_1)|^2\, d\xi_1 \tag{39}$$

among all m satisfying (38). Here, we have used the thin–film approximation (33) of the magnetostatic energy generated by volume charges. Also, we have neglected the contribution of anisotropy and external field. This seems justified, since in view of (36), these terms are of leading order only as (two–dimensional) bulk terms and not via their contribution in (essentially one–dimensional) walls. It is convenient to non–dimensionalize specific energy and length as follows

$$E_{Neel} \;=\; t^2\,\hat E_{Neel}, \quad x_1 \;=\; \ell\,\hat x_1.$$

Then (39) and (38) turn into

$$\hat E_{Neel} \;:=\; \frac{d^2}{\ell\,t}\int_0^1 |\frac{dm'}{d\hat x_1}|^2\,d\hat x_1 + \int_{-\infty}^{\infty}|\hat\xi_1|\,|\mathcal{F}(m_1)(\hat\xi_1)|^2\,d\hat\xi_1 \tag{40}$$

resp.

$$m \;=\; \begin{pmatrix} m' \\ 0\end{pmatrix}, \quad m \;=\; m(\hat x_1), \quad \left\{\begin{array}{rcl} m_2(\hat x_1=0) &=& -1 \\ m_2(\hat x_1=1) &=& 1\end{array}\right\}. \tag{41}$$

The exchange energy term in (40) is a singular perturbation: Only this term enforces continuity of m. If it were not there, we could choose

$$m \;=\; \left\{\begin{array}{ll} \binom{0}{-1} & \text{for } \hat x_1 \;<\; \frac12 \\ \binom{0}{1} & \text{for } \hat x_1 \;>\; \frac12 \end{array}\right\},$$

in particular $m_1=0$, so that there is no magnetostatic contribution in (40). Thus we expect

$$\min \hat E_{Neel} \;\longrightarrow\; 0 \quad \text{as} \quad \frac{d^2}{\ell\,t} \;\longrightarrow\; 0. \tag{42}$$

Hence even in the weaker regime (37), the energy of Néel walls of a total length $O(\ell)$ scales as $o(\ell\,t^2)$ and thus is of higher order.

In fact, (42) is a bit subtle: Continuity of m', the constraint $|m'|^2=1$, and the boundary conditions in (41) enforce

$$\|m_1\|_{L^\infty(\mathbb{R})} \;=\; \sup_{\hat x_1}|m_1(\hat x_1)| \;=\; 1.$$

On the other hand, we notice that the magnetostatic energy is the square of the homogeneous $H^{\frac12}(\mathbb{R})$–norm, i. e.

$$\|\left(\frac{d}{dx_1}\right)^{\frac12} m_1\|_{L^2(\mathbb{R})} \;=\; \left(\int_{-\infty}^{\infty}|\hat\xi_1|\,|\mathcal{F}(m_1)(\hat\xi_1)|^2\,d\hat\xi_1\right)^{\frac12},$$

and that the embedding $H^{\frac12}(\mathbb{R})\subset L^\infty(\mathbb{R})$ is critical and *barely* fails. This suggests that (42) is *logarithmically* slow. Indeed, one can show

$$\min \hat E_{Neel} \;\sim\; \log^{-1}\frac{\ell\,t}{d^2}. \tag{43}$$

We will use the construction leading to the upper bound in (43) implicitly in our rigorous analysis in Subsection 7.2.

Summing up, we have argued that the energetic cost of Néel walls of total length of $O(\ell)$ is $O(\ell\, t^2 \log^{-1} \frac{\ell\, t}{d^2})$.

4.4 Soft boundary condition

If (35), that is,

$$m' \cdot \nu' \;=\; 0 \quad \text{on } \partial\Omega'$$

is violated, $\nabla' \cdot m'$ has a distributional component (since we have to think of m' as being extended by zero on $I\!R^2$). Even if it is true that m' varies on scales at least of $O(d)$ in the interior of Ω', a violation of (35) means that the trivially extended m' varies on a much smaller scale at $\partial\Omega'$. Hence the approximation which led from (32) to (33) is not valid if we want to measure how much a violation of (35) is penalized.

Our goal is to infer the scaling of the specific energy (i. e. the energy per unit length in tangential direction) of $m' \cdot \nu' = O(1)$. For this purpose, it suffices to consider the prototypical situation

$$m' \;=\; \left\{ \begin{array}{cc} \binom{1}{0} & \text{for } x_1 \in (0,\ell) \\ 0 & \text{else} \end{array} \right\}.$$

and the related magnetostatic energy (per length)

$$\begin{aligned} E_{bc} \;&=\; t \int_{-\infty}^{\infty} f(\frac{t}{2}\,|\xi_1|)\, |\mathcal{F}(m')|^2\, d\xi_1 \\ &\overset{(32)}{\sim}\; \frac{t^2}{2} \int_{\{t\,|\xi_1|<1\}} |\xi_1|\, |\mathcal{F}(m')|^2\, d\xi_1 + t \int_{\{t\,|\xi_1|\geq 1\}} |\mathcal{F}(m')|^2\, d\xi_1. \end{aligned}$$

Since

$$|\mathcal{F}(m')| \;\sim\; \left\{ \begin{array}{cc} \ell & \text{for } \ell\,|\xi_1| \leq 1 \\ \frac{1}{|\xi_1|} & \text{for } \ell\,|\xi_1| \geq 1 \end{array} \right\},$$

we obtain a logarithmic divergence

$$E_{bc} \;\sim\; t^2 \log \frac{\ell}{t},$$

which reflects the fact that a step function *barely* fails to be in $H^{\frac{1}{2}}(I\!R)$.

To conclude, we have argued that an $O(1)$ normal component $m' \cdot \nu'$ over an $O(\ell)$ length of $\partial\Omega'$ costs $O(\ell\, t^2 \log \frac{\ell}{t})$.

4.5 Bloch lines

In the previous section we made the point that the cost of an $O(1)$ normal component $m' \cdot \nu'$ over a substantial part $\partial\Omega'$ of the boundary is $O(\ell\, t^2 \log\frac{\ell}{t})$ and thus beats the leading order energy scaling $O(\ell\, t^2)$ by a logarithmic factor. Loosely speaking, this enforces

$$m' \cdot \nu' \;\approx\; 0 \quad \text{on } \partial\Omega'.$$

This would impose a *topological* constraint on a continuous m': m' must vanish somewhere in Ω', hence $m_3 = 1 - |m'|^2 = 1$ somewhere in Ω'. This is not quite true: The penalization of $m' \cdot \nu'$ is only in $L^2(\partial\Omega')$, and, on the other hand, control of $\int |\nabla' m'|^2\, dx'$ fails to give a good control of the modulus of continuity of $m'|_{\partial\Omega'}$. Hence it is conceivable that the penalization of $m' \cdot \nu'$ is compatible with a topological degree zero of $m'|_{\partial\Omega'}$. We will not pursue this possibility of a "vortex just outside Ω'" here.

Experiments suggest that the magnetization accommodates such topological constraints by so–called Bloch lines. A Bloch line is a regularization of a vortex

$$m'(x') \;=\; \frac{x'^{\perp}}{r}, \tag{44}$$

where $\perp$ denote the in–plane rotation by 90^o and $r = |x'|$. Away from the core of the Bloch line, the magnetization looks like such a vortex. When approaching the core of the Bloch line, the magnetization avoids the singularity by turning out–of–plane; it is completely normal to the plane in the center of the Bloch line. This out–of–plane component is a surface charge, hence the magnetostatic energy keeps vortices localized, while the exchange energy wants to spread them. In our two–dimensional setting (29), a Bloch *line* is really an essentially zero–dimensional object. Figure 3 and 4 show a sketch of the magnetization and the surface charge within a Bloch line

A prototype of a Bloch line in a sample of diameter ℓ is of the following form

$$\nabla' \cdot m' \;=\; 0 \quad \text{and} \quad m(x') \;=\; \frac{1}{\ell}\, x'^{\perp} \;\text{ for }\; r := |x'| = \ell. \tag{45}$$

Our goal is to infer the scaling of the minimum of

$$E_{Bloch} \;=\; d^2\, t \int_{\{r<\ell\}} |\nabla' m|^2\, dx' + t \int_{\{r<\ell\}} m_3^2\, dx' \tag{46}$$

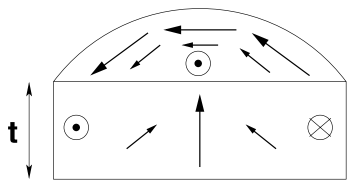

Figure 3

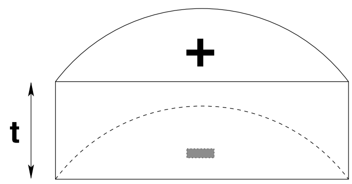

Figure 4

among all m satisfying (45). Here, we have used the approximation (33) of the magnetostatic energy coming from surface charges m_3. As for Néel walls, we have neglected the contribution of anisotropy and external field. This seems justified, since in view of (36), these terms are of leading order only as (two–dimensional) bulk terms and not via their contribution in (zero–dimensional) Bloch lines.

It is convenient to non–dimensionalize energy and length as follows

$$E_{Bloch} \;=\; d^2\, t\, \hat{E}_{Bloch}, \quad x' \;=\; \ell\, \hat{x}'.$$

Then (46) and (45) turn into

$$\hat{E}_{Bloch} \;=\; \int_{\{\hat{r}<1\}} |\hat{\nabla}' m|^2 \, d\hat{x}' + \left(\frac{\ell}{d}\right)^2 \int_{\{\hat{r}<1\}} m_3^2 \, d\hat{x}'$$

resp.

$$\hat{\nabla}' \cdot m' \;=\; 0 \quad \text{and} \quad m(\hat{x}') \;=\; \hat{x}'^{\perp} \;\text{ for }\; \hat{r} = 1. \tag{47}$$

There is no two–dimensional vector field m' on $\{\hat{r} < 1\}$ which satisfies the boundary condition in (47), which is of unit length, i. e. $|m'|^2 = 1$, and which has finite Dirichlet integral. Hence we expect

$$\min \hat{E}_{Bloch} \;\to\; +\infty \quad \text{as} \quad \frac{\ell}{d} \;\to\; \infty.$$

On the other hand, the vortex (44), which satisfies (45) and is of unit length, *barely* fails to have finite Dirichlet integral, it is only *logarithmically* divergent. In fact, one can show

$$\min \hat{E}_{Bloch} \;\sim\; \log \frac{\ell}{d}. \tag{48}$$

We will use the construction leading to the upper bound in (48) implicitly in our rigorous analysis in Subsection 7.3.

Summing up, we have argued that an $O(1)$ number of Bloch lines in Ω contribute $O(d^2\, t\, \log \frac{\ell}{d})$ to the energy.

4.6 Separation of energy scales

The conclusion of this section are summarized in the table

out–of–plane component m_3	$\ell^2\, t$
non–tangential component $m' \cdot \nu'$	$\ell\, t^2 \, \log \frac{\ell}{t}$
in–plane divergence $\nabla' \cdot m'$	$\ell\, t^2$
external field H_{ext}	$\lvert H_{ext}\rvert\, \ell^2\, t$
anisotropy Q	$Q\, \ell^2\, t$
Néel wall	$\ell\, t^2 \, \log^{-1} \frac{\ell\, t}{d^2}$
Bloch line	$d^2\, t \, \log \frac{\ell}{d}$

Let us now argue that the regime (5), (24) & (25) is just what is needed to ensure clearly *separated energy scales*, as indicated in the table: The upper tier scales $\gg \ell\, t^2$, the middle tier scales as $\ell\, t^2$ and the lower tier scales $\ll \ell\, t^2$.

First of all, (5) ensures that the first tier of the table, that is

- the penalization of an out–of–plane component over an $O(1)$–fraction of the sample's cross–section and
- a non–tangential component over an $O(1)$–fraction of the boundary of the sample's cross section

is stronger than $\ell\, t^2$. Hence the penalization in the full model turns into a constraint in the reduced model.

As already pointed out in Subsection 4.2, (25) makes sure that the terms in the middle tier of the table, that is,

- the energy contributions of the in–plane divergence,

- the external field and
- the anisotropy

are all of the same order, namely $\ell\, t^2$. Hence we expect that the leading order energy scaling is determined by a competition of penalization of in–plane divergence with the energy contributions from the external field and anisotropy, as expressed in the reduced model.

Finally, (24) just ensures that

- the cost of Néel walls of a total length of the order of the sample diameter and
- the cost of an $O(1)$ number of Bloch–lines

are much less than $\ell\, t^2$. Indeed, (5) and (24) in particular imply

$$\frac{\ell\, t}{d^2} \gg 1, \tag{49}$$

so that the cost of Néel walls is always small

$$\ell\, t^2 \log^{-1} \frac{\ell\, t}{d^2} \ll \ell\, t^2.$$

Furthermore, we observe that

$$\left(\frac{\ell}{d}\right)^2 \overset{(49)}{\gg} \frac{\ell}{t} \overset{(5)}{\gg} 1,$$

so that the argument of the logarithm which appears in the Bloch line energy is indeed large. The cost of a Bloch line is higher order if and only if

$$d^2\, t \log \frac{\ell}{d} \ll \ell\, t^2,$$

which is equivalent to

$$\left(\frac{d}{\ell}\right)^2 \log \frac{\ell}{d} \ll \frac{t}{\ell}. \tag{50}$$

Since both $\frac{d}{\ell}$ and $\frac{t}{\ell}$ are small, (50) is equivalent[1] to

$$\left(\frac{d}{\ell}\right)^2 \ll \frac{t}{\ell}\log^{-1}\frac{\ell}{t}, \tag{53}$$

which is just a reformulation of (24).

