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NOTE ON LAGRANGIAN TORUS FIBRATIONS WITH SIMPLE SINGULARITIES

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ABSTRACT. In this note we study the geometry of Lagrangian torus fibrations with simple singularities in the case of SL-fibrations and in the case when the total space of fibration is of dimension 4.

1. Introduction.

Lagrangian fibrations are important subjects in geometric quantization, completely integrable systems and in the context of mirror symmetry according to [SYZ]. In this note we study Lagrangian torus fibrations with simple singularities. Our first result concerns regular special Lagrangian torus fibrations. We recall that a submanifold L^n in a Calabi-Yau manifold X^{2n} is called special Lagrangian, if there is a holomorphic complex volume form Ω^n on X^{2n} such that $\Omega^n_{|T_xL} = vol(L)$ for all $x \in L$ (see [H-L]). $H_*(X^{2n}, \mathbf{R})$.[1]

Theorem 1. *If the bas B^n of a SL torus fibration $T^n \to X^{2n} \to B^{2n}$ is compact and all the fibers are regular then B^n is a torus and $X^{2n} = T^{2n}$.*

We would like to notice that the condition of fiber being special Lagrangian in Theorem 1 is important as we know that the Kodaira-Thurson symplectic manifold also admits a regular Langrangian torus fibration.

Next we study the Lagrangian torus fibrations with simple singularities. In higher dimension ($n \ge 3$) the structure of singularities of a SL-fibration can be very complicated as it was shown by Joyce in [J].

[1]Equivalently we have for SL-manifolds L the identity: $\Im\Omega^n_{|L} = 0$ which can be taken as a definition (see e.g. [J]). With this definition we can extend the notion of SL- submanifold to the class of almost C-Y manifolds [J]. Theorem 1 is still valid for this bigger class of ACY).

In dimension 4 it is reasonable to define that *a Lagrangian torus fibration is said to have simple singularities*, if it is good torus fibration in the definition of Matsumoto [Ma] (see also section 3 below). The class of singularities which can occur in a good torus fibration is lager than the class of of singularities which occur in an elliptic fibration ([Ma]) in the category of complex surfaces. We prove

Theorem 2. *Let $(M^4, \omega) \to B$ be a Lagrangian torus fibration with simple singularities. If the fiber T^2 realizes a nontrivial element in $H_2(M^4, \mathbf{R})$, then there is a perturbation ω' of the original symplectic structure ω on M^4 such that (M^4, ω') becomes a symplectic fibration.*

Here we say that a fibration $F : (M, \omega) \to B$ is *symplectic*, if the restriction of ω to each fiber outside the singular set is symplectic.

Theorem 3. *Let (M^4, ω) is a compact symplectic manifold. Suppose that M^4 admits a symplectic torus fibration with simple singularites. Then (M^4, ω) is either a regular torus symplectic fibration or the symplectic form ω is deformation equivalent to a Kahler form through a family of symplectic forms which are compatible to the given fibration.*

Regular symplectic torus fibrations over T^2 has been classified by Geiges [G]. Thurston's argument shows that a regular torus fibration over a closed surface Σ_g admits a compatible symplectic structure, if and only if the fiber T^2 represents a non-trivial homology class. Regular torus fibrations over Σ_g are defined by their Chern class and monodromies.

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2. Regular SL torus fibrations.

We denote by X^{2n} a Calabi-Yau manifold of complex dimension n. In this note we assume that X^{2n} admits a fibration of SL-torus T^n. We denote by B^n the base of the fibration. Let us recall some important notions.

2.1. Let $\mathcal{A}$ be the space of all differentiable mappings $f : T^n \to X^{2n}$ such that f is almost everywhere diffeomorphism and the image $f(T^n)$ is special Lagrangian in X. The quotient $\mathcal{M} = \mathcal{A}/Diff(T^n)$ is called **the moduli space of the SL submanifold T^n.** A tangent vector in the tangent space $T^n_t\mathcal{M}$ clearly can be identified with a normal vector field to the image $\tilde{t}T^n \subset X$ (in the case of f being an immersion), where $\tilde{t}$ is a representative of t in $\mathcal{A}$.

