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A Class of Special Matrices and Quantum Entanglement

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Abstract

We present a kind of construction for a class of special matrices with at most two different eigenvalues, in terms of some interesting multiplicators which are very useful in calculating eigenvalue polynomials of these matrices. This class of matrices defines a special kind of quantum states — d-computable states. The entanglement of formation for a large class of quantum mixed states is explicitly presented.

Quantum entangled states are playing an important role in quantum communication, information processing and quantum computing [1], especially in the investigation of quantum teleportation [2, 3], dense coding [5], decoherence in quantum computers and the evaluation of quantum cryptographic schemes [6]. To quantify entanglement, a number of entanglement measures such as the entanglement of formation and distillation [7, 8, 9], negativity [10, 11], relative entropy [9, 12] have been proposed for bipartite states [6, 8] [11-13]. Most of these measures of entanglement involve extremizations which are difficult to handle analytically. For instance the entanglement of formation [7] is intended to quantify the amount of quantum communication required to create a given state. For the entanglement of a pair of qubits, it has been shown that the entanglement of formation can be expressed as a monotonically increasing function of the "concurrence", which can be taken as a measure of entanglement in its own right [15]. From the expression of this concurrence, the entanglement of formation for mixed states of a pair of qubits is calculated [15]. Although entanglement of formation is defined for arbitrary dimension, so far no explicit analytic

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formulae for entanglement of formation have been found for systems larger than a pair of qubits, except for some special symmetric states [18].

For a multipartite quantum system, the degree of entanglement will neither increase nor decrease under local unitary transformations on a subquantum system. Therefore the measure of entanglement must be an invariant of local unitary transformations. The entanglements have been studied in the view of this kind of invariants and a generalized formula of concurrence for high dimensional bipartite and multipartite systems is derived from the relations among these invariants [19]. The generalized concurrence can be used to deduce necessary and sufficient separability conditions for some high dimensional mixed states [20]. However in general the generalized concurrence is not a suitable measure for N-dimensional bipartite quantum pure states, except for $N = 2$. And it does not help in calculating the entanglement of formation for bipartite mixed states.

Let $\mathcal{H}$ be an N-dimensional complex Hilbert space with orthonormal basis e_i, $i = 1, ..., N$. A general bipartite pure state on $\mathcal{H} \otimes \mathcal{H}$ is of the form,

$$|\psi >= \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j, \qquad a_{ij} \in \mathbb{C} \tag{1}$$

with normalization $\sum_{i,j=1}^{N} a_{ij} a_{ij}^* = 1$. The entanglement of formation is given by

$$E(|\psi >) = -Tr(\rho_0 \log_2 \rho_0), \tag{2}$$

where ρ_0 is the partial trace of $|\psi >< \psi|$ over one of the subsystems. Let A denote the matrix given by $(A)_{ij} = a_{ij}$, then $\rho_0 = AA^\dagger$.

For a given density matrix of a pair of quantum systems on $\mathcal{H} \otimes \mathcal{H}$, consider all possible pure-state decompositions of ρ, i.e., all ensembles of states $|\Psi_i\rangle$ of the form (1) with probabilities $p_i \geq 0$,

$$\rho = \sum_{i=1}^{l} p_i |\psi_i\rangle\langle\psi_i|, \qquad \sum_{i=1}^{l} p_i = 1$$

for some $l \in \mathbb{N}$. The entanglement of formation for the mixed state ρ is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ,

$$E(\rho) = \min \sum_{i=1}^{l} p_i E(|\psi_i\rangle). \tag{3}$$

It is a challenge to calculate (3) for general N. Till now a general explicit formula of $E(\rho)$ is obtained only for the case $N=2$. In this case (2) can be written as

$$E(|\psi\rangle)|_{N=2} = h(\frac{1+\sqrt{1-C^2}}{2}),$$

where

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x),$$

C is called concurrence [15]:

$$C(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where $|\tilde{\psi}\rangle = \sigma_y \otimes \sigma_y |\psi^*\rangle$, $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$, σ_y is the Pauli matrix, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

As E is a monotonically increasing function of C, C can be also taken as a kind of measure of entanglement. Calculating (3) is reduced to calculate the corresponding minimum of $C(\rho) = \min \sum_{i=1}^{M} p_i C(|\psi_i\rangle)$, which simplifies the problems.

For $N \geq 3$, there is no such concurrence C in general. The concurrences discussed in [19] can be only used to judge whether a pure state is separable (or maximally entangled) or not [20]. The entanglement of formation is no longer a monotonically increasing function of these concurrences. Nevertheless, for a special class of quantum states such that $AA^\dagger$ has only two non-zero eigenvalues, we can find certain quantities (generalized concurrence) to simplify the calculation of the corresponding entanglement of formation [21].

Let λ_1 (resp. λ_2) be the two non-zero eigenvalues of $AA^\dagger$ with degeneracy n (resp. m), $n+m \leq N$, and the maximal non-zero diagonal determinant D,

$$D = \lambda_1^n \lambda_2^m \,. \tag{4}$$

¿From the normalization of $|\psi\rangle$, one has $Tr(AA^\dagger) = 1$, i.e.,

$$n\lambda_1 + m\lambda_2 = 1 \,. \tag{5}$$

λ_1 (resp. λ_2) takes values $(0, \frac{1}{n})$ (resp. $(0, \frac{1}{m})$). In this case the entanglement of formation of $|\psi\rangle$ is given by

$$E(|\psi\rangle) = -n\lambda_1 \log_2 \lambda_1 - m\lambda_2 \log_2 \lambda_2 \,. \tag{6}$$

According to (4) and (5) we get

$$\frac{\partial E}{\partial D} = \frac{m\lambda_1^{1-n}}{1 - n\lambda_1 - m\lambda_1} \left(\frac{1 - n\lambda_1}{m\lambda_1}\right)^{1-m} \log_2 \frac{1 - n\lambda_1}{m\lambda_1}\,, \tag{7}$$

which is positive for $\lambda_1 \in (0, \frac{1}{n})$. Therefore $E(|\psi\rangle)$ is a monotonically increasing function of D. D is a generalized concurrence and can be taken as a kind of measure of entanglement in this case. Here we have assumed that λ_1, $\lambda_2 \neq 0$. In fact the right hand side of (7) keeps positive even when λ_1 (or equivalently λ_2) goes to zero. Hence $E(|\psi\rangle)$ is a monotonically increasing function of D for $\lambda_1 \in [0, \frac{1}{n}]$ (resp. $\lambda_2 \in [0, \frac{1}{m}]$) satisfying the relation (5). Nevertheless if $\lambda_1 = 0$ (or $\lambda_2 = 0$), from (4) one gets $D = 0$, which does not necessarily mean that the corresponding state $|\psi\rangle$ is separable. As $E(|\psi\rangle)$ is just a monotonically increasing function of D, D only characterizes the relative degree of the entanglement among the class of these states.

