Max-Planck-Institut
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Symplectic structures on moduli spaces
of parabolic Higgs and Hilbert scheme

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by

Indranil Biswas and Avijit Mukherjee

SYMPLECTIC STRUCTURES ON MODULI SPACES OF PARABOLIC HIGGS BUNDLES AND HILBERT SCHEME

INDRANIL BISWAS AND AVIJIT MUKHERJEE

ABSTRACT. Parabolic triples of the form (E_*, θ, σ) are considered, where (E_*, θ) is a parabolic Higgs bundle on a given compact Riemann surface X with parabolic structure on a fixed divisor S, and σ is a nonzero section of the underlying vector bundle. Sending such a triple to the Higgs bundle (E_*, θ) a map from the moduli space of stable parabolic triples to the moduli space of stable parabolic Higgs bundles is obtained. The pull back, by this map, of the symplectic form on the moduli space of stable parabolic Higgs bundles will be denoted by $d\Omega'$. On the other hand, there is a map from the moduli space of stable parabolic triples to a Hilbert scheme $\mathrm{Hilb}^\delta(Z)$, where Z denotes the total space of the line bundle $K_X \otimes \mathcal{O}_X(S)$, that sends a triple (E_*, θ, σ) to the divisor defined by the section σ on the spectral curve corresponding to the parabolic Higgs bundle (E_*, θ). Using this map and a meromorphic one–form on $\mathrm{Hilb}^\delta(Z)$, a natural two–form on the moduli space of stable parabolic triples is constructed. It is shown here that this form coincides with the above mentioned form $d\Omega'$.

1. INTRODUCTION

In [2], we proved that the two–form on the moduli space of triples of the form (E, θ, σ), where (E, θ) is a stable Higgs bundle on Riemann surface X and σ a nonzero section of E, obtained by pulling back the natural symplectic form on the moduli space of stable Higgs bundles coincides with the pullback of the symplectic structure on the Hilbert scheme of zero dimensional subschemes of the total space of of the holomorphic cotangent bundle K_X of X. Our aim here is to establish an analogous result in the context of parabolic triples.

Let X be a compact connected Riemann surface of genus at least two. Fix a finite subset S of X. A parabolic vector bundle of rank two with parabolic structure over S consists of a holomorphic vector bundle E, a line $F_s \subset E_s$, and a real number $0 < \lambda_s < 1$ for each $s \in S$. A Higgs field on this parabolic bundle is a holomorphic section θ of the vector bundle $\mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)$ such that $\theta(s)$ is nilpotent with respect to the flag $0 = F^0 \subset F^1 = F_s \subset F^2 = E_s$ of the fiber E_s for each $s \in S$.

We fix the parabolic weights $\{\lambda_s\}_{s \in S}$ and consider parabolic Higgs bundles of rank two and fixed degree d, with $d > 6(\mathrm{genus}(X) - 1) + \#S$. Let $\mathcal{M}_H^s$ denote the moduli space of stable parabolic Higgs bundles. For such stable parabolic Higgs bundles the dimension of the space of all holomorphic sections of the underlying vector bundle remains constant

over the moduli space (Lemma 3.1). Therefore, there is a projective bundle

$$\phi \,:\, \mathbb{P}_H \,\longrightarrow\, \mathcal{M}_H^s$$

whose fiber over any point representing a parabolic Higgs bundle $((E, \{F_s\}), \theta)$ is the projective space $\mathbb{P}H^0(X, E)$. In other words, $\mathbb{P}_H$ is the moduli space of triples of the form $((E, \{F_s\}), \theta, \sigma)$, where $((E, \{F_s\}), \theta)$ is a stable parabolic Higgs bundle and σ is a nonzero section of E. See [1] and [5], where the notion of triples were introduced, for a detailed study and many interesting results on triples.

The moduli space $\mathcal{M}_H^s$ has a natural holomorphic one–form, which we call Ω, such that $d\Omega$ is a symplectic form on $\mathcal{M}_H^s$. The total space of the holomorphic cotangent bundle of the moduli space of stable parabolic bundles sits inside $\mathcal{M}_H^s$ as a Zariski open dense subset. The restriction of Ω to this open subset coincides with the canonical one–form on the total space of any cotangent bundle. The main result proved here relates the form $\phi^*\Omega$ on $\mathbb{P}_H$ with a certain one–form on a Hilbert scheme.

