Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 84 2001

The Speed of Travelling Waves for Convective Reaction-Diffusion Equations

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Abstract

Explicit upper and lower estimates for the speed of fronts for reaction- diffusion equation with a convective shear flow are derived. The enhancement of the speed is proportional to the amplitude of the convection and proportional to the square root of the amplitude of the reaction. Several asymptotic regimes will be considered, i.e. large Peclet numbers, rapidly oscillating convection, and small diffusivity.

1 Introduction

We consider the following reaction-diffusion equation with a shear flow convection in an infinite cylinder $(x,y) \in I\!R \times \Omega$:

$$\partial_t u(t,x,y) = d\Delta_{x,y} u(t,x,y) + b(y)\partial_x u(t,x,y) + f(u(t,x,y)). \tag{1}$$

For the cross section Ω we either assume $\Omega = (0,L)^n$ and periodic boundary conditions or Ω is a bounded domain in $I\!R^n$ with Neumann boundary conditions. For the nonlinearity $f(u)$ we always require $f \in C^1([0,1])$ and $f(0) = f(1) = 0$. Tow cases will be considered:

A: KPP-type

$$\dot{f}(0) > 0, \quad f(u) > 0, \quad \text{for } 0 < u < 1.$$

B: Combustion type

$$\begin{aligned} f(0) = 0 \quad &\text{for} \quad 0 \le u \le \theta, \\ f(u) > 0 \quad &\text{for} \quad \theta < u < 1. \end{aligned}$$

Let $F(u)$ denote the primitive of $f(u)$. The shear flow is given by a Lipschitz continuous function $b(y)$ on Ω with mean value zero. The amplitude of the flow is called the Peclet number.
It is shown in [8] that the large time behavior of (1) for a large class of initial data is described by travelling waves. In particular the asymptotic speed of propagation is given by the speed c of a travelling wave. With the moving coordinate $\xi = x + ct$ a travelling wave $u(\xi, y)$ satisfies

$$c\partial_\xi u(\xi,y) = d\Delta_{\xi,y} u(\xi,y) + b(y)\partial_\xi u(\xi,y) + f(u(\xi,y)) \tag{2}$$

$$u(\infty, y) = 0, \quad u(\infty, y) = 1.$$

In [2], [3] the existence of travelling waves is shown for $c \ge c_0$ in case **A** and for a unique $c = c_0$ in case **B**. It is important to know how the speed depends on the diffusivity d, the convection $b(y)$ and the reaction $f(u)$. Physically the convection results in an enlargement of the effective reaction zone. Therefor an increase of the propagation speed is expected. Formal arguments in [1] indicate a linear growth of the front speed with the amplitude of the shear flow. For shear flows a lower bound of the speed of propagation growing linear with the Peclet number has been obtained in [4], [7]. Here we will derive a simplified proof of this result with an explicit and easy computable lower bound. Furthermore explicit upper bounds are derived, which have the same scaling with respect to the diffusivity, the convection and the reaction.
The general estimates give bounds for the following asymptotic regimes:
strong convection (large Peclet number),

strong convection and weak reaction,
rapidly oscillation convection (homogenization),
small diffusivity.
The case of large Peclet numbers is the most interesting case in applications and is close to the regime of turbulent combustion.

2 Lower bounds for the speed

An explicit lower bound for c_0 depending linearly on the amplitude of b will be derived. We will make use of the solution with zero mean value of

$$\Delta_y \chi = -b(y) \quad in \quad \Omega \tag{3}$$

with periodic or Neumann boundary conditions. Observe that if Ω is a cube then this would be the cell problem used in homogenization.