5 Some regularity

This section establishes a basic regularity result for the reduced variational problem. The methods used here are quite different from those in the rest of the paper. Hence this section may be skipped in a first reading.

Proof of Proposition 1. We start with a remark on the choice of our method. Our reduced variational problem can be reformulated as saddle point problem in (m', u) for the functional

$$q\int_{\omega'} m_2^2\,dx' - \int_{\boldsymbol{R}^3}|\nabla u|^2\,dx - 2\int_{\omega'}(h'_{ext}-\nabla' u)\cdot m'\,dx'$$

This functional has to be minimized among all m' with (6) and $\|(\nabla')^{-\frac{1}{2}}\nabla'\cdot m'\|_{L^2(\boldsymbol{R}^2)} < \infty$ and maximized among all u with $\int_{\boldsymbol{R}^3}|\nabla u|^2\,dx < \infty$. Hence u minimizes

$$\int_{\boldsymbol{R}^3}\frac{1}{2}|\nabla u|^2\,dx + \int_{\omega'} H^*(h'_{ext}-\nabla' u)\,dx',$$

[1]In fact, the equivalence of (50) and (53) reduces to the fact that for $x, y \ll 1$,

$$y = x^2\log\frac{1}{x} \tag{51}$$

implies

$$x^2 \sim y\log^{-1}\frac{1}{y}. \tag{52}$$

Indeed, (51) in particular yields

$$x^2 \ll y \ll x^3,$$

so that

$$\log\frac{1}{y} \sim \log\frac{1}{x},$$

which implies (52).

where H^* is the Legendre transform of

$$H(m') \;:=\; \left\{ \begin{array}{cc} \frac{q}{2}\, m_2^2 & \text{if } |m'|^2 \le 1 \\ +\infty & \text{else} \end{array} \right\}.$$

This is the "dual problem". If the anisotropy is negligible, i. e. $q = 0$, we have

$$H^*(p') \;=\; |p'|.$$

In this case, fairly standard local elliptic regularity theory (differentiate the Euler–Lagrange equation in direction of $x_i, i = 1, 2$ and test with $\eta^2 \frac{\partial u}{\partial x_i}$) would yield Proposition 1.

However, in case of $q \neq 0$, H^* is irregular not just in $p' = 0$. At the same time, it is not homogeneous. There seems to be no simple fix of the standard argument in this case. Therefore, we take a different approach and work directly on the primal problem.

In general terms, the strategy will be to construct several (to be more precise: three) smooth one–parameter groups of diffeomorphisms $\{\Phi_\tau^{(i)}\}_{\tau \in \mathbf{R}}$ $(i = 1, 2, 3)$ of $I\!R^2$ with the following two properties

- The transformed potentials $u_\tau^{(i)}$, i. e.

$$u_\tau^{(i)} \circ \Phi_\tau^{(i)} \;=\; u, \tag{54}$$

  satisfy

$$\int_{\mathbf{R}^3} |\nabla(u_\tau^{(i)} - u)|^2 \, dx \;\le\; O(\tau^2). \tag{55}$$

  Here, the diffeomorphisms are naturally extended to $I\!R^3$.

- The generating vector fields, i. e.

$$\xi'^{(i)} \;:=\; \left.\frac{\partial \Phi^{(i)}}{\partial \tau}\right|_{\tau=0},$$

  satisfy

$$\left\{\xi'^{(i)}(x')\right\}_{i=1,2,3} \quad \text{span } I\!R^2 \text{ in every } x' \;\in\; \omega'. \tag{56}$$

By the standard difference quotient argument, this implies

$$\nabla\,\nabla' u \;\in\; L^2_{loc}(\omega' \times \mathbb{R}),$$

which in turn yields (16) by the embedding $H^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$.

Our Ansatz is to use one–parameter subgroups of the group of conformal diffeomorphisms of ω' onto ω' (automorphisms). We recall the following results from complex variable calculus

- Since ω' is simply connected, there exist three one–parameter subgroups $\{\Phi^{(i)}_\tau\}_{\tau\in\mathbf{R}}$, $i = 1,2,3$, with (56). Recall that $\{\Phi^{(i)}_\tau\}_{\tau\in\mathbf{R}}$ is the flow generated by $\xi^{(i)}$.

- Since ω' has a smooth boundary, $\mathbb{R}\times\omega' \ni (\tau,x') \mapsto \Phi^{(i)}_\tau(x') \in \omega'$ is smooth up to the boundary. In particular, ${\xi'}^{(i)}\colon \omega' \to \mathbb{R}^2$ is smooth up to the boundary. Hence we can smoothly extend ${\xi'}^{(i)}$ to a ${\xi'}^{(i)}\colon \mathbb{R}^2 \to \mathbb{R}^2$ s. t.
$$ {\xi'}^{(i)}(x') \;=\; 0 \quad \text{for } |x'| \;\geq\; R$$
for some $R<\infty$. This extends the flow $\{\Phi^{(i)}_\tau\}_{\tau\in\mathbf{R}}$ smoothly to diffeomorphisms of $\mathbb{R}^2$ with
$$\Phi^{(i)}(x') \;=\; \text{id} \quad \text{for } |x'| \;\geq\; R.$$

We will use these one–parameter groups of diffeomorphisms to define the following variations $\{{m'}^{(i)}_\tau\}_{\tau\in\mathbf{R}}$ of m'

$$\sqrt{\det \mathrm{D}\Phi^{(i)}_\tau}\,(\mathrm{D}\Phi^{(i)}_\tau)^{-1}\,({m'}^{(i)}_\tau \circ \Phi^{(i)}_\tau) \;=\; m'. \tag{57}$$

Since $\Phi^{(i)}_\tau$ maps ω' onto ω', this indeed defines an ${m'}^{(i)}_\tau\colon \omega' \to \mathbb{R}^2$. Since $\Phi^{(i)}_\tau$ is conformal, we have

$$\sqrt{\det \mathrm{D}\Phi^{(i)}_\tau}\,(\mathrm{D}\Phi^{(i)}_\tau)^{-1} \;\in\; SO(2)$$

and thus
$$|{m'}^{(i)}_\tau \circ \Phi^{(i)}_\tau|^2 \;=\; |m'|^2 \;\leq\; 1.$$

We define $v^{(i)}_\tau$ to satisfy (7), i. e.

$$\int_{\mathbf{R}^3} \nabla v^{(i)}_\tau \cdot \nabla\zeta\,dx \;=\; \int_{\omega'} m^{(i)}_\tau \cdot \nabla'\zeta\,dx' \quad \text{for all } \zeta \in C^0_\infty(\mathbb{R}^3). \tag{58}$$

By construction, $(m_\tau^{(i)}, v_\tau^{(i)})$ is admissible.

Since the reduced variational problem is convex, its solution is characterized by the variational inequality

$$\begin{aligned} q \int_{\omega'} m_2\,(m_2 - \tilde{m}_2)\,dx' - \int_{\omega'} h'_{ext} \cdot (m' - \tilde{m}')\,dx' & \\ + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla \tilde{u})\,dx \ \leq \ 0 \quad \text{for all admissible } (\tilde{m}', \tilde{u}). & \end{aligned}$$

We use this variational inequality for $(\tilde{m}', \tilde{u}) = ({m'}_\tau^{(i)}, v_\tau^{(i)})$ and $(\tilde{m}', \tilde{u}) = ({m'}_{-\tau}^{(i)}, v_{-\tau}^{(i)})$. We add both resulting inequalities and will prove the following estimates for the individual energy contributions:

$$q \int_{\omega'} m_2\,(m_2 - m_{\tau,2}^{(i)})\,dx' + q \int_{\omega'} m_2\,(m_2 - m_{-\tau,2}^{(i)})\,dx' \ \geq \ O(\tau^2), \tag{59}$$

$$\left| \int_{\omega'} h'_{ext} \cdot (m' - {m'}_\tau^{(i)})\,dx' + \int_{\omega'} h'_{ext} \cdot (m' - {m'}_{-\tau}^{(i)})\,dx' \right| \ \leq \ O(\tau^2), \tag{60}$$

$$\begin{aligned} \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla v_\tau^{(i)})\,dx + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla v_{-\tau}^{(i)})\,dx & \\ \geq \ \frac{1}{4} \int_{\boldsymbol{R}^3} |\nabla (u_\tau^{(i)} - u)|^2\,dx + O(\tau^2), & \end{aligned} \tag{61}$$

where $u_\tau^{(i)}$ is defined in (54). This evidently implies (55).

A little remark before proving (59,60,61). Our choice of variations has the following advantages

- The variations are of the form

  $$\begin{aligned} \mathrm{det}\mathrm{D}\Phi_\tau^{(i)}\,(\mathrm{D}\Phi_\tau^{(i)})^{-1}\,({m'}_\tau^{(i)} \circ \Phi_\tau^{(i)}) \ &= \ a_\tau^{(i)}\,m \\ \text{with a smooth scalar field } a_\tau^{(i)}, & \end{aligned}$$

  so that the divergence transforms nicely. More precisely, $\nabla' \cdot {m'}_\tau^{(i)}$ depends on $\mathrm{D}{m'}_\tau^{(i)}$ only via $\nabla' \cdot m'$:

  $$\mathrm{det}\mathrm{D}\Phi_\tau^{(i)}\,(\nabla' \cdot {m'}_\tau^{(i)}) \circ \Phi_\tau^{(i)} \ = \ \nabla' \cdot (a_\tau^{(i)}\,m') \ = \ a_\tau^{(i)}\,\nabla' \cdot m' + \nabla a_\tau^{(i)} \cdot m'.$$

  This is important for (61).

- The smooth scalar field $a_\tau^{(i)}$ enforces

  $$|{m'}_\tau^{(i)} \circ \Phi_\tau^{(i)}|^2 \ = \ |m'|^2.$$

- Each family $\{\Phi_\tau^{(i)}\}_{\tau\in\boldsymbol{R}}$ of diffeomorphisms has the group–morphism property
$$\Phi_{\tau+\sigma}^{(i)} \;=\; \Phi_\tau^{(i)}\circ\Phi_\sigma^{(i)}.$$
- At the same time, the three families $\{\Phi_\tau^{(i)}\}_{\tau\in\boldsymbol{R}}$, $i=1,2,3$, are just rich enough in the sense of (56).

We shall now establish (59,60,61). To simplify notation,we drop the superscripts (i). We start with the term (60) coming from the external field. It is convenient to introduce the following transformation of vector fields
$$\sqrt{\mathrm{det}\mathrm{D}\Phi_\tau}\,(h'_{ext,\tau}\circ\Phi_\tau) \;=\; \mathrm{D}\Phi_\tau^{-*}\,h'_{ext}, \tag{62}$$
related to the transformation (57) of the magnetizations in the following sense
$$\int_{\omega'} m'_\tau\cdot h'_{ext,\tau}\,dx' \;=\; \int_{\omega'} m'\cdot h'_{ext}\,dx'. \tag{63}$$
Hence
$$\begin{aligned}&\int_{\omega'} h'_{ext}\cdot(m'-m'_\tau)\,dx'+\int_{\omega'} h'_{ext}\cdot(m'-m'_{-\tau})\,dx'\\ &\overset{(63)}{=}\;\int_{\omega'}(2\,h'_{ext}-h'_{ext,\tau}-h'_{ext,-\tau})\cdot m'\,dx'\\ &=\;O(\tau^2),\end{aligned}$$
since $m'\in L^\infty(\omega')$ and (62) defines a smooth variation $\{h'_{ext,\tau}\}_{\tau\in\boldsymbol{R}}$ of $h'_{ext}\in H^{2,1}(\omega')$. (Recall that $h'_{ext}\in H^{2,1}(\omega')$ means that all partial derivatives up to the second order are in $L^1(\omega')$).