The quantum states with the measure of entanglement characterized by D are generally entangled. They are separable when $n = 1$, $\lambda_1 \to 1$ ($\lambda_2 \to 0$) or $m = 1$, $\lambda_2 \to 1$ ($\lambda_1 \to 0$). For the case $n = m > 1$, all the pure states in this class are non-separable. In this case, we have

$$E(|\psi\rangle) = n\left(-x \log_2 x - (\frac{1}{n} - x)\log_2(\frac{1}{n} - x)\right),$$

where

$$x = \frac{1}{2}\left(\frac{1}{n} + \sqrt{\frac{1}{n^2}(1 - d^2)}\right)$$

and

$$d \equiv 2nD^{\frac{1}{2n}} = 2n\sqrt{\lambda_1\lambda_2}. \tag{8}$$

We define d to be the generalized concurrence in this case. Instead of calculating $E(\rho)$ directly, one may calculate the minimum decomposition of $D(\rho)$ or $d(\rho)$ to simplify the calculations.

In [21] a class of pure states (1) with the matrix A given by

$$A = \left(\begin{array}{cccc} 0 & b & a_1 & b_1 \\ -b & 0 & c_1 & d_1 \\ a_1 & c_1 & 0 & -e \\ b_1 & d_1 & e & 0 \end{array}\right), \tag{9}$$

$a_1, b_1, c_1, d_1, b, e \in \mathbb{C}$, is considered. The matrix $AA^\dagger$ has two eigenvalues with degeneracy two, i.e., $n = m = 2$.

$$|AA^\dagger| = |b_1c_1 - a_1d_1 + be|^4. \tag{10}$$

The generalized concurrence is given by

$$d = 4|b_1 c_1 - a_1 d_1 + be|. \tag{11}$$

Let p be a 16×16 matrix with only non-zero entries $p_{1,16} = p_{2,15} = -p_{3,14} = p_{4,10} = p_{5,12} = p_{6,11} = p_{7,13} = -p_{8,8} = -p_{9,9} = p_{10,4} = p_{11,6} = p_{12,5} = p_{13,7} = -p_{14,3} = p_{15,2} = p_{16,1} = 1$. d in (11) can be written as

$$d = |\langle\psi|p\psi^*\rangle| \equiv |\langle\langle\psi|\psi\rangle\rangle|, \tag{12}$$

where $\langle\langle\psi|\psi\rangle\rangle = \langle\psi|p\psi^*\rangle$.

Let Ψ denote the set of pure states (1) with A given as the form of (9). Consider all mixed states with density matrix ρ such that its decompositions are of the form

$$\rho = \sum_{i=1}^{M} p_i |\psi_i\rangle\langle\psi_i|, \qquad \sum_{i=1}^{M} p_i = 1, \qquad |\psi_i\rangle \in \Psi. \tag{13}$$

The minimum decomposition of the generalized concurrence is [21]

$$d(\rho) = \Lambda_1 - \sum_{i=2}^{16} \Lambda_i, \tag{14}$$

where Λ_i, in decreasing order, are the square roots of the eigenvalues of the Hermitian matrix $R \equiv \sqrt{\sqrt{\rho}p\rho^* p\sqrt{\rho}}$, or, alternatively, the square roots of the eigenvalues of the non-Hermitian matrix $\rho p\rho^* p$. Similar to the case $N = 2$, there are decompositions such that the generalized concurrence of each individual state is equal to $d(\rho)$. Therefore the average entanglement is $E(d(\rho))$. Different from the case $N = 2$, the entanglement of formation of density matrices (13) can not be zero in general. As every individual pure state in the decompositions is generally an entangled one, this class of mixed states are not separable.

In the following we call an N-dimensional pure state (1) <u>d-computable</u> if A satisfies the following relations:

$$\begin{array}{l} |AA^\dagger| = ([A][A]^*)^{N/2}, \\ |AA^\dagger - \lambda Id_N| = (\lambda^2 - \|A\|\lambda + [A][A]^*)^{N/2}, \end{array} \tag{15}$$

where $[A]$ and $\|A\|$ are quadratic forms of a_{ij}, Id_N is the $N \times N$ identity matrix. We denote $\mathcal{A}$ the set of matrices satisfying (15), which implies that for $A \in \mathcal{A}$, $AA^\dagger$ has at most two different eigenvalues and each one has order $N/2$. Formula (14) can be generalized to general $N^2 \times N^2$ density matrices with decompositions on N-dimensional d-computable pure states.

We first present a kind of construction for a class of N-dimensional, $N = 2^k$, $2 \le k \in I\!N$, d-computable states. Set

$$A_2 = \begin{pmatrix} a & -c \\ c & d \end{pmatrix},$$

where $a, c, d \in \mathbb{C}$. For any $b_1, c_1 \in \mathbb{C}$, a 4×4 matrix $A_4 \in \mathcal{A}$ can be constructed in the following way,

$$A_4 = \begin{pmatrix} B_2 & A_2 \\ -A_2^t & C_2^t \end{pmatrix} = \begin{pmatrix} 0 & b_1 & a & -c \\ -b_1 & 0 & c & d \\ -a & -c & 0 & -c_1 \\ c & -d & c_1 & 0 \end{pmatrix}, \tag{16}$$

where

$$B_2 = b_1 J_2, \quad C_2 = c_1 J_2, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A_4 satisfies the relations in (15):

$$\begin{aligned} \left|A_4 A_4^\dagger\right| &= [(b_1c_1 + ad + c^2)(b_1c_1 + ad + c^2)^*]^2 = ([A_4][A_4]^*)^2, \\ \left|A_4 A_4^\dagger - \lambda I d_4\right| &= (\lambda^2 - (b_1 b_1^* + c_1 c_1^* + aa^* + 2cc^* + dd^*)\lambda \\ &\quad + (b_1c_1 + ad + c^2)(b_1c_1 + ad + c^2)^*)^2 \\ &= (\lambda^2 - \|A_4\|\lambda + [A_4][A_4]^*)^2, \end{aligned}$$

where

$$[A_4] = (b_1c_1 + ad + c^2), \quad \|A_4\| = b_1 b_1^* + c_1 c_1^* + aa^* + 2cc^* + dd^*. \tag{17}$$

$A_8 \in \mathcal{A}$ can be obtained from A_4,

$$A_8 = \begin{pmatrix} B_4 & A_4 \\ -A_4^t & C_4^t \end{pmatrix}, \tag{18}$$

where

$$B_4 = b_2 J_4, \quad C_4 = c_2 J_4, \quad J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad b_2,\ c_2 \in \mathbb{C}. \tag{19}$$

For general construction of high dimensional matrices $A_{2^{k+1}} \in \mathcal{A}$, $2 \le k \in I\!N$, we have

$$A_{2^{k+1}} = \begin{pmatrix} B_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A_{2^k}^t & C_{2^k}^t \end{pmatrix} \equiv \begin{pmatrix} b_k J_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A_{2^k}^t & c_k J_{2^k}^t \end{pmatrix}, \tag{20}$$

$$J_{2^{k+1}}=\begin{pmatrix} 0 & J_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^t & 0 \end{pmatrix}, \tag{21}$$

where $b_k, c_k \in \mathbb{C}$, $B_{2^k}=b_kJ_{2^k}$, $C_{2^k}=c_kJ_{2^k}$. We call $J_{2^{k+1}}$ multipliers. Before proving that $A_{2^{k+1}}\in\mathcal{A}$, we first give the following lemma.