Theorem 1 *The speed c of any travelling wave solution of (1) is bounded from below by*

$$c \geq \max\left\{ \int_0^1 \sqrt{2d\, f(s)}\, ds, \quad \frac{|\nabla_y \chi|_2^2\, F(1)}{|\Omega|} \left(|\chi|_\infty \left(\frac{1}{2} |\dot{f}|_\infty + |f|_\infty \right) + \sqrt{\frac{d}{2}} |\nabla_y \chi|_\infty |f|_\infty^{1/2} \right)^{-1} \right\}. \tag{4}$$

Proof: It is known [3] that the travelling waves above approach its values at $\pm\infty$ exponentially and all derivatives of u tend to zero exponentially. This means, that all integrals below exist. Furthermore $\partial_\xi u > 0$ and hence $0 < u < 1$ holds.
Integrate (2) over $\mathbf{R} \times \Omega$:

$$c|\Omega| = \int_{I\!R \times \Omega} f(u)\, d\xi\, dy, \tag{5}$$

since $b(y)$ has mean value zero. Multiply (2) by u and integrate:

$$c\frac{|\Omega|}{2} = -d \int_{I\!R \times \Omega} |\nabla u|^2\, d\xi\, dy + \int_{I\!R \times \Omega} u f(u)\, d\xi\, dy$$

$$\leq -d \int_{I\!R \times \Omega} |\nabla u|^2\, d\xi\, dy + c|\Omega|,$$

since $u < 1$ and $f(u) \geq 0$. This implies

$$d \int_{I\!R \times \Omega} |\nabla u|^2\, d\xi\, dy \leq c\frac{|\Omega|}{2}. \tag{6}$$

Since $u(x, y)$ is monotone increasing we obtain from (5) and (6)

$$\int_0^1 \sqrt{2d\, f(s)}\, ds = \frac{1}{|\Omega|} \int_{I\!R \times \Omega} \sqrt{2d\, f(u)} \partial_x u\, dx\, dy$$

$$\leq \frac{1}{|\Omega|} \left(2d \int_{I\!R \times \Omega} f(u)\, dx\, dy \int_{I\!R \times \Omega} |\partial_x u|^2\, dx\, dy \right)^{1/2} \leq c.$$

This is the first inequality in (4). In order to derive a lower bound, which depends on the convection the equation (2) has to be tested by some function, s.t. the resulting integral over the convective term gives a positive contribution and the other integrals are bounded by a multiple of c. This is achieved by the test function $\chi(y)f(u(\xi,y))$ where $\chi(y)$ is defined in (3). We obtain

$$\int\limits_{I\!R\times\Omega} \left(-d\,|\nabla u|^2\chi\dot f(u)\;d\xi\;dy - d\;\nabla_y\chi\nabla_y u f(u) + f(u)^2\chi\right)\;d\xi\;dy + F(1)\int\limits_{\Omega} b\;\chi\;dy$$

$$= c\;F(1)\int\limits_{\Omega}\chi\;dy = 0$$

or using the definition of χ in (3)

$$F(1)\int_{\Omega}|\nabla_y\chi|^2\;dy = \int\limits_{I\!R\times\Omega}\left(d\,|\nabla u|^2\chi\dot f(u) + d\;\nabla_y\chi\nabla_y u f(u) - f(u)^2\chi\right)\;d\xi\;dy$$

$$\leq d\;|\chi\dot f|_\infty\int\limits_{I\!R\times\Omega}|\nabla u|^2\;d\xi\;dy + d\;|\nabla_y\chi|_\infty\left(\int\limits_{I\!R\times\Omega}|\nabla_y u|^2\;d\xi\;dy\int\limits_{I\!R\times\Omega}f(u)^2\;d\xi\;dy\right)^{1/2}$$

$$+|\chi|_\infty\int\limits_{I\!R\times\Omega}f(u)^2\;d\xi\;dy$$

$$\leq \frac{c|\Omega|}{2}|\chi|_\infty|\dot f|_\infty + \frac{c\sqrt{d}|\Omega|}{\sqrt{2}}|\nabla_y\chi|_\infty|f|_\infty^{1/2} + c|\Omega||\chi|_\infty|f|_\infty,$$

where (5) and (6) have been used. Hence we get the result

$$c \geq \frac{|\nabla_y\chi|_2^2\;F(1)}{|\Omega|}\left(\frac{1}{2}|\chi|_\infty|\dot f|_\infty + \sqrt{\frac{d}{2}}|\nabla_y\chi|_\infty|f|_\infty^{1/2} + |\chi|_\infty|f|_\infty\right)^{-1}.$$

This completes the proof.