2.2. In the case that a map $f : T^n \to X$ is a diffeomorphism we can also compose f with a diffeomorphism $g : f(T^n) \to f(T^n)$ to get another map $T^n \to X$. But this new map differs from f exactly by the diffeomorphism $h = f^{-1}gf : T^n \to T^n$.

2.3. If X admits a SL torus fibration with a base B, then B is the moduli space M by dimension and local structure argument based on the following theorem of McLean([McL]). Let us denote by $(V)^\flat$ the 1-form on a special Lagrangian submanifold L^n which is dual to vector V w.r. to the metric on L^n.

2.4. Theorem [McL] *A normal vector field V to a compact special Lagrangian submanifold $L \subset X^n$ is the deformation vector field to a normal deformation through special Lagrangian submanifolds, if and only if the corresponding 1-form $(JV)^\flat$ is closed and coclosed i.e. harmonic on L^n.*

2.5. It is easy to see that Theorem 2.4 implies the existence of a local section $M \to \mathcal{A}$. Namely for any $x \in M_B$ we define a local section

$$s : U(x) \to \mathcal{A}$$

by the identification $U(x)$ with the normal deformation of the special Lagrangian submanifold $f_x(T^n)$, where $f_x \in \mathcal{A}$ is a choosen representation of x.

Thus we identify B^n with the moduli space of the SL torus fibers.

Proof of Theorem 1. We shall show that the universall covering $\tilde{X}_u$ of X is homeomorphic to $\mathbf{R}^{2n}$. Then our Proposition follows from the classification Theorem of Beauville [B. Thm.1] which says that $\tilde{X}_u$ must be the standard $\mathbf{C}^n$ and from the oberservation, that if a discrete group $\Gamma \subset Aut(\mathbf{C}^n, holomorphic\ J)$ acts on C^n freely then it must be a subgroup of the translationsgroup $\mathbf{C}^n$ (in our case Γ is the fundamental group acting on $\tilde{X}_u$ by covering transformation).

Let us denote by $\tilde{B}$ the universal covering over B. Denote by $\tilde{X}$ the covering over X induced from the covering $\tilde{B} \to B$. Clearly $\tilde{X}$ is fibered over $\tilde{B}$ with T^n fiber. Moreover $\tilde{X}$ has an induced Calabi-Yau structure from X and the fiber T^n in $\tilde{X}_1$ is also special Lagrangian. Since $\tilde{B}$ is simply connected, the monodromy of the fibration $\tilde{X} \to \tilde{B}$ is trivial.

Let $\{A_i\}$ be a basis of $H_1(T^n, \mathbf{Z})$, and $\alpha_i \in H^1(T^n, \mathbf{R})$ its dual basis : $< A_i, \alpha_j >= \delta_{ij}$. Since $\tilde{B}$ is simply connected, we can extend the frame A_i smoothly over $\tilde{B}$. For each $x \in \tilde{B}$ we denote by $\alpha(x)$ the unique harmonic 1-form on the Lagrangian submanifold $T^n(x) \subset \tilde{X}$ representing the class $\alpha \in H^1(T^n, \mathbf{R})$. We can see $\alpha(x), x \in \tilde{B}$, as a restriction of a 1-form $\bar{\alpha}$ on $\tilde{X}$, requiring that the restriction of $\bar{\alpha}_i$ to the normal bundle of the fiber $T^n(x)$ vanishes.

We denote by V_i the vector field on $\tilde{X}$ such that $V_i = \Phi_\omega(\bar{\alpha}_i)$ where Φ_ω is the linear trasformation from $T^*\tilde{X}$ to $T_*\tilde{X}$ satisfying

$$\omega(V_i, W) = \bar{\alpha}_i(W). \tag{2.6}$$

Here abusing notation we also denote by ω the induced symplectic form on $\tilde{X}$.

Substituting a normal vector field W on $T^n(x) \subset \tilde{X}$ into (2.6), we conclude that V_i is also normal to the fiber $T^n(x)$. We note that $(JV_i)^\flat = \alpha_i$.

Thus the vector field $V_i(\tilde{x})$ is the infinitesimal deformation of the fiber $T^n(x)$ generated by the 1-form α_i, which can be identified with the vector field $V_i(x)$ on the base $\tilde{B}$. in other words we have

$$V_i(s(x)) = s_*(V_i(x)).$$

Thus there must be no confusion in the notation of V_i, considered as a vector field on $\tilde{B}$ as well as a vector field on $\tilde{X}$.