Let Z denote the surface defined by the total space of the line bundle $K_X \otimes \mathcal{O}_X(S)$ over X. Given a stable parabolic Higgs bundle $((E, \{F_s\}), \theta)$, there is a spectral curve, which is a divisor on Z and a rank one torsionfree sheaf $\mathcal{L}$ on it such that $\gamma_*\mathcal{L} \cong E$, where γ is the projection map of the spectral curve to X. Since γ is a finite map, we have $H^0(X, E)$ identified with the space of all sections of $\mathcal{L}$. Considering the divisor of the section of $\mathcal{L}$ corresponding to $\sigma \in H^0(X, E) \setminus \{0\}$ we get a map from $\mathbb{P}_H$ to the Hilbert scheme $\mathrm{Hilb}^\delta(Z)$ of zero dimensional subschemes of Z of length $\delta = d + 2(\mathrm{genus}(X) - 1) + \#S$. This map, which we will denote by f, sends a parabolic triple $((E, \{F_s\}), \theta, \sigma)$ to the divisor on the spectral curve for $((E, \{F_s\}), \theta)$ defined by σ. Note that using the inclusion map of a spectral curve in Z, a divisor on a spectral curve is a zero dimensional subscheme of Z. (See Section 3.1 for the details.)

Using the fact that Z is the total space of $K_X \otimes \mathcal{O}_X(S)$, there is a natural meromorphic one–form Ω_δ on $\mathrm{Hilb}^\delta(Z)$. The pullback $f^*\Omega_\delta$ is a holomorphic one–form on $\mathbb{P}_H$.

We prove that $f^*\Omega_\delta$ coincides with $\phi^*\Omega$ (Theorem 3.2).

Although we have restricted ourselves to rank two parabolic bundles, the extension of Theorem 3.2 to higher rank case is straight forward. The reason for restriction to rank two case is the ensuing notational simplification.

2. Preliminaries

2.1. Parabolic bundles.
Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix a finite subset

$$S \,:=\, \{s_1, s_2, \cdots, s_n\} \,\subset\, X\,.$$

A *parabolic vector bundle of rank two* over X with parabolic structure over S consists of the following [8]:

(1) a holomorphic vector bundle E of rank two over X;
(2) for each point $s \in S$, a line $F_s \subset E_s$ of the fiber E_s;
(3) for each point $s \in S$, a real number λ_s with $0 < \lambda_s < 1$.

The numbers $\{\lambda_s\}$ are called *parabolic weights*.

For a parabolic vector bundle $E_* := (E, \{F_s\}, \{\lambda_s\})$ as above, the *parabolic degree* is defined to be

$$\text{par-deg}(E_*) := \deg(E) + \sum_{s \in S} \lambda_s$$

where $\deg(E)$ is the degree of E [8, Definition 1.11]. The parabolic vector bundle E_* is called *stable* if for every holomorphic line subbundle L of E, the inequality

$$(2.1) \qquad \deg(L) + \sum_{s \in S'} \lambda_s < \frac{\text{par-deg}(E_*)}{2}$$

where $S' := \{s \in S \,|\, L_s = F_s\}$ [8, Definition 1.13(ii)]. If an weaker condition, namely

$$(2.2) \qquad \deg(L) + \sum_{s \in S'} \lambda_s \leq \frac{\text{par-deg}(E_*)}{2}$$

is valid, then E_* is called *semistable* [8].

We fix real numbers $\{\lambda_s\}_{s \in S}$ and an integer d. Henceforth, by a parabolic vector bundle we will always mean a parabolic vector bundle over X of rank two and degree d with parabolic structure over S and with parabolic weight λ_s for each $s \in S$.

The moduli space of all semistable parabolic bundles is an irreducible normal projective variety of dimension $4g - 3 + n$, where $n = \#S$ [8], [4], and the moduli space of all stable parabolic bundles is a Zariski open smooth subvariety of it.

For a general rank two parabolic bundle E_* as in [8] at each point $s \in S$ over which parabolic structure is defined there are two parabolic weights. However given a general set of weights, say $0 \leq \alpha_s < \beta_s < 1$, if we replace it by the single weight $\beta_s - \alpha_s$, then the parabolic (semi)stability condition remains unchanged. Hence there is no loss of generality if we assume that at each $s \in S$ there is exactly one parabolic weight.

