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On solutions to the Navier-Stokes
equations with lower bounds on pressure

by

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Abstract We prove that weak solutions of the three-dimensional incompressible Navier-Stokes equations are smooth if the negative part of the pressure is controlled, or if the positive part of the quantity $|v|^2 + 2p$ is controlled.

1   Introduction

We consider the Cauchy problem for the three-dimensional Navier-Stokes equations

$$\partial_t v + \operatorname{div} v \otimes v - \Delta v + \nabla p = 0, \qquad \operatorname{div} v = 0$$

in $\mathbb{R}^3 \times]0, \infty[$, with $v(x, 0)$ smooth (or "sufficiently regular") and decaying sufficiently fast at infinity. Our main goal is to study the regularity of solutions of to the Navier-Stokes equations under certain assumptions on the pressure p. The pressure p is a relatively well-defined quantity in real fluids. In the Navier-Stokes system, p is determined only up to an arbitrary function of t, due to the idealized assumption of incompressibility. A way to remove this ambiguity is to specify p at infinity. In the context of this work there will be no loss of generality in assuming that p vanishes at infinity. (See Section 2 for a precise definition.) The pressure defined in this way will be called the *normalized pressure*. In what follows p will always denote the normalized pressure.

Our work was motivated by the following question.

(Q) If a solution to the Navier-Stokes equations develops a singularity, must the normalized pressure become unbounded from below?

One of the main results in this work is a positive answer to this question (Theorem 2.2).

Considering a flow of water under some standard conditions, one can speculate that if p becomes very low, one will encounter the phenomenon of cavitation. This means that in areas of very low pressure bubbles of water vapor will form in the fluid. Since the areas of very low pressure must have small volume, one can expect that eventually the bubbles will be carried into an area where the pressure is not so low, and will collapse. The collapse of even very small bubbles should create observable effects (e. g. popping sounds). As far as we know, cavitation is not observed in reasonable flows, such as flows in pipes under normal temperature and pressure, even when the Reynolds number is high. Therefore, for such flows one can expect that p does not become exceedingly low. Hence, by the result above, v should be smooth. One can further speculate that this means that all singularities of solutions to the Navier-Stokes must be unstable, if they at all exist. This was conjectured in [14].

We prove a slightly stronger statement than suggested by (Q), in that we do not need a point-wise condition $p(x, t) \geq -C$ to get regularity, but only a weaker integral condition is necessary (see (2.7) and (2.8)).

It turns out that our method also gives a proof of the following statement, which is of independent interest. If the quantity $|v|^2 + 2p$ is bounded from above, the solution must be regular. This is related to the works [6] and [19] on the five-dimensional steady-state Navier-Stokes equations.

We briefly outline the main idea of the proof. The key is the following identity:

$$\int\limits_{B(x_0,R)} \frac{1}{|y-x_0|} \left(2p(y,t) + |\widehat{v}^{x_0}(y,t)|^2 \right) dy =$$

$$= \frac{1}{R} \int\limits_{B(x_0,R)} \left(3p(y,t) + |v(y,t)|^2 \right) dy =$$

$$= R^2 \int\limits_{\mathbb{R}^3 - B(x_0,R)} \nabla_y^2 \left(\frac{1}{|y-x_0|} \right) : \left(v(y,t) \otimes v(y,t) \right) dy,$$

where $\hat{v}^{x_0}(x, t)$ is the orthogonal projection of $v(x, t)$ into the two-dimensional subspace of $\mathbb{R}^3$ perpendicular to $x - x_0$. One can see that bounds for the

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negative part of p or the positive part of $|v|^2 + 2p$ give non-trivial estimates for v. A key feature of these estimates is that the controlled quantities are invariant under the natural scaling of the equation $v(x,t) \to \lambda v(\lambda x, \lambda^2 t)$. In the language of regularity theory, under our assumptions the above identity gives estimates which move the equation from the realm of "super-critical" to the realm of "critical". This makes the problem manageable.

Other papers where regularity for weak solutions to the Navier-Stokes equations is studied under various assumptions on pressure include [1], [2], [3], [5], and [16] .

2　Notation and Main Results

We denote by $\mathbb{M}^3$ the space of all real 3×3 matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we shall use the following notation:

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3,\, v = (v_i) \in \mathbb{R}^3;$$

$$A : B = \mathrm{tr}A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$

$$A^* = (A_{ji}), \quad \mathrm{tr}A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3,\, B = (B_{ij}) \in \mathbb{M}^3;$$

$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3,\, A \in \mathbb{M}^3.$$

Let ω be a domain in some finite-dimensional space. We denote by $L_m(\omega; \mathbb{R}^l)$ and $W_m^1(\omega; \mathbb{R}^l)$ the known Lebesgue and Sobolev spaces of functions from ω into $\mathbb{R}^l$. The norm of the space $L_m(\omega; \mathbb{R}^l)$ is denoted by $\|\cdot\|_{m,\omega}$. If $m = 2$, then we use the abbreviation $\|\cdot\|_\omega \equiv \|\cdot\|_{2,\omega}$.

Let T be a positive parameter, Ω be a domain in $\mathbb{R}^3$. We denote by $Q_T \equiv \Omega \times]0, T[$ the space-time cylinder. Space-time points are denoted by $z = (x, t)$, $z_0 = (x_0, t_0)$, etc. Let $L_{m,n}(Q_T; \mathbb{R}^l)$ be the space of measurable $\mathbb{R}^l$-valued functions with the following norm

$$\|f\|_{m,n,Q_T} = \begin{cases} \left(\int\limits_0^T \|f(\cdot,t)\|_{m,\Omega}^n \, dt \right)^{\frac{1}{n}}, & n \in [1, +\infty[\\[4mm] \operatorname*{ess\,sup}_{t \in [0,T]} \|f(\cdot,t)\|_{m,\Omega}, & n = +\infty. \end{cases}$$

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In the special case $\Omega = \mathbb{R}^3$ and $T = +\infty$, we abbreviate

$$L_m(\Omega; \mathbb{R}^3) = L_m, \quad W_2^1(\Omega; \mathbb{R}^3) = H^1, \quad L_{m,n}(Q_T; \mathbb{R}^3) = L_{m,n},$$

$$L_m(0, T; W_2^1(\Omega; \mathbb{R}^3)) = L_m(H^1).$$

For integrable in Q_T scalar-valued, vector-valued, and tensor-valued functions, we shall use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_{,i}), \quad \nabla u = (u_{i,j}),$$

$$\operatorname{div} v = v_{i,i}, \quad \operatorname{div} \tau = (\tau_{ij,j}), \quad \Delta u = \operatorname{div} \nabla u,$$

which are understood in the sense of distributions. Here x_i, $i = 1, 2, 3$, are the Cartesian coordinates of a point $x \in \mathbb{R}^3$, and $t \in]0, T[$ is the time variable.

