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by

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Abstract

It is proven, that a single semilinear parabolic equation in an unbounded cylinder with cubic-like source or boundary flux admit travelling waves. The problem is reformulated as a constrained minimization problem, where the wave velocity is related to the infimum. This characterization implies the monotone dependence of the velocity on the domain, the nonlinearity and the boundary conditions. Using rearrangement of the minimizer the monotonicity of the wave profile is proven.

1 Introduction

Our starting point is the search for travelling wave solutions of a single semilinear parabolic equation in infinite cylinders $\mathbf{R} \times \Omega$, where Ω is a bounded domain in $\mathbf{R}^n$ with C^1 boundary and outer normal ν. The boundary $\partial\Omega$ consists of two parts Γ_1, Γ_2 corresponding to different boundary conditions. Γ_1 or Γ_2 may be empty. Using the notation $\xi \in \mathbf{R}$, $y \in \Omega$ we consider

$$\begin{array}{rcll} \partial_t u(t,\xi,y) &=& \Delta u(t,\xi,y) + f(u(t,\xi,y),y), & (t,\xi,y) \in \mathbf{R}^+ \times \mathbf{R} \times \Omega, \\ \partial_\nu u(t,\xi,y) &=& g(u(t,\xi,y),y), & (t,\xi,y) \in \mathbf{R}^+ \times \mathbf{R} \times \Gamma_1 \\ u(t,\xi,y) &=& 0, & (t,\xi,y) \in \mathbf{R}^+ \times \mathbf{R} \times \Gamma_2 \end{array} \quad (1.1)$$

Here Δ denotes the Laplacian w.r.t. $(\xi, y) \in \mathbf{R} \times \Omega$. For different kinds of nonlinearities equation (1.1) is used in some simplified biological models and in combustion in order to explain propagation phenomena. The nonlinear boundary conditions have not been studied before in connection with moving fronts. In [5] a model for the heater in boiling systems has been proposed. The heat flux through the boundary is given by a cubic shaped function of the temperature. Heat fronts along the surface of the heater separating the regimes of nucleate and film boiling have been observed. They are often responsible for the damage of the heater due to the sudden increase of the temperature. We will prove that such front solutions exist for the proposed model. The last chapter gives more details about this model and some asymptotic results for thin heaters.
Let $x = c(\xi + ct \in \mathbf{R})$ with $c \neq 0$ as the unknown wave velocity. The scaling factor c is introduced in order to derive a variational problem. After a possible change of ξ to $-\xi$ we can assume $c > 0$. With $\lambda = 1/c^2$ a travelling wave solution $u = u(x,y)$ of (1.1) satisfies

$$\begin{array}{rcll} \partial_x u(x,y) &=& \partial_{xx} u(x,y) + \lambda \left(\Delta_y u(x,y) + f(u(x,y),y)\right), & (x,y) \in \mathbf{R} \times \Omega \\ \partial_\nu u(x,y) &=& g(u(x,y),y), & (x,y) \in \mathbf{R} \times \Gamma_1 \\ u(x,y) &=& 0, & (x,y) \in \mathbf{R} \times \Gamma_2 \\ u(\pm\infty,y) &=& v^\pm(y), & y \in \Omega. \end{array} \quad (1.2)$$

where $v^\pm(y)$ solve

$$\begin{array}{rcll} 0 &=& \Delta_y v(y) + f(v(y),y), & y \in \Omega \\ \partial_\nu v(y) &=& g(v(y),y), & y \in \Gamma_1 \\ v(y) &=& 0, & y \in \Gamma_2 \end{array} \quad (1.3)$$

Replacing u by $u - v^-$ and redefining the nonlinearities f and g we can assume

$$u(-\infty, y) = v^-(y) = 0,$$

$$f(0, y) = 0, \quad g(0, y) = 0$$

In the language of dynamical systems $v^\pm$ correspond to equilibria. But since (1.3) is an elliptic equation it does not define a dynamical system and usual methods for detecting travelling waves are difficult to apply.

We mention that the one-dimensional, i.e. y-independent, problem has been extensively studied, e.g. [11], [6]. We review the well known results in the scalar case:

$$u_{xx} - cu_x + f(u) = 0, \quad f(0) = f(1) = 0.$$

Three cases have to be distinguished:
1. KPP type:

$$f'(0) > 0, \quad 0 < f(u) < 1 \quad \text{for } 0 < u < 1,$$

e.g. $f(u) = u(1-u)$. In this case there exists

$$c^* \geq 2\sqrt{f'(0)}, \tag{1.4}$$

s.t. there are travelling waves u connecting 0 to 1 if and only if $c \geq c^*$ and for fixed $c \geq c^*$ the solution u is unique. Equality holds in (1.4) for nonlinearities which satisfy

$$\sup_{0<u<1} \frac{f(u)}{u} = f'(0).$$

In this case the linearization implies for the solution with the minimal speed $c = c^*$

$$u^*(x) \sim e^{c^* x/2}$$

near $-\infty$. Therefor $u(x)$ is not in the weighted space $L^2(\mathbf{R}; e^{-c^* x})$. But this space will be essential for the variational formulation, that will be described below. So we will discard such kind of nonlinearities from our consideration.
2. Combustion type:

$$f(u) = 0 \quad \text{for } 0 \leq u \leq \theta, \quad f(u) > 0 \quad \text{for } \theta \leq u < 1.$$