We now consider the term (59) coming from anisotropy. It is convenient to slightly generalize (59) to
$$\int_{\omega'} m'\cdot B\,(m'-m'_\tau)\,dx'+\int_{\omega'} m'\cdot B\,(m'-m'_{-\tau})\,dx' \;\geq\; O(\tau^2),$$
where B is a positive semi–definite 2×2–matrix. The transformation of tensor fields
$$B_\tau\circ\Phi_\tau \;=\; (\mathrm{D}\Phi_\tau)^{-*}\,B\,(\mathrm{D}\Phi_\tau)^{-1} \tag{64}$$
is related to the transformation (57) of magnetizations in the following way:
$$\int_{\omega'} m'_\tau\cdot B_\tau\,\tilde m'_\tau\,dx' \;=\; \int_{\omega'} m'\cdot B\,\tilde m'\,dx'. \tag{65}$$

Here, the star $*$ in the exponent indicates the transpose and $-*$ denotes the transpose of the inverse. We thus have

$$
\begin{aligned}
&\int_{\omega'} m' \cdot B\,(m' - m'_\tau)\,dx' + \int_{\omega'} m' \cdot B\,(m' - m'_{-\tau})\,dx' \\
&\stackrel{(65)}{=} \int_{\omega'} m' \cdot B\,(m' - m'_\tau)\,dx' + \int_{\omega'} m'_\tau \cdot B_\tau\,(m'_\tau - m')\,dx' \\
&= \int_{\omega'} (m'_\tau - m') \cdot B_{\tau/2}\,(m'_\tau - m')\,dx' \\
&+ \int_{\omega'} m' \cdot (B - B_{\tau/2})\,(m' - m'_\tau)\,dx' \\
&+ \int_{\omega'} m'_\tau \cdot (B_\tau - B_{\tau/2})\,(m'_\tau - m')\,dx' \\
&\geq \int_{\omega'} m' \cdot (B - B_{\tau/2})\,m'\,dx' - \int_{\omega'} m' \cdot (B - B_{\tau/2})\,m'_\tau\,dx' \\
&+ \int_{\omega'} m'_\tau \cdot (B_\tau - B_{\tau/2})\,m'_\tau\,dx' - \int_{\omega'} m'_\tau \cdot (B_\tau - B_{\tau/2})\,m'\,dx' \\
&\stackrel{(65)}{=} \int_{\omega'} m' \cdot (2\,B - B_{\tau/2} - B_{-\tau/2})\,m'\,dx' \\
&+ \int_{\omega'} m' \cdot (2\,B_{\tau/2} - B_\tau - B)\,m'_\tau\,dx' \\
&= O(\tau^2),
\end{aligned}
$$

since $m' \in L^\infty(\omega')$ and (64) defines a smooth variation $\{B_\tau\}_{\tau \in \boldsymbol{R}}$ of the constant tensor B.

We finally consider the term (61) coming from the magnetostatic energy. It is helpful to introduce $(\tilde{m}_\tau, \tilde{v}_\tau)$ via

$$\mathrm{det} \mathrm{D}\Phi_\tau\,(\mathrm{D}\Phi_\tau)^{-1}\,(\tilde{m'}_\tau \circ \Phi_\tau) \;=\; m', \tag{66}$$

$$\int_{\boldsymbol{R}^3} \nabla \tilde{v}_\tau \cdot \nabla \zeta\,dx \;=\; \int_{\omega'} \tilde{m'}_\tau \cdot \nabla' \zeta\,dx' \qquad \text{for all } \zeta \;\in\; C_0^\infty(I\!R^3). \tag{67}$$

We split (61) into

$$
\begin{aligned}
&\int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla \tilde{v}_\tau)\,dx + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla \tilde{v}_{-\tau})\,dx \\
&\qquad \geq \frac{1}{2} \int_{\boldsymbol{R}^3} |\nabla (u_\tau - u)|^2\,dx + O(\tau^2),
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
&\int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla \tilde{v}_\tau - \nabla v_\tau)\,dx + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla \tilde{v}_{-\tau} - \nabla v_{-\tau})\,dx \\
&\qquad \geq -\frac{1}{4} \int_{\boldsymbol{R}^3} |\nabla (u_\tau - u)|^2\,dx + O(\tau^2).
\end{aligned} \tag{69}
$$

We start with (68). We recall the transformation (54) of scalar fields on $I\!R^3$

$$\zeta_\tau \circ \Phi_\tau \;=\; \zeta.$$

We introduce the tensor field $\{A_\tau\}_{\tau\in\boldsymbol{R}}$ of 3×3–matrices

$$\mathrm{det}\mathrm{D}\Phi_\tau\, A_\tau \circ \Phi_\tau \;=\; \mathrm{D}\Phi_\tau\,(\mathrm{D}\Phi_\tau)^*. \tag{70}$$

These are adapted to (66) and (67):

$$\int_{\boldsymbol{R}^3} \nabla u_\tau \cdot A_\tau \nabla\zeta_\tau\, dx \;=\; \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla\zeta\, dx, \tag{71}$$

$$\int_{\omega'} a_\tau\, \tilde m'_\tau \cdot \nabla'\zeta_\tau\, dx' \;=\; \int_{\omega'} a\, m' \cdot \nabla'\zeta\, dx'. \tag{72}$$

Hence $\tilde v_\tau$ and u_τ are related via

$$\begin{aligned}
\int_{\boldsymbol{R}^3} \nabla u_\tau \cdot A_\tau \nabla\zeta_\tau\, dx &\overset{(71)}{=} \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla\zeta\, dx \\
&\overset{(7)}{=} \int_{\omega'} m' \cdot \nabla'\zeta\, dx' \\
&\overset{(72)}{=} \int_{\omega'} \tilde m'_\tau \cdot \nabla'\zeta_\tau\, dx' \\
&\overset{(67)}{=} \int_{\boldsymbol{R}^3} \nabla\tilde v \cdot \nabla\zeta_\tau\, dx \quad \text{for all } \zeta_\tau \in C_0^\infty(I\!R^3).
\end{aligned} \tag{73}$$

Thus we obtain

$$\begin{aligned}
&\int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla\tilde v_\tau)\, dx + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla\tilde v_{-\tau})\, dx \\
&\quad\overset{(73)}{=} \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla u\, dx - \int_{\boldsymbol{R}^3} \nabla u \cdot A_\tau \nabla u_\tau\, dx \\
&\quad\quad + \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla u\, dx - \int_{\boldsymbol{R}^3} \nabla u \cdot A_{-\tau} \nabla u_{-\tau}\, dx \\
&\quad\overset{(71)}{=} \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla u\, dx - \int_{\boldsymbol{R}^3} \nabla u \cdot A_\tau \nabla u_\tau\, dx \\
&\quad\quad + \int_{\boldsymbol{R}^3} \nabla u_\tau \cdot A_\tau \nabla u_\tau\, dx - \int_{\boldsymbol{R}^3} \nabla u_\tau \cdot \nabla u\, dx \\
&\quad = \int_{R^3} |\nabla(u-u_\tau)|^2\, dx \\
&\quad\quad + \int_{\boldsymbol{R}^3} \nabla(u_\tau - u) \cdot (A_\tau - \mathrm{id}) \nabla u_\tau\, dx \\
&\quad \ge \frac{1}{2} \int_{\boldsymbol{R}^3} |\nabla(u-u_\tau)|^2\, dx - \frac{1}{2} \int_{\boldsymbol{R}^3} |(A_\tau - \mathrm{id})\nabla u_\tau|^2\, dx.
\end{aligned}$$

Now (68) follows from

$$\begin{aligned}
\int_{\boldsymbol{R}^3} |(A_\tau - \mathrm{id})\nabla u_\tau|^2 \, dx &\le \sup_{\boldsymbol{R}^3} |A_\tau - \mathrm{id}|^2 \int_{\boldsymbol{R}^3} |\nabla u_\tau|^2 \, dx \\
&\le C \sup_{\boldsymbol{R}^3} |A_\tau - \mathrm{id}|^2 \int_{\boldsymbol{R}^3} |\nabla u|^2 \, dx \\
&= O(\tau^2),
\end{aligned}$$

the latter being true because the tensor field (70) is a smooth variation of the constant identity matrix and since $\int_{\boldsymbol{R}^3} |\nabla u|^2 \, dx < \infty$.

We finally address (69). For ease of notation, we introduce

$$a_\tau \;:=\; \sqrt{\mathrm{det} \mathrm{D}\Phi_\tau},$$

so that the relation between m'_τ and $\tilde{m}'_\tau$ (which follows from the definitions (57) and (66)) can be restated as

$$m'_\tau \;=\; (a_\tau \circ \Phi_{-\tau})\, \tilde{m}'_\tau. \tag{74}$$

We split the l. h. s. of (69) into three parts as follows

$$\begin{aligned}
&\int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla \tilde{v}_\tau - \nabla v_\tau) \, dx + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla \tilde{v}_{-\tau} - \nabla v_{-\tau}) \, dx \\
&\quad \overset{(67,58)}{=} \int_{\omega'} \nabla' u \cdot (\tilde{m}'_\tau - m'_\tau) \, dx' + \int_{\omega'} \nabla' u \cdot (\tilde{m}'_{-\tau} - m'_{-\tau}) \, dx' \\
&\quad \overset{(74)}{=} \int_{\omega'} (1 - a_\tau \circ \Phi_{-\tau}) \, \nabla' u \cdot \tilde{m}'_\tau \, dx' + \int_{\omega'} (1 - a_{-\tau} \circ \Phi_\tau) \, \nabla' u \cdot \tilde{m}'_{-\tau} \, dx' \\
&\quad \overset{(72)}{=} \int_{\omega'} (1 - a_\tau) \, \nabla' u_{-\tau} \cdot m' \, dx' + \int_{\omega'} (1 - a_{-\tau}) \, \nabla' u_\tau \cdot m' \, dx' \\
&\quad = \int_{\omega'} (1 - a_\tau) \, \nabla' (u_{-\tau} - u) \cdot m' \, dx' + \int_{\omega'} (1 - a_{-\tau}) \, \nabla' (u_\tau - u) \cdot m' \, dx' \\
&\quad \quad + \int_{\omega'} (2 - a_\tau - a_{-\tau}) \, \nabla' u \cdot m' \, dx' \\
&\quad =: T_1 + T_2 + T_3.
\end{aligned}$$

We now make use of the following embedding resp. interpolation estimate

$$\begin{aligned}
\|v\|_{L^4(\boldsymbol{R}^2)} &\le C \, \|(\nabla')^{\frac{1}{2}} v\|_{L^2(\boldsymbol{R}^2)}, \\
\|(\nabla')^{\frac{1}{2}} (\alpha \, v)\|_{L^2(\boldsymbol{R}^2)} &\le C \, \|\alpha\|^{\frac{1}{3}}_{L^3(\boldsymbol{R}^2)} \, \|\nabla' \alpha\|^{\frac{2}{3}}_{L^3(\boldsymbol{R}^2)} \, \|(\nabla')^{\frac{1}{2}} v\|_{L^2(\boldsymbol{R}^2)},
\end{aligned} \tag{75}$$

(the first inequality is standard, we will give the short argument for the second one at the end of the proof of this proposition) which we combine into

$$\begin{aligned}
\left|\int_{\mathbf{R}^2} \alpha\, \nabla' v \cdot m'\, dx'\right| &\le \left|\int_{\mathbf{R}^2} \alpha\, v\, \nabla' \cdot m'\, dx'\right| + \left|\int_{\mathbf{R}^2} \nabla'\alpha \cdot m'\, v\, dx'\right| \\
&\le \|(\nabla')^{\frac12}(\alpha\, v)\|_{L^2(\mathbf{R}^2)} \|(\nabla')^{-\frac12}\nabla' \cdot m'\|_{L^2(\mathbf{R}^2)} \\
&\quad + \|\nabla'\alpha\|_{L^{\frac43}(\mathbf{R}^2)} \|m'\|_{L^\infty(\mathbf{R}^2)} \|v\|_{L^4(\mathbf{R}^2)} \\
&\le C\Big\{ \|\alpha\|_{L^3(\mathbf{R}^2)}^{\frac13} \|\nabla'\alpha\|_{L^3(\mathbf{R}^2)}^{\frac23} \|(\nabla')^{-\frac12}\nabla' \cdot m'\|_{L^2(\mathbf{R}^2)} \\
&\quad + \|\nabla'\alpha\|_{L^{\frac43}(\mathbf{R}^2)} \|m'\|_{L^\infty(\mathbf{R}^2)} \Big\} \|(\nabla')^{\frac12} v\|_{L^2(\mathbf{R}^2)}. \quad (76)
\end{aligned}$$

Here, we use the notation

$$\|(\nabla')^\beta v\|_{L^2(\mathbf{R}^2)}^2 \;=\; \int_{\mathbf{R}^2} |\xi'|^{2\,\beta}\, |\mathcal{F}(v)|^2\, d\xi'.$$

In case of T_1,

$$\alpha \;=\; 1 - a_\tau$$

is of first order in τ; more precisely

$$\|\alpha\|_{L^3(\mathbf{R}^2)}, \|\nabla'\alpha\|_{L^3(\mathbf{R}^2)}, \|\nabla'\alpha\|_{L^{\frac43}(\mathbf{R}^2)} \;=\; O(\tau),$$

so that (76) yields

$$|T_1| \;\le\; C\,|\tau| \left(\|(\nabla')^{-\frac12}\nabla' \cdot m'\|_{L^2(\mathbf{R}^2)} + \|m'\|_{L^\infty(\mathbf{R}^2)} \right) \|(\nabla')^{\frac12}(u_{-\tau} - u)\|_{L^2(\mathbf{R}^2)}.$$

Since $m' \in L^\infty(\mathbb{R}^2)$ and $\|(\nabla')^{-\frac12}\nabla' \cdot m'\|_{L^2(\mathbf{R}^2)}^2 = \int_{\mathbf{R}^3} |\nabla u|^2\, dx < \infty$,

$$\begin{aligned}
|T_1| &\le C\,|\tau| \int_{\mathbf{R}^3} |\nabla(u_{-\tau} - u)|^2\, dx \\
&\le C\,|\tau| \int_{\mathbf{R}^3} |\nabla(u_\tau - u)|^2\, dx \\
&\le \frac18 \int_{\mathbf{R}^3} |\nabla(u_\tau - u)|^2\, dx + O(\tau^2).
\end{aligned}$$