For balls and parabolic cylinders, we use the standard notation:

$$B(x_0, R) \equiv \{x \in \mathbb{R}^3 \mid |x - x_0| < R\}, \quad Q(z_0, R) \equiv B(x_0, R) \times]t_0 - R^2, t_0[,$$

where $z_0 = (x_0, t_0)$.

Let us formulate the main results of the paper. To this end, we recall to the reader that for the initial data satisfying the conditions

$$v_0 \in H^1, \quad \operatorname{div} v_0 = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2.1}$$

the Cauchy problem

$$\left.\begin{array}{r} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v + \nabla p = 0, \\[2mm] \operatorname{div} v = 0 \end{array}\right\} \quad \text{in} \quad \mathbb{R}^3 \times]0, +\infty[, \tag{2.2}$$

$$v(\cdot, 0) = v_0(\cdot) \quad \text{in} \quad \mathbb{R}^3 \tag{2.3}$$

always has a so-called Leray-Hopf weak solution (see [12], [8], [9], and [10]). This means that there exists at least one function v with the following properties:

$$v \in L_{2,\infty} \cap L_2(H^1), \quad \operatorname{div} v(\cdot, t) = 0 \quad \text{in} \quad \mathbb{R}^3 \quad \text{for all} \quad t \geq 0;$$

the function $t \mapsto \int_{\mathbb{R}^3} v(\cdot, t) \cdot w(\cdot)\, dx \quad$ is continuous on $\quad [0, +\infty[$

$$\text{for all} \quad w \in L_2;$$

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$$\int\limits_{\mathbb{R}^3 \times]0,\infty[} \Big\{ - v \cdot \partial_t w - v \otimes v : \nabla w + \nabla v : \nabla w \Big\} dx = 0$$

for any $w \in C_0^\infty(\mathbb{R}^3 \times]0, \infty[; \mathbb{R}^3)$ such that $\operatorname{div} w(\cdot, t) = 0$ for all $t > 0$;

$$\|v(\cdot, t) - v_0(\cdot)\|_{\mathbb{R}^3} \to 0 \quad \text{as} \quad t \to 0 + 0;$$

$$\int\limits_{\mathbb{R}^3} |v(x,t)|^2 dx + 2 \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla v(x,t')|^2 \, dx dt' \leq \int\limits_{\mathbb{R}^3} |v_0(x)|^2 dx$$

for all $t \geq 0$.

In this formulation, no information about the pressure p is given.

However, using the uniqueness theorem and the coercive $L_{s,l}$-estimates of solutions to the Cauchy problem for the Stokes equations (see, for example, [7], [15] and [9], [10], and [18] in the case $s = l$), one can introduce pressure in a natural way. More precisely, one can prove (see, for instance [4] and [11]) that there exists a function $p \in L_{1,\text{loc}}$ such that, for $O < \delta < T < +\infty$,

$$\nabla p \in L_{s,l}(\mathbb{R}^3 \times]\delta, T[; \mathbb{R}^3), \tag{2.4}$$

where

$$\frac{3}{s} + \frac{2}{l} \geq 4.$$

Moreover,

$$\partial_t v \in L_{s,l}(\mathbb{R}^3 \times]\delta, T[; \mathbb{R}^3), \quad \nabla^2 v \in L_{s,l}(\mathbb{R}^3 \times]\delta, T[; \mathbb{M}^3 \times \mathbb{R}^3),$$

and the equation

$$\partial_t v + \operatorname{div}(v \otimes v) - \Delta v + \nabla p = 0 \tag{2.5}$$

holds a.e. in $\mathbb{R}^3 \times]0, \infty[$.

The pressure p is determined up to an arbitrary function of t. We fix a representative for p by setting

$$p(x,t) \equiv \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} \operatorname{div} \operatorname{div} \Big(v(y,t) \otimes v(y,t) \Big) dy. \tag{2.6}$$

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To show that this function satisfies (2.5), let us denote the function on the right hand side of (2.6) by p_0. It is known that

$$\Delta\, p_0(x,t) = -\operatorname{div}\operatorname{div}\Big(v(x,t) \otimes v(x,t)\Big).$$

Differentiation in x gives us:

$$\nabla\, p_0 = \frac{1}{3}G + T(G),$$

where $G \equiv v_k v_{,k} = (v_k v_{i,k})$ and

$$T(G)(x,t) \equiv -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \nabla_x^2\Big(\frac{1}{|x-y|}\Big)G(y,t)dy$$

$$= -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\Big(\Big(\frac{\delta_{ij}}{|x-y|^3} - \frac{3(x_i-y_i)(x_j-y_j)}{|x-y|^5}\Big)G_j(y,t)\Big)dy$$

is a singular integral. According to the boundedness of singular integrals in L_s, we have the estimate

$$\int\limits_{\mathbb{R}^3}|\nabla\, p_0(x,t)|^{\frac{9}{8}}dx \le c_1\int\limits_{\mathbb{R}^3}\Big(|v(x,t)|\,|\nabla\, v(x,t)|\Big)^{\frac{9}{8}}dx$$

for all positive t and for some absolute constant c_1.

Next, by Hölder's inequality and by the multiplicative inequality, we obtain

$$\int\limits_{\mathbb{R}^3}|v|^{\frac{9}{8}}|\nabla\, v|^{\frac{9}{8}}dx \le \Big(\int\limits_{\mathbb{R}^3}|\nabla\, v|^2dx\Big)^{\frac{9}{16}}\Big(\int\limits_{\mathbb{R}^3}|v|^{\frac{18}{7}}dx\Big)^{\frac{7}{18}}$$

$$\le c_2\Big(\int\limits_{\mathbb{R}^3}|\nabla\, v|^2dx\Big)^{\frac{3}{4}}\Big(\int\limits_{\mathbb{R}^3}|v|^2dx\Big)^{\frac{3}{2}}.$$

Thus,

$$\nabla\, p_0(\cdot,t) \in L_{\frac{9}{8}}$$

for a. e. $t > 0$

On the other hand, it follows from equation (2.5) that

$$\Delta\, p(x,t) = -\operatorname{div}\operatorname{div}\Big(v(x,t) \otimes v(x,t)\Big).$$

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Therefore, $q \equiv p - p_0$ is a harmonic function in x for a.a. $t > 0$. But, by (2.4),

$$\nabla p \in L_{\frac{9}{8}, \frac{3}{2}}(\mathbb{R}^3 \times]\delta, T[; \mathbb{R}^3).$$

This means that

$$\nabla q(\cdot, t) \in L_{\frac{9}{8}}(\mathbb{R}^3 \times]\delta, T[; \mathbb{R}^3)$$

for a.a $t > 0$. Since q is harmonic in x, we obtain that $\nabla q(\cdot, t) = 0$ in $\mathbb{R}^3$ and, therefore, q is a function of t only.