3. Bistable type:

$$f'(0) < 0, \quad \text{there exists a unique } \theta \in (0,1) \text{ with } f(\theta) = 0,$$

e.g. $f(u) = u(u-\theta)(1-u)$, $\mu < 1/2$.
In the cases 2. and 3. there exists a unique velocity c and a unique profile u connecting 0 to 1. Linearization at $-\infty$ shows now $u \in L^2(\mathbf{R}; e^{-cx})$. We remark, that u connects two minima of $-F(u)$ and $-F(1) < -F(0)$ holds. This means that $v^+ = 1$ has smaller energy than $v^- = 0$. It is precisely this result, that we will generalize to higher dimensions.
Results in higher space dimension have been obtained in the papers [2], [3], [4]. There problem (1.1) is treated including a drift term with zero Neumann boundary conditions. Existence of fronts is obtained by degree theory and an approximation on bounded domains. In [7] the case of a cubic nonlinearity with Dirichlet boundary conditions is treated in an infinite strip using the Conley index for discretized problems. This method is very involved. This was extended in [13] to higher dimensions and more general nonlinearities. All these existence results rely heavily on the use of the maximum principle and give no formulas for the wave speed. Our result generalizes the result in [7] and [13] to nonlinear boundary conditions. Also our method does not require the a priori knowledge of all rest states, i.e. x-independent solutions of (1.2), which is in general not available. But more important is our variational characterization of the wave velocity given

by a constrained minimization problem. In [8], [9] variational principles for the speed based on the maximum principle are given for multidimensional waves. Since these formulas are of saddle point type, they give only limited information about the speed. Also the front itself cannot be obtained from this variational formulas. The minimization problem which we will develop here gives the speed and the profile of the front. It also allows to derive some qualitative properties of the wave velocity. Using rearrangement arguments we will prove the monotonicity of the wave. This can also be done in a more complicated way using the moving hyperplane method as in [4]. We also obtain monotone dependence of the wave velocity on the nonlinearity, the domain and the boundary conditions. Furthermore for $\Gamma_1 = \emptyset$ and $\partial_y f(u,y) = 0$ we show, that the wave velocity is largest for a ball compared to all domains Ω with the same volume.

2 Variational Formulation

For the nonlinearities $f(u,y), g(u,y)$ we require the conditions: There exist constants $m < M$ s.t.

$$\begin{aligned} f \in C^1(\mathbf{R}\times\Omega), \quad & f(0,y) = 0, \\ uf(u,y) \le 0 \quad & \text{for } |u| > M \text{ and } |u| < m, y \in \Omega, \\ g \in C^1(\mathbf{R}\times\Gamma_1), \quad & g(0,y) = 0, \\ ug(u,y) \le 0 \quad & \text{for } |u| > M \text{ and } |u| < m, y \in \Gamma_1 \end{aligned} \tag{2.5}$$

The maximum principle implies that all solutions of (1.2) are bounded by M in L^∞. The assumptions in (2.5) on the behavior of the nonlinearities near $u = 0$ correspond to the combustion or bistable case.
Define the Hilbert space X consisting of functions in the weighted space $H^1(\mathbf{R}\times\Omega, e^{-x})$ which vanish on Γ_2. On X we define the following two functionals

$$I(u) = \frac{1}{2} \int\limits_{\mathbf{R}\times\Omega} |\partial_x u(x,y)|^2 e^{-x}\, dx\, dy \tag{2.6}$$

and

$$J(u) = \int\limits_{\mathbf{R}\times\Omega} \left(\frac{1}{2}|\nabla_y u(x,y)|^2 - F(u(x,y),y)\right) e^{-x}\, dx\, dy - \int\limits_{\mathbf{R}\times\partial\Gamma_1} G(u(x,y),y) e^{-x}\, dx\, dT_y, \tag{2.7}$$

where $F(u,y) = \int_0^u f(s,y)ds, \quad G(u,y) = \int_0^u g(s,y)ds.$
Consider the following minimization problem

$$\inf_{\{u\in X \,|\, J(u)=b\}} I(u). \tag{2.8}$$

It is easy to check that (1.2) is precisely the variational equation corresponding to (2.8) with λ as the Lagrange multiplier. Testing the Euler-Lagrange equation (1.2) with $\partial_x u e^{-x}$ we deduce easily

$$I(u) + \lambda J(u) = 0.$$

Since we want $\lambda = 1/c^2 > 0$ and $I(u) > 0$, we have to require $J(u) < 0$. Hence after a suitable shift in x it is enough to consider $b = -1$ in (2.8). This also shows, that the Lagrange multiplier satisfies

$$\frac{1}{c^2} = \lambda = \inf_{\{u\in X \,|\, J(u)=-1\}} I(u), \tag{2.9}$$

which is a variational characterization of the wave speed. The problem (2.9) is unusual in two aspects. At first both functionals $I(u)$ and $J(u)$ involve some part of the gradient. Thus the gradient term in the constraint is only lower semicontinuous. Second, due to the unbounded domain, also the other terms in $J(u)$ are not continuous with respect to weak convergence in H^1. In theorem 2.4 we show the existence of a minimizer for the problem (2.9).