T_2 is treated similarly:

$$|T_2| \;\le\; \frac18 \int_{\mathbf{R}^3} |\nabla(u_\tau - u)|^2\, dx + O(\tau^2).$$

In the case of T_3,

$$\alpha \;=\; 2 - a_\tau - a_{-\tau}$$

is of second order in τ:

$$\|\alpha\|_{L^3(\boldsymbol{R}^2)}, \|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)}, \|\nabla'\alpha\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} \;=\; O(\tau^2).$$

Therefore (76) yields

$$|T_3| \;\leq\; C\,\tau^2\,\left(\|(\nabla')^{-\frac{1}{2}}\nabla'\cdot m'\|_{L^2(\boldsymbol{R}^2)} + \|m'\|_{L^\infty(\boldsymbol{R}^2)}\right)\,\|(\nabla')^{\frac{1}{2}}u\|_{L^2(\boldsymbol{R}^2)}.$$

Since $\|(\nabla')^{-\frac{1}{2}}\nabla'\cdot m'\|^2_{L^2(\boldsymbol{R}^2)} = \|(\nabla')^{\frac{1}{2}}u\|^2_{L^2(\boldsymbol{R}^2)} = \int_{\boldsymbol{R}^3}|\nabla u|^2\,dx < \infty$ and $m' \in L^\infty(\mathbb{R}^2)$, we obtain as desired

$$T_3 \;\leq\; O(\tau^2).$$

We close by establishing (75). For any extension $\bar\alpha$ of α into $\mathbb{R}^3$ we have

$$\|(\nabla')^{\frac{1}{2}}(\alpha\,v)\|_{L^2(\boldsymbol{R}^2)} \;\leq\; C\,\left(\|\bar\alpha\|_{L^\infty(\boldsymbol{R}^3)} + \|\nabla\bar\alpha\|_{L^3(\boldsymbol{R}^3)}\right)\,\|(\nabla')^{\frac{1}{2}}v\|_{L^2(\boldsymbol{R}^2)}. \tag{77}$$

Indeed, let $\bar v$ denote the harmonic extension of v into $\mathbb{R}^3$, i. e. the extension which satisfies

$$\|\nabla\bar\alpha\|_{L^2(\boldsymbol{R}^3)} \;=\; \|(\nabla')^{\frac{1}{2}}\alpha\|_{L^2(\boldsymbol{R}^2)}. \tag{78}$$

We have

$$\nabla(\bar\alpha\,\bar v) \;=\; \bar\alpha\,\nabla\bar v + \bar v\,\nabla\bar\alpha$$

and thus

$$\|\nabla(\bar\alpha\,\bar v)\|_{L^2(\boldsymbol{R}^3)} \;\leq\; \|\bar\alpha\|_{L^\infty(\boldsymbol{R}^3)}\,\|\nabla\bar v\|_{L^2(\boldsymbol{R}^3)} + \|\bar v\|_{L^6(\boldsymbol{R}^3)}\,\|\nabla\bar\alpha\|_{L^3(\boldsymbol{R}^3)}. \tag{79}$$

Since $\bar\alpha\,\bar v$ is an extension of $\alpha\,v$, and in view of (78) and the standard embedding

$$\|\bar v\|_{L^6(\boldsymbol{R}^3)} \;\leq\; C\,\|\nabla\bar v\|_{L^2(\boldsymbol{R}^3)},$$

(79) turns into (77).

Because of (77), (75) reduces to constructing an extension $\bar\alpha$ of α such that

$$\|\bar\alpha\|_{L^\infty(\boldsymbol{R}^3)} \;\leq\; C\,\|\alpha\|^{\frac{1}{3}}_{L^3(\boldsymbol{R}^2)}\,\|\nabla'\alpha\|^{\frac{2}{3}}_{L^3(\boldsymbol{R}^3)}, \tag{80}$$

$$\|\nabla\bar\alpha\|_{L^3(\boldsymbol{R}^3)} \;\leq\; C\,\|\alpha\|^{\frac{1}{3}}_{L^3(\boldsymbol{R}^2)}\,\|\nabla'\alpha\|^{\frac{2}{3}}_{L^3(\boldsymbol{R}^3)}. \tag{81}$$

Our Ansatz is the following: We select an $[0,1]$–valued $\eta \in C_0^\infty(I\!R)$ with $\eta(\hat{x}_3=0)=1$ and set

$$\bar{\alpha}(x) \;=\; \eta(\frac{x_3}{\ell})\,\alpha(x'). \tag{82}$$

Since

$$\nabla\bar{\alpha}(x) \;=\; \begin{pmatrix} \eta(\frac{x_3}{\ell})\,\nabla'\alpha(x') \\ \frac{1}{\ell}\,\frac{d\eta}{d\hat{x}_3}(\frac{x_3}{\ell})\,\alpha(x') \end{pmatrix},$$

we have

$$\begin{aligned}
\|\nabla\bar{\alpha}\|_{L^3(\boldsymbol{R}^3)} \;&\le\; \left(\int_{\boldsymbol{R}} |\eta(\frac{x_3}{\ell})|^3\,dx_3\right)^{\frac{1}{3}}\,\|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)} \\
&\quad + \left(\frac{1}{\ell^3}\int_{\boldsymbol{R}} |\frac{d\eta}{d\hat{x}_3}(\frac{x_3}{\ell})|^3\,dx_3\right)^{\frac{1}{3}}\,\|\alpha\|_{L^3(\boldsymbol{R}^2)} \\
&\le\; C\left(\ell^{\frac{1}{3}}\,\|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)} + \ell^{-\frac{2}{3}}\,\|\alpha\|_{L^3(\boldsymbol{R}^2)}\right).
\end{aligned}$$

Optimizing in the decay length ℓ yields (81).

Since the extension (82) does not increase the supremum, the estimate (80) reduces to

$$\|\alpha\|_{L^\infty(\boldsymbol{R}^2)} \;\le\; C\,\|\alpha\|_{L^3(\boldsymbol{R}^2)}^{\frac{1}{3}}\,\|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)}^{\frac{2}{3}},$$

which follows from the standard embedding

$$\sup_{x_1'\neq x_2'}\frac{|\alpha(x_1')-\alpha(x_2')|}{|x_1'-x_2'|^{\frac{1}{3}}} \;\le\; C\,\|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)}$$

and the elementary interpolation

$$\|\alpha\|_{L^\infty(\boldsymbol{R}^2)} \;\le\; C\,\|\alpha\|_{L^3(\boldsymbol{R}^2)}^{\frac{1}{3}}\,\sup_{x_1'\neq x_2'}\frac{|\alpha(x_1')-\alpha(x_2')|}{|x_1'-x_2'|^{\frac{1}{3}}}. \tag{83}$$

(83) follows via optimizing the averaging length λ in

$$\begin{aligned}
|\alpha(0)| \;&\le\; \frac{1}{\pi\,\lambda^2}\int_{B_\lambda'}|\alpha|\,dx' + \frac{1}{\pi\,\lambda^2}\int_{B_\lambda'}|\alpha(0)-\alpha(x')|\,dx' \\
&\le\; C\left(\lambda^{-\frac{2}{3}}\,\|\alpha\|_{L^3(\boldsymbol{R}^2)} + \lambda^{\frac{1}{3}}\,\sup_{x_1'\neq x_2'}\frac{|\alpha(x_1')-\alpha(x_2')|}{|x_1'-x_2'|^{\frac{1}{3}}}\right).
\end{aligned}$$

6 Convexity and lower semicontinuity

This section combines the soft analysis parts of Theorem 1 and Corollary 1. By “soft analysis”, we understand methods involving compactness, convexity and lower semicontinuity arguments.

Proof of Theorem 1 i). Our starting point is an admissible sequence $\{(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ for the full problem with bounded energy, i. e.

$$\{e(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty} \quad \text{is bounded.} \tag{84}$$

According to (9) and (10), the linear part in the energy (12) is always bounded

$$\left| \left(\frac{1}{t^{(\nu)}}\right)^2 \int_\omega H_{ext}^{(\nu)} \cdot m^{(\nu)}\, dx \right| \;\leq\; \int_{\omega'} |h'_{ext}|\, dx'.$$

Together with (84), we deduce that the remaining three positive quadratic terms in (12) are bounded. In particular,

$$\left\{ \left(\frac{1}{t^{(\nu)}}\right)^2 \int_{\mathbb{R}^3} |\nabla U^{(\nu)}|^2\, dx \right\}_{\nu\uparrow\infty} \quad \text{is bounded.} \tag{85}$$

In view of (10) and (85), it is well–known that there exist $m\colon \omega' \to \mathbb{R}^3$, $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\frac{1}{t^{(\nu)}} \int_0^{t^{(\nu)}} m^{(\nu)}(\cdot, x_3)\, dx_3 \;\stackrel{w}{\rightharpoonup}\; m \quad \text{in } L^2(\omega')^3, \tag{86}$$

$$\frac{1}{t^{(\nu)}} \nabla U^{(\nu)} \;\stackrel{w}{\rightharpoonup}\; \nabla u \quad \text{in } L^2(\mathbb{R}^3)^3 \tag{87}$$

for a subsequence. From elementary lower semicontinuity arguments, it is obvious that under (86,87) we have

$$e'(m', u) \;\leq\; \liminf_{\nu\uparrow\infty} e(m^{(\nu)}, U^{(\nu)}),$$

where m' denotes the in–plane components of m. It is also standard that under (86), the nonconvex constraint (10) for $m^{(\nu)}$ deteriorates into

$$|m'|^2 \;\leq\; |m|^2 \;\leq\; 1 \quad \text{in } \omega'.$$

On the other hand, the linear relation (11) for $(m^{(\mu)}, U^{(\nu)})$ is preserved under (86,87). More precisely, it simplifies to

$$\int_{\boldsymbol{R}^3} \nabla u \cdot \nabla \zeta \, dx \;=\; \int_{\omega'} m \cdot \nabla \zeta \, dx' \quad \text{for all } \zeta \in C_0^\infty(I\!R^3). \tag{88}$$

It thus remains to argue that

$$m_3 \;=\; 0, \tag{89}$$

so that we can identify (88) with (7).

Fix an $\eta \in C_0^\infty(I\!R)$ with

$$\eta|_{\hat{x}_3=0} \;=\; 0 \quad \text{and} \quad \left.\frac{d\eta}{d\hat{x}_3}\right|_{\hat{x}_3=0} \;=\; 1. \tag{90}$$

Let $\psi \in C_0^\infty(I\!R^2)$ and $\tau > 0$ be arbitrary and consider

$$\zeta(x) \;=\; \tau\, \eta(\frac{x_3}{\tau})\, \psi(x')$$

in (88). Since

$$\nabla \zeta(x) \;=\; \begin{pmatrix} \tau\, \eta(\frac{x_3}{\tau})\, \nabla' \psi(x') \\ \frac{d\eta}{d\hat{x}_3}(\frac{x_3}{\tau})\, \psi(x') \end{pmatrix},$$

we obtain thanks to (90)

$$\begin{aligned} &\left|\int_{\omega'} m_3\, \psi\, dx'\right| \\ &\quad\le\; \left|\int_{\boldsymbol{R}^3} \nabla u \cdot \nabla \zeta\, dx\right| \\ &\quad\le\; \left(\tau^3 \int_{\boldsymbol{R}} \eta^2\, d\hat{x}_3 \int_{\boldsymbol{R}^2} |\nabla' \psi|^2\, dx' + \tau \int_{\boldsymbol{R}} |\frac{d\eta}{d\hat{x}_3}|^2\, d\hat{x}_3 \int_{\boldsymbol{R}^2} |\psi|^2\, dx'\right)^{\frac{1}{2}} \\ &\qquad \times \left(\int_{\boldsymbol{R}^3} |\nabla u|^2\, dx\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\tau > 0$ was arbitrary, we obtain

$$\int_{\omega'} m_3\, \psi\, dx' \;=\; 0.$$

Since $\psi \in C_0^\infty(I\!R^2)$ was arbitrary, this shows (89).

Proof of Corollary 1. We start with (17). Since our reduced variational problem is convex, its minimizers are characterized by the variational inequality

$$q \int_{\omega'} m_2\,(m_2 - \tilde m_2)\,dx' + \int_{\boldsymbol{R}^3} \nabla u \cdot (\nabla u - \nabla \tilde u)\,dx \qquad (91)$$
$$- \int_{\omega'} h'_{ext} \cdot (m' - \tilde m')\,dx' \;\le\; 0 \quad \text{for all admissible } (\tilde m', \tilde u).$$

Thanks to the additional regularity stated in Proposition 1, we may integrate by parts in the magnetostatic term and obtain

$$\int_{\omega'} \left(q \begin{pmatrix} 0 \\ m_2 \end{pmatrix} + \nabla' u - h'_{ext} \right) \cdot (m' - \tilde m')\,dx' \;\le\; 0 \qquad (92)$$
$$\text{for all } \tilde m' \text{ with (6) and } \operatorname{supp}(\tilde m' - m') \subset \omega'.$$

Indeed, since (m', u) and $(\tilde m', \tilde u)$ are admissible, we have

$$\int_{\boldsymbol{R}^3} (\nabla u - \nabla \tilde u) \cdot \nabla \zeta\,dx \;=\; \int_{\omega'} (m' - \tilde m') \cdot \nabla' \zeta\,dx' \quad \text{for all } \zeta \in C_0^\infty(I\!R^3).$$

Since $\nabla u - \nabla \tilde u \in L^2(I\!R^3)$, $m' - \tilde m' \in L^\infty(\omega')$ and since $m' - \tilde m'$ is compactly supported in ω', i. e. $\operatorname{supp}(m' - \tilde m') \subset \omega'$, this variational formulation is valid for a larger class of test functions:

$$\int_{\boldsymbol{R}^3} (\nabla u - \nabla \tilde u) \cdot \nabla \zeta\,dx \;=\; \int_{\omega'} (m' - \tilde m') \cdot \nabla' \zeta\,dx' \qquad (93)$$
$$\text{for all } \zeta \in L^6(I\!R^3) \quad \text{with} \quad \nabla \zeta \in L^2(I\!R^3) \text{ and } \nabla' \zeta \in L^1_{loc}(\omega').$$

According to Proposition 1 and the Sobolev embedding, we may choose

$$\zeta \;=\; u,$$

in (93). Hence (91) yields (92). (92) in turn implies

$$h'_{pen} \;:=\; q \begin{pmatrix} 0 \\ m_2 \end{pmatrix} + \nabla' u - h'_{ext} \;=\; 0 \quad \text{a. e. on } \{|m'|^2 < 1\},$$

which is just a reformulation of (17).