3 Upper bounds for the speed

We will derive upper estimates for the (minimal) speed showing the same asymptotic scaling with respect to the diffusivity, the flow field and the reaction term as the lower bound.
We will need the one dimensional front w with (minimal) speed γ_0:

$$\gamma_0 w' = dw'' + f(w). \tag{7}$$

Theorem 2 *The (minimal) speed is estimated by*

$$c_0 \leq \min\left\{2(d\;|\dot f|_\infty)^{1/2} + \sup_{y\in\Omega} b(y),\; 2\left((d + \frac{1}{d}|\nabla_y\chi|_\infty^2)\;|\dot f|_\infty\right)^{1/2}\right\}, \tag{8}$$

where $\chi(y)$ is defined in (3).

Proof: Upper bounds for c_0 will be derived from the variational characterization of c_0 given in [5], [6]:

$$c_0 = \inf_{v\in E} \sup_{\xi,y} \left(\frac{d\Delta_{\xi,y} v + f(v)}{\partial_\xi v} + b \right) \tag{9}$$

with $E = \{v \in C^2(I\!R \times \Omega) \mid \partial_\xi v > 0,\ v(-\infty,y) = 0,\ v(\infty,y) = 1\}$. Also v satisfies periodic or Neumann boundary conditions. Choosing for $v(\xi,y)$ the one dimensional front $w(\xi)$ from (7) gives

$$c_0 \le \gamma_0 + \sup_{y\in\Omega} b(y). \tag{10}$$

The bound for γ_0 follows from the second estimate in (8) with $\chi = 0$.
For the second estimate it is convenient to perform a change of coordinates in (9):

$$\eta := \alpha\xi + \beta\chi(y),$$

where $\chi(y)$ is defined in (5). The factors α, β will be chosen below. The variational formula (9) transforms to

$$c_0 = \inf_{v\in E} \sup_{\eta,y} \left(\frac{d(\alpha^2 + \beta^2|\nabla_y\chi|^2)\partial_{\eta\eta} v + 2d\beta\nabla_y\chi\nabla_y\partial_\eta v + d\Delta_y v + (d\beta\Delta_y\chi + \alpha b)\partial_\eta v + f(v)}{\alpha\partial_\eta v} \right).$$

With $\beta = \alpha/d$ the definition of χ in (3) implies

$$c_0 = \inf_{v\in E} \sup_{\eta,y} \left(\frac{\alpha^2(d + |\nabla_y\chi|^2/d)\partial_{\eta\eta} v 2\alpha\nabla_y\chi\nabla_y\partial_\eta v + d\Delta_y v + f(v)}{\alpha\partial_\eta v} \right). \tag{11}$$

Restricting $v \in E$ to functions with $\nabla_y v = 0$ we obtain from (11)

$$c_0 \le \alpha(d + \frac{1}{d}|\nabla_y\chi|^2_\infty) \left| \frac{v''}{v'} \right|_\infty + \frac{1}{\alpha} \left| \frac{f(v)}{v'} \right|_\infty .$$

Choosing the optimal α gives

$$c_0 \le 2 \left((d + \frac{1}{d}|\nabla_y\chi|^2_\infty) \left| \frac{v''}{v'} \right|_\infty \left| \frac{f(v)}{v'} \right|_\infty \right)^{1/2} . \tag{12}$$

For $\tau > 0$ let $v(\eta)$ be the solution on $I\!R$ of

$$v' = f(v) + \tau v(1 - v) \tag{13}$$

with $v(-\infty) = 0$, $v(\infty) = 1$. The perturbation of f is introduced in order to have the right hand side positive in the combustion case. With $v'' = (\dot f(v) + \tau(1 - 2v))v'$ we get from (12)

$$c_0 \le 2 \left((d + \frac{1}{d}|\nabla_y\chi|^2_\infty) \left| \dot f(v) + \tau v(1 - v) \right|_\infty \left| \frac{f(v)}{f(v) + \tau v(1 - v)} \right|_\infty \right)^{1/2} .$$

Letting $\tau \to 0$ we finally arrive at

$$c_0 \le 2(d + \frac{1}{d}|\nabla_y\chi|^2_\infty)^{1/2} |\dot f|^{1/2}_\infty .$$

Remarks:
If we assume $\sup_{0<u<1} \dot f(u) = \dot f(0)$, e.g. $f(u) = u(1 - u)$, then the speed is estimated by

$$c_0 \le 2(d + \frac{1}{d}|\nabla_y\chi|^2_\infty)^{1/2} (\dot f(0))^{1/2} .$$

In this case the (minimal) speed of the one dimensional front is $2\sqrt{d\dot{f}(0)}$. Hence the estimate is exact for $\chi(y) = 0$, i.e. $b(y) = 0$.