2.2. **Parabolic Higgs bundles.** Let K_X denote the holomorphic cotangent bundle of X. A Higgs structure of a parabolic vector bundle $E_* := (E, \{F_s\})$ is a holomorphic section

$$(2.3) \qquad \theta \in H^0(X, \text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S))$$

with the property that for each $s \in S$, the image of the homomorphism

$$\theta(s) : E_s \longrightarrow (E \otimes K_X \otimes \mathcal{O}_X(S))_s$$

is contained in the subspace $F_s \otimes (K_X \otimes \mathcal{O}_X(S))_s \subset (E \otimes K_X \otimes \mathcal{O}_X(S))_s$ and $\theta(s)(F_s) = 0$ [6], [3], [4]. In other words, $\theta(s)$ is nilpotent with respect to the flag $0 \subset F_s \subset E_s$.

A parabolic Higgs bundle (E_*, θ) as above is called *stable* if for every line subbundle L of E with

$$\theta(L) \subseteq L \otimes K_X \otimes \mathcal{O}_X(S) \subset E \otimes K_X \otimes \mathcal{O}_X(S)$$

the inequality (2.1) is satisfied; if the weaker inequality (2.2) is valid, then (E_*, θ) is called *semistable*. In particular, if E_* is stable (respectively, semistable), then (E_*, θ) is stable (respectively, semistable). However, (E_*, θ) can be stable (respectively, semistable) without E_* being stable (respectively, semistable).

The moduli space of stable parabolic Higgs bundles is an irreducible smooth quasiprojective variety of dimension $8g - 6 + 2n$. This moduli space will be denoted by $\mathcal{M}_H^s$. There is a natural inclusion of the total space of the holomorphic cotangent bundle of the moduli space of stable parabolic bundles into $\mathcal{M}_H$. Given a stable parabolic bundle E_*, the holomorphic cotangent space at the point E_* of the moduli space of stable parabolic bundles is the space of all sections θ as in the definition of Higgs structure in (2.3) [3, Section 6]. Therefore, we have a tautological map from the moduli space of stable parabolic bundles to $\mathcal{M}_H$. The image of this map is a Zariski open subset.

The moduli space of semistable parabolic Higgs bundles will be denoted by $\mathcal{M}_H$. It is an irreducible normal quasiprojective variety, and $\mathcal{M}_H^s$ is a Zariski open smooth subvariety of it.

2.3. Symplectic structure on moduli space.

The natural symplectic form on the total space of the cotangent bundle of the moduli space of stable parabolic bundles extends to a symplectic form on the moduli space $\mathcal{M}_H^s$ of stable parabolic Higgs bundles. The symplectic form on $\mathcal{M}_H^s$ is described in [3, Section 6]; we will briefly recall it here.

Take a stable parabolic Higgs bundle $(E_*, \theta) := ((E, \{F_s\}), \theta)$ represented by a point in $\mathcal{M}_H^s$. Define

$$\mathrm{End}^1(E) \subset \mathrm{End}(E)$$

by the condition that for any $s \in S$ and $v \in \mathrm{End}^1(E)_s$ we have $v(F_s) \subseteq F_s$. Let

$$\mathrm{End}^0(E) \subset \mathrm{End}^1(E)$$

be defined by the condition $v(F_s) = 0$. So $\mathrm{End}^1(E)$ and $\mathrm{End}^0(E)$ coincide with $\mathrm{End}(E)$ on the complement of S, and the Higgs field θ is a section of $\mathrm{End}^0(E) \otimes K_X \otimes \mathcal{O}_X(S)$. Consider the two term complex $C_{\bullet}$ of sheaves defined by

(2.4) $$C_0 := \mathrm{End}^1(E) \xrightarrow{[-,\theta]} C_1 := \mathrm{End}^0(E) \otimes K_X \otimes \mathcal{O}_X(S)$$

where C_i is at the ith position and the homomorphism is defined by $v \longmapsto [v, \theta] = v\theta - \theta v$. Note that the defining condition on θ ensures that $[C_0, \theta] \subset C_1$.

The tangent space $T_{(E_*,\theta)}\mathcal{M}_H^s$ of the variety $\mathcal{M}_H^s$ at the point represented by the parabolic Higgs bundle (E_*,θ) is identified with the hypercohomology $\mathbb{H}^1(C_\cdot)$ [3, Lemma 6.1]. We will show that the Higgs field θ defines a functional on $\mathbb{H}^1(C_\cdot)$. Consider the homomorphism

$$
\begin{array}{ccc}
C_0 & \xrightarrow{[-,\theta]} & C_1 \\
\downarrow & & \downarrow \\
\mathrm{End}^1(E) & \longrightarrow & 0
\end{array}
$$