[Lemma 1]. $A_{2^{k+1}}$ and $J_{2^{k+1}}$ satisfy the following relations:

$$\begin{aligned} J_{2^{k+1}}^tJ_{2^{k+1}} &= J_{2^{k+1}}J_{2^{k+1}}^t = Id_{2^{k+1}},\\ J_{2^{k+1}}^tJ_{2^{k+1}}^t &= J_{2^{k+1}}J_{2^{k+1}} = (-1)^{\frac{(k+1)(k+2)}{2}}Id_{2^{k+1}}, \end{aligned} \tag{22}$$

$$\begin{aligned} J_{2^{k+1}}^\dagger &= J_{2^{k+1}}^t, & J_{2^{k+1}}^t &= (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^{k+1}},\\ A_{2^{k+1}}^t &= (-1)^{\frac{k(k+1)}{2}}A_{2^{k+1}}, & A_{2^{k+1}}^\dagger &= (-1)^{\frac{k(k+1)}{2}}A_{2^{k+1}}^*. \end{aligned} \tag{23}$$

[Proof]. One easily checks that relations in (22) hold for $k=1$. Suppose (22) hold for general k. We have

$$\begin{aligned} J_{2^{k+1}}^tJ_{2^{k+1}} &= \begin{pmatrix} 0 & (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^t \\ J_{2^k} & 0 \end{pmatrix}\begin{pmatrix} 0 & J_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^t & 0 \end{pmatrix}\\ &= \begin{pmatrix} (-1)^{(k+1)(k+2)}J_{2^k}J_{2^k}^t & 0 \\ 0 & J_{2^k}^tJ_{2^k} \end{pmatrix} = Id_{2^{k+1}} \end{aligned}$$

and

$$\begin{aligned} J_{2^{k+1}}^tJ_{2^{k+1}}^t &= \begin{pmatrix} 0 & (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^t \\ J_{2^k} & 0 \end{pmatrix}\begin{pmatrix} 0 & (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^t \\ J_{2^k} & 0 \end{pmatrix}\\ &= \begin{pmatrix} (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}J_{2^k}^t & 0 \\ 0 & (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}^tJ_{2^k} \end{pmatrix} = (-1)^{\frac{(k+1)(k+2)}{2}}Id_{2^{k+1}}. \end{aligned}$$

Therefore the relations for $J_{2^{k+1}}^tJ_{2^{k+1}}$ and $J_{2^{k+1}}^tJ_{2^{k+1}}^t$ are valid also for $k+1$. The cases for $J_{2^{k+1}}J_{2^{k+1}}^t$ and $J_{2^{k+1}}J_{2^{k+1}}$ can be similarly treated.

The formula $J_{2^{k+1}}^t=(-1)^{\frac{(k+1)(k+2)}{2}}J_{2^{k+1}}$ in (23) is easily deduced from (22) and the fact $J_{2^{k+1}}^\dagger=J_{2^{k+1}}^t$.

The last two formulae in (23) are easily verified for $k=1$. If it holds for general k, we have then,

$$A^t_{2^{k+1}} = \left(\begin{array}{cc} B^t_{2^k} & (-1)^{\frac{k(k+1)}{2}} A_{2^k} \\ A^t_{2^k} & C_{2^k} \end{array} \right) = \left(\begin{array}{cc} (-1)^{\frac{k(k+1)}{2}} B_{2^k} & (-1)^{\frac{k(k+1)}{2}} A_{2^k} \\ A^t_{2^k} & (-1)^{\frac{k(k+1)}{2}} C^t_{2^k} \end{array} \right) = (-1)^{\frac{k(k+1)}{2}} A_{2^{k+1}},$$

i.e., it holds also for $k+1$. The last equality in (23) is obtained from the conjugate of the formula above. ■

¿From Lemma 1 the following equations can be deduced:

[Lemma 2].

$$\begin{array}{c} B^t_{2^k} = (-1)^{\frac{k(k+1)}{2}} B_{2^k}, \quad C^t_{2^k} = (-1)^{\frac{k(k+1)}{2}} C_{2^k}, \\ B^\dagger_{2^k} = (-1)^{\frac{k(k+1)}{2}} B^*_{2^k}, \quad C^\dagger_{2^k} = (-1)^{\frac{k(k+1)}{2}} C^*_{2^k}. \end{array} \tag{24}$$

$$B^{-1}_{2^{k+1}} = \frac{1}{b_k^2} B^t_{2^{k+1}} = \frac{1}{b_k b_k^*} B^\dagger_{2^{k+1}}, \quad C^{-1}_{2^{k+1}} = \frac{1}{c_k^2} C^t_{2^{k+1}} = \frac{1}{c_k c_k^*} C^\dagger_{2^{k+1}}. \tag{25}$$

$$\begin{array}{c} B^t_{2^{k+1}} B_{2^{k+1}} = B_{2^{k+1}} B^t_{2^{k+1}} = b_k^2 Id_{2^{k+1}}, \quad C^t_{2^{k+1}} C_{2^{k+1}} = C_{2^{k+1}} C^t_{2^{k+1}} = c_k^2 Id_{2^{k+1}}, \\ B^\dagger_{2^{k+1}} B_{2^{k+1}} = B_{2^{k+1}} B^\dagger_{2^{k+1}} = b_k b_k^* Id_{2^{k+1}}, \quad C^\dagger_{2^{k+1}} C_{2^{k+1}} = C_{2^{k+1}} C^\dagger_{2^{k+1}} = c_k c_k^* Id_{2^{k+1}}. \end{array} \tag{26}$$