At first we have to ensure that there are functions u satisfying the constraint. In lemma 2.1 we give conditions for this to be the case.
We give conditions, s.t. the constraint $J(u)$ in (2.9) is satisfied for some function $u \in X$. Let Y be the space of $H^1(\Omega)$ functions which vanish on Γ_2. For $v \in Y$ define the functional

$$K(v) = \int\limits_{\Omega} \left(\frac{1}{2} |\nabla_y v(y)|^2 - F(v(y), y) \right) dy - \int\limits_{\Gamma_1} G(v(y), y) \, dT_y. \tag{2.10}$$

Lemma 2.1 *A necessary and sufficient condition for the existence of a function in $u \in X$ with $J(u) = -1$ is $\inf_{v \in Y} K(v) < 0$.*

Proof. The functional $K(v)$ is weakly lower semicontinuous and bounded from below on Y. Thus the infimum $\underline{K}$ is attained for some function $v \in Y$. Since f and g are dissipative, we have $|v|_\infty \leq M$. We define $u(x, y) = \psi(x) v(y)$, where

$$\psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/\delta & \text{if } 0 \leq x \leq \delta \\ 1 & \text{if } x > \delta \end{cases}$$

Clearly u is in the space X.

$$\begin{aligned} J(u) = & \int\limits_{(0,\infty)\times\Omega} \left(\frac{1}{2} \psi^2 |\nabla_y v|^2 - F(\psi v, y) \right) e^{-x} \, dx \, dy - \int\limits_{(0,\infty)\times\Gamma_1} G(\psi v, y) e^{-x} \, dx \, dT_y \\ & = \underline{K} \int\limits_0^\infty \psi^2 e^{-x} \, dx + \int\limits_{(0,\delta)\times\Omega} \left(\psi^2 F(v, y) - F(\psi v, y) \right) e^{-x} \, dx \, dy \\ & \quad + \int\limits_{(0,\delta)\times\Gamma_1} \left(\psi^2 G(v, y) - G(\psi v, y) \right) e^{-x} \, dx \, dy \\ & \leq \underline{K} + 2 \left(|\Omega| \sup_{0<u<M, y\in\Omega} |F(u, y)| + |\Gamma_1| \sup_{0<u<M, y\in\Gamma_1} |G(u, y)| \right) (1 - e^{-\delta}). \end{aligned}$$

This is negative for δ sufficiently small, since $\underline{K}$ is negative by assumption. After a suitable shift in x we can now achieve $J(u) = -1$.
For the reverse implication let $J(u) = -1$ and use the representation

$$-1 = J(u) = \int\limits_{\mathbf{R}} K(u(x, \cdot)) e^{-x} \, dx.$$

Next we give a condition for $\inf_{v \in Y} K(v) < 0$. It is clear from the definition of $K(v)$, that the nonlinearities $F(v, y)$ and $G(v, y)$ have to be positive on a large enough set. We formulate a result for the special case of Dirichlet boundary conditions, i.e. $\Gamma_1 = \emptyset$. Similar results can be obtained in the general case. In the last chapter we will treat the nonlinear Neumann condition. We replace $F(u, y)$ by $\rho F(u, y)$ in $K(v)$.

Lemma 2.2 *Assume that $f(v, y)$ satisfies (2.5) and $F(a, y) \geq \eta > 0$ for some $a > 0$. If ρ is sufficiently large, then $K(v)$ has a negative infimum.*

Proof. For simplicity let $a = 1$. It suffices to show $K(v) < 0$ for some $v \in H_0^1(\Omega)$. Define

$$\Omega_\delta = \{ y \in \Omega | \text{dist}(y, \partial\Omega) < \delta \} \text{ and } v(y) = \min \{ 1, 1/\delta \text{ dist}(y, \partial\Omega) \}.$$

Then we calculate

$$K(v) = \int\limits_{\Omega_\delta} \left(\frac{1}{2}|\nabla_y v|^2 + \rho(F(1,y) - F(v,y)) \right) \, dy - \int\limits_{\Omega} \rho F(1,y) \, dy \leq$$

$$\leq \rho \left(\sup_{0<v<1, y\in\Omega} (F(1,y) - F(v,y))|\Omega_\delta| - |\Omega|\eta \right) + \frac{C(\partial\Omega)}{\delta}.$$

Now choose δ small, but fixed, s.t. the term in brackets is negative and then choose ρ sufficiently large. This gives $K(v) < 0$, completing the proof.

The next lemma is a substitute for the lack of compactness of the embedding H^1 into L^2 in unbounded domains.

Lemma 2.3 *Assume that the nonlinearities f and g satisfy (2.5). Let u_i be a bounded sequence in $X \cap L^\infty(\mathbf{R} \times \Omega)$ with $J(u_i) = -1$ Then there is a subsequence, still labeled u_i which converges weakly in X to some $u \in X$ with $J(u) \leq -1$.*

Proof. Let $|u_i|_X \leq M_1$ and let $|u_i|_{L^\infty} \leq M$. We may assume, that u_i converges weakly in X to $u \in X$. Since $\int\limits_{\mathbf{R}\times\Omega} |\nabla_y u_i|^2 e^{-x} \, dx \, dy$ is lower semicontinuous on X we concentrate on the other two terms in $J(u_i)$.

Let X_R be the space of restrictions to $(-R, -\infty) \times \Omega$ of functions from X. The embedding

$$X_R \cap L^\infty((-R,\infty) \times \Omega) \to L^2((-R,\infty) \times \Omega, e^{-x})$$

and the trace map

$$X_R \cap L^\infty((-R,\infty) \times \Omega) \to L^2((-R,\infty) \times \partial\Omega, e^{-x})$$

are compact for all $R < \infty$. Hence we have for a subsequence, still labeled u_i, and for all R

$$\lim_{i\to\infty} \int\limits_{(-R,\infty)\times\Omega} F(u_i,y)e^{-x} \, dx \, dy = \int\limits_{(-R,\infty)\times\Omega} F(u,y)e^{-x} \, dx \, dy. \tag{2.11}$$

$$\lim_{i\to\infty} \int\limits_{(-R,\infty)\times\Gamma_1} G(u_i,y)e^{-x} \, dx \, dT_y = \int\limits_{(-R,\infty)\times\Gamma_1} G(u,y)e^{-x} \, dx \, dT_y. \tag{2.12}$$

For the part of the integral near $-\infty$ we only consider $F(u,y)$ since the boundary integral is treated similarly.