Definition 2.1 *We say that a function $g : \mathbb{R}^3 \times]0, +\infty[\rightarrow [0, +\infty[$ satisfies condition (C) if, for any $t_0 > 0$, there exists a positive number $R_0 = R_0(t_0)$ such that*

$$A(t_0) \equiv \sup_{x_0 \in \mathbb{R}^3} \sup_{t_0 - R_0^2 \le t \le t_0} \int_{B(x_0, R_0)} \frac{g(x, t)}{|x - x_0|} dx < +\infty \qquad (2.7)$$

and,

for each fixed $x_0 \in \mathbb{R}^3$ and for each fixed $R \in]0, R_0]$,
the function $t \mapsto \int_{B(x_0, R)} \frac{g(x,t)}{|x-x_0|} dx$ is continuous at t_0 from the left. $\qquad (2.8)$

Our main result is as follows

Theorem 2.2 *Let v be a Leray-Hopf solution to the Cauchy problem (2.1)-(2.3) and let p be the normalized pressure associated with v. Assume that there exists a function g satisfying condition (C) such that*

$$|v(x, t)|^2 + 2p(x, t) \le g(x, t) \qquad x \in \mathbb{R}^3, \ t \in]0, +\infty[\qquad (2.9)$$

or

$$p(x, t) \ge -g(x, t) \qquad x \in \mathbb{R}^3, \ t \in]0, +\infty[. \qquad (2.10)$$

Then v is Hölder continuous on $\mathbb{R}^3 \times]0, +\infty[$ and therefore smooth and unique.

Remark 2.3 *Obviously, conditions (2.7) and (2.8) are satisfied if $g \equiv$ constant > 0 in $\mathbb{R}^3 \times]0, +\infty[$.*

3 Remarks on suitable weak solutions to the Navier-Stokes equations

In this section, we are going to discuss some facts about the so-called suitable weak solutions to the Navier-Stokes equations:

$$\partial_t v + \operatorname{div} v \otimes v - \Delta v = f - \nabla p$$
$$\operatorname{div} v = 0 \qquad \text{in} \quad Q_T, \qquad (3.1)$$

where $Q_T \equiv \Omega \times]0, T[$, Ω is a domain in $\mathbb{R}^3$, and T is a positive parameter. We always assume that f lies in the Morrey space

$$M_{2,\gamma}(Q_T; \mathbb{R}^3) \equiv \left\{ f \in L_2(Q_T; \mathbb{R}^3) \parallel d_\gamma(f; Q_T) < +\infty \right\}$$

for some positive number γ, where

$$d_\gamma(f; Q_T) \equiv \sup \left\{ \frac{1}{R^{\gamma+1/2}} \Big(\int\limits_{Q(x_0, R)} |f|^2 dx \Big)^{\frac{1}{2}} \parallel Q(x_0, R) \Subset Q_T, \ R > 0 \right\}.$$

We say that a pair of functions v and p is suitable weak solutions of the Navier-Stokes equations if the following conditions hold (see [17], [4], [13], and [11] for details):

$$v \in L_{2,\infty}(Q_T; \mathbb{R}^3) \cap L_2(0, T; W_2^1(\Omega; \mathbb{R}^3)), \quad p \in L_{\frac{3}{2}}(Q_T); \qquad (3.2)$$

equations (3.1) are satisfied in Q_T in the sense of distributions; $\qquad$ (3.3)

$$\int\limits_\Omega |v(x, t)|^2 \phi(x, t) \, dx + 2 \int\limits_0^t \int\limits_\Omega |\nabla v(x, t')|^2 \phi(x, t') \, dx dt'$$

$$\leq \int\limits_0^t \int\limits_\Omega \left\{ |v(x, t')|^2 \Big(\partial_t \phi(x, t') + \Delta \phi(x, t') \Big) \right.$$

$$+ 2 f(x, t') \cdot v(x, t') \, \phi(x, t')$$

$$+ \left. \Big(|v(x, t')|^2 + 2 p(x, t') \Big) v(x, t') \cdot \nabla \phi(x, t') \right\} dx dt' \qquad (3.4)$$

for a. a. $t \in]0, T[$ and for all $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ vanishing in a neighborhood of the parabolic boundary $\partial' Q_T \equiv \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T]$.

As in [11], we call a point $z_0 \in Q_T$ regular for v if there exists a non-empty neighborhood $\mathcal{O}_{z_0}$ of this point where the function $z \mapsto v(z)$ has a Hölder continuous representative. It can be proved that there exists a representative of v such that (see [17], [4], [13], and [11] for details)

$$\mathcal{H}^1(\Sigma) = 0, \tag{3.5}$$

where Σ is the set of all singular points of v and $\mathcal{H}^1$ is the one-dimensional parabolic Hausdorff measure. By definition,

$$\mathcal{H}^1(\Sigma) \equiv \lim_{\delta \to 0} \inf \left\{ \sum_i R_i \, \| \, \Sigma \subset \sum_i Q(z_i, R_i), \, 0 < R_i \leq \delta \right\}.$$

In what follows, we shall fix a representative of v such that

$$\liminf_{t \to t_0} \int_\Omega |v(x,t)|^2 \, dx \geq \int_\Omega |v(x,t_0)|^2 \, dx \quad \text{for all } 0 < t_0 < T \tag{3.6}$$

and, for each $w \in L_2(\Omega, \mathbb{R}^3)$,

$$t \in]0, T[\mapsto \int_\Omega v(x,t) \cdot w(x) \, dx \ \text{ is a continuous function.} \tag{3.7}$$

To see that this is possible we note that, by (3.5),

$$\mathcal{H}^1(\Omega \times \{t = t_0\} \cap \Sigma) = 0$$

and, according to the definition of regular points,

$$v(x,t) \to v(x,t_0) \quad \text{for a. e. } x \in \Omega. \tag{3.8}$$

Therefore (3.6) follows from Fateau's lemma. On the other hand, by (3.2), we have

$$\|v(\cdot, t_0)\|_{2,\Omega} \leq \|v\|_{2,\infty,Q_T} \tag{3.9}$$

for all $t_0 \in]0, T[$, and thus, by (3.8) and (3.9),

$$v(\cdot, t) \to v(\cdot, t_0) \quad \text{in} \quad L_r(\Omega; \mathbb{R}^3) \tag{3.10}$$

for any $r \in [1, 2[$. In turn, (3.9) and (3.10) imply (3.7).