For any $A_{2^{k+1}} \in \mathcal{A}$, $k \geq 2$, we define

$$\begin{array}{rcl} ||A_{2^{k+1}}|| & =: & b_k b_k + c_k c_k + ||A_{2^k}||, \\ [A_{2^{k+1}}] & =: & (-1)^{k(k+1)/2} b_k c_k - [A_{2^k}]. \end{array} \tag{27}$$

[Lemma 3]. For any $k \geq 2$, we have,

$$\begin{array}{rcl} (A_{2^{k+1}} J_{2^{k+1}})(J_{2^{k+1}} A_{2^{k+1}})^t & = & (A_{2^{k+1}} J_{2^{k+1}})^t (J_{2^{k+1}} A_{2^{k+1}}) \\ & = & ((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}]) Id_{2^{k+1}} = [A_{2^{k+1}}] Id_{2^{k+1}}, \\ (A^*_{2^{k+1}} J_{2^{k+1}})(J_{2^{k+1}} A^*_{2^{k+1}})^t & = & (A^*_{2^{k+1}} J_{2^{k+1}})^t (J_{2^{k+1}} A^*_{2^{k+1}}) = [A_{2^{k+1}}]^* Id_{2^{k+1}}. \end{array} \tag{28}$$

[Proof]. One can verify that Lemma 3 holds for $k=2$. Suppose it is valid for k, we have

$$(A_{2^{k+1}} J_{2^{k+1}})(J_{2^{k+1}} A_{2^{k+1}})^t$$

$$= \left(\begin{array}{cc} (-1)^{\frac{(k+1)(k+2)}{2}} A_{2^k} J^t_{2^k} & B_{2^k} J_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}} C^t_{2^k} J^t_{2^k} & (-1)^{\frac{k(k+1)}{2}} A^t_{2^k} J_{2^k} \end{array} \right) \left(\begin{array}{cc} (-1)^{\frac{k(k+1)}{2}} J_{2^k} A^t_{2^k} & J_{2^k} C^t_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}} J^t_{2^k} B_{2^k} & (-1)^{\frac{(k+1)(k+2)}{2}} J^t_{2^k} A_{2^k} \end{array} \right)^t$$

$$= \left(\begin{array}{cc} e_{11} & e_{12} \\ e_{21} & e_{22} \end{array} \right),$$

where

$$\begin{aligned}
e_{11} &= (-1)^{\frac{(k+1)(k+2)+k(k+1)}{2}} A_{2^k} J^t_{2^k} A_{2^k} J^t_{2^k} + (-1)^{\frac{k(k+1)}{2}} b_k c_k Id_{2^k} \\
&= (-1)^{\frac{(k+1)(k+2)+k(k-1)}{2}} (A_{2^k} J^t_{2^k})(J_{2^k} A^t_{2^k})^t + (-1)^{\frac{k(k+1)}{2}} b_k c_k Id_{2^k} \\
&= ((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}]) Id_{2^k}, \\
e_{12} &= b_k A_{2^k} J^t_{2^k} + (-1)^{\frac{(k+1)(k+2)+k(k+1)}{2}} b_k A^t_{2^k} J_{2^k} \\
&= b_k A_{2^k} J^t_{2^k} (1 + (-1)^{\frac{(k+1)(k+2)+k(k-1)}{2}}) = 0, \\
e_{21} &= (-1)^{\frac{(k+1)(k+2)}{2}} c_k A_{2^k} J^t_{2^k} + (-1)^{\frac{k(k+1)}{2}} c_k A^t_{2^k} J_{2^k} = 0, \\
e_{22} &= (-1)^{\frac{k(k+1)}{2}} b_k c_k Id_{2^k} + (-1)^{\frac{(k+1)(k+2)+k(k+1)}{2}} A^t_{2^k} J_{2^k} A^t_{2^k} J_{2^k} \\
&= (-1)^{\frac{k(k+1)}{2}} b_k c_k Id_{2^k} + (-1)^{\frac{(k+1)(k+2)+k(k-1)}{2}} (A_{2^k} J_{2^k})(J_{2^k} A_{2^k})^t \\
&= ((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}]) Id_{2^k}.
\end{aligned}$$

Hence

$$(A_{2^{k+1}} J_{2^{k+1}})(J_{2^{k+1}} A_{2^{k+1}})^t = ((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}]) Id_{2^{k+1}} = [A_{2^{k+1}}]) Id_{2^{k+1}}.$$

Similar calculations apply to $(A_{2^{k+1}} J_{2^{k+1}})^t (J_{2^{k+1}} A_{2^{k+1}})$. Therefore the Lemma holds for $k+1$. The last equation can be deduced from the first one.

[Theorem 2]. A_{2^k} satisfies the following relation:

$$|A_{2^{k+1}} A^\dagger_{2^{k+1}}| = ([A_{2^{k+1}}][A_{2^{k+1}}]^*)^{2^k} = [((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}])((-1)^{\frac{k(k+1)}{2}} b^*_k c^*_k - [A_{2^k}]^*)]^{2^k}. \tag{29}$$

[Proof]. By using Lemma 1-3, we have

$$\begin{aligned}
|A_{2^{k+1}}| &= \left| \begin{pmatrix} B_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A^t_{2^k} & C^t_{2^k} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} Id_{2^k} & -A_{2^k}(C^t_{2^k})^{-1} \\ 0 & Id_{2^k} \end{pmatrix} \begin{pmatrix} B_{2^k} & A_{2^k} \\ (-1)^{\frac{k(k+1)}{2}} A^t_{2^k} & C^t_{2^k} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} B_{2^k} - (-1)^{\frac{k(k+1)}{2}} A_{2^k}(C^t_{2^k})^{-1} A^t_{2^k} & 0 \\ (-1)^{\frac{k(k+1)}{2}} A^t_{2^k} & C^t_{2^k} \end{pmatrix} \right| \\
&= |b_k c_k J_{2^k} J^t_{2^k} - (-1)^{\frac{k(k+1)}{2}} \frac{1}{c_k^2} A_{2^k} C_{2^k} A^t_{2^k} C^t_{2^k}| \\
&= |b_k c_k Id_{2^k} - (-1)^{\frac{k(k+1)}{2}} (A_{2^k} J_{2^k})(J_{2^k} A_{2^k})^t| \\
&= |(-1)^{\frac{k(k+1)}{2}} b_k c_k Id_{2^k} - [A_{2^k}] Id_{2^k}| = ((-1)^{\frac{k(k+1)}{2}} b_k c_k - [A_{2^k}])^{2^k}.
\end{aligned}$$