The assumption (2.5) implies $F(u,y) \leq 0$, for $0 < |u| < m$. This gives for the positive part F^+ of F, for $|u| \leq M$

$$F^+(u,y) \leq M_2 \max(|u| - m, 0).$$

Let

$$A_i = \{(x,y) | x < -R, y \in \Omega, |u_i(x,y)| > m\}.$$

We will show, that A_i has a small measure uniformly in i. We estimate

$$\int\limits_{A_i} e^{-x} \leq 1/m^2 \int\limits_{\mathbf{R}\times\Omega} u_i^2 e^{-x} \leq M_1^2/m^2 \tag{2.13}$$

and

$$|A_i| \leq 1/m^2 e^{-R} \int\limits_{A_i} u_i^2 e^{-x} \leq M_1^2/m^2 e^{-R}. \tag{2.14}$$

Now calculate using (2.13), (2.14) and the Poincare inequality with the constant $M_3|A_i|^{\frac{1}{n+1}}$

$$\int\limits_{(-\infty,-R)\times\Omega} F(u_i,y)e^{-x} \le \int\limits_{A_i} F^+(u_i,y)e^{-x} \le M_2 \left(\int\limits_{A_i} (|u_i|-m)^2 e^{-x} \int\limits_{A_i} e^{-x} \right)^{1/2}$$

$$\le \frac{M_1M_2M_3}{m}|A_i|^{\frac{1}{n+1}} \left(\int\limits_{A_i} |\nabla((|u_i|-m)e^{-x/2})|^2 \right)^{1/2}$$

$$= \frac{M_1M_2M_3}{m}|A_i|^{\frac{1}{n+1}} \left(\int\limits_{A_i} (|\nabla u_i|^2 - \frac{(|u_i|-m|)^2}{4})e^{-x} \right)^{1/2} \le \left(\frac{M_1}{m}\right)^{1+\frac{2}{n+1}} M_1M_2M_3e^{-\frac{R}{n+1}}.$$

Now choose R, s.t. the last expression is less than ϵ and s.t.

$$-\int\limits_{(-\infty,-R)\times\Omega} F(u,y)e^{-x} < \epsilon.$$

Hence we have

$$\int\limits_{(-\infty,-R)\times\Omega} F(u_i,y)e^{-x} \le \int\limits_{(-\infty,-R)\times\Omega} F(u,y)e^{-x} + 2\epsilon.$$

Together with (2.11), (2.12) and since $\epsilon > 0$ was arbitrary, we have proved the lemma.

Theorem 2.4 *Assume (2.5) and let* $\inf\limits_{v\in Y} K(v) < 0$. *Then problem (2.9) has a minimizer* $u \in X$. *The corresponding Lagrange multiplier* λ *is positive and coincides with the infimum in (2.9). There are at most two minimizers and they have constant sign. If there are two minimizers, they have opposite sign. All minimizers are strictly monotone w.r.t.* x. *Minimizers are classical solution of (1.2) with* $u(-\infty,y) = 0$. *The limit* $u(\infty,y) = v^+(y)$ *exists and satisfies* $K(v^+) < 0$. *The second variation of* $K(v)$ *at* v^+ *has a nonnegative spectrum.*

Proof. Let u_i be a minimizing sequence for $I(u)$ under the constraint $J(u) = -1$:

$$I(u_i) \to \lambda.$$

Let

$$\tilde{u}_i(x,y) = \max(-M, \min(M,u_i))(x,y).$$

Since f and g are dissipative we get

$$J(\tilde{u}_i) \le J(u_i) = -1, \quad I(\tilde{u}_i) \le I(u_i).$$

With $\tilde{u}_i^a(x,y) = \tilde{u}_i(x+a,y)$ we have for some $a \le 0$:

$$J(\tilde{u}_i^a) = e^a J(\tilde{u}_i) = -1.$$

Therefor

$$I(\tilde{u}_i^a) = e^a I(\tilde{u}_i) \le I(u_i)$$

and we can assume $|u_i|_\infty \le M$. We claim, that u_i is uniformly bounded in X. Indeed $I(u_i) \le C$ implies, using Hardy's inequality,

$$\int\limits_{\mathbf{R}\times\Omega} u_i^2 e^{-x}\, dx\, dy \le 4 \int\limits_{\mathbf{R}\times\Omega} |\partial_x u_i|^2 e^{-x}\, dx\, dy \le 8C. \tag{2.15}$$

The trace inequality applied to each cross section $\{\xi\} \times \Omega$ implies

$$\int\limits_{\mathbf{R}\times\Gamma_1} u_i^2 e^{-x}\, dx\, dT_y \leq \epsilon \int\limits_{\mathbf{R}\times\Omega} |\nabla_y u_i|^2 e^{-x}\, dx\, dy + C(\epsilon) \int\limits_{\mathbf{R}\times\Omega} u_i^2 e^{-x}\, dx\, dy. \tag{2.16}$$

Now $J(u_i) = -1$ gives

$$\frac{1}{2} \int\limits_{\mathbf{R}\times\Omega} |\nabla_y u_i|^2 e^{-x}\, dx\, dy \leq -1 + \left|\frac{F(u,y)}{u^2}\right|_\infty \int\limits_{\mathbf{R}\times\Omega} u_i^2 e^{-x}\, dx\, dy + \left|\frac{G(u,y)}{u^2}\right|_\infty \int\limits_{\mathbf{R}\times\Gamma_1} u_i^2 e^{-x}\, dx\, dT_y. \tag{2.17}$$