2.7. Lemma. *The vector fields V_i on $(\tilde{B})$ commute each with other.*

Proof of Lemma 2.7. Our argument is inspired by Hitchin's argument in [Hit].

Let β_i be the 1-forms on $\tilde{B}$ which is dual to vector field frame $\{V_i\}$. It suffices to show that all the form β_i are closed.

We denote by s^{-1} the diffeomorphism between the image $s(U(x))$ and $U(x)$. We consider the evaluation map

$$ev : s(U(x)) \times T^n \to \tilde{X}$$

and the projections

$$p : s(U(x)) \times T^n \to \tilde{B},$$

$$\pi_1 : s(U(x)) \times T^n \to s(U(x)).$$

2.8. Proposition. *We have*

$$(ev)^*\omega = \sum_i ((\pi_1)^*(s^{-1})^*)\beta_i \wedge (ev)^*(\bar{\alpha}_i).$$

Proof. It suffices to show that for a local frame fields $\{s_*(V_i), \partial t_j\}$ on $s(U(x)) \times T^n$, where t_i are coordinates on T^n, we have

$$(ev)^*(\omega)(X, Y) = \sum_i (\pi_1^*(s^{-1})^*)\beta_i \wedge (ev)^*(\bar{\alpha}_i)(X, Y) \tag{2.9}$$

for any $X, Y \in \{s_*(V_i), \partial t_j\}$.

Since the image $f(T^n)$ is Lagrangian and $\pi_1(T^n) = pt$ we have

$$(ev)^*(\omega)_{|T^n} = 0 = \sum_i (\pi_1^*(s^{-1})^*)\beta_i \wedge (ev)^*(\bar{\alpha}_i)_{|T^n}.$$

By the construction we also have that the image $f_*(s_*T_x(\tilde{B}))$ is orthogonal to $T_{\tilde{x}}T^n$. Thus we also have

$$(ev)^*(\omega)_{|s(U(x))} = 0 = \sum_i (\pi_1^*(s^{-1})^*)\beta_i \wedge f^*(\bar{\alpha}_i)_{|s(U(x))}.$$

Therefore it suffices to check the identity (2.9) for $X = s_*(V_i), Y = \partial t_j$. Now we observe that

$$(ev)^*(\omega)(s_*(V_i), \partial t_j) = \omega((ev)_* s_*(V_i), (ev)_*(\partial t_j))$$
$$= \beta_i((ev)^*(\partial t_j)) = \sum_i (\pi_1^*(s^{-1})^*)\beta_i \wedge (ev)^*(\bar{\alpha}_i)(s_*(V_i), \partial t_j).$$

This completes the proof. □

We now represent the homology class A_i by a cicle S_i in T^n and let

$$\mathcal{M}_i := s(U(x)) \times S_i \subset s((U(x)) \times T^n. \tag{2.10}$$

We also denote by (ev) the restriction of the evaluation map to $\mathcal{M}_i$.

From Proposition 2.8 we get immediately

$$s^{-1}(\beta_i) = p_i * (ev)^*\omega, \tag{2.11}$$

where p_i* is the push-down (intergation over the fiber) map in (2.10). Since ω is a closed form we must have that $(s^{-1})^*\beta_i$ is closed and therefore β_i is also closed. □

Continuation of the proof of Theorem 1. Lemma 2.7 says that $\tilde{B}$ is diffeomorphic to $\mathbf{R}^n$. Hence $\tilde{X}$ must be diffeomorphic to $\mathbf{R}^n \times T^n$. It implies that the universal covering of X is diffeomorphic to $\mathbf{R}^{2n}$. □

3. Good Lagrangian torus fibrations in dimension 4.

3.1. Good torus fibrations. Torus fibrations are important subject of 4-dimension topology. They are intensively studied in both categories of complex surfaces and differentiable manifolds [Mo, Ko, Zi, Ue, Ma]. The most general notion "torus fibration" which we follow in this note was introduced by Matsumoto [Ma]. The class of 4-manifolds which admit a torus fibration is large enough as Masumoto proves that any 4-manifold with a handlebody decompostion without one or three handles admits a structure of a torus fibration (TF) over S^2.