Therefore

$$|A_{2^{k+1}}A^{\dagger}_{2^{k+1}}| = ([A_{2^{k+1}}][A_{2^{k+1}}]^*)^{2^k}.$$

■

[Lemma 4]

$$\begin{aligned}
&(A_{2^{k+1}}J_{2^{k+1}})(J_{2^{k+1}}A_{2^{k+1}})^{\dagger} + (J_{2^{k+1}}A^*_{2^{k+1}})(J_{2^{k+1}}A_{2^{k+1}})^t \\
&\qquad = A_{2^{k+1}}A^{\dagger}_{2^{k+1}} + J_{2^{k+1}}A^*_{2^{k+1}}A^t_{2^{k+1}}J^t_{2^{k+1}} = ||A_{2^{k+1}}||Id_{2^{k+1}}, \\
&(A_{2^{k+1}}J_{2^{k+1}})^t(A^*_{2^{k+1}}J_{2^{k+1}}) + (J_{2^{k+1}}A_{2^{k+1}})^{\dagger}(J_{2^{k+1}}A_{2^{k+1}}) \\
&\qquad = A^{\dagger}_{2^{k+1}}A_{2^{k+1}} + J^t_{2^{k+1}}A^t_{2^{k+1}}A^*_{2^{k+1}}J_{2^{k+1}} = ||A_{2^{k+1}}||Id_{2^{k+1}}.
\end{aligned}$$

It can be verified that the first formula holds for $k = 2$, if it holds for k, we have

$$\begin{aligned}
&(A_{2^{k+1}}J_{2^{k+1}})(A_{2^{k+1}}J_{2^{k+1}})^{\dagger} + (J_{2^{k+1}}A^*_{2^{k+1}})(J_{2^{k+1}}A_{2^{k+1}})^t \\
&= \begin{pmatrix} (-1)^{\frac{(k+1)(k+2)}{2}}A_{2^k}J^t_{2^k} & B_{2^k}J_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}}C^t_{2^k}J^t_{2^k} & (-1)^{\frac{k(k+1)}{2}}A^t_{2^k}J_{2^k} \end{pmatrix} \begin{pmatrix} (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}A^{\dagger}_{2^k} & (-1)^{\frac{(k+1)(k+2)}{2}}J_{2^k}C^*_{2^k} \\ J^t_{2^k}B^{\dagger}_{2^k} & (-1)^{\frac{k(k+1)}{2}}J^t_{2^k}A^*_{2^k} \end{pmatrix} \\
&+ \begin{pmatrix} (-1)^{\frac{k(k+1)}{2}}J_{2^k}A^{\dagger}_{2^k} & J_{2^k}C^{\dagger}_{2^k} \\ (-1)^{\frac{(k+1)(k+2)}{2}}J^t_{2^k}B^*_{2^k} & (-1)^{\frac{(k+1)(k+2)}{2}}J^t_{2^k}A^*_{2^k} \end{pmatrix} \begin{pmatrix} (-1)^{\frac{k(k+1)}{2}}A_{2^k}J^t_{2^k} & (-1)^{\frac{(k+1)(k+2)}{2}}B^t_{2^k}J_{2^k} \\ C_{2^k}J^t_{2^k} & (-1)^{\frac{(k+1)(k+2)}{2}}A^t_{2^k}J_{2^k} \end{pmatrix} \\
&= \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},
\end{aligned}$$

where, by using Lemma 1 and 2,

$$\begin{aligned}
f_{11} = f_{22} &= A_{2^k}A^{\dagger}_{2^k} + J_{2^k}A^{\dagger}_{2^k}A_{2^k}J^t_{2^k} + BB^{\dagger} + J_{2^k}C^{\dagger}CJ^t_{2^k} \\
&= A_{2^k}A^{\dagger}_{2^k} + J_{2^k}(-1)^{\frac{k(k+1)}{2}}A^*_{2^k}(-1)^{\frac{k(k+1)}{2}}A^t_{2^k}J^t_{2^k} + (b_kb^*_k + c_kc^*_k)Id_{2^k} \\
&= (||A_{2^k}|| + b_kb^*_k + c_kc^*_k)Id_{2^k} = ||A_{2^{k+1}}||Id_{2^k},
\end{aligned}$$

$$\begin{aligned}
f_{12} &= A_{2^k}C^*_{2^k} + (-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + (-1)^{\frac{k(k+1)+(k+1)(k+2)}{2}}b_kJ_{2^k}A^{\dagger}_{2^k} + (-1)^{\frac{(k+1)(k+2)}{2}}c^*_kA^t_{2^k}J_{2^k} \\
&= (-1)^{\frac{k(k+1)}{2}}(B_{2^k}A^*_{2^k} + (-1)^{\frac{k(k-1)+(k+1)(k+2)}{2}}B_{2^k}A^*_{2^k}) \\
&\quad + A_{2^k}C^*_{2^k} + (-1)^{\frac{k(k-1)+(k+1)(k+2)}{2}}A_{2^k}C^*_{2^k} = 0, \\
f_{21} &= C^t_{2^k}A^{\dagger}_{2^k} + (-1)^{\frac{k(k+1)}{2}}A^t_{2^k}B^{\dagger}_{2^k} + (-1)^{\frac{k(k+1)+(k+1)(k+2)}{2}}b^*_kA_{2^k}J^t_{2^k} + (-1)^{\frac{(k+1)(k+2)}{2}}c_kJ^t_{2^k}A^*_{2^k} \\
&= (-1)^{\frac{k(k+1)}{2}}(b_kA^t_{2^k}J^t_{2^k} + (-1)^{\frac{k(k-1)+(k+1)(k+2)}{2}}b_kA^t_{2^k}J^t_{2^k}) \\
&\quad + c^*_kJ^t_{2^k}A^t_{2^k} + (-1)^{\frac{k(k-1)+(k+1)(k+2)}{2}}c^*_kJ^t_{2^k}A^t_{2^k} = 0.
\end{aligned}$$

Hence the first formula holds also for $k + 1$. The second formula can be verified similarly. ■

[Lemma 5].

$$((-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* + A_{2^k} C_{2^k}^*)((-1)^{\frac{k(k+1)}{2}} A_{2^k}^* B_{2^k} + C_{2^k}^* A_{2^k})^t = F(A_{2^{k+1}}) Id_{2^k}, \qquad (30)$$

where

$$F(A_{2^{k+1}}) = c_k^{*2}[A_{2^k}] + b_k^2 [A_{2^k}]^* + (-1)^{\frac{k(k+1)}{2}} b_k c_k^* \| A_{2^k} \|. \qquad (31)$$