Using (2.15), (2.16) and choosing ϵ sufficiently small implies a uniform bound for the left hand side in (2.17). Hence u_i is uniformly bounded in $X \cap L^\infty$. Now take as u_i the sequence given in lemma 2.3, with weak limit u and $J(u) \leq -1$. Since I is lower semicontinuous we also have $I(u) \leq \lambda$. For $u^a(x,y) = u(x+a,y)$ we have

$$J(u^a) = e^a J(u) \quad \text{and} \quad I(u^a) = e^a I(u).$$

Now choose $a \leq 0$ s.t. $J(u^a) = -1$. Since λ is the infimum of I we get

$$\lambda \leq I(u_a) = e^a I(u) \leq e^a \lambda$$

Hence $a = 0$ and u is a minimizer of (2.9). It is standard to check, that u is a weak solution of (1.2). Since f and g are C^1, elliptic regularity theory implies that all derivatives of u up to order three are bounded. In particular $u(-\infty, y) = 0$ follows.
Next we show that minima have a constant sign. Multiply (1.2) by $u_x e^{-x}$. After some manipulations we arrive at

$$0 < I(u) = -\lambda J(u) = \lambda.$$

This implies

$$I(w) + \lambda J(w) \geq 0, \tag{2.18}$$

for all $w \in X$ and equality only holds for $w = 0$ and for any shifted minimizer of (2.9). Let u^+ be the positive part of u and let $u^- = u - u^+$. Then we have by (2.18)

$$0 = I(u) + \lambda J(u) = I(u^+) + \lambda J(u^+) + I(u^-) + \lambda J(u^-) \geq 0.$$

and therefor

$$I(u^\pm) + \lambda J(u^\pm) = 0.$$

Thus u^+ and u^- are minimizers of $I + \lambda J$. The maximum principle implies, that one of them vanishes identically and the vanishes nowhere. Hence u has a constant sign.
Uniqueness of minimizers with a fixed sign will be proven in theorem 3.1. Monotonicity of u is shown in theorem 3.2. The monotonicity and the boundedness of u implies that $\lim\limits_{x\to\infty} u(x,y) = v^+(y)$ exists and $v^+(y)$ solves (1.3). From the equation (1.2) we conclude

$$\frac{d}{dx} \int\limits_\Omega \left(\frac{1}{2} u_x^2 - \lambda K(u(x,\cdot)\right)\, dy = \int\limits_\Omega u_x^2\, dy > 0.$$

Integration gives $K(v^+) < K(0) = 0$.
Now let $w(x) = \int_\Omega \partial_x u(x,y)\phi(y) dy$, where ϕ is the first eigenfunction with eigenvalue μ of the second variation of K at v^+:

$$\begin{aligned} \Delta_y \phi + \partial_u f(v^+,\cdot)\phi &= -\mu\phi, && \text{in } \Omega \\ \partial_\nu \phi &= \partial_u g(v^+,\cdot)\phi, && \text{on } \Gamma_1 \\ \phi &= 0, && \text{on } \Gamma_2 \end{aligned}$$

Let sign ϕ = sign $\partial_x u$ and suppose $\mu < 0$. We have

$$w_{xx} - w_x = \int_\Omega (u_{xxx} - u_{xx})\phi = \int_\Omega (-\Delta_y u_x - \partial_u f(u,y)u_x)\phi\, dy$$

$$= \int_\Omega (\partial_u f(v^+,y) - \partial_u f(u,y) + \mu)\phi u_x\, dy \int_{\Gamma_1} (\partial_u g(v^+,y) - \partial_u g(u,y))\phi u_x\, dT_y < 0$$

for $x > x^*$, since $u(x,y) \to v^+(y)$, as $x \to \infty$. Integration over (x,∞) shows $w_x > w > 0$ for large x and w would grow exponentially. This contradicts the boundedness of w. Hence μ is nonnegative, There are two minimizers, if and only if the infimum in (2.9) is the same when restricted to positive and negative functions respectively.
This proves all the assertions in the theorem.

Remarks:
1. Minimizing (2.9) over positive and negative functions respectively, we obtain two solutions of (1.2), but possibly not for the same λ.
2. Our solution connects a given relative Minimum of $K(v)$ at $-\infty$ to another relative Minimum with lower energy by a monotone travelling wave. If there are several possible such states, it would be interesting to know which one is selected by our solution.
3. Find travelling waves between not necessarily relative minima of $K(v)$. They cannot be found by our variational approach, since they are not in the space X. This can be seen by linearization at $\pm\infty$.
4. One-dimensional examples show the possible existence of sign changing travelling waves between relative minima of $K(v)$. In these examples one can also prescribe the number of sign changes. They correspond to higher critical points of our variational problem (2.9). It might be possible to apply some version of the Liusterik-Schnirelmann category theory.

3 Uniqueness and Properties of Minimizers

One could use the sliding domain method as in [4] to prove uniqueness and monotonicity of travelling waves. This would require precise information about the asymptotic behavior of the front for large $|x|$. Instead we will use the variational formulation. Hence we can only prove uniqueness and monotonicity for fronts lying in the weighted space X. This leaves open the possibility of solutions which are not in X, e.g. having only an algebraic decay near $-\infty$. Linearization at $u = 0$ shows however that all possible fronts are in the space X, if $v = 0$ is a nondegenerate local minimum of $K(v)$. In this case it is enough to prove uniqueness in the space X.
We show at first the uniqueness of minimizers of (2.9).