Let us recall that [Ma], for a given smooth 4-manifold M^4 and a smooth 2-manifold B the triple $(f : M \to B)$ is called a torus fibration if the following conditions hold.

1. f is a continuous map and there is a finite set of points $\Gamma \subset Int\, B$ so that the restriction of f onto $B \setminus \Gamma$ from $M \setminus f^{-1}(\Gamma) \to B \setminus \Gamma$ is a smooth fiber bundle with a 2-torus T^2 as fiber.

2. For each $x \in M$ we have that the germ (f, y) is smoothly equivalent to to $(f', 0)$, the germ at 0 of a cone extension $f' : \mathbf{R}^4 \to \mathbf{C}$ of a multiple fibered link $g' : S^3 \to \mathbf{C}$, where g' may vary depending on p.

Here we recall that a smooth map $g : S^3 \to C$ is called a multiple fibered link if it has the following properties

(i) $g^{-1}(0) \neq \emptyset$;

(ii) the map $\phi = g/|g| : S^3 \setminus g^{-1}(0) \to S^1$ is the projection of a fiber bundle over S^1.

(iii) around each point $x_0 \in \{g^{-1}(0)\}$ there exits local coordinates u_1, u_2 and u_3 of S^3 such that $g(x) = (u_2(x) + \sqrt{-1}u_3(x))^m$ holds, m being a positive integer. (We calll m the multiplicity of g at x_0.)

We recall that the function $f : \mathbf{R}^4 \to \mathbf{C}$ is called the dth cone extension of $g : S^3 \to \mathbf{C}$ if $f(0) = 0$ and

$$f(x) = ||x||^d g(x/||x||) \text{ for } x \neq 0$$

for a positive integer d.

Furthermore, we say that the germ (f_1, x_1) is smoothly equivalent to the germ (f_2, x_2) if they are conjugate through orientation-preserving local diffeomorphisms around x_i and $f_i(x_i)$

In particular a singular fiber of a good torus fibration is a 2-dimensional smooth submanifold of M except at a finite number of points. If we remove the "nonsmooth" points from F_y the singular fiber splits into some connected components which are called irreducible components of F_y.

3. A torus fibartion is a GTF (good torus fibration) if the only singularities of the singular fiber are normal crossing.

3.2. Proof of Theorem 2. We modify Gompf's idea in [Go], where he proves a similar perturbation result for a single embedded Lagrangian torus. We emphasize that Gompf proof uses the smoothing outside a tubular neighborhood of a submanifold and therefore works only for an isolated set of embedded Langrangian tori. One important ingridient of our proof is the use of Poincare Lemma with singularities. We also use the language of fiber bundle to get a parametrized version of Poincare's lemma.

3.2.1. Lemma (Poincare Lemma with singularities).*Let M_g be a surface with finite singularity points $\{x_1, \cdots, x_p\}$. Suppose that ω is a 2-form on M_g with*

$$\int_{M_g} \omega = 0, \text{ and } \lim_{x \to x_i} \omega(x) = 0.$$

Then there is a 1-form γ on Σ_g such that

$$d\gamma(x) = \omega(x) \text{ if } x \neq 0, \text{ and } \gamma(x_i) = 0.$$

Proof. First we choose some bounded Riemann metric g on Σ_g outside a singular set compatible with the topology on Σ_g, such that the induced metric can be completed to define a metric on Σ_g. Next we triangulize Σ_g into k -starlike domain $A_1, \cdots, A_k$ with a choosen interior point $z_i \in A_i$ with the following properties

1. Each domain A_i contains at most one point x_j.
2. If $x_j \in A_i$ then $x_j = z_i$.
3. For any point $z' \in A_i$ there is a unique geodesic ray $r(z_i, z')$ which goes from z_i to z' and lies entirely in A_i.

Now for any path $x(t)$, $t = 0, 1$ we can define the function $\gamma_i([x(t)])$ as follows. We denote the intersection $\{x(t)\} \cap A_i$ by $\{x_i(t)\}, t \in [\varepsilon_1, \varepsilon_2], \cdots, [\varepsilon_{2n-1}, \varepsilon_{2n}]$. We denote by D_i the domain inside the closed curve $r(z_i, x_{\varepsilon_{2i-1}}) \cup x(t), t \in [\varepsilon_{2i-1}, \varepsilon_{2i}], \cup r(z_i, \varepsilon_{2i})$. we put

$$\gamma_i([x(t)]) = \int_{D_i} \omega.$$

$$\gamma([x(t)]) = \sum_i \gamma_i([x(t)]).$$

Now it is easy to check that γ defines a 1-form also denoted by γ which satisfies all the required condition in our Lemma.