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Remark 3.1 *Following the arguments in [11], one can see that all the above statements remain to be valid for $t_0 = T$.*

Lemma 3.2 *Let v be as above. Given $\Omega_0 \Subset \Omega$, $0 < t_0 \leq T$, and $0 < \delta_0 < \sqrt{t_0}$, assume that*

$$a(\Omega_0, t_0, \delta_0) \equiv \sup \left\{ \frac{1}{R} \int\limits_{B(x_0, R)} |v(x,t)|^2 \, dx \;\|\; x_0 \in \Omega_0, \right.$$

$$\left. t \in [t_0 - \delta_0^2, t_0], \; 0 < R \leq d_0 \equiv \tfrac{1}{2}\text{dist}(\partial\Omega, \Omega_0) \right\} < +\infty. \tag{3.11}$$

Then,

$$\lim_{t \to t_0 - 0} \int\limits_{\Omega_0} |v(x,t) - v(x,t_0)|^2 \, dx = 0. \tag{3.12}$$

Proof Taking into account (3.7), we see that it is enough to prove

$$\lim_{t \to t_0 - 0} \int\limits_{\Omega_0} |v(x,t|^2 \, dx = \int\limits_{\Omega_0} |v(x,t_0|^2 \, dx. \tag{3.13}$$

We first note that (3.5) implies the following fact. For each γ, there exists a countable family of sets of the form

$$b_i^{\gamma, t_0} \equiv B(x_i^\gamma, R_{\gamma i}) \times \{t = t_0\}$$

such that

$$R_{\gamma i} \leq d_0, \quad \Sigma \cap \left(\overline{\Omega}_0 \times \{t = t_0\} \right) \subset \sum_i b_i^{\gamma, t_0}, \quad \sum_i R_{\gamma i} < \gamma. \tag{3.14}$$

Let us fix $\varepsilon > 0$ and let

$$\gamma = \frac{\varepsilon}{8 \, a(\Omega_0, t_0, \delta_0)}.$$

Then, by (3.11) and (3.14), we obtain

$$\left| \int\limits_{\sum_i B(x_i^\gamma, R_{\gamma i})} |v(x,t)|^2 \, dx - \int\limits_{\sum_i B(x_i^\gamma, R_{\gamma i})} |v(x,t_0)|^2 \, dx \right| \leq$$

$$\leq \sum_i \int\limits_{B(x_i^\gamma, R_{\gamma i})} |v(x,t)|^2 \, dx + \sum_i \int\limits_{B(x_i^\gamma, R_{\gamma i})} |v(x,t_0)|^2 \, dx \leq \tag{3.15}$$

$$\leq 2 \, a(\Omega_0, t_0, \delta_0) \sum_i R_{\gamma i} < 2 \, \gamma \, a(\Omega_0, t_0, \delta_0) \leq \tfrac{\varepsilon}{4}$$

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for all $t \in [t_0 - \delta_0^2, t_0]$.

We let
$$\omega^\gamma \equiv \overline{\Omega}_0 \times \{t = t_0\} - \sum_i b_i^{\gamma, t_0}.$$

For each $z \in \omega^\gamma$, there exists a non-empty neighborhood $\mathcal{O}_z$ such that the function $z \mapsto v(z)$ is Hölder continuous on $\mathcal{O}_z \cap \overline{Q}_T$. Since ω^γ is compact, there exists a non-empty neighborhood $\mathcal{O}_\omega^\gamma$ of the set ω^γ such that

$$\omega^\gamma \subset \mathcal{O}_\omega^\gamma$$

and the function $z \to v(z)$ is continuous in $\overline{\mathcal{O}}_\omega^\gamma \cap \overline{Q}_T$. Hence,

$$\left| \int_{\omega^\gamma} |v(x,t)|^2 \, dx - \int_{\omega^\gamma} |v(x,t_0)|^2 \, dx \right| < \frac{\varepsilon}{2} \tag{3.16}$$

for all $0 \le t_0 - t < \mu = \mu(\varepsilon, \Omega_0, t_0, \delta_0) \le \delta_0^2$. Combining (3.15) and (3.16), we obtain

$$\left| \int_{\Omega_0} |v(x,t)|^2 \, dx - \int_{\Omega_0} |v(x,t_0)|^2 \, dx \right| \le \left| \int_{\omega^\gamma} |v(x,t)|^2 \, dx - \int_{\omega^\gamma} |v(x,t_0)|^2 \, dx \right| +$$

$$+ \left| \int_{\sum_i B(x_i^\gamma, R_{\gamma i})} |v(x,t)|^2 \, dx - \int_{\sum_i B(x_i^\gamma, R_{\gamma i})} |v(x,t_0)|^2 \, dx \right| < \varepsilon$$

for all $0 \le t_0 - t < \mu$. Therefore (3.13) and Lemma (3.2) are proved.

In what follows, we going to use the following condition for local Hölder continuity of v.

Lemma 3.3 *Let a pair of v and p be an arbitrary suitable weak solution to the Navier-Stokes equations in Q_T with external force $f \in M_{2,\gamma}(Q_T; \mathbb{R}^3)$ for some positive number γ. There exists a positive number $\varepsilon_\star$, depending on γ only and having the following property. Assume that, for some positive $R_\star$, $Q(z_0, R_\star) \in Q_T$ and*

$$\sup_{0 < R < R_\star} A(z_0, R) < \varepsilon_\star, \tag{3.17}$$

where

$$A(z_0, R) \equiv \sup_{t_0 - R^2 \le t \le t_0} \frac{1}{R} \int_{B(x_0, R)} |v(x,t)|^2 \, dx.$$

Then, z_0 is a regular point of v.

Proof. Our proof is mostly based on the method developed by F.-H. Lin in [13] (see also [11]). As in [11], we introduce the following functionals:

$$A(\rho) \equiv A(z_0, \rho), \quad E(\rho) \equiv \frac{1}{r} \int\limits_{Q(z_0, \rho)} |\nabla v|^2 \, dz,$$

$$C(r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0, \, rho)} |v|^3 \, dz, \quad D(r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0, \rho)} |p|^{\frac{3}{2}} \, dz.$$

We have assumed that $Q(z_0, \rho) \subset Q_T$.