Theorem 3.1 *Minimizers of (2.9) with the same sign are unique.*

Proof. For definiteness assume, that u_1, and u_2 are positive minimizers of (2.9). We claim the existence of $a^* \in \mathbf{R}$, $(x^*, y^*) \in \mathbf{R} \times \Omega$, s.t. $u_1(x^* + a^*, y^*) = u_2(x^*, y^*)$. Otherwise we can assume w.l.o.g. $u_1(x + a, y) > u_2(x, y)$ for all (a, x, y). This implies

$$e^a \int_{\mathbf{R}\times\Omega} u_1(x,y)^2 e^{-x} \geq \int_{\mathbf{R}\times\Omega} u_2(x,y)^2 e^{-x}$$

for all $a \in \mathbf{R}$. Hence $u_2 = 0$, contradicting $J(u_2) = -1$.
Now we define $u_3(x, y) = u_1(x + a^*, y)$ and we claim that

$$\underline{u} = \min(u_2, u_3) \text{ and } \overline{u} = \max(u_2, u_3)$$

are shifted minimizers of (2.9). We calculate using (2.18)

$$I(\underline{u}) + \lambda J(\underline{u}) + I(\overline{u}) + \lambda J(\overline{u}) = I(u_2) + I(u_3) + \lambda(J(u_2) + J(u_3)) = 0$$

Using again (2.18) we conclude

$$I(\underline{u}) + \lambda J(\underline{u}) = 0,$$
$$I(\overline{u}) + \lambda J(\overline{u}) = 0.$$

and $\underline{u}$, $\overline{u}$ are shifted minimizers of (2.9) and hence solutions of (1.2). Thus the difference $w = \overline{u} - \underline{u}$ satisfies,

$$w_{xx} - w_x + \lambda(\Delta_y w - bw) = 0$$

for some continuous function b. Since $w \geq 0$, $w(x^*, y^*) = 0$ the maximum principle now implies $w = 0$ everywhere and $u_2(x,y) = u_3(x,y) = u_1(x + a^*, y)$ holds. Now $a^* = 0$ follows, since $J(u_1) = J(u_2) = -1$.

Now we prove the monotonicity of the minimizer using a suitable rearrangement.

Theorem 3.2 *Positive minimizers u of (2.9) are strictly increasing and negative minimizer strictly decreasing.*

Proof. Let $u(x,y)$ be the positive minimizer obtained in theorem 2.4. The proof of the monotonicity uses monotone rearrangement, but not in x. Instead we introduce $z = -e^{-x}$. Observe that monotonicity in z implies monotonicity in x. In this new coordinate the functionals $I(u)$ and $J(u)$ can be written as

$$I(u) = \frac{1}{2} \int\limits_{(-\infty,0)\times\Omega} z^2 |u_z|^2 \, dz \, dy$$

$$J(u) = \int\limits_{(-\infty,0)\times\Omega} \left(\frac{1}{2} |\nabla_y u|^2 - F(u,y) \, dz \, dy \right) + \int\limits_{(-\infty,0)\times\Gamma_1} G(u,y) \, dz \, dT_y.$$

Let u^* be the monotone increasing rearrangement of u. In the proof of theorem 4.1 in [1] it is shown that $I(u)$ and the first term in $J(u)$ do not increase under this rearrangement, while the terms involving F and G is preserved. Hence there is a shift $a \leq 0$ s.t. for $u_a^*(x,y) = u^*(x+a,y)$

$$J(u_a^*) = e^a J(u^*) = -1$$

and

$$I(u_a^*) = e^a I(u^*) \leq e^a I(u)$$

holds. Since u is the minimizer we get $a = 0$ and u^* coincides with u by uniqueness of the minimizer. This implies $u_x \geq 0$. Differentiate (1.2) w.r.t. x and apply the maximum principle to u_x and conclude $u_x > 0$.

Now we will prove some qualitative properties of the minimal value $\lambda = 1/c^2$ of the variational problem. Since λ is a Lagrange multiplier of a minimization problem, it depends monotonically on $F(u,y)$, Ω and the boundary conditions. Using $c^2 = 1/\lambda$ this implies the corresponding properties for the wave velocity.

Theorem 3.3 *Let $u(-\infty, y) = 0$ and assume, that all minimization problems occuring below have nontrivial solutions.*

1. *$F_1(u,y) \leq F_2(u,y)$ implies $\lambda_1 \geq \lambda_2$.*
2. *$G_1(u,y) \leq G_2(u,y)$ implies $\lambda_1 \geq \lambda_2$.*
3. *Assume $\Gamma_1 = \emptyset$ and $\partial_y f(u,y) = 0$. Then $\Omega_1 \subset \Omega_2$ implies $\lambda_1 \geq \lambda_2$.*
4. *With the same assumptions as in 3, λ is smallest for the ball compared to all domains Ω with the same volume.*

Proof. For proving claim 1 let u be a minimizer of (2.9) corresponding to F_1. Then $-1 = J_1(u) \geq J_2(u)$ and there is a shift $a \leq 0$, s.t. $u_a(x,y) = u(x+a,y)$ satisfies

$$J_2(u_a) = e^a J_2(u) = -1 \text{ and } I(u_a) = e^a I(u) \leq \lambda_1.$$

This implies $\lambda_2 \leq \lambda_1$. The proof for claim 2 is identical. The third assertion follows byextending the minimizer in Ω_1 by 0 to Ω_2. Then one and concludes as above. The proof for claim 4 uses spherical rearrangement in the coordinate y. It is well known [10], that this process decreases the functional and $J(u)$ and preserves $I(u)$. After a suitable shift we have proved the claim.