Let us to continue the proof of Theorem 1. Since the fiber is homologically nontrivial there is a closed 2-form $\beta \in \Omega^2(M)$ such that

$$\int_{T^2} \beta = 1.$$

Here we take integration with counting the multiplicity of singular fibers. Moreover using the Darboux theorem we can assume that $\beta(x_i) = 0$ (if not we just modify β by an exact form $d\alpha_i$ with support around the singular set and $d\alpha_i(x_i) = \beta_i$).

Next we associate to each Lagrangian torus bundle $F : M^4 \to B^2$ a bundle Ω_F over B^2, whose fiber over $y \in B^2$ is the set of of all symplectic forms Ω_y over $T^2(y)$:

$$\int_{T^2_y} \Omega_y = 1.$$

Here we assume that if T^2_y is a singular fiber then the symplectic form vanishes at these singular points. The bundle Ω_F has a natural topology induced from that one on M. Clearly each fiber $\Omega_F^{-1}(y)$ ia a convex set. Now we shall show the exitstence of a section $s : B^2 \to \Omega_F$. First we construct a section over a small disk $D_\varepsilon(F(x_i))$ which contains a unique critical value $F(x_i)$ in the following way. Here the disk $D_\varepsilon(F(x_i))$ is the geodesics disk with the center in $F(x_i)$ and of radius ε. Let $f(T^2 \times D(F(x_i))$ be a continous map from $T^2 \times D^2$ to $F^{-1}(D^2)$ such that $f(T^2, z) = F^{-1}(z)$ is a diffeomorphism, if $z \neq F(x_i)$. Abusing notation we denote by f_z the map from $f(T^2, z)$ to $f(T^2, F(x_i))$. Then we construct a section s over $D(F(x_i))$ as follows.

$$s_z = e^t \cdot f_z^*(\Omega_{F(x_i)} + (1 - e^t)\omega_0$$

hall pull back a symplectic form form a singular fiber to the nearby regular fiber. To smoothing up the value where t denotes the distance $|z - x_i|$, and ω_0 is a canonical symplectic form on $T^2_z, z \neq F(x_i)$ i.e. $< \omega, T^2 >= 1$. Outside the disks $D_\varepsilon(F(x_i))$ we can extend our section easily since each fiber is convex and hence contractible.

Next for each torus fiber T^2_y we define the set

$$P(\beta, s, y) = \{\gamma \in \Omega^1(T^2) \,|\beta_{|T^2(y)} - s(y) = d\gamma_y, \}.$$

Here we assume again that the value of γ at singular points is zero. Lemma 3.2.1 says that this set is nonempty. Moreover it is an affine space. We denote by $\mathcal{P}(\beta, s)$ the total space of $P(\beta, s, y)$. Since $P(\beta, s, y)$ is contractible, using the same argument as above we can show that there is a section s_2 from $B^2 \to \mathcal{P}(\beta, s)$, $s_2(y) = \gamma_y$.

Now we equippe M with a Riemannian metric. Using this metric we can extend γ_y to a global defined 1-form γ on M by letting γ on the normal direction equal to zero and letting γ at the singular points equal zero. For a positive and small enough t clearly the 2-form $\omega' = \omega - t(\beta - d\gamma)$ is a symplectic form and the restriction of ω' to each fiber is symplectic, since by construction $\beta - d\gamma_{|T^2_y} = s(y)$ which is a symplectic form. □

The following question seems to me interesting. Given a class $x \in H_n(M^{2n}, \mathbf{Z})$ such that $\omega(x) = 0$. What is an obstruction to the existence of an embedded Lagrangian tori realizing x?

If $x = 0$, then we can use the Darboux theorem to show the existence of such a Langrangian torus. If $x \neq 0$ and $n = 4$, then Gompf shows that [Go] there exists a symplectic form ω' on (M^4, ω) such that our Lagrangian torus is symplectic with respect to ω'. The adjunction formula shows that

$$-c_1(M)x + x^2 + 2 = 0 \qquad (3.3)$$

This theorem holds because the perturbed symplectic form is deformation equivalent to the original one. The Taubes result [Ta] shows a strong topological restriction on M^4 which has a symplectic torus.