The convexity of the constraint (6) implies that also the pair $(\frac{1}{2}\,(m+\tilde m), \frac{1}{2}\,(u+\tilde u))$ is admissible. Now the convexity of e' implies

$$e'((\frac{1}{2}\,(m + \tilde m), \frac{1}{2}\,(u + \tilde u)) \;\le\; \frac{1}{2}\,(e'(m', u) + e'(\tilde m', \tilde u')) \;\le\; \inf e' \qquad (94)$$

with equality if and only if

$$q \int_{\omega'} (m_2 - \tilde{m}_2)^2 \, dx' + \int_{\mathbb{R}^3} |\nabla(u - \tilde{u})|^2 \, dx \;=\; 0.$$

This establishes (18), which in turn implies that

$$\omega'_{pen} \text{ is unique.} \tag{95}$$

Finally, (94) implies that $(\frac{1}{2}(m + \tilde{m}), \frac{1}{2}(u + \tilde{u}))$ is also a minimizer. In view of (17) and (95), we conclude

$$|\frac{1}{2}(m' + \tilde{m}')|^2 \;=\; 1 \quad \text{a. e. on } \omega'_{pen}.$$

This yields (19). Part i) of Corollary 1 is therefore established.

We now address part ii) of Corollary 1. We observe that $\inf e' \leq 0$. According to Theorem 1 ii), this implies

$$\limsup_{\nu\uparrow\infty} e(m^{(\nu)}, U^{(\nu)}) \overset{(20)}{\leq} \limsup_{\nu\uparrow\infty} \inf e^{(\nu)} \;\leq\; 0.$$

In particular, $\{e(m^{(\nu)}, U^{(\nu)})\}_{\nu\uparrow\infty}$ is bounded. According to Theorem 1 i), we conclude that there exists an admissible (m', u) such that

$$\frac{1}{t^{(\nu)}} \int_0^{t^{(\nu)}} m^{(\nu)}(\cdot, x_3) \, dx_3 \;\overset{w}{\rightharpoonup}\; \begin{pmatrix} m' \\ 0 \end{pmatrix} \quad \text{in } L^2(\omega')^3, \tag{96}$$

$$\frac{1}{t^{(\nu)}} \nabla U^{(\nu)} \;\overset{w}{\rightharpoonup}\; \nabla u \quad \text{in } L^2(\mathbb{R}^3)^3 \tag{97}$$

and

$$e'(m', u) \;\leq\; \liminf_{\nu\uparrow\infty} e(m^{(\nu)}, U^{(\nu)})$$

for a subsequence. According to Theorem 1 ii), for any other admissible $(\tilde{m'}, \tilde{u})$, there exists an admissible sequence $\{(\tilde{m}^{(\nu)}, \tilde{u}^{(\nu)})\}_{\nu\uparrow\infty}$ with

$$e'(\tilde{m'}, \tilde{u}) \;\geq\; \limsup_{\nu\uparrow\infty} e(\tilde{m}^{(\nu)}, \tilde{u}^{(\nu)}).$$

Hence we have

$$e'(m', u) \;\leq\; e'(\tilde{m'}, \tilde{u}).$$

Thus (m', u) is a minimizer of the reduced variational problem and

$$\inf e' \;=\; e'(m', u) \;=\; \lim_{\nu\uparrow\infty} e(m^{(\nu)}, U^{(\nu)}). \tag{98}$$

According to part i) of Corollary 1, we have to show (21) & (22) only for this particular minimizer and subsequence. The convergence (96) implies

$$\lim_{\nu\uparrow\infty} \frac{1}{(t^{(\nu)})^2} \int_\omega H_{ext}^{(\nu)} \cdot m^{(\nu)}\, dx \;=\; \int_{\omega'} h'_{ext} \cdot m'\, dx'. \tag{99}$$

By standard lower semicontinuity arguments, we obtain for the quadratic parts of the energy

$$q \int_{\omega'} m_2^2\, dx' \;\le\; q \liminf_{\nu\uparrow\infty} \frac{1}{t^{(\nu)}} \int_\omega (m_2^{(\nu)})^2\, dx, \tag{100}$$

$$\int_{\boldsymbol{R}^3} |\nabla u|^2\, dx \;\le\; \liminf_{\nu\uparrow\infty} \int_{\boldsymbol{R}^3} |\frac{1}{t^{(\nu)}} \nabla U^{(\nu)}|^2\, dx. \tag{101}$$

The convergence of energy (98), which is the sum of the three contributions (99,100,101) implies that the two liminfs turn into proper limits:

$$q \int_{\omega'} m_2^2\, dx' \;=\; q \lim_{\nu\uparrow\infty} \frac{1}{t^{(\nu)}} \int_\omega (m_2^{(\nu)})^2\, dx, \tag{102}$$

$$\int_{\boldsymbol{R}^3} |\nabla U|^2\, dx \;=\; \lim_{\nu\uparrow\infty} \int_{\boldsymbol{R}^3} |\frac{1}{t^{(\nu)}} \nabla U^{(\nu)}|^2\, dx. \tag{103}$$

This norm convergence together with the weak convergence implies strong convergence. This is standard for u: Since

$$\begin{aligned} &\int_{\boldsymbol{R}^3} \left| \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} - \nabla u \right|^2 dx \\ &\quad = \int_{\boldsymbol{R}^3} \left| \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} \right|^2 dx - 2 \int_{\boldsymbol{R}^3} \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} \cdot \nabla u\, dx + \int_{\boldsymbol{R}^3} |\nabla u|^2\, dx \end{aligned}$$

we have

$$\begin{aligned} &\lim_{\nu\uparrow\infty} \int_{\boldsymbol{R}^3} \left| \frac{1}{t^{(\nu)}} \nabla U^{(\nu)} - \nabla u \right|^2 dx \\ &\quad \overset{(97),(103)}{=} \int_{\boldsymbol{R}^3} |\nabla u|^2\, dx - 2 \int_{\boldsymbol{R}^3} \nabla u \cdot \nabla u\, dx + \int_{\boldsymbol{R}^3} |\nabla u|^2\, dx \;=\; 0. \end{aligned}$$

The argument for $q\,m_2$ is similar:

$$\frac{q}{t^{(\nu)}}\int_0^{t^{(\nu)}}\int_{\omega'}|m_2^{(\nu)}(x',x_3)-m_2(x')|^2\,dx'\,dx_3$$
$$= \frac{q}{t^{(\nu)}}\int_\omega (m_2^{(\nu)})^2\,dx - 2\,q\int_{\omega'}\left(\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}} m_2^{(\nu)}(x',x_3)\,dx_3\right)m_2(x')\,dx'$$
$$+q\int_{\omega'} m_2^2\,dx,$$

and thus

$$\lim_{\nu\uparrow\infty}\frac{q}{t^{(\nu)}}\int_0^{t^{(\nu)}}\int_{\omega'}|m_2^{(\nu)}(x',x_3)-m_2(x')|^2\,dx'\,dx_3$$
$$\overset{(96),(102)}{=}\; q\int_{\omega'} m_2^2\,dx' - 2\,q\int_{\omega'} m_2\,m_2\,dx' + q\int_{\omega'} m_2^2\,dx' \;=\; 0.$$

This establishes (22). Also the argument for (21) is not very different. Since

$$\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}}\int_{\omega'_{pen}}\left|m^{(\nu)}(x',x_3)-\begin{pmatrix}m'(x')\\0\end{pmatrix}\right|^2\,dx'\,dx_3$$
$$=\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}}\int_{\omega'_{pen}}|m^{(\nu)}(x',x_3)|^2\,dx'\,dx_3$$
$$-2\int_{\omega'_{pen}}\left(\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}} m^{(\nu)}(x',x_3)\,dx_3\right)\cdot\begin{pmatrix}m'(x')\\0\end{pmatrix}\,dx'$$
$$+\int_{\omega'_{pen}}|m'(x')|^2\,dx'$$
$$\overset{(1)}{=}1\;-2\int_{\omega'_{pen}}\left(\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}} m^{(\nu)}(x',x_3)\,dx_3\right)\cdot\begin{pmatrix}m'(x')\\0\end{pmatrix}\,dx'$$
$$+\int_{\omega'_{pen}}|m'(x')|^2\,dx',$$

we obtain

$$\lim_{\nu\uparrow\infty}\frac{1}{t^{(\nu)}}\int_0^{t^{(\nu)}}\int_{\omega'_{pen}}\left|m^{(\nu)}(x',x_3)-\begin{pmatrix}m'(x')\\0\end{pmatrix}\right|^2\,dx'\,dx_3$$
$$\overset{(96)}{=}\;1-2\int_{\omega'_{pen}} m'\cdot m'\,dx' + \int_{\omega'_{pen}}|m'|^2\,dx' \;\overset{(17)}{=}\; 0.$$

7 The construction

This section deals with the construction of appropriate magnetizations. The mathematical construction is motivated by the physical intuition outlined in Subsections 4.1, 4.3 and 4.5, which we suggest to read beforehand.

Proof of Theorem 1 ii). The sequence $\{m^{(\nu)}\}_{\nu\uparrow\infty}$ we construct will consist of magnetizations m which do not depend on the thickness variable

$$m \;=\; m(x').$$

In this case, the full energy (12) simplifies to

$$\begin{aligned} e \;=\; & \frac{d^2}{t}\int_{\omega'}|\nabla' m|^2\,dx' + q\int_{\omega'}(m_2^2+m_3^2)\,dx' \\ & +\frac{1}{t^2}\int_{\boldsymbol{R}^3}|\nabla U|^2\,dx - 2\int_{\omega'}h'_{ext}\cdot m'\,dx'. \end{aligned}$$

We now consider the Fourier representation (32) of $\int_{\boldsymbol{R}^3}|\nabla U|^2\,dx$. Since the Fourier multipliers satisfy the inequalities

$$f(z)\;\le\; z \quad \text{and} \quad g(z)\;\le\; 1,$$

we actually have

$$\begin{aligned} \int_{\boldsymbol{R}^3}|\nabla U|^2\,dx \;\le\; & \frac{t^2}{2}\int_{\boldsymbol{R}^2}|\xi'|\left|\frac{\xi'}{|\xi'|}\cdot\mathcal{F}(m')\right|^2 d\xi' + t\int_{\boldsymbol{R}^2}|\mathcal{F}(m_3)|^2\,d\xi' \\ =\; & t^2\int_{\boldsymbol{R}^3}|\nabla u|^2\,dx + t\int_{\omega'}m_3^2\,dx', \end{aligned} \tag{104}$$

where u and m' are related via (7). Hence the two energies from Definition (1) are related by

$$e \;\le\; e' + \frac{d^2}{t}\int_{\omega'}|\nabla' m|^2\,dx' + (\frac{1}{t}+q)\int_{\omega'}m_3^2\,dx' \;=:\; \tilde{e}. \tag{105}$$

In the sequel, it is more suggestive to express $\int_{\boldsymbol{R}^3}|\nabla u|^2$ in terms of m': We write

$$e'(m') \;=\; q\int_{\omega'}m_2^2\,dx' + \frac{1}{2}\int_{\boldsymbol{R}^2}|(\nabla')^{-\frac{1}{2}}\,\nabla'\cdot m'|^2\,dx' - 2\int_{\omega'}h'_{ext}\cdot m'\,dx' \tag{106}$$

with the notation

$$\|(\nabla')^\alpha v\|^2_{L^2(\mathbb{R}^2)} \;=\; \int_{\mathbb{R}^2} |(\nabla')^\alpha v|^2\, dx' \;=\; \int_{\mathbb{R}^2} |\xi'|^{2\,\alpha}\, |\mathcal{F}(v)|^2\, d\xi'.$$

For (106) and in the sequel, we think of m' as trivially extended on all of $\mathbb{R}^2$, so that a non–vanishing normal component $m'\cdot\nu'$ would appear as a singular contribution to $\nabla'\cdot m'$ on $\partial\omega'$.

In view of (105) and (104), it is sufficient to prove the following proposition.