In [11], the following decay estimates involving the above functionals are proved:

$$C(r) \le c_1 \left[\left(\frac{r}{\rho} \right)^3 A^{\frac{3}{2}}(\rho) + \left(\frac{\rho}{r} \right)^3 A^{\frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho) \right] \tag{3.18}$$

for all $0 < r \le \rho$ (see Lemma 5.1 in [11]),

$$A(\tfrac{\rho}{2}) + E(\tfrac{\rho}{2}) \le c_1 \left[C^{\frac{2}{3}}(\rho) + C^{\frac{1}{3}}(\rho) D^{\frac{2}{3}}(\rho) + C(\rho) + d_\gamma^2 \rho^{2(\gamma+1)} \right] \tag{3.19}$$

(see inequality (5.4) in [11]),

$$D(r) \le c_1 \left[\frac{r}{\rho} D(\rho) + \left(\frac{\rho}{r} \right)^2 \left(A^{\frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \right) \right] \tag{3.20}$$

for all $r \in]0, \rho]$ (see Lemma 5.3 in [11]). Here $d_\gamma \equiv d_\gamma(f; Q_T)$ and c_1 is an absolute positive constant.

In contrast to [11], we focus on the functional

$$\mathcal{F}(R) \equiv C(R) + D(R).$$

Let $\theta \in]0, 1/2[$ and $Q(z_0, \rho) \subset Q_T$. We shall fix numbers θ and ρ later. From Young's inequality and from (3.19), we can derive

$$A(\tfrac{\rho}{2}) + E(\tfrac{\rho}{2}) \le c_2 \left[\mathcal{F}^{\frac{2}{3}}(\rho) + \mathcal{F}(\rho) + d_\gamma^2 \rho^{2(\gamma+1)} \right], \tag{3.21}$$

where c_2 is an absolute constant. Combining estimates (3.18) and (3.21), we obtain

$$C(r) \le c_1 \left[\left(\frac{2r}{\rho} \right)^3 A^{\frac{3}{2}}(\tfrac{\rho}{2}) + \left(\frac{\rho}{2r} \right)^3 A^{\frac{3}{4}}(\tfrac{\rho}{2}) E^{\frac{3}{4}}(\tfrac{\rho}{2}) \right] \le c_3 \left[\left(\frac{r}{\rho} \right)^3 A^{\frac{3}{2}}(\rho) \right.$$

$$\left. + \left(\frac{\rho}{r} \right)^3 A^{\frac{3}{4}}(\rho) \left(\mathcal{F}^{\frac{2}{3}}(\rho) + \mathcal{F}(\rho) + d_\gamma^2 \rho^{2(\gamma+1)} \right)^{\frac{3}{4}} \right] \tag{3.22}$$

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for all $0 < r \leq \rho/2$, with c_3 an absolute constant. The same can be done with estimate (3.20). As a result, we have

$$D(r) \leq c_4 \left[\tfrac{r}{\rho} \mathcal{F}(\rho) + \left(\tfrac{\rho}{r} \right)^2 \left(A^{\frac{3}{4}}(\rho) \left(\mathcal{F}^{\frac{2}{3}}(\rho) + \mathcal{F}(\rho) + d_\gamma^2 \rho^{2(\gamma+1)} \right)^{\frac{3}{4}} \right. \right.$$
$$\left. \left. + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \right) \right], \tag{3.23}$$

for all $0 < r \leq \rho/2$, where c_4 is an absolute constant.

Setting $\theta = r/\rho$, we observe that from (3.22) and (3.23) it follows the following estimate:

$$\mathcal{F}(\theta\rho) \leq c_5 \left[\theta \mathcal{F}(\rho) + \theta^3 A^{\frac{3}{2}}(\rho) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} + \right.$$

$$\left. + (\theta^{-3} + \theta^{-2}) A^{\frac{3}{4}}(\rho) \left(\mathcal{F}^{\frac{2}{3}}(\rho) + \mathcal{F}(\rho) + d_\gamma^2 \rho^{2(\gamma+1)} \right)^{\frac{3}{4}} \right] \leq$$

$$\leq c_6 \left[\theta \mathcal{F}(\rho) + \tfrac{1}{\theta^{15}} \left(A^3(\rho) + A^{\frac{3}{2}}(\rho) + A^{\frac{3}{4}}(\rho) d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \right) + d_\gamma^{\frac{3}{2}} \rho^{\frac{3}{2}(\gamma+1)} \right]. \tag{3.24}$$

Here, c_5 and c_6 are absolute constants and $\theta \in]0, 1/2[$.

Let us fix $\theta \in]0, 1/2[$ and $\rho_0 \in]0, R_\star]$ in such a way that

$$\theta c_6 \leq \frac{1}{2}, \qquad d_\gamma^{\frac{3}{2}} \rho_0^{\frac{3}{2}(\gamma+1)} \leq \varepsilon_\star. \tag{3.25}$$

Without loss of generality, one may assume that $\varepsilon_\star \leq 1$. Then, (3.24) and (3.25) imply the bound

$$\mathcal{F}(\theta\rho) \leq \frac{1}{2} \mathcal{F}(\rho) + c_7 \varepsilon_\star, \tag{3.26}$$

for any $\rho \in]0, \rho_0]$, with c_7 an absolute constant. Iterating (3.26), we obtain

$$\mathcal{F}(\frac{\rho}{2^k}) \leq \frac{1}{2^k} \mathcal{F}(\rho) + 2 c_7 \varepsilon_\star$$

for all natural k. The last estimate implies

$$\liminf_{R \to 0+0} \mathcal{F}(R) \leq 2 c_7 \varepsilon_\star. \tag{3.27}$$

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According to Proposition 2.8 in [11], there exists $\overline{\varepsilon}_0(\gamma)$ such that if

$$\liminf_{R \to 0+0} \left\{ \left(\frac{3}{4\pi} C(R) \right)^{\frac{1}{3}} + \left(\frac{3}{4\pi} D(R) \right)^{\frac{2}{3}} \right\} < \overline{\varepsilon}_0(\gamma), \tag{3.28}$$

then z_0 is regular point. Choosing $\varepsilon_\star$ in an appropriate way, we deduce the statement of the lemma from (3.27) and (3.28). Lemma 3.3 is proved.