The proof above can be modified to cover also bistable nonlinearities, i.e. f changes sign. In (13) the right hand side has to be replaced by any KPP nonlinearity $g(v)$. This gives

$$c_0 \leq 2(d + \frac{1}{d}|\nabla_y \chi|_\infty^2)^{1/2} |\dot{g}|_\infty^{1/2} |f/g|_\infty^{1/2},$$

which has the same scaling behavior as (8) with respect to the nonlinearity $f(u)$.

4 Asymptotic results

We will consider different asymptotic scalings in the general estimates above.
At first assume a strong convection by replacing $b(y)$ by $Mb(y)$ in (2), i.e.

$$c\partial_\xi u(\xi, y) = d\Delta u(\xi, y) + Mb(y)\partial_\xi u(\xi, y) + f(u(\xi, y)). \tag{14}$$

Since the lower bound on c is homogeneous of degree one in χ and therefor also in b we have a linear growth of the front speed in the amplitude of the shear flow. We have specifically

Corollary 1 *Let $\chi(y)$ be the solution of (3). The (minimal) speed c_0 satisfies for all $M \in I\!R$ the following estimates*

$$c_0(M) \geq \max\left\{ \int_0^1 \sqrt{2d\, f(s)}\, ds,\ M\frac{|\nabla_y \chi|_2^2\, F(1)}{|\Omega|} \left(|\chi|_\infty \left(\frac{1}{2}|\dot{f}|_\infty + |f|_\infty \right) + \sqrt{\frac{d}{2}} |\nabla_y \chi|_\infty |f|_\infty^{1/2} \right)^{-1} \right\},$$

$$c_0(M) \leq \min\left\{ 2(d|\dot{f}|_\infty)^{1/2}| + M \sup_{y\in\Omega} b(y),\ 2(d + \frac{M^2}{d}|\nabla_y \chi|_\infty^2)^{1/2} |\dot{f}|_\infty^{1/2} \right\}.$$

This shows the linear growth of the speed with the Peclet number. In [1] it was conjectured based on some formal arguments that $\lim\limits_{M\to\infty} \frac{c_0(M)}{M}$ should be equal to $\sup\limits_{y\in\Omega} b(y)$. From the corollary we get

$$\limsup_{M\to\infty} \frac{c_0(M)}{M} \leq \min\left\{ \sup_{y\in\Omega} b(y),\quad \frac{2}{\sqrt{d}} |\nabla_y \chi|_\infty |\dot{f}|_\infty^{1/2} \right\}.$$

Hence the conjecture can only be true if the first bound is smaller than the second, which is

$$\frac{|\nabla_y \chi|_\infty}{\sup\limits_{y\in\Omega} b(y)} \geq \frac{1}{2} \left(\frac{d}{|\dot{f}|_\infty} \right)^{1/2}. \tag{15}$$

The left hand side is a typical length scale of the convective flow and the right hand side corresponds to the thickness of the reaction zone for the one dimensional front (7). Therefor (15) is precisely the assumption on which the formal derivation in [1] is based.
Next we will show, that the estimates in theorem 1 and 2 remain useful even in the high oscillation limit which corresponds to homogenization. Let $\Omega = (0, L)^n$ and consider (2) with periodic boundary conditions. Assume that the flow field $b(y)$ in (2) is replaced by $\frac{m}{L} b(\frac{my}{L})$ with a 1-periodic $b(z)$. The function $\chi(y)$ in (3) can be written as $\chi(y) = \frac{L}{m}\widetilde{\chi}(z)$ with $z = \frac{my}{L}$. Now $\widetilde{\chi}(z)$ is 1-periodic and satisfies

$$\Delta_z \widetilde{\chi}(z) = -b(z).$$

With $C = (0,1)^n$ this implies for the lower estimate in theorem 1:

$$c_0(m) \geq \frac{F(1) \int_C |\nabla_z \widetilde{\chi}(z)|^2 \, dz}{\frac{L}{2m} |\widetilde{\chi}|_\infty \left(|\dot{f}|_\infty + 2|f|_\infty \right) + \sqrt{\frac{d}{2}} |\nabla_z \widetilde{\chi}|_\infty |f|_\infty^{1/2}}.$$