Proposition 2 . *Let $m'\colon\omega'\to\mathbb{R}^2$ satisfy*

$$|m'|^2 \;\le\; 1 \quad in\, \omega'.$$

Let the sequences $\{t^{(\nu)}\}_{\nu\uparrow\infty}$ and $\{d^{(\nu)}\}_{\nu\uparrow\infty}$ satisfy (13). Then there exists a sequence $\{m^{(\nu)}\colon\omega'\to\mathbb{R}^3\}_{\nu\uparrow\infty}$ with

$$|m^{(\nu)}|^2 \;=\; 1 \quad in\, \omega'.$$

and such that

$$\begin{aligned} m^{(\nu)} &\;\stackrel{w}{\rightharpoonup}\; \begin{pmatrix} m' \\ 0 \end{pmatrix} \quad in\; L^2(\omega')^3, \\ \limsup_{\nu\uparrow\infty} \tilde{e}(m^{(\nu)}) &\;\le\; e(m'). \end{aligned}$$

This construction will be carried through in three steps, each corresponding to one of the lemmas below. In Lemma 1, an appropriate domain pattern is constructed. In Lemma 2, the line singularities are replaced by Néel walls. In Lemma 3, the point singularities are replaced by Bloch lines.

Lemma 1 . *Given an m' as in Proposition 2 and an $\epsilon>0$, there exists an $\widetilde{m}'\colon\omega'\to\mathbb{R}^2$ with the following four properties*

- $|\widetilde{m}'|^2 \;=\; 1 \quad in\, \omega'$,
- *$\widetilde{m}'$ is smooth apart from points in a finite union of smooth curves,*
- *these curves only intersect in their end points and they do so in a transversal way,*

- $\widetilde{m}'$ *is smooth up to these curves and the normal component does not jump, whereas the magnitude of the jump of the tangential component is bounded from below,*

such that

$$\|(\nabla')^{-1}(\widetilde{m}' - m')\|_{L^2(\mathbb{R}^2)} \;<\; \epsilon \quad \textit{and} \quad e'(\widetilde{m}') \;<\; e'(m') + \epsilon. \tag{107}$$

Lemma 2 . *Given an m' as constructed in Lemma 1 and an $\epsilon > 0$, there exists an $\widetilde{m}' \colon \omega' \to I\!R^2$ with the following properties*

- $|\widetilde{m}'|^2 \;=\; 1 \quad \textit{in}\, \omega'$,
- $\widetilde{m}'$ *is Lipschitz on $\overline{\omega'}$ apart from a finite number of points,*
- *if x_0' is such a point, the blow–up of the derivative of $\widetilde{m}'$ is controlled as follows*
$$|\nabla' \widetilde{m}'(x')| \;\leq\; C\,\frac{1}{|x' - x_0'|},$$

such that

$$\|(\nabla')^{-1}(\widetilde{m}' - m')\|_{L^2(\mathbb{R}^2)} \;<\; \epsilon \quad \textit{and} \quad e'(\widetilde{m}') \;<\; e'(m') + \epsilon.$$

Lemma 3 . *Let m' be as constructed in Lemma 2. Let the sequences $\{t^{(\nu)}\}_{\nu\uparrow\infty}$ and $\{d^{(\nu)}\}_{\nu\uparrow\infty}$ satisfy (13). Then there exists $\{m^{(\nu)} \colon \omega' \to I\!R^3\}_{\nu\uparrow\infty}$ with*

$$|m^{(\nu)}|^2 \;=\; 1 \quad \textit{in}\, \omega',$$

such that

$$\begin{aligned} m^{(\nu)} \quad &\stackrel{w}{\rightharpoonup} \quad \begin{pmatrix} m' \\ 0 \end{pmatrix} \quad \textit{in}\; L^2(\omega')^3, \\ \limsup_{\nu\uparrow\infty} \tilde{e}(m^{(\nu)}) \quad &\leq \quad e'(m'). \end{aligned}$$

7.1 Domain pattern

Proof of Lemma 1. Since this construction is entirely two–dimensional, we drop all primes. We start by arguing that we may assume the following additional regularity properties for the given m

$$\sup|m|^2 \;<\; 1 \quad \text{in } \mathbb{R}^2, \tag{108}$$

$$m \text{ has compact support in } \omega, \tag{109}$$

$$m \text{ is smooth in } \mathbb{R}^2. \tag{110}$$

(108) is easy to achieve: Replace m by $\tilde{m} = (1-\epsilon)\, m$. This affects the energy e', which consists of three homogeneous terms of degree 1 resp. 2, by $O(\epsilon)$:

$$e'(\tilde{m}) \;=\; e'(m) + O(\epsilon). \tag{111}$$

For (109), we choose a diffeomorphism Φ of $\mathbb{R}^2$ with $\Phi(\omega) \subset\subset \omega$ such that Φ is (at least) C^2–close to the identity

$$\|\Phi - \mathrm{id}\|_{C^2(\mathbb{R}^2)} \;=\; O(\epsilon). \tag{112}$$

Replace m by $\tilde{m}$ defined through

$$\mathrm{detD}\Phi \;\; \mathrm{D}\Phi^{-1}\,(\tilde{m}\circ\Phi) \;=\; m, \tag{113}$$

which satisfies (109) by construction. For $\epsilon \ll 1$, (108) is not destroyed. The transformation (113) has the advantage that

$$\mathrm{detD}\Phi \;\; (\nabla\cdot\tilde{m})\circ\Phi \;=\; \nabla\cdot m,$$

so that by (112)

$$\|\nabla^{-\frac{1}{2}}\nabla\cdot\tilde{m}\|_{L^2(\mathbb{R}^2)} \;=\; \|\nabla^{-\frac{1}{2}}\nabla\cdot m\|_{L^2(\mathbb{R}^2)} \;+\; O(\epsilon).$$

Furthermore, (113) and (112) also imply $\|\tilde{m}-m\|_{L^2(\mathbb{R}^2)} = O(\epsilon)$, so that (111) holds.

(110) is standard: Set

$$\varphi_\epsilon(x) \;=\; \frac{1}{\epsilon^2}\,\varphi(\frac{x}{\epsilon}),$$

where φ is a smooth non–negative function of unit integral with support in the unit disk. Consider the convolution

$$\tilde{m} \;=\; \varphi_\epsilon * m,$$

which yields a smooth $\tilde{m}$. This does not affect (108) and, provided $\epsilon \ll 1$, does not destroy (109). Also here, we have (111). This establishes (108), (109) & (110).

We now come to the main part of the construction. The Ansatz for Lemma 1 is

$$\widetilde{m} \;:=\; m + \nabla^{\perp}\psi, \tag{114}$$

where ψ solves the Hamilton–Jacobi equation

$$|\nabla^{\perp}\psi - m|^2 \;=\; 1 \quad \text{in } \omega \tag{115}$$

with homogeneous boundary conditions

$$\psi \;=\; 0 \quad \text{on } \partial\omega. \tag{116}$$

Here $\perp$ denotes the rotation of a two–dimensional vector by 90^o. The Ansatz (114) does not affect the divergence of the magnetization, and (116) ensures that the normal component is not altered at the boundary. Both make sure that the delicate stray–field energy, that is, $\|\nabla^{-\frac{1}{2}}\nabla\cdot m\|_{L^2(\mathbb{R}^2)}$, is unchanged. On the other hand, (115) enforces unit length for $\tilde{m}$.

It is well–known that the boundary value problem (116) for the Hamilton–Jacobi equation (115) does not admit a smooth solution (think for instance of $m = 0$, in which case (115) turns into the eikonal equation). We will construct a "piecewise smooth" ψ, which we extend by zero on all of $\mathbb{R}^2$. By piecewise smooth, we understand the following: There exist pairwise open disjoint sets $\tilde{\omega}_1, \ldots, \tilde{\omega}_K \subset \mathbb{R}^2$ and curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_L \subset \mathbb{R}^2 - \bigcup_k \tilde{\omega}_k$ such that $\mathbb{R}^2 = \bigcup_k \tilde{\omega}_k \,\cup\, \bigcup_\ell \tilde{\gamma}_\ell$ with

- ψ is smooth up to the boundary of each open set,
- the curves are smooth up to their end points.

We also include the following non–degeneracy conditions in our notion of "piecewise smooth":

- The magnitude of the jump of $\nabla\psi$ along each curve is bounded from below,
- the curves only intersect in their end points and do so in a transversal way.

These properties ensure that the $\tilde{m}$ defined in (114) meets the regularity and non–degeneracy requirements listed in Lemma 1. In addition, given our small ϵ, we construct ψ such that it additionally satisfies

$$\sup_{\boldsymbol{R}^2} |\psi| \;\leq\; O(\epsilon), \tag{117}$$

$$\int_{\boldsymbol{R}^2} |\partial_1 \psi|^2 \, dx \;\leq\; O(\epsilon), \tag{118}$$

which obviously yields (107) for $\tilde{m}$ defined as in (114).

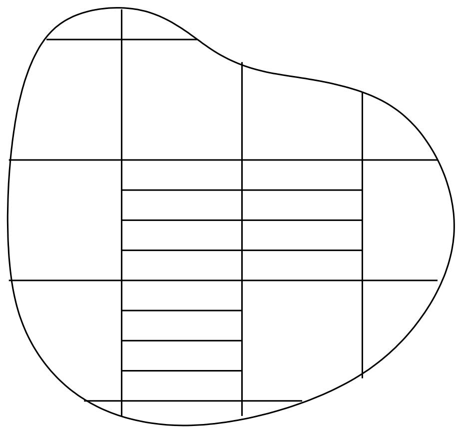

Figure 5

In order to construct this ψ we will subdivide ω into subdomains $\tilde{\omega}$ (see Figure 5, which will be explained below) and use local constructions $\tilde{\psi}$. We need a few preparations. For notational convenience, we introduce the following class $\mathcal{C}$ of smooth curves γ in $I\!R^2$ endowed with a unit normal ν

$$\mathcal{C} \;=\; \{\text{straight lines, } \partial\omega \text{ with inner normal}\}\,.$$

We also introduce a class $\mathcal{D}$ of domains $\tilde{\omega}$ in $I\!R^2$. For $r_0 > 0$ to be fixed later, we say that an open $\tilde{\omega}$ belongs to $\mathcal{D}$ if

$$\partial\tilde{\omega} \;=\; \textstyle\bigcup_{i=1}^N \tilde{\gamma}_i \quad \text{where } \tilde{\gamma}_i \text{ is a closed segment of a } \gamma_i \in \mathcal{C}, \tag{119}$$

$$\tilde{\omega} \;\subset\; \{y + r\,\nu_i(y) \,|\, y \in \gamma_i,\ 0 < r < r_0\} \quad \text{for } i, \ldots, N, \tag{120}$$

$$\{\,(y, \nu_i(y)) \,|\, y \in \gamma_i\,\}\ ,\ i \;=\; 1, \ldots, N, \text{ are pairwise disjoint.} \tag{121}$$

For any such $\tilde{\omega} \in \mathcal{D}$, we will construct a piecewise smooth (in the above sense) function $\tilde{\psi}$ on $\tilde{\omega}$ with

$$|\nabla\tilde{\psi} - m^\perp|^2 \ = \ 1 \qquad \text{a. e. in } \tilde{\omega}, \tag{122}$$

$$\tilde{\psi} \ = \ 0 \ \text{ on } \partial\tilde{\omega}, \qquad \tilde{\psi} \ \geq \ 0 \ \text{ in } \tilde{\omega}. \tag{123}$$

Introduce the Hamiltonian

$$H(x,p) \ := \ |p - m(x)^\perp|^2.$$

Consider the flow $\{\Phi_s = (\Phi_s^1, \Phi_s^2)\}_{s\in\mathbb{R}}$ on the phase space $(x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$ generated by the corresponding Hamiltonian system, i. e.

$$\dot{x} \ = \ H_p(x,p), \quad \dot{p} \ = \ -H_x(x,p).$$

According to (108) we have

$$\inf\left\{ \nu \cdot H_p(x,\nu)\,\nu \,\middle|\, \nu \in \mathbb{R}^2,\ |\nu|^2 = 1 \text{ and } x \in \mathbb{R}^2 \right\} \ > \ 0. \tag{124}$$

Now let a curve $(\gamma,\nu) \in \mathcal{C}$ be given. According to (124), there exist $s_0, r_0 > 0$ such that

$$\left\{ \Phi_s^1(y,\nu(y)) \,\middle|\, y \in \gamma,\ 0 < s < s_0 \right\} \\ \subset \ \left\{ y + r\,\nu(y) \,\middle|\, y \in \gamma,\ 0 < r < r_0 \right\} \ =: \ \omega_\gamma$$

and such that the projection is one–to–one, so that

$$\psi(\Phi_s^1(y,\nu(y))) \ = \ s, \quad y \in \gamma,\ 0 < s < s_0 \tag{125}$$

defines a smooth function ψ on ω_γ. It obviously satisfies

$$\psi \ = \ 0 \ \text{ on } \gamma, \quad \psi \ \geq \ 0 \ \text{ in } \omega_\gamma, \tag{126}$$

and it is well–known and easy to check that ψ also fulfills

$$\nabla\psi(\Phi_s^1(y,\nu(y))) \ = \ \Phi_s^2(y,\nu(y)) \quad \text{for all } y \in \gamma,\ 0 < s < s_0, \tag{127}$$

$$H(x,\nabla\psi(x)) \ = \ 1 \quad \text{for all } x \in \omega_\gamma. \tag{128}$$