4 Proof of Theorem 2.2

First, let us prove that, for functions v and p connected by relation (2.6), and for any $x_0 \in \mathbb{R}^3$, for any $t > 0$, and for any $R > 0$, the following identities are valid:

$$\int\limits_{B(x_0,R)} \frac{1}{|y-x_0|} \left(2p(y,t) + |\widehat{v}^{x_0}(y,t)|^2 \right) dy =$$

$$= \int\limits_{B(x_0,R)} \frac{1}{R} \left(3p(y,t) + |v(y,t)|^2 \right) dy = \tag{4.1}$$

$$= R^2 \int\limits_{\mathbb{R}^3 - B(x_0,R)} \nabla_y^2 \left(\frac{1}{|y-x_0|} \right) : \left(v(y,t) \otimes v(y,t) \right) dy,$$

where

$$\widehat{v}^{x_0}(y,t) \equiv v(y,t) - \widetilde{v}^{x_0}(y,t), \quad \widetilde{v}^{x_0}(y,t) \equiv \frac{v(y,t) \cdot (y-x_0)(y-x_0)}{|y-x_0|^2}.$$

To this end, we take a sufficiently regular function $g :]0, +\infty[\to [0, +\infty[$ and observe that, by (2.6),

$$\int\limits_{B(x_0,R)} g(|x_0 - y|) p(y,t) \, dy$$

$$= \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} f(x,t) \int\limits_{B(x_0,R)} g(|x_0 - y|) \frac{1}{|x-y|} \, dy \, dx \,, \tag{4.2}$$

where $f(x,t) \equiv \operatorname{div} \operatorname{div} \left(v(x,t) \otimes v(x,t) \right)$. It is easy to check that

$$\int\limits_{B(x_0,R)} g(|x_0 - y|) \frac{1}{|x-y|} \, dy =$$

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$$
= 4\pi \begin{cases} \dfrac{1}{|x-x_0|} \displaystyle\int_0^{|x-x_0|} \rho^2 g(\rho)d\rho + \displaystyle\int_{|x-x_0|}^R \rho\, g(\rho)d\rho & \text{if} \quad |x-x_0| \le R \\[3em] \dfrac{1}{|x-x_0|} \displaystyle\int_0^R \rho^2 g(\rho)d\rho & \text{if} \quad |x-x_0| > R \end{cases}
$$

Integration by parts in (4.2) leads to the identity

$$
\int_{B(x_0,R)} g(|x_0 - y|)p(y,t)\, dy
$$

$$
= \int_{B(x_0,R)} \Big(v(y,t) \otimes v(y,t) \Big) : \nabla_y^2 \Big(\frac{1}{|y-x_0|} \int_0^{|y-x_0|} \rho^2 g(\rho)d\rho
$$

$$
+ \int_{|y-x_0|}^R \rho\, g(\rho)d\rho \Big)
$$

$$
+ \int_0^R \rho^2 g(\rho)d\rho \int_{\mathbb{R}^3 - B(x_0,R)} \Big(v(y,t) \otimes v(y,t) \Big) : \nabla_y^2 \Big(\frac{1}{|y-x_0|} \Big)\, dy.
$$

Taking $g(\rho) = \frac{1}{\rho}$ and then $g(\rho) = 1$, we arrive at identities (4.1).

Arguing by contradiction, let us denote by t_0 the first moment of time when singular points of v appear. It is known that $t_0 > 0$ and for any $0 < T < t_0$, our solution v is smooth on $\mathbb{R}^3 \times]0,T]$ (see [12]). In particular, for any domain $\Omega \subset \mathbb{R}^3$ and for any $0 < \delta < t_0$, the function v together with the associated pressure p forms a suitable weak solution to the Navier-Stokes equations in the space-time cylinder $Q_{\delta,t_0} \equiv \Omega \times]\delta, t_0[$. Moreover, for any $0 \le t < t_0$, the following two identities hold:

$$
\int_{\mathbb{R}^3} |v(x,t)|^2\, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla\, v(x,t')|^2\, dxdt' = \int_{\mathbb{R}^3} |v_0(x)|^2\, dx
$$

and

$$
\int_{\mathbb{R}^3} |v(x,t)|^2\phi(x)\, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla\, v(x,t')|^2\phi(x)\, dxdt'
$$

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$$= \int\limits_{\mathbb{R}^3} |u_0(x)|^2 \phi(x)\, dx + \int\limits_0^t \int\limits_{\mathbb{R}^3} |v(x,t')|^2 \Delta\phi(x)\, dxdt'$$

$$+ \int\limits_0^t \int\limits_{\mathbb{R}^3} \Big(|v(x,t')|^2 + 2p(x,t') \Big) v(x,t') \cdot \nabla\, \phi(x)\, dxdt'$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$. They imply

$$\int\limits_{\mathbb{R}^3} |v(x,t)|^2 \Big(1 - \phi(x)\Big)\, dx + 2 \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla\, v(x,t')|^2 \Big(1 - \phi(x)\Big)\, dxdt'$$

$$= \int\limits_{\mathbb{R}^3} |u_0(x)|^2 \Big(1 - \phi(x)\Big)\, dx - \int\limits_0^t \int\limits_{\mathbb{R}^3} |v(x,t')|^2 \Delta\phi(x)\, dxdt' \tag{4.3}$$

$$- \int\limits_0^t \int\limits_{\mathbb{R}^3} \Big(|v(x,t')|^2 + 2p(x,t') \Big) v(x,t') \cdot \nabla\, \phi(x)\, dxdt'$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$ and for all $0 \leq t < t_0$. We note that, by the multiplicative inequality, we have

$$\|u\|_{3,Q_{t_0}}^3 \leq c_1 t_0^{\frac{1}{4}} \|u\|_{2,\infty,Q_{t_0}}^{\frac{3}{2}} \|\nabla u\|_{2,Q_{t_0}}^{\frac{3}{2}} \leq c_1 t_0^{\frac{1}{4}} \|u_0\|_{2,Q_{t_0}}^3, \tag{4.4}$$

where $Q_{t_0} = \mathbb{R}^3 \times]0, t_0[$ and c_1 is an absolute constant. Dividing (4.1) by $\frac{4\pi}{3}R^2$ and taking the limit as $R \to 0 + 0$, we obtain

$$3p(x,t) + |u(x,t)|^2 = \frac{3}{4\pi} \int\limits_{\mathbb{R}^3} \nabla_y^2 \Big(\frac{1}{|y-x|} \Big) : \Big(v(y,t) \otimes v(y,t) \Big)\, dy.$$

The theory of singular integrals and (4.4) tells us that

$$\|p\|_{\frac{3}{2},Q_{t_0}} < +\infty. \tag{4.5}$$

Thus, by an appropriate choice of the cut-off function ϕ, we obtain from (4.3)-(4.5) that

$$\lim_{R \to 0+0} \sup_{0 \leq t < t_0} \int\limits_{\mathbb{R}^3 - B(0,R)} |u(x,t)|^2\, dx = 0.$$