4. Symplectic torus fibration.

4.1. Symplectic Lefschetz fibrations.

In the recent year powerful methods of complex surfaces have been applied in the study of symplectic 4-manifolds [Do, Sm, Au, S-T]. (Our result presented here was conceived before the devolopment of these methods. It was motivated by the relation between Kähler geometry and symplectic geometry in dimension 4. We were amazed by the fact that up to deformation equivalence any symplectic structure on rational or ruled surface is Kähler).

One of the central notion of the complex category which has been applied to the symplectic category is the notion of Lefschetz fibration which was introduced by Moishezon [Mo] in order to study topology of elliptic surfaces. Roughly speaking a Lefschetz fibration $M^4 \xrightarrow{\pi} \Sigma$ is a locally trivial fibration outside a finite set of singular value $\{y_i\}$ in Σ and the preimage of $\pi^{-1}(y_i)$ is a immersed complex surface in certain local complex coordinates in M. More precisely

4.1.a. Definition. [Mo]. Let $f : M^4 \to \Sigma$ be a differentiable map of connected compact oriented differentiable manifolds (which may have boundaries). We say that $f : M \to \Sigma$ is **a Lefschetz fibration**, if the following conditions hold

a) $\partial M = f^{-1}(\partial\Sigma)$,

b) there is a finite set of **critical points** $a_1, \cdots, a_n \in S \setminus \partial S$ such that the restricriction of f on $M_{reg} = f^{-1}(\Sigma \setminus \cup\{a_i\})$ is a differentiable fiber bundle with connected fibers,

c) for any critical point a_i we have $H_2(f^{-1}(a_i), \mathbf{Z}) = \mathbf{Z}$ and there exists a single point $c_i \in f^{-1}(a_i)$ such that

c_1) $(df)_x$ is an epimorphism for any $x \in f^{-1}(a_i) \setminus c_i$

c_2) there is exists neighborhood B_i of a_i in S, U_i of c_i in M and complex coordinates λ_i in B_i and z_{i1}, z_{i2} in U_i, which define in B_i and U_i the same orientations as global orientations of S and M restricted to B_i and U_i correspondingly, $f(U_i) = B_i$ and $f_{|U_i} : U_i \to B_i$ is given by the following formula:

$$\lambda_i = z_{i1}^2 + z_{i2}^2$$

4.1.b Definition. Let ω be a symplectic form on M^4. We say that ω is compatible to a Lefschetz fibration $f : M \to \Sigma$, if all the regular fibers are symplectic submanifolds.

4.1.c. Example. Let $E(n)$ be the fiber-connected sum of n copies of rational elliptic surface $K(1) = \mathbf{C}P^2 \# 9\overline{\mathbf{C}P^2}$. Then $E(n)$ is a Lefschetz fibration over S^2 and the Kähler form on the simply-connected elliptic surface $E(n)$ is compatible with this fibration.

4.1.d. Example. Thurston-Kodaira symplectic torus fibration over a torus is a Lefschetz fibration which is compatible with a symplectic structure. More generally in [G] Geiges classifies all symplectic manifolds admitting torus fibration over a torus and show which symplectic structures is compatible with the given torus fibration.

4.1.e. Example. (Gluing symplectic fibrations.) Let M_i^4, ω_i, π_i, $i = 1, 2$ be a symplectic fibration over $B_i \setminus \{pt\}$ such that there is a bundle isomorphism $\partial M_1^4 \to \partial M_2^4$ which sends the cohomology class $[\omega_1]$ restricted to ∂M_1^4 to $[\omega_2]$ restricted to ∂M_2^4. Then there exists a compatible sympletic form ω on the fiber connected sum $M_1^4 \# M_2^4$.

Proof. First we want to show the existence of a closed 2-form ω' on the connected sum such that the restricted of ω on each fiber is non-degenerate. After that, immitating Thurston's argument, we choose a new closed 2-form $\omega'_K = \omega' + K\pi^*\omega_B$ on the connected sum. Here ω_B is a symplectic form on the base B. Clearly for K big enough the closed 2-form ω_K will be non-degenerate and compatible with π.