Now let $\tilde{\omega}$ be of class $\mathcal{D}$ and $\gamma_1, \ldots, \gamma_N$ like in (119). Like above, we construct a ψ_i for every γ_i. Thanks to (120),

$$\tilde{\psi} \ := \ \min\{\psi_1, \ldots, \psi_N\} \tag{129}$$

defines a Lipschitz function $\tilde\psi$ on all of $\tilde\omega$. By hypothesis (119), (126) yields

$$\tilde\psi \;=\; 0 \;\text{ on } \partial\tilde\omega, \quad \tilde\psi \;\geq\; 0 \;\text{ in } \tilde\omega.$$

Furthermore, (128) turns into

$$H(x,\nabla\tilde\psi(x)) \;=\; 1 \quad \text{for a. e. } \; x\in\tilde\omega.$$

It remains to show that $\tilde\psi$ is piecewise smooth. This amounts to the following statement: For any $i\neq j$ and any $x\in\tilde\omega$, we have

$$\psi_i(x) \;=\; \psi_j(x) \quad\Longrightarrow\quad |\nabla\psi_i(x)-\nabla\psi_j(x)| \;\geq\; \frac{1}{C}.$$

Indeed, set $s:=\psi_i(x)=\psi_j(x)\in(0,s_0)$. By definition (125) of ψ_i,ψ_j, there exist $y_i\in\gamma_i, y_j\in\gamma_j$ such that

$$\Phi_s^1(y_i,\nu_i(y_i)) \;=\; x \;=\; \Phi_s^1(y_j,\nu_j(y_j)). \tag{130}$$

By hypothesis (121), the closed sets in phase space

$$\left\{\,(y,\nu_i(y))\,\Big|\,y\in\gamma_i\,\right\},\;\left\{\,(y,\nu_j(y))\,\Big|\,y\in\gamma_j\,\right\}\quad\text{have positive distance.}$$

Therefore, also

$$\left\{\,\Phi_s(y,\nu_i(y))\,\Big|\,y\in\gamma_i\,\right\},\;\left\{\,\Phi_s(y,\nu_j(y))\,\Big|\,y\in\gamma_j\,\right\}\quad\text{have positive distance}$$

uniformly in $0<s<s_0$. In view of (130), this implies

$$|\Phi_s^2(y_i,\nu_i(y_i))-\Phi_s^2(y_j,\nu_j(y_j))| \;\geq\; \frac{1}{C}.$$

According to (130) and (127), this translates into

$$|\nabla\psi_i(x)-\nabla\psi_j(x)| \;\geq\; \frac{1}{C}.$$

This finishes the construction of a piecewise smooth $\tilde\psi$ with (122) & (123).

In order to treat the anisotropy term, we need an additional information on our above construction: In case of

$$\tilde\omega \;=\; (x_1^-,x_1^+)\times(x_2^-,x_2^+) \quad\text{with}\quad \triangle x_2 \;:=\; x_2^+-x_2^- \;\ll\; x_1^+-x_1^-,$$

we want to conclude

$$\int_{\tilde\omega} |\partial_1 \tilde\psi|^2\,dx \;=\; O((\triangle x_2)^2). \tag{131}$$

Indeed, consider the curves $\gamma_1^\pm = I\!R \times \{x_2^\pm\}$, $\gamma_2^\pm = \{x_1^\pm\} \times I\!R$. Let $\psi_1^\pm, \psi_2^\pm$ be constructed as above. Since $\nu_1^\pm = (0,\mp 1)$, it follows from (127)

$$\nabla \psi_1^\pm \;=\; (0,\mp 1) + O(\triangle x_2). \tag{132}$$

Furthermore, we have on $\tilde\omega$

$$\psi_1^\pm \;=\; O(\triangle x_2),\;\; \psi_2^- \;=\; O(x_1 - x_1^-),\;\; \psi_2^+ \;=\; O(x_1^+ - x_1)$$

and thus in view of (129)

$$\tilde\psi \;=\; \min\{\psi_1^+, \psi_1^-\} \quad \text{for } x_1 - x_1^-, x_1^+ - x_1 \;\gg\; \triangle x_2. \tag{133}$$

Together with the L^∞–bound on $\nabla\tilde\psi$, (132) and (133) imply (131).

We now specify how to subdivide ω into $\tilde\omega \in \mathcal{D}$. Consider the cubic grid in direction of the coordinate axis x_1, x_2 of grid size $\epsilon \ll r_0$. By possibly shifting the grid a bit, we can always achieve that the horizontal and vertical lines forming the grid intersect $\partial\omega$ transversally. This grid cuts many domains $\tilde\omega$ out of ω. By construction, each $\tilde\omega$ satisfies (119). Our assumption $\epsilon \ll r_0$ also ensures (120). The above mentioned transversality just means (121). Hence indeed $\tilde\omega \in \mathcal{D}$. Those $\tilde\omega$ which lie entirely in ω will be subdivided further by horizontal lines of distance ϵ^2, as indicated in Figure 5. Also the resulting $\tilde\omega$ are in the class $\mathcal{D}$. We can concatenate the above constructed $\tilde\psi$ on each $\tilde\omega$ to obtain a piecewise smooth ψ on ω with (115). Indeed, the sign $\tilde\psi \geq 0$ together with $|m|^2 < 1$ ensures that the various boundaries $\partial\tilde\omega$ become non–degenerate discontinuity lines. For the same reason the trivial extension of ψ on $I\!R^2$ has a non–degenerate discontinuity line $\partial\omega$ and satisfies (116).

For (117), we observe that since the diameter of all $\tilde\omega$ is of order ϵ and the Lipschitz constant of all $\tilde\psi$ is of order 1 (as a consequence of (122)), the boundary condition (123) implies that all $\tilde\psi$ are of order ϵ. For (118) finally we notice that all $\tilde\omega$ which are entirely in ω are rectangular with x_1–width ϵ and x_2–width ϵ^2. Hence we conclude from (131)

$$\int_{\tilde\omega} |\partial_1 \tilde\psi|^2\,dx \;=\; O(\epsilon^2) \;=\; O(\epsilon) \times (\text{area of } \tilde\omega).$$

On the other hand, the total area of those $\tilde\omega$ which cut into the boundary of ω is of order ϵ. Together with the fact that $\nabla\psi$ is of order 1, we obtain (118).

7.2 Néel walls

PROOF OF LEMMA 2. Since all vector fields in this construction are in–plane, we drop the primes on the vector fields. However, we will consider functions which depend on $x \in I\!R^3$, so that it is advisable to keep the primes on x' and ∇'. Let m be as constructed in Lemma 1. We consider a single discontinuity curve γ which connects two discontinuity points. We parameterize γ by arc length with parameter $-S_0 \leq s \leq S_0$. The two geometric quantities

- the directed distance t to γ,
- the parameter s of the closest point on γ,

define smooth coordinates $(s,t) \in [-S_0, S_0] \times [-T_0, T_0]$ in a tubular neighborhood of γ for $T_0 \ll 1$. We denote by τ and ν the corresponding tangential and normal vector fields. We will now construct an $\widetilde{m}$ with

$$\left.\begin{array}{llll} \widetilde{m} & = \; m & \text{on} & \{|s| < S_0, |t| = T(s)\}, \\ \widetilde{m} & \text{is smooth} & \text{in} & \{|s| < S_0, |t| \leq T(s)\}, \\ |\nabla'\widetilde{m}| & \leq \; O(\frac{1}{|s|-S_0}) & \text{in} & \{|s| < S_0, |t| \leq T(s)\}, \end{array}\right\} \tag{134}$$

where the width T of the neighborhood $\{|s| < S_0, |t| < T(s)\}$ of γ linearly approaches zero toward the end points of γ:

$$T(s) \; := \; T_0 \left(1 - (\frac{s}{S_0})^2\right),$$

see Figure 6, which indicates the set $\{|s| < S_0, |t| < T(s)\}$ through shading. The construction will depend on a parameter $0 < \delta \ll 1$ and will be carried out such that

$$\left.\begin{array}{rcl} \|(\nabla')^{-\frac{1}{2}}\,\nabla' \cdot (\widetilde{m} - m)\|_{L^2(\mathbf{R}^2)} & \leq & O(\log^{-\frac{1}{2}} \frac{1}{\delta}), \\ \|\widetilde{m} - m\|_{L^2(\mathbf{R}^2)} & \leq & O(\log^{-\frac{1}{2}} \frac{1}{\delta}), \end{array}\right\} \tag{135}$$

where we think of $\widetilde{m} - m$ as being trivially extended on $I\!R^2$. This construction is to be carried out for every curve. Since by assumption, the discontinuity curves meet transversally (themselves and the boundary), the constructions can be concatenated, to yield the desired global $\widetilde{m}$.

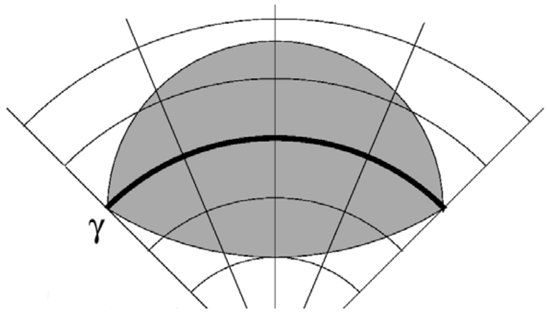

Figure 6

Since by assumption, the magnitude of the jump of the tangential component $m\cdot\tau$ is bounded from below, the normal component — which does not jump — must satisfy

$$|m\cdot\nu| \;<\; 1 \quad \text{uniformly along } \gamma.$$

This allows us to make the following Ansatz

$$\begin{array}{rcl} \widetilde{m} & := & (\eta + (1-\eta)\, m\cdot\nu)\,\nu \;\; + \;\; \lambda\, m\cdot\tau\,\tau \quad \text{hence} \\ \widetilde{m} - m & = & \eta\,(1 - m\cdot\nu)\,\nu \;\; + \;\; (\lambda - 1)\, m\cdot\tau\,\tau, \end{array}$$

where $\eta \in [0,1]$ and where λ is defined as to meet the requirement $|\widetilde{m}|^2 = 1$, that is,

$$\lambda \;:=\; \sqrt{\tfrac{1-(\eta+(1-\eta)\,m\cdot\nu)^2}{1-(m\cdot\nu)^2}} \;=\; \sqrt{(1-\eta)\,(1+\eta\,\tfrac{1-m\cdot\nu}{1+m\cdot\nu})}.$$

Before making an Ansatz for η, we derive the requirements on η. We observe that expressions in λ can be estimated against η with help of

$$\left.\begin{array}{rcl} |\lambda - 1| & \le & C\,\eta, \\ |\partial_\alpha\lambda| & \le & C\,(|\partial_\alpha\eta| + |\partial_\alpha\sqrt{1-\eta}|), \end{array}\right\} \tag{136}$$

where $\alpha = s, t$ and C is a generic constant only depending on m. In order to meet the regularity requirements (134), η should satisfy

$$\left.\begin{array}{rcll} \eta & = 0 & \text{on} & \{|s| < S_0, |t| = T(s)\} \\ \eta & = 1 & \text{on} & \{|s| < S_0, t = 0\} \\ |\nabla'\eta|, |\nabla'\sqrt{1-\eta}| & \le & O(\frac{1}{|s|-S_0}) \quad \text{in} & \{|s| < S_0, |t| \le T(s)\} \end{array}\right\}. \tag{137}$$

Using $\nabla'\cdot\tau = 0$, we observe that

$$\nabla'\cdot(\widetilde{m} - m) \;=\; \nabla'\cdot(\eta\,(1 - m\cdot\nu)\,\nu) + \partial_s\lambda\; m\cdot\tau + (\lambda - 1)\,\partial_s(m\cdot\tau). \tag{138}$$

We will use the embedding and interpolation estimates

$$\|(\nabla')^{-\frac{1}{2}}\partial_\alpha\phi\|_{L^2(\boldsymbol{R}^2)} \leq C\,\|(\nabla')^{\frac{1}{2}}\phi\|_{L^2(\boldsymbol{R}^2)} \quad \text{for } \alpha=1,2, \tag{139}$$

$$\|\phi\|_{L^4(\boldsymbol{R}^2)} \leq C\,\|(\nabla')^{\frac{1}{2}}\phi\|_{L^2(\boldsymbol{R}^2)} \quad \text{and thus} \tag{140}$$

$$\|(\nabla')^{-\frac{1}{2}}\phi\|_{L^2(\boldsymbol{R}^2)} \leq C\,\|\phi\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}, \tag{141}$$

$$\|(\nabla')^{\frac{1}{2}}(\alpha\,\phi)\|_{L^2(\boldsymbol{R}^2)} \leq C\,\|\alpha\|_{L^3(\boldsymbol{R}^2)}^{\frac{1}{3}}\,\|\nabla'\alpha\|_{L^3(\boldsymbol{R}^2)}^{\frac{2}{3}}\,\|(\nabla')^{\frac{1}{2}}\phi\|_{L^2(\boldsymbol{R}^2)}. \tag{142}$$