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Finally, since

$$\liminf_{t \to t_0 - 0} \int\limits_{\mathbb{R}^3 - B(0,R)} |u(x,t)|^2 \, dx \geq \int\limits_{\mathbb{R}^3 - B(0,R)} |u(x,t_0)|^2 \, dx,$$

we have

$$\lim_{R \to 0+0} \sup_{0 \leq t \leq t_0} \int\limits_{\mathbb{R}^3 - B(0,R)} |u(x,t)|^2 \, dx = 0. \tag{4.6}$$

Assume first that condition (2.9) holds. Then, (4.1) can be transformed to the form

$$-\frac{1}{2R} \int\limits_{B(x_0,R)} |v(x,t)|^2 \, dx + \frac{3}{2R} \int\limits_{B(x_0,R)} \Big(|v(x,t)|^2 + 2p(x,t) \Big) \, dx$$

$$= \int\limits_{B(x_0,R)} \frac{1}{|x - x_0|} \Big(|v(x,t)|^2 + 2p(x,t) \Big) \, dx - \int\limits_{B(x_0,R)} \frac{1}{|x - x)|} |\widetilde{v}^{x_0}(x,t)|^2 \, dx$$

$$= R^2 \int\limits_{\mathbb{R}^3 - B(x_0,R)} K(x, x_0) : \Big(v(x,t) \otimes v(x,t) \Big) \, dx, \tag{4.7}$$

where

$$K(x, x_0) \equiv \nabla_x^2 \Big(\frac{1}{|x - x_0|} \Big).$$

From (4.7), it follows that

$$\frac{1}{2R} \int\limits_{B(x_0,R)} |v(x,t)|^2 \, dx = \frac{3}{2R} \int\limits_{B(x_0,R)} \Big(|v(x,t)|^2 + 2p(x,t) \Big) \, dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x - x_0|} |\widetilde{v}^{x_0}(x,t)|^2 \, dx - \int\limits_{B(x_0,R)} \frac{1}{|x - x_0|} \Big(|v(x,t)|^2 + 2p(x,t) \Big) \, dx$$

$$\leq \frac{3}{2R} \int\limits_{B(x_0,R)} g(x,t) \, dx - \int\limits_{B(x_0,R)} \frac{1}{|x - x_0|} g(x,t) \, dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x - x_0|} |\widetilde{v}^{x_0}(x,t)|^2 \, dx$$

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$$\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}\Big[g(x,t) - \Big(|v(x,t)|^2 + 2p(x,t)\Big)\Big]\,dx$$

and thus

$$\frac{1}{2R}\int\limits_{B(x_0,R)} |v(x,t)|^2\,dx \le \frac{1}{2}\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}g(x,t)\,dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}|\widetilde{v}^{x_0}(x,t)|^2 \qquad (4.8)$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}\Big[g(x,t) - \Big(|v(x,t)|^2 + 2p(x,t)\Big)\Big]\,dx$$

In addition, we are going to use the identity

$$\frac{1}{2}\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}g(x,t)\,dx + \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}|\widetilde{v}^{x_0}(x,t)|^2$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}\Big[g(x,t) - \Big(|v(x,t)|^2 + 2p(x,t)\Big)\Big]\,dx \qquad (4.9)$$

$$= \frac{3}{2}\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}g(x,t)\,dx - R^2 \int\limits_{\mathbb{R}^3-B(x_0,R)} K(x,x_0) : \Big(v(x,t)\otimes v(x,t)\Big)\,dx.$$

According to (2.7), we can obtain from (4.8) and (4.9) that, for any $x_0 \in \mathbb{R}^3$ and for any $R \in\,]0, R_0(t_0)]$, the following bound is valid:

$$\frac{1}{2R}\int\limits_{B(x_0,R)} |v(x,t_0)|^2\,dx \le \frac{1}{2}\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}g(x,t_0)\,dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}|\widetilde{v}^{x_0}(x,t_0)|^2$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|}\Big[g(x,t_0) - \Big(|v(x,t_0)|^2 + 2p(x,t_0)\Big)\Big]\,dx$$

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$$\leq \frac{1}{2} \int\limits_{B(x_0,R_0)} \frac{1}{|x-x_0|} g(x,t_0)\, dx + \int\limits_{B(x_0,R_0)} \frac{1}{|x-x_0|} |\widetilde{v}^{x_0}(x,t_0)|^2$$

$$+ \int\limits_{B(x_0,R_0)} \frac{1}{|x-x_0|} \left[g(x,t_0) - \Big(|v(x,t_0)|^2 + 2p(x,t_0) \Big) \right] dx$$

$$= \frac{3}{2} \int\limits_{B(x_0,R_0)} \frac{1}{|x-x_0|} g(x,t_0)\, dx$$

$$- R_0^2 \int\limits_{\mathbb{R}^3 - B(x_0,R_0)} K(x,x_0) : \Big(v(x,t_0) \otimes v(x,t_0) \Big)\, dx$$

$$\leq \frac{3}{2} A(t_0) + \frac{c_2}{R_0(t_0)} \|v(\cdot,t_0)\|_{2,\mathbb{R}^3}^2,$$

where c_2 is an absolute constant. The last estimate together with the statement of Lemma 3.2 implies that

$$\lim_{t \to t_0 - 0} \int\limits_{B(0,R)} |v(x,t) - v(x,t_0)|^2\, dx = 0$$

for any $R > 0$. Using (4.6), we see that

$$\lim_{t \to t_0 - 0} \int\limits_{\mathbb{R}^3} |v(x,t) - v(x,t_0)|^2\, dx = 0. \tag{4.10}$$

Let $\varepsilon_\star = \varepsilon_\star(0)$ be the number of Lemma 3.3. Fix an arbitrary x_0 in $\mathbb{R}^3$. There exists a positive number $R_\star \leq R_0(t_0)$ such that

$$\frac{\varepsilon_\star}{2} > \frac{1}{2} \int\limits_{B(x_0,R_\star)} \frac{1}{|x-x_0|} g(x,t_0)\, dx + \int\limits_{B(x_0,R_\star)} \frac{1}{|x-x_0|} |\widetilde{v}^{x_0}(x,t_0)|^2\, dx +$$

$$+ \int\limits_{B(x_0,R_\star)} \frac{1}{|x-x_0|} \left[g(x,t_0) - \Big(|v(x,t_0)|^2 + 2p(x,t_0) \Big) \right] dx$$

$$= \frac{3}{2} \int\limits_{B(x_0,R_\star)} \frac{1}{|x-x_0|} g(x,t_0)\, dx$$

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$$-R_\star^2 \int_{\mathbb{R}^3 - B(x_0, R_\star)} K(x, x_0) : \Big(v(x, t_0) \otimes v(x, t_0) \Big) \, dx.$$