[Proof]. By using Lemma 3 and 4, we have

$$\begin{aligned}
&((-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* + A_{2^k} C_{2^k}^*)((-1)^{\frac{k(k+1)}{2}} A_{2^k}^* B_{2^k} + C_{2^k}^* A_{2^k})^t \\
&= b_k^2 (J_{2^k} A_{2^k}^*)(A_{2^k}^* J_{2^k})^t + c_k^{*2} (A_{2^k} J_{2^k})(J_{2^k} A_{2^k})^t \\
&\quad + (-1)^{\frac{k(k+1)}{2}} b_k c_k^* [(A_{2^k}^* J_{2^k})(A_{2^k}^* J_{2^k})^t + (J_{2^k} A_{2^k}^*)(J_{2^k} A_{2^k})^t] \\
&= (c_k^{*2}[A_{2^k}] + b_k^2 [A_{2^k}]^* + (-1)^{\frac{k(k+1)}{2}} b_k c_k^* \| A_{2^k} \|) Id_{2^k} = F(A_{2^{k+1}}) Id_{2^k}.
\end{aligned}$$

[Lemma 6].

$$\| A_{2^k} \| J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t = [A_{2^k}][A_{2^k}]^* Id_{2^k} + J_{2^k} A_{2^k}^* A_{2^k}^t A_{2^k}^* A_{2^k}^t J_{2^k}^t. \qquad (32)$$

[Proof]. From (30) we have the following relation:

$$\begin{aligned}
&F(A_{2^{k+1}}) J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t \\
&= ((-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* + A_{2^k} C_{2^k}^*)(-1)^{\frac{k(k+1)}{2}} b_k (A_{2^k}^* J_{2^k})^t J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t \\
&\quad + ((-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* + A_{2^k} C_{2^k}^*) c_k^* (J_{2^k} A_{2^k})^t J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t \\
&= (-1)^{\frac{k(k+1)}{2}} b_k ((-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* + A_{2^k} C_{2^k}^*)[(A_{2^k}^* J_{2^k})^t (J_{2^k} A_{2^k}^*)] A_{2^k}^t J_{2^k}^t \\
&\quad + c_k^* [(-1)^{\frac{k(k+1)}{2}} B_{2^k} A_{2^k}^* (J_{2^k} A_{2^k})^t J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t + A_{2^k} C_{2^k}^* (J_{2^k} A_{2^k})^t J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t] \\
&= (-1)^{\frac{k(k+1)}{2}} b_k ((-1)^{\frac{k(k+1)}{2}} b_k J_{2^k} A_{2^k}^* [A_{2^k}] A_{2^k}^t J_{2^k}^t + c_k^* A_{2^k} J_{2^k} [A_{2^k}]^* A_{2^k}^t J_{2^k}^t) \\
&\quad + c_k^* [(-1)^{\frac{k(k+1)}{2}} b_k J_{2^k} A_{2^k}^* A_{2^k}^t A_{2^k}^* A_{2^k}^t J_{2^k}^t + c_k^* [A_{2^k}] J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t] \\
&= b_k^2 [A_{2^k}]^* J_{2^k} A_{2^k}^* A_{2^k}^t J_{2^k}^t + (-1)^{\frac{k(k+1)}{2}} b_k c_k^* [A_{2^k}][A_{2^k}]^* Id_{2^k}) \\
&\quad + (-1)^{\frac{k(k+1)}{2}} c_k^* b_k J_{2^k} A_{2^k}^* A_{2^k}^t A_{2^k}^* A_{2^k}^t J_{2^k}^t + c_2^2 [A_{2^k}] J_{2^k} A_{2^k} A_{2^k}^t J_{2^k}^t] \\
&= (b_k^2 + c_k^2)[A_{2^k}] J_{2^k} A_{2^k} A_{2^k}^t J_{2^k}^t + (-1)^{\frac{k(k+1)}{2}} b_k c_k ([A_{2^k}]^2 Id_{2^k} \\
&\quad + (-1)^{\frac{k(k+1)}{2}} c_k b_k J_{2^k} A_{2^k} A_{2^k}^t A_{2^k} A_{2^k}^t J_{2^k}^t).
\end{aligned}$$

Using (31) we have

$$\|A_{2^k}\|J_{2^k}A^*_{2^k}A^t_{2^k}J^t_{2^k} = [A_{2^k}][A_{2^k}]^* Id_{2^k} + J_{2^k}A^*_{2^k}A^t_{2^k}A^*_{2^k}A^t_{2^k}J^t_{2^k}.$$

$\blacksquare$

[Theorem 3]. The eigenvalue polynom of $A_{2^{k+1}}A^\dagger_{2^{k+1}}$ satisfies the following relations:

$$\begin{aligned} |A_{2^{k+1}}A^\dagger_{2^{k+1}} - \lambda Id_{2^{k+1}}| &= (\lambda^2 - ||A_{2^{k+1}}||\lambda + [A_{2^{k+1}}][A_{2^{k+1}}]^*)^{2^k}, \\ |A^\dagger_{2^{k+1}}A_{2^{k+1}} - \lambda Id_{2^{k+1}}| &= (\lambda^2 - ||A_{2^{k+1}}||\lambda + [A_{2^{k+1}}][A_{2^{k+1}}]^*)^{2^k}. \end{aligned} \tag{33}$$

[Proof]. Let

$$\Lambda_k = -[(c_kc^*_k - \lambda)Id_{2^k} + A^t_{2^k}A^*_{2^k}][(-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k}]^{-1}.$$

$$\left|A_{2^{k+1}}A^\dagger_{2^{k+1}} - \lambda Id_{2^{k+1}}\right| = \left|\begin{pmatrix} (-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k} & (b_kb^*_k - \lambda)Id_{2^k} + A_{2^k}A^\dagger_{2^k} \\ (c_kc^*_k - \lambda)Id_{2^k} + A^t_{2^k}A^*_{2^k} & (-1)^{\frac{k(k+1)}{2}}A^t_{2^k}B^\dagger_{2^k} + C^t_{2^k}A^\dagger_{2^k} \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} Id_{2^k} & 0 \\ \Lambda_k & Id_{2^k} \end{pmatrix}\begin{pmatrix} (-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k} & (b_kb^*_k - \lambda)Id_{2^k} + A_{2^k}A^\dagger_{2^k} \\ (c_kc^*_k - \lambda)Id_{2^k} + A^t_{2^k}A^*_{2^k} & (-1)^{\frac{k(k+1)}{2}}A^t_{2^k}B^\dagger_{2^k} + C^t_{2^k}A^\dagger_{2^k} \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} (-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k} & (b_kb^*_k - \lambda)Id_{2^k} + A_{2^k}A^\dagger_{2^k} \\ 0 & -\Lambda_k[(b_kb^*_k - \lambda)Id_{2^k} + A_{2^k}A^\dagger_{2^k}] + (-1)^{\frac{k(k+1)}{2}}A^t_{2^k}B^\dagger_{2^k} + C^t_{2^k}A^\dagger_{2^k} \end{pmatrix}\right|$$