4 Applications

In this chapter we consider equation (1.2) with $f(u,y)$ identically zero and $\Gamma_1 = \partial\Omega$. This describes the heat distribution in a heater, where some part of the heater is in contact with a boiling liquid. On this part the heat flux, the so called boiling curve, $-g(u,y) > 0$ has a cubic like shape. At lower temperatures almost all of the surfaces of the heater is in contact with the fluid and the heat flux grows with the temperature. At higher temperatures more and more vapor bubbles occur and these lower the heat flux due to their lower heat conductivity. For even higher temperatures a film of vapor covers the heater and the flux increases again. For further details we refer to [5].
At first we will derive the formal asymptotics of a front as the cross section shrinks to a point. In the limit we will recover the one dimensional front with an averaged nonlinearity. This shows, that the one dimensional problem is a good approximation of an electrically heated wire in a surrounding liquid.
Consider the travelling wave equation in the unscaled moving frame coordinate $x = \xi + xt$. In the small cross section $\sqrt{\epsilon}\Omega$ correct scaling demands that the boundary flux is rescaled by $\sqrt{\epsilon}$.

$$\begin{array}{rcll} c\,\partial_x u(x,y) & = & \partial_{xx} u(x,y) + \Delta_y u(x,y), & (x,y) \in \mathbf{R} \times \sqrt{\epsilon}\,\Omega \\ \partial_\nu u(x,y) & = & \sqrt{\epsilon}\; g(u(x,y), \frac{y}{\sqrt{\epsilon}}), & (x,y) \in \mathbf{R} \times \sqrt{\epsilon}\,\partial\Omega. \end{array} \tag{4.19}$$

In the rescaled variable $z = \frac{y}{\sqrt{\epsilon}}$ we get

$$\begin{array}{rcll} c\,\partial_x u(x,z) & = & \partial_{xx} u(x,z) + \frac{1}{\epsilon}\Delta_z u(x,z), & (x,y) \in \mathbf{R} \times \Omega \\ \frac{1}{\epsilon}\partial_\nu u(x,z) & = & g(u(x,z),z), & (x,z) \in \mathbf{R} \times \partial\Omega. \end{array} \tag{4.20}$$

We assume that for each ϵ we have a front as in theorem 2.4. We postulate a formal expansion of the fronts in the form

$$u = u_0 + \epsilon u_1 + \cdots, \quad c = c_0 + \epsilon c_1 + \cdots .$$

If we use this ansatz in (4.20) and equate powers of ϵ we get at the order of ϵ^{-1}

$$\Delta_z u_0 = 0, \quad \partial_\nu u_0 = 0,$$

which implies $\nabla_z u_0 = 0$. Order ϵ gives

$$c_0 \partial_x u_0 = \partial_{xx} u_0 + \Delta_z u_1, \quad \partial_\nu u_1 = g(u_0, z). \tag{4.21}$$

Averaging over Ω results in

$$c_0 \partial_x u_0 = \partial_{xx} u_0 + \overline{g}(u_0) \tag{4.22}$$

with $\overline{g}(s) := \frac{1}{|\Omega|} \int\limits_{\partial\Omega} g(s,z)\, dT_z$. From (4.21) and (4.22) we have

$$\Delta_z u_1 = \overline{g}(u_0), \quad \partial_\nu u_1 = g(u_0, z) \tag{4.23}$$

(4.22) gives the averaged one dimensional front. The influence of a heater with a finite thickness is seen in the next higher order problem.

$$c_1 \partial_x u_0 + c_0 \partial_x u_1 = \partial_{xx} u_1 + \Delta_z u_2, \quad \partial_\nu u_2 = \partial_u g(u_0, z) u_1.$$

Let $\overline{u}_1(x) = \frac{1}{|\Omega|}\int\limits_{\Omega} u_1(x,z)\,dz$ and average:

$$c_1\partial_x u_0 + c_0\partial_x\overline{u}_1 = \partial_{xx}\overline{u}_1 + \frac{1}{|\Omega|}\int\limits_{\partial\Omega} \partial_u g(u_0,z)u_1\,dT_z$$

or

$$\partial_{xx}\overline{u}_1 - c_0\partial_x\overline{u}_1 + \partial_u\overline{g}(u_0)\overline{u}_1 = c_1\partial_x u_0 - \frac{1}{|\Omega|}\int\limits_{\partial\Omega} \partial_u g(u_0,z)(u_1-\overline{u}_1)\,dz.$$

The right hand side has to be orthogonal to the kernel of the adjoint of the linear operator on the left. The kernel of the adjoint is spanned by $\partial_x u_0 e^{-c_0x}$. Denoting $v = u_1-\overline{u}_1$ and $v^-(z) = v(-\infty,z)$ we obtain

$$c_1\int_{\mathbf{R}} |\partial_x u_0|^2 e^{-c_0x}\,dx = \frac{1}{|\Omega|}\int\limits_{\partial\Omega}\int\limits_{\mathbf{R}} \partial_u g(u_0,z)\partial_x u_0 v\,e^{-c_0x}\,dx = \frac{1}{|\Omega|}\int\limits_{\partial\Omega}\int\limits_{\mathbf{R}} \partial_x g(u_0,z)v\,e^{-c_0x}\,dx. \tag{4.24}$$

Differentiating (4.23) with respect to x gives

$$\Delta_z\partial_x(v-v^-) = \partial_x\overline{g}(u_0), \quad \partial_\nu\partial_x(v-v^-) = \partial_x g(u_0,z).$$

Multiply by $(v-v^-)e^{-c_0x}$ and integrate

$$\frac{1}{2}\int\limits_{\Omega} \partial_x|\nabla_z(v-v^-)|^2 e^{-c_0x}\,dz\,dx = \int\limits_{\partial\Omega} \partial_x g(u_0,z)(v-v^-)e^{-c_0x}\,dz\,dx,$$

since $v-v^-$ has mean value zero. Multiply by e^{-c_0x} and integrate over $\mathbf{R}$:

$$\frac{c_0}{2}\int\limits_{\mathbf{R}\times\Omega} |\nabla_z(v-v^-)|^2 e^{-c_0x}\,dz\,dx = \frac{1}{2}\int\limits_{\mathbf{R}\times\Omega} \partial_x|\nabla_z(v-v^-)|^2 e^{-c_0x}\,dz\,dx$$

$$= \frac{1}{|\Omega|}\int\limits_{\mathbf{R}\times\partial\Omega} \partial_x g(u_0,z)(v-v^-)e^{-c_0x}\,dz = \frac{1}{|\Omega|}\int\limits_{\mathbf{R}\times\partial\Omega} \partial_x g(u_0,z)ve^{-c_0x}\,dz.$$