But, by the continuity condition (2.8) and by (4.10), the function

$$t \mapsto \frac{3}{2} \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t) \, dx$$

$$-R_\star^2 \int_{\mathbb{R}^3 - B(x_0, R_\star)} K(x, x_0) : \Big(v(x, t) \otimes v(x, t) \Big) \, dx$$

is continuous from the left at the point t_0. Therefore there exists a positive number $\delta_\star \leq \sqrt{t_0/2}$ such that

$$\frac{\varepsilon_\star}{2} > \frac{1}{2} \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t) \, dx + \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} |\widetilde{v}^{x_0}(x, t)|^2$$

$$+ \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} \Big[g(x, t) - \Big(|v(x, t)|^2 + 2p(x, t) \Big) \Big] \, dx$$

for all $t \in [t_0 - \delta_\star^2, t_0]$. Then (4.8) leads to the estimate

$$\frac{1}{2R} \int_{B(x_0, R)} |v(x, t)|^2 \, dx \leq \frac{1}{2} \int_{B(x_0, R)} \frac{1}{|x - x_0|} g(x, t) \, dx$$

$$+ \int_{B(x_0, R)} \frac{1}{|x - x_0|} |\widetilde{v}^{x_0}(x, t)|^2$$

$$+ \int_{B(x_0, R)} \frac{1}{|x - x_0|} \Big[g(x, t) - \Big(|v(x, t)|^2 + 2p(x, t) \Big) \Big] \, dx$$

$$\leq \frac{1}{2} \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t) \, dx + \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} |\widetilde{v}^{x_0}(x, t)|^2$$

$$+ \int_{B(x_0, R_\star)} \frac{1}{|x - x_0|} \Big[g(x, t) - \Big(|v(x, t)|^2 + 2p(x, t) \Big) \Big] \, dx < \frac{\varepsilon_\star}{2}$$

being valid for all $R \in]0, R_\star]$ and for all $t \in [t_0 - \delta_\star^2, t_0]$. The last bound and Lemma 3.3 imply that $z_0 = (x_0, t_0)$ is a regular point. Since x_0 was chosen arbitrarily we get a contradiction with the definition of t_0.

Assume now that conditions (2.10) holds. This case is treated more or less in the same way as the previous one. In particular, it follows from (4.1) that

$$\frac{1}{R} \int\limits_{B(x_0,R)} \left(|v(x,t)|^2 + 3(p(x,t) + g(x,t)) \right) dx$$

$$= \frac{3}{R} \int\limits_{B(x_0,R)} g(x,t)\, dx - 2 \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} g(x,t)\, dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} \left(|\widehat{v}^{x_0}(x,t)|^2 + 2(p(x,t) + g(x,t)) \right) dx \qquad (4.11)$$

$$\leq \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} g(x,t)\, dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} \left(|\widehat{v}^{x_0}(x,t)|^2 + 2(p(x,t) + g(x,t)) \right) dx$$

and

$$\int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} g(x,t)\, dx$$

$$+ \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} \left(|\widehat{v}^{x_0}(x,t)|^2 + 2(p(x,t) + g(x,t)) \right) dx \qquad (4.12)$$

$$= 3 \int\limits_{B(x_0,R)} \frac{1}{|x-x_0|} g(x,t)\, dx + R^2 \int\limits_{\mathbb{R}^3 - B(x_0,R)} K(x,x_0) : \left(v(x,t) \otimes v(x,t) \right) dx.$$

Next, by (2.7), (4.11) and 4.12) that, for any $x_0 \in \mathbb{R}^3$ and for any $R \in]0, R_0(t_0)]$, the following bound is valid:

$$\frac{1}{R} \int\limits_{B(x_0,R)} |v(x,t_0)|^2\, dx \leq 3\, A(t_0) + \frac{c_2}{R_0(t_0)} \|v(\cdot,t_0)\|_{2,\mathbb{R}^3}^2.$$

This estimate, Lemma 3.2, and (4.6) imply (4.10).

Let $\varepsilon_\star = \varepsilon_\star(0)$ be the number of Lemma 3.3. Fix an arbitrary x_0 in $\mathbb{R}^3$. There exists a positive number $R_\star \leq R_0(t_0)$ such that

$$\frac{\varepsilon_\star}{2} > \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t_0)\, dx$$

$$+ \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} \left(|\widehat{v}^{x_0}(x, t_0)|^2 + 2(p(x, t_0) + g(x, t_0)) \right) dx \qquad (4.13)$$

$$= 3 \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t_0)\, dx$$

$$+ R_\star^2 \int\limits_{\mathbb{R}^3 - B(x_0, R_\star)} K(x, x_0) : \Big(v(x, t_0) \otimes v(x, t_0) \Big)\, dx.$$

By continuity condition (2.8) and by (4.10)-(4.13), the function

$$t \mapsto 3 \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t)\, dx$$

$$+ R_\star^2 \int\limits_{\mathbb{R}^3 - B(x_0, R_\star)} K(x, x_0) : \Big(v(x, t) \otimes v(x, t) \Big)\, dx$$

is continuous from the left at the point t_0. Therefore there exists a positive number $\delta_\star \leq \sqrt{t_0/2}$ such that

$$\frac{1}{R} \int\limits_{B(x_0, R)} |v(x, t)|^2\, dx \leq \int\limits_{B(x_0, R)} \frac{1}{|x - x_0|} g(x, t)\, dx$$

$$+ \int\limits_{B(x_0, R)} \frac{1}{|x - x_0|} \left(|\widehat{v}^{x_0}(x, t)|^2 + 2(p(x, t) + g(x, t)) \right) dx \leq$$

$$\leq \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} g(x, t)\, dx$$

$$+ \int\limits_{B(x_0, R_\star)} \frac{1}{|x - x_0|} \left(|\widehat{v}^{x_0}(x, t)|^2 + 2(p(x, t) + g(x, t)) \right) dx < \frac{\varepsilon_\star}{2}$$

22

for all $R \in]0, R_\star]$ and for all $t \in [t_0 - \delta_\star^2, t_0]$. And, again, this estimate together with Lemma 3.3 leads to the same contradiction with the definition of t_0. Theorem 2.2 is proved.

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