$$= |I + II|,$$

where

$$\begin{aligned} I &= ((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})((-1)^{\frac{k(k+1)}{2}}B^*_{2^k}A_{2^k} + A^*_{2^k}C_{2^k})^t \\ &= (-1)^{\frac{k(k+1)}{2}}b_kc_k[A_{2^k}]^*Id_{2^k} + (-1)^{\frac{k(k+1)}{2}}b^*_kc^*_k[A_{2^k}]Id_{2^k} + b_kb^*_kJ_{2^k}A^*_{2^k}A^t_{2^k}J^t_{2^k} + c_kc^*_kA_{2^k}A^\dagger_{2^k} \end{aligned}$$

and, by using Lemma 5,

$$\begin{aligned} II &= -((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})\Lambda_k[(b_kb^*_k - \lambda)Id_{2^k} + A_{2^k}A^\dagger_{2^k}] \\ &= [(c_kc^*_k - \lambda)((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k}) + ((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})A^t_{2^k}A^*_{2^k}] \\ &\quad [(b_kb^*_k - \lambda)((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})^{-1} + ((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})^{-1}A_{2^k}A^\dagger_{2^k}] \\ &= (b_kb^*_k - \lambda)(c_kc^*_k - \lambda)Id_{2^k} + (b_kb^*_k - \lambda)((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} \\ &\quad + A_{2^k}C^*_{2^k})A^t_{2^k}A^*_{2^k}((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})^{-1} + (c_kc^*_k - \lambda)A_{2^k}A^\dagger_{2^k} \\ &\quad + ((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})A^t_{2^k}A^*_{2^k}((-1)^{\frac{k(k+1)}{2}}B_{2^k}A^*_{2^k} + A_{2^k}C^*_{2^k})^{-1}A_{2^k}A^\dagger_{2^k} \\ &= (b_kb^*_k - \lambda)(c_kc^*_k - \lambda)Id_{2^k} + (c_kc^*_k - \lambda)A_{2^k}A^\dagger_{2^k} + \frac{b_kb^*_k - \lambda}{F(A_{2^{k+1}})}III + \frac{1}{F(A_{2^{k+1}})}IIIA_{2^k}A^\dagger_{2^k}, \end{aligned}$$

where

$$
\begin{aligned}
III &= ((-1)^{\frac{k(k+1)}{2}} B_{2^k} A^*_{2^k} + A_{2^k} C^*_{2^k}) A^t_{2^k} A^*_{2^k} ((-1)^{\frac{k(k+1)}{2}} A^*_{2^k} B_{2^k} + C^*_{2^k} A_{2^k})^t \\
&= ((-1)^{\frac{k(k+1)}{2}} B_{2^k} A^*_{2^k} + A_{2^k} C^*_{2^k}) A^t_{2^k} J^t_{2^k} J_{2^k} A^*_{2^k} ((-1)^{\frac{k(k+1)}{2}} A^*_{2^k} B_{2^k} + C^*_{2^k} A_{2^k})^t \\
&= [(-1)^{\frac{k(k+1)}{2}} b_k (J_{2^k} A^*_{2^k})(J_{2^k} A_{2^k})^t + c^*_k (A_{2^k} J_{2^k})(J_{2^k} A_{2^k})^t] \\
&\quad [(-1)^{\frac{k(k+1)}{2}} b_k (J_{2^k} A^*_{2^k})(A^*_{2^k} J_{2^k})^t + c^*_k (J_{2^k} A^*_{2^k})(J_{2^k} A_{2^k})^t] \\
&= [(-1)^{\frac{k(k+1)}{2}} b_k (J_{2^k} A^*_{2^k})(J_{2^k} A_{2^k})^t + c^*_k [A_{2^k}] Id_{2^k}] \cdot \\
&\quad\ [(-1)^{\frac{k(k+1)}{2}} b_k [A_{2^k}]^* Id_{2^k} + c^*_k (J_{2^k} A^*_{2^k})(J_{2^k} A_{2^k})^t] \\
&= (b_k^2 [A_{2^k}]^* + c^{*2}_k [A_{2^k}]) J_{2^k} A^*_{2^k} A^t_{2^k} J^t_{2^k} + (-1)^{\frac{k(k+1)}{2}} b_k c^*_k J_{2^k} A^*_{2^k} A^t_{2^k} A^*_{2^k} A^t_{2^k} J^t_{2^k} \\
&\quad + (-1)^{\frac{k(k+1)}{2}} b_k c^*_k [A_{2^k}][A_{2^k}]^* Id_{2^k}.
\end{aligned}
$$

¿From Lemma 6, we get

$$
\begin{aligned}
III &= (b_k^2 [A_{2^k}]^* + c^{*2}_k [A_{2^k}]) J_{2^k} A^*_{2^k} A^t_{2^k} J^t_{2^k} + (-1)^{\frac{k(k+1)}{2}} b_k c^*_k ||A_{2^k}|| J_{2^k} A^*_{2^k} A^t_{2^k} J^t_{2^k} \\
&= F(A_{2^{k+1}}) J_{2^k} A^*_{2^k} A^t_{2^k} J^t_{2^k}.
\end{aligned}
$$

¿From Lemma 3 we also have

$$
\begin{aligned}
III A_{2^k} A^\dagger_{2^k} &= III A_{2^k} J_{2^k} J^t_{2^k} A^\dagger_{2^k} \\
&= F(A_{2^{k+1}}) J_{2^k} A^*_{2^k} (J_{2^k} A_{2^k})^t (A_{2^k} J_{2^k})(A^*_{2^k} J_{2^k})^t = F(A_{2^{k+1}})[A_{2^k}][A_{2^k}]^* Id_{2^k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left| A_{2^{k+1}} A^\dagger_{2^{k+1}} - \lambda Id_{2^{k+1}} \right| = |I + II| \\
&= | -\lambda^2 Id_{2^k} + \lambda (b_k b^*_k + c_k c^*_k + ||A_{2^k}||) Id_{2^k} - (b_k b^*_k c_k c^*_k - (-1)^{\frac{k(k+1)}{2}} b^*_k c^*_k [A_{2^k}] \\
&\quad - (-1)^{\frac{k(k+1)}{2}} b_k c_k [A_{2^k}]^* + [A_{2^k}][A_{2^k}]^*) Id_{2^k} | \\
&= (\lambda^2 - ||A_{2^{k+1}}|| \lambda + [A_{2^{k+1}}][A_{2^{k+1}}]^*)^{2^k},
\end{aligned}
$$

where the first formula in Lemma 4 is used. The second formula in Theorem 3 is obtained from the fact that $A_{2^{k+1}} A^\dagger_{2^{k+1}}$ and $A^\dagger_{2^{k+1}} A_{2^{k+1}}$ have the same eigenvalue set. ■