As m tends to infinity we arrive at

$$\liminf_{m \to \infty} c_0(m) \geq \max \left\{ \int_0^1 \sqrt{2d \, f(s)} \, ds, \quad \frac{\sqrt{2} \, F(1) \int_C |\nabla_z \widetilde{\chi}(z)|^2 \, dz}{\sqrt{d} \, |f|_\infty^{1/2} |\nabla_z \widetilde{\chi}|_\infty} \right\}. \tag{16}$$

The upper bound simply rescales to

$$c_0(m) \leq 2(d + \frac{1}{d} |\nabla_z \widetilde{\chi}|_\infty^2)^{1/2} |\dot{f}|_\infty^{1/2}. \tag{17}$$

This can be compared to the homogenized problem:

$$c_0^h u' = d_h u'' + f(u). \tag{18}$$

The effective diffusion coefficient is given by

$$d_h = d + \frac{1}{d} \int_C |\nabla_z \widetilde{\chi}(z)|^2 \, dz.$$

The first estimate in theorem 1 and the second estimate in theorem 2 with $\chi(y) = 0$ give:

$$\int_0^1 \sqrt{2 d_h f(s)} \, ds \leq c_0^h \leq 2 \sqrt{d_h \, |\dot{f}|_\infty}. \tag{19}$$

Hence the estimates (16) and (17) have the same scaling behavior with respect to the convection and the reaction as the homogenized problem. This shows that homogenization gives qualitatively the correct behavior of the wave speed for arbitrary convection and reaction.

Another scaling corresponds to a strong convection and a weak reaction, i.e.

$$c\partial_\xi u(\xi, y) = d\Delta u(\xi, y) + M b(y) \partial_\xi u(\xi, y) + \frac{1}{M^2} f(u(\xi, y)). \tag{20}$$

Replacing $\chi(y)$ by $M\chi(y)$ and $f(u)$ by $f(u)/M^2$ in theorem 1 and 2 we get

Corollary 2 *The (minimal) speed $c_0(M)$ for (20) satisfies*

$$\begin{aligned} \liminf_{M \to \infty} c_0(M) &\geq \frac{\sqrt{2} \, |\nabla_y \chi|_2^2 \, F(1)}{|\Omega| \, \sqrt{d} \, |\nabla_y \chi|_\infty \, |f|_\infty^{1/2}}, \\ \limsup_{M \to \infty} c_0(M) &\leq \frac{2}{\sqrt{d}} \, |\nabla_y \chi|_\infty \, |\dot{f}|_\infty^{1/2} \end{aligned}$$

Heuristically this result means, that a strong wind still sustains a front with speed of order one for a very weak reaction. Without convection the front velocity would be of order $1/M$.
At last we let the diffusivity tend to zero in (2). This time the second estimate in theorem 2 explodes but the first estimate stays bounded.

Corollary 3 *The (minimal) speed $c_0(d)$ for (2) satisfies*

$$\liminf_{d\to 0} c_0(d) \geq \frac{2\,|\nabla_y \chi|_2^2\, F(1)}{|\Omega|\,|\chi|_\infty \left(|\dot{f}|_\infty + 2|f|_\infty\right)},$$

$$\limsup_{d\to 0} c_0(d) \leq \sup_{y\in\Omega} b(y).$$

In particular $c_0(d)$ stays bounded from above and below. Without convection the speed would be of order $\sqrt{d}$.
The formal limit problem as $d \to 0$ is

$$(c_0(0) - b(y)\partial_\xi u = f(u)$$

Since u is nondecreasing and $f(u)$ is nonnegative we obtain

$$c_0(0) \geq \sup_{y\in\Omega} b(y).$$

Therefor one expects

$$\lim_{d\to 0} c(d) = \sup_{y\in\Omega} b(y).$$

With all these asymptotic estimates at hand it would be interesting to study the limits of suitably rescaled solutions of (2) and to derive limit problems for the various asymptotic cases.

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