In the last equality we used

$$\int\limits_{\mathbf{R}} \partial_x\overline{g}(u_0)e^{-c_0x}\,dx = c_0\int\limits_{\mathbf{R}} \overline{g}(u_0)e^{-c_0x}\,dx = -\int\limits_{\mathbf{R}} \partial_x\left(\partial_x u_0 e^{-c_0x}\right)\,dx = 0$$

Plug this into (4.24) gives the final result for c_1:

$$c_1\int_{\mathbf{R}} |\partial_x u_0|^2 e^{-c_0x}\,dx = \frac{c_0}{2|\Omega|}\int\limits_{\mathbf{R}\times\Omega} |\nabla_z(v-v^-)|^2 e^{-c_0x}\,dz\,dx.$$

Since c_1 is positive this implies an increase of the speed for thin heaters compared to the one dimensional problem.
The expression for c_1 can be simplified for the following special class of nonlinearities

$$g(u,z) = a(z)h(u) + b(z).$$

This kind of nonlinearities occurs in the model for the heater, where a constant heat flux is supplied on some part of the boundary and another part is in contact with the fluid. Averaging gives

$$\overline{g}(u) = \overline{a}h(u) + \overline{b}, \quad \text{and} \quad 0 = \overline{a}h(u_0^-) + \overline{b}.$$

v can be written in the following form:

$$v(x,z) = \chi(z)h(u_0(x)) + \eta(z), \quad v^-(z) = \chi(z)h(u_0^-) + \eta(z)$$

with

$$\Delta_z \chi = \overline{a} = \frac{1}{|\Omega|}\int_{\partial\Omega} a, \quad \partial_\nu \chi = a, \quad \int_\Omega \chi = 0$$

$$\Delta_z \eta = \overline{b} = \frac{1}{|\Omega|}\int_{\partial\Omega} b, \quad \partial_\nu \eta = b, \quad \int_\Omega \eta = 0$$

$$\int_\Omega |\nabla_z(v - v^-)|^2 \, dz = (h(u_0) - h(u_0^-))^2 \int_\Omega |\nabla_z \chi|^2 \, dz = \frac{\overline{g}(u_0)^2}{\overline{a}^2}\int_\Omega |\nabla_z \chi|^2 \, dz$$

If the cross section is a one dimensional interval this can be calculated explicitly: $\Omega = (0,1)$, $a(1) = 1$, $a(0) = 0$, $\overline{a} = 1$, $b(1) = 0$, $b(0) = \beta$, $\overline{b} = \beta$

$$\chi = \frac{z^2}{2} - \frac{1}{6}, \quad \eta = \beta\left(\frac{1}{2}z^2 - z + \frac{1}{3}\right)$$

$$\int_\Omega |\nabla_z(v - v^-)|^2 \, dz = \frac{1}{3}\overline{g}(u_0)^2$$

$$c_\epsilon = c_0 + \epsilon\frac{c_0}{6|\Omega|}\int_{\mathbf{R}} \overline{g}(u_0)^2 e^{-c_0 x} \, dx \left(\int_{\mathbf{R}} |\partial_x u_0|^2 e^{-c_0 x} \, dx\right)^{-1} + o(\epsilon)$$

Also the rest states can be computed explicitly for all ϵ:

$$u_i(z) = \alpha_i + \epsilon\beta(1 - z)$$

where α_i are the solutions of $-\beta = h(\alpha)$. Since $h(u)$ has a cubic shape we assume that there are exactly three solutions $\alpha_1 < \alpha_2 < \alpha_3$. Their energy is easily calculated:

$$K(u_i) = \frac{1}{2}\int_0^1 |\partial_z u_i|^2 \, dz - \epsilon G(u_i(1), 1) - \epsilon G(u_i(0), 0) = \frac{\epsilon^2\beta^2}{2} - \epsilon H(\alpha_i) - \epsilon\beta(\epsilon\beta + \alpha_i),$$

where $H(u) = \int_0^u h(s) \, ds$. The second variation is

$$\int_0^1 |\partial_z w|^2 \, dz - \epsilon\partial_u g(u_i(1), 1)w(1)^2 - \epsilon\partial_u g(u_i(0), 0)w(0)^2 = \int_0^1 |\partial_z w|^2 \, dz - \epsilon\partial_u h(\alpha_i)w(1)^2.$$

The stable solutions correspond to $\partial_u h(\alpha_i) < 0$. We assume this for $i = 1, 3$. The energy difference is

$$K(u_3) - K(u_1) = -\epsilon\int_{\alpha_1}^{\alpha_3} h(s) \, ds - \epsilon\beta(\alpha_3 - \alpha_1).$$

If $K(u_1) > K(u_3)$ then all assumptions of the existence theorem are satisfied and there exists a front u with $u^- = u_1$ and $u^+ = u_3$, since u_1, u_3 are the only stable states. If $K(u_3) > K(u_1)$ then $u^- = u_3$, $u^+ = u_1$.

In the applications the stability and the domain of attraction of these fronts is an important problem in order to control the heater. It is conceivable that this could investigated by the methods developed in [12] for a related problem.

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