Let $X_i^3 = \partial M_i^4$. Then X_i^3 be a (regular) torus fibration over S^1. Since the monodromies of X_i^3 over S^1 are the same there is a torus fibration M_0^4 on $S^1 \times [0,1]$ such that its restriction over $S^1 \times \{0\}$ (correspondingly $\{1\}$) agrees with X_1^3 (correspondingly $-X_2^3$). Identify X_1^3 with X_2^3 (via cobordism X_0^4), by the assumption, the images $[\omega_1]$ and $[\omega_2]$ in $H^2(X^3, \mathbf{R})$ coincide. Let $\omega_1 - \omega_2 = d\eta$. We define a 2-form on X_0^4 as follows: $\tilde{\omega}(x,t) = \omega_1 + f(t)d\eta + df(t) \wedge \eta$. Here $f(t)$ is a smooth monotone function which equals 0 nearby 0 and equals 1 nearby 1. Then $\tilde{\omega}(x,0) = \omega_1$, $\tilde{\omega}(x,1) = \omega_2$ and $d\tilde{\omega}(x,t) = 0$. Clearly the torus bundle $M_1^4 \# M_2^4$ is isomorphic to $M_1^4 \# M_0^4 \# M_2^4$. Since the form $\tilde{\omega}$ is nondegenerate nearby X_1^3 and X_2^3, according to Weinstein's theorem we can define a form $\omega' := \omega_1 \# \tilde{\omega} \# \omega_2$ on this connected sum. Since the restriction of $\tilde{\omega}$ to each fiber $\pi^{-1}(x,t)$ equals $(1-t)\omega_1 + t\omega_2$ is positive, the restriction of ω' to each fiber of $M_1^4 \# M_0^4 \# M_2^4$ is non-degenerate.□

Using perturbation theory and the observation that if a symplectic form is compatible with a good torus fibration then all self-intersection number of the singular fiber must be positive we come to the following simple observation.

4.1.f. Lemma. *A symplectic good torus fibration is a symplectic Leftschetz torus fibration.*

4.2. Proof of Theorem 3.

Using Lemma 4.1.f we reduce our problem to the case of symplectic Lefschetz torus fibration. It is well-known fact [Mo-Ma] that if the diffeomorphism type of a Lefschetz fibration depends only on its homotopy type (more precisely the fundamental group and the Euler class). Thus we can assume that our Lefshetz torus fibration is either a locally trivial fibration or a fiber-connected sum of a trivial fibration over a Riemmannian surface Σ_g and a fibration $E(n)$.

It is a well-know fact that there exists a Kähler form ω_{can} on $E_n \# (\Sigma_g \times T^2)$ which is compatible with the given Lefschetz fibration. First we claim that for any K positive the forms ω and $\omega + K\pi^*\omega_B$ are deformation equivalent. Let W denote the symplectic orthogonal complement of a tangent plane V to a fiber $\pi^{-1}(y)$. Since $(\omega + K\pi^*\omega_B)_{|V} = \omega_{|V}$ and $\omega + K\pi^*\omega_B(w,v) = 0$ for any $w \in W$, $v \in V$ it follows that $\omega + K\pi^*\omega_B$ is also a symplectic form. To complete the proof of theorem 3 it suffices to show that for K big enough the two forms $\omega + K\pi^*\omega_B$ and $\omega_{can} + K\pi^*\omega_B$ are deformation equivalent. This claim follows from the fact that for K big enough all the 2-closed forms $t\omega+(1-t)\omega_{can}+K\pi^*\omega_B$ is non-degenerate if $[\omega](\pi^{-1}(y)) = \omega_{can}(\pi^*{}_1(y))$. (We notice that our proof uses Thurston's argument in the case of presence of immersred singular fibers.) □

4.2.a. Remark. (i) Theorem 3 shows the striking stabilizing effect of the fiber-connected summing. That is if we take a non-Kähler symplectic form on a locally trivial torus bundle over T^2 with another symplectic torus Lefschetz bundle over S^2 we will get a symplectic form which is deformation equivalent to aKáhler form.

(ii) If M is simply-connected elliptic surface then we can use the result in [CLO] to show that there exists a unique homotopy class of almost complex structure on M which is compatible with a symplectic structure on M.

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