(139) is obvious from the Fourier representation, (141) (and its dual) are standard; we gave the argument for (142)=(75) at the end of the proof of Proposition 1. In particular, we have the following obvious estimates for our η with compact support

$$\|\eta\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} \leq C\,\|\eta\|_{L^2(\boldsymbol{R}^2)} \leq C\,\|\eta\|_{L^4(\boldsymbol{R}^2)} \overset{(140)}{\leq} C\,\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)}. \tag{143}$$

From these, we now obtain

$$\begin{aligned}
&\|(\nabla')^{-\frac{1}{2}}\nabla'\cdot(\widetilde{m}-m)\|_{L^2(\boldsymbol{R}^2)}\\
&\overset{(138)}{\leq} \|(\nabla')^{-\frac{1}{2}}\nabla'\cdot(\eta\,(1-m\cdot\nu)\,\nu)\|_{L^2(\boldsymbol{R}^2)}\\
&\qquad +\|(\nabla')^{-\frac{1}{2}}(\partial_s\lambda\; m\cdot\tau)\|_{L^2(\boldsymbol{R}^2)} + \|(\nabla')^{-\frac{1}{2}}((\lambda-1)\,\partial_s(m\cdot\tau))\|_{L^2(\boldsymbol{R}^2)}\\
&\overset{(139,141)}{\leq} C\,\Big(\|(\nabla')^{\frac{1}{2}}(\eta\,(1-m\cdot\nu)\,\nu)\|_{L^2(\boldsymbol{R}^2)}\\
&\qquad +\|\partial_s\lambda\; m\cdot\tau\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} + \|(\lambda-1)\,\partial_s(m\cdot\tau)\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}\Big)\\
&\overset{(142)}{\leq} C\,\Big(\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)} + \|\partial_s\lambda\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} + \|\lambda-1\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}\Big)\\
&\overset{(136)}{\leq} C\,\Big(\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)} + \|\partial_s\eta\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} + \|\partial_s\sqrt{1-\eta}\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} + \|\eta\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}\Big)\\
&\overset{(143)}{\leq} C\,\Big(\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)} + \|\partial_s\eta\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} + \|\partial_s\sqrt{1-\eta}\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}\Big).
\end{aligned}$$

In the third inequality, we have that $(1-m\cdot\nu)\,\nu$ does not jump across the discontinuity curve, so that it can be extended on $I\!R^2$ as a smooth function α. In the third inequality, we have also used that $m\cdot\tau$ is smooth in tangential direction. Likewise, we obtain

$$\begin{aligned}
\|\widetilde{m}-m\|_{L^2(\boldsymbol{R}^2)} &\leq C\,\Big(\|\eta\|_{L^2(\boldsymbol{R}^2)} + \|\lambda-1\|_{L^2(\boldsymbol{R}^2)}\Big)\\
&\overset{(136)}{\leq} C\,\|\eta\|_{L^2(\boldsymbol{R}^2)} \overset{(143)}{\leq} C\,\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)}.
\end{aligned}$$

Hence we need

$$\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)} \;\leq\; O(\log^{-\frac{1}{2}}\tfrac{1}{\delta}) \tag{144}$$

$$\|\partial_s\sqrt{1-\eta}\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} \;\leq\; O(\log^{-\frac{1}{2}}\tfrac{1}{\delta}) \tag{145}$$

$$\|\partial_s\eta\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)} \;\leq\; O(\log^{-\frac{1}{2}}\tfrac{1}{\delta}) \tag{146}$$

in order to meet the smallness requirements (135).

Our Ansatz for η is

$$\eta \;=\; \left\{\begin{array}{cc} \dfrac{\log\left((\frac{t}{T(s)})^2+\delta^2\right)}{\log(\delta^2)} & \text{for } |t|\leq T(s)\sqrt{1-\delta^2} \;\text{ and }\; |s|<S_0 \\ 0 & \text{else} \end{array}\right\},$$

which obviously satisfies (137). We now argue that it also satisfies (144), (145) & (146). For the first estimate, i. e. (144), we write η as the trace of a suitable function $\overline{\eta}$ in $I\!R^3$, since then we have

$$\|(\nabla')^{\frac{1}{2}}\eta\|_{L^2(\boldsymbol{R}^2)} \;\leq\; \|\nabla\overline{\eta}\|_{L^2(\boldsymbol{R}^3)}.$$

In order to define $\overline{\eta}$, we introduce curvilinear cylindrical coordinates (s,r,θ) around γ, which we view as a curve in $I\!R^3$. We set

$$\overline{\eta} \;=\; \left\{\begin{array}{cc} \dfrac{\log\left((\frac{r}{T(s)})^2+\delta^2\right)}{\log(\delta^2)} & \text{for } 0\leq r\leq T(s)\sqrt{1-\delta^2} \text{ and } |s|<S_0 \\ 0 & \text{else} \end{array}\right\}.$$

For the squared gradient of this function, we obtain for all $0\leq r\leq T(s)$ and $|s|<S_0$ the estimate

$$|\nabla\overline{\eta}|^2 \;\leq\; C\,\left((\frac{\partial\overline{\eta}}{\partial r})^2+(\frac{\partial\overline{\eta}}{\partial s})^2\right) \;\leq\; \frac{C}{\log^2\frac{1}{\delta}}\,\left(\frac{r^2}{(r^2+(\delta\,T(s))^2)^2}+(\frac{\frac{dT}{ds}(s)}{T(s)})^2\right).$$

Hence

$$\begin{aligned}\int_{\boldsymbol{R}^3}|\nabla\overline{\eta}|^2\,dx \;&\leq\; C\int_0^{S_0}\int_0^{T(s)}|\nabla\overline{\eta}|^2\,r\,dr\,ds\\ &\leq\; C\,\log^{-2}\tfrac{1}{\delta}\,\left(S_0\int_0^{\frac{1}{\delta}}\frac{\hat{r}^3}{(\hat{r}^2+1)^2}\,d\hat{r}+\int_0^{S_0}(\frac{dT}{ds}(s))^2\,ds\right)\\ &\leq\; C\,\log^{-1}\tfrac{1}{\delta}.\end{aligned}$$

For the second estimate (145), we observe for $|t| \le T(s)$ and $|s| < S_0$ we have

$$\begin{aligned}
\sqrt{1-\eta} &= \sqrt{\frac{\log\left(1+(\frac{t}{\delta\,T(s)})^2\right)}{2\,\log\frac{1}{\delta}}}, \\
\partial_s\sqrt{1-\eta} &= -\frac{1}{\sqrt{2\,\log\frac{1}{\delta}}}\,\frac{1}{\sqrt{\log\left(1+(\frac{t}{\delta\,T(s)})^2\right)}}\,\frac{(\frac{t}{\delta\,T(s)})^2}{1+(\frac{t}{\delta\,T(s)})^2}\,\frac{\frac{dT}{ds}(s)}{T(s)}, \\
\left|\partial_s\sqrt{1-\eta}\right| &\le C\,\log^{-\frac{1}{2}}\frac{1}{\delta}\left|\frac{\frac{dT}{ds}(s)}{T(s)}\right|,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{\boldsymbol{R}^2}\left|\partial_s\sqrt{1-\eta}\right|^{\frac{4}{3}}\,dx' &\le C\int_0^{S_0}\int_0^{T(s)}|\partial_s\sqrt{1-\eta}|^{\frac{4}{3}}\,dt\,ds \\
&\le C\,\log^{-\frac{2}{3}}\frac{1}{\delta}\int_0^{S_0}\frac{|\frac{dT}{ds}(s)|^{\frac{4}{3}}}{T(s)^{\frac{1}{3}}}\,ds \\
&\le C\,\log^{-\frac{2}{3}}\frac{1}{\delta}.
\end{aligned}$$

The last estimate (146) (in a stronger version with $O(\log^{-1}\frac{1}{\delta})$) follows along the same lines:

$$|\partial_s\eta| \le C\,\log^{-1}\frac{1}{\delta}\left|\frac{\frac{dT}{ds}(s)}{T(s)}\right|, \quad \int_{\boldsymbol{R}^2}|\partial_s\eta|^{\frac{4}{3}}\,dx' \le C\,\log^{-\frac{4}{3}}\frac{1}{\delta}.$$

7.3 Bloch lines

Proof of Lemma 3. Let m' be as constructed in Lemma 3. We consider a single discontinuity point x_0'. x_0' might be on the boundary of ω', but there is no other discontinuity point within a distance $R > 0$ of x_0'. According to Lemma 3, the blow–up of the gradient of m' within the ball $B_R'(x_0')$ of radius R is controlled as follows

$$|\nabla' m'(x')| \le \frac{C}{|x'-x_0'|}. \tag{147}$$

Our Ansatz for $m^{(\nu)}$ in $B_R'(x_0')$ is

$$(m^{(\nu)})'(x') := \eta(\tfrac{|x'-x_0'|}{t^{(\nu)}})\,m'(x') \quad \text{and} \quad m_3^{(\nu)}(x') := \sqrt{1-\eta^2(\tfrac{|x'-x_0'|}{t^{(\nu)}})},$$

where we have fixed a smooth function $\eta = \eta(r)$ of $r \geq 0$ with

$$\eta(r) \left\{ \begin{array}{ccl} = & 0 & \text{for } r \leq 1 \\ = & 1 & \text{for } r \geq 2 \\ \in & [0,1] & \text{for all } r \end{array} \right\}.$$

Since

$$\begin{aligned} \nabla'(m^{(\nu)})'(x') &= \frac{1}{t^{(\nu)}} \frac{d\eta}{dr}(\tfrac{|x'-x_0'|}{t^{(\nu)}}) \frac{x'-x_0'}{|x'-x_0'|} \times m'(x') + \eta(\tfrac{|x'-x_0'|}{t^{(\nu)}}) \nabla' m'(x'), \\ \nabla' m_3^{(\nu)}(x') &= -\frac{1}{t^{(\nu)}} \frac{d\eta}{dr}(\tfrac{|x'-x_0'|}{t^{(\nu)}}) \frac{\eta}{\sqrt{1-\eta^2(\frac{|x'-x_0'|}{t^{(\nu)}})}} \frac{x'-x_0'}{|x'-x_0'|}, \end{aligned} \qquad (148)$$

we obtain with help of (147)

$$|\nabla' m^{(\nu)}(x')| \;\leq\; \frac{C}{|x'-x_0'| + t^{(\nu)}}$$

and thus for $t^{(\nu)} \ll 1$

$$\int_{B'_R(x_0')} |\nabla' m^{(\nu)}|^2 \, dx' \;\leq\; C \log \frac{1}{t^{(\nu)}}.$$

On the other hand

$$\int_{B'_R(x_0')} (m_3^{(\nu)})^2 \, dx' \;\leq\; C\,(t^{(\nu)})^2.$$

Hence in view of (13), the construction satisfies as desired

$$\lim_{d\downarrow 0} \left(\frac{(d^{(\nu)})^2}{t^{(\nu)}} \int_{B'_R(x_0')} |\nabla' m^{(\nu)}|^2 \, dx' + \frac{1}{t^{(\nu)}} \int_{B'_R(x_0')} (m_3^{(\nu)})^2 \, dx' \right) \;=\; 0.$$

It remains to show that

$$\left. \begin{array}{rcl} \lim_{\nu\uparrow 0} \|(\nabla')^{-\frac{1}{2}} \nabla' \cdot ((m^{(\nu)})' - m')\|_{L^2(\boldsymbol{R}^2)} &=& 0, \\ \lim_{\nu\uparrow 0} \|m^{(\nu)} - m'\|_{L^2(\boldsymbol{R}^2)} &=& 0. \end{array} \right\} \qquad (149)$$

We observe that

$$\begin{aligned} (m^{(\nu)} - m')(x') &= \begin{pmatrix} (\eta(\frac{|x'-x_0'|}{t^{(\nu)}}) - 1)\, m'(x') \\ \sqrt{1-\eta^2} \end{pmatrix}, \\ \nabla' \cdot ((m^{(\nu)})' - m')(x') &\overset{(148)}{=} \frac{1}{t^{(\nu)}} \frac{d\eta}{dr}(\frac{|x'-x_0'|}{t^{(\nu)}}) \frac{x'-x_0'}{|x'-x_0'|} \cdot m'(x') \\ &\quad + (\eta(\frac{|x'-x_0'|}{t^{(\nu)}}) - 1) \, \nabla' \cdot m'(x'), \end{aligned}$$

so that according to (147)

$$
\begin{aligned}
|(m^{(\nu)} - m')(x')| &\leq \left\{ \begin{array}{ll} 1 & \text{for } |x' - x'_0| \leq 2\,t^{(\nu)} \\ 0 & \text{else} \end{array} \right\}, \\
|\nabla' \cdot ((m^{(\nu)})' - m')(x')| &\leq \tfrac{C}{t^{(\nu)}} \left\{ \begin{array}{ll} 1 & \text{for } |x' - x'_0| \leq 2\,t^{(\nu)} \\ 0 & \text{else} \end{array} \right\}.
\end{aligned}
$$

This implies (149) in view of the estimate (141)

$$
\|(\nabla')^{-\frac{1}{2}} \nabla' \cdot ((m^{(\nu)})' - m')\|_{L^2(\boldsymbol{R}^2)} \;\leq\; C\,\|\nabla' \cdot ((m^{(\nu)})' - m')\|_{L^{\frac{4}{3}}(\boldsymbol{R}^2)}.
$$

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