¿From Theorem 2 and 3 the states given by (20) are d-computable. In terms of (8) the generalized concurrence for these states is given by

$$
d_{2^{k+1}} = 2^{k+1} |[A_{2^{k+1}}]| = 2^{k+1} |b_k c_k + b_{k-1} c_{k-1} + ... + b_1 c_1 + ad + c^2|.
$$

Let $p_{2^{k+1}}$ be a symmetric anti-diagonal $2^{2k+2} \times 2^{2k+2}$ matrix with all the anti-diagonal elements 1 except for those at rows $2^{k+1} - 1 + s(2^{k+2} - 2)$, $2^{k+1} + s(2^{k+2} - 2)$, $2^{k+2} - 1 +$

$s(2^{k+2}-2)$, $2^{k+2}+s(2^{k+2}-2)$, $s=0,...,2^{k+1}-1$, which are -1. $d_{2^{k+1}}$ can then be written as

$$d_{2^{k+1}}=|\langle\psi_{2^{k+1}}|p_{2^{k+1}}\psi^*_{2^{k+1}}\rangle|\equiv|\langle\langle\psi_{2^{k+1}}|\psi_{2^{k+1}}\rangle\rangle|, \tag{34}$$

where

$$|\psi_{2^{k+1}}\rangle=\sum_{i,j=1}^{2^{k+1}}(A_{2^{k+1}})_{ij}\,e_i\otimes e_j. \tag{35}$$

For a $2^{2k+2}\times 2^{2k+2}$ density matrix $\rho_{2^{k+2}}$ with decompositions on pure states of the form (35), its entanglement of formation, by using a similar caculation in obtaining formula (14) [21], is given by $E(d_{2^{k+1}}(\rho_{2^{2k+2}}))$, where

$$d_{2^{k+1}}(\rho_{2^{2k+2}})=\Omega_1-\sum_{i=2}^{2^{2k+2}}\Omega_i, \tag{36}$$

and Ω_i, in decreasing order, are the the square roots of the eigenvalues of the matrix $\rho_{2^{2k+2}}p_{2^{k+1}}\rho^*_{2^{2k+2}}p_{2^{k+1}}$.

Therefore from high dimensional d-computable states $A_{2^{k+1}}$ in (20), $2\leq k\leq N$, the entanglement of formation for a class of density matrices whose decompositions lie in these d-computable quantum states can be obtained analytically.

Remarks Besides the d-computable states constructed above, from (16) we can also construct another class of high dimensional d-computable states given by $2^{k+1}\times 2^{k+1}$ matrices $A_{2^{k+1}}$, $2\leq k\in I\!N$,

$$A_{2^{k+1}}=\begin{pmatrix} B_k & A_k\\ -A_k^t & C_k\end{pmatrix}\equiv\begin{pmatrix} b_kI_{2^k} & A_{2^k}\\ -A_{2^k}^t & c_kI_{2^k}\end{pmatrix}, \tag{37}$$

where $b_k,\ c_k\in \mathbb{C}$, $I_4=J_4$,

$$I_{2^{k+1}}=\begin{pmatrix} 0 & I_{2^k}\\ -I_{2^k} & 0\end{pmatrix} \tag{38}$$

for $k+2\ mode\ 4=0$,

$$I_{2^{k+1}}=\begin{pmatrix} 0 & I_{2^k}\\ I_{2^k} & 0\end{pmatrix} \tag{39}$$

for $k+2\ mode\ 4=1$,

$$I_{2^{k+1}}=\begin{pmatrix} 0 & 0 & 0 & I_{2^{k-1}}\\ 0 & 0 & -I_{2^{k-1}} & 0\\ 0 & I_{2^{k-1}} & 0 & 0\\ -I_{2^{k-1}} & 0 & 0 & 0\end{pmatrix} \tag{40}$$

for $k+2$ *mode* $4=2$, and

$$I_{2^{k+1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{2^{k-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{2^{k-2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{2^{k-2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{2^{k-2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{2^{k-2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{2^{k-2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (41)$$

for $k+2$ *mode* $4=3$.

One can prove that the matrices in (37) also give rise to d-computable states:

$$|A_{2^{k+1}}A^\dagger_{2^{k+1}}| = [(c^2+ad-\sum_{i=1}^k b_ic_i)(c^2+ad-\sum_{i=1}^k b_ic_i)^*]^{2^k},$$

$$\begin{aligned} |A_{2^{k+1}}A^\dagger_{2^{k+1}} - \lambda Id_{2^{k+1}}| \;&=\; [\lambda^2 - (aa^*+2cc^*+dd^*+\sum_{i=1}^k b_ib_i^* + \sum_{i=1}^k c_ic_i^*)\lambda \\ &\quad +(c^2+ad-\sum_{i=1}^k b_ic_i)(c^2+ad-\sum_{i=1}^k b_ic_i)^*]^{2^k}. \end{aligned}$$

The entanglement of formation for a density matrix with decompositions in these states is also given by a formula of the form (36).

In addition, the results obtained above may be used to solve linear equation systems, e.g., in the analysis of data bank, described by $A\mathbf{x}=\mathbf{y}$, where A is a $2^k\times 2^k$ matrix, $k\in I\!N$, $\mathbf{x}$ and $\mathbf{y}$ are 2^k-dimensional column vectors. When the dimension 2^k is large, the standard methods such as Gauss elimination to solve $A\mathbf{x}=\mathbf{y}$ could be not efficient. From our Lemma 3, if the matrix A is of one of the following forms: A_{2^k}, $B_{2^k}A_{2^k}$, $A^t_{2^k}$ or $A^t_{2^k}B^t_{2^k}$, the solution $\mathbf{x}$ can be obtained easily by applying the matrix multiplicators. For example, $A_{2^k}\mathbf{x}=\mathbf{y}$ is solved by

$$\mathbf{x} = \frac{1}{[A_{2^k}]}(A_{2^k}J_{2^k})^tJ_{2^k}\mathbf{y}.$$

The solution to $B_{2^k}A_{2^k}\mathbf{x}=\mathbf{y}$ is given by

$$\mathbf{x} = \frac{1}{b_k[A_{2^k}]}(A_{2^k}J_{2^k})^tJ_{2^k}\mathbf{y}.$$

We have presented a kind of construction for a class of special matrices with at most two different eigenvalues. This class of matrices defines a special kind of d-computable states. The entanglement of formation for these d-computable states is a monotonically increasing function of a the generalized concurrence. From this generalized concurrence the entanglement of formation for a large class of density matrices whose decompositions lie in these d-computable quantum states is obtained analytically. Besides the relations to the quantum entanglement, the construction of d-computable states has its own mathematical interests.

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