This induces a homomorphism

(2.5)
$$
f \;:\; H^1(C_\cdot) \longrightarrow H^1(X, \mathrm{End}^1(E)).
$$

Now observe that $\mathrm{End}^1(E)^* \cong \mathrm{End}^0(E) \otimes \mathcal{O}_X(S)$ with the duality pairing defined by

$$
(\omega, \alpha) \longmapsto \mathrm{trace}(\omega\alpha) \in \mathbb{C},
$$

where $\alpha \in \mathrm{End}^1(E)_x$ and $\omega \in (\mathrm{End}^0(E) \otimes \mathcal{O}_X(S))_x$, and x is any point of X. Now the Serre duality says

$$
H^1(X, \mathrm{End}^1(E))^* \cong H^0(X, \mathrm{End}^0(E) \otimes K_X \otimes \mathcal{O}_X(S)).
$$

Consequently, we have a functional $\theta' \in H^1(C_\cdot)^*$ defined by $\beta \longmapsto (\theta, f(\beta))$, where f is constructed in (2.5).

Let Ω denote the one–form on the variety $\mathcal{M}_H^s$ that sends any tangent vector $\beta \in T_{(E_*,\theta)}\mathcal{M}_H^s$, where $(E_*,\theta) \in \mathcal{M}_H^s$, to $\theta'(\beta)$ constructed above.

The two–form $\mathrm{d}\Omega$ is a symplectic form on $\mathcal{M}_H^s$. The restriction of $\mathrm{d}\Omega$ to the Zariski open subset of $\mathcal{M}_H^s$ defined by the total space of the cotangent bundle of the moduli space of parabolic bundles coincides with the canonical symplectic structure on cotangent bundles. (See [3], [4].)

2.4. **Spectral data for parabolic Higgs bundles.** Let (E_*,θ) be a parabolic Higgs bundle. So we have $\mathrm{trace}(\theta) \in H^0(X, K_X)$ and

$$
\mathrm{trace}(\theta^2) \in H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S))
$$

since $\theta(s)$ is nilpotent for each $s \in S$. Set

(2.6)
$$
\mathcal{H} \;:=\; H^0(X, K_X) \oplus H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).
$$

Hitchin defined a map

(2.7)
$$
\psi \;:\; \mathcal{M}_H \longrightarrow \mathcal{H}
$$

that sends any semistable parabolic Higgs bundle (E_*,θ) to $(\mathrm{trace}(\theta), \mathrm{trace}(\theta^2)) \in \mathcal{H}$ [7], [6], [4]. This map ψ is known as the *Hitchin map*, and $\mathcal{H}$ is known as the *Hitchin space*.

Given any point of $\mathcal{H}$, there is an algebraic curve on the total space of the line bundle $K_X \otimes \mathcal{O}_X(S)$ which is known as a *spectral curve* [7], [4]. To describe its construction, let Z denote the total space of the line bundle $K_X \otimes \mathcal{O}_X(S)$, which is a quasiprojective

complex surface. The natural projection of Z to X will be denoted by γ. Since S is an effective divisor, there is a natural homomorphism from $K_X^{\otimes 2} \otimes \mathcal{O}_X(S)$ to $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$; this homomorphism will be denoted by q.

Take any point $(\alpha\,,\beta) \in \mathcal{H}$. Consider the map from Z to the total space of the line bundle $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$ defined by

$$(2.8) \qquad z \longmapsto z \otimes z + q(z \otimes \alpha(\gamma(z))) + q(\beta(\gamma(z))) \in (K_X^{\otimes 2} \otimes \mathcal{O}_X(2S))_{\gamma(z)}\,.$$

The inverse image (in Z) by this map of the zero section of $K_X^{\otimes 2} \otimes \mathcal{O}_X(2S)$ is the spectral curve associated to the point $(\alpha\,,\beta)$ of the Hitchin space. In other words, a point $z \in Z$ lies in the spectral curve associated to $(\alpha\,,\beta)$ if and only if we have $z \otimes z + q(z \otimes \alpha(\gamma(z))) + q(\beta(\gamma(z))) = 0$.

For any point $h = (\alpha\,,\beta) \in \mathcal{H}$, the corresponding spectral curve will be denoted by Y_h. The restriction to Y_h of the projection γ to X will also be denoted by γ. Note that the map

$$(2.9) \qquad\qquad\qquad\qquad \gamma : Y_h \longrightarrow X$$

is of degree two. This map is evidently ramified over every point of S. Indeed, this is an immediate consequence of the fact that the homomorphism q vanishes at any point of S. Note that for any point of $s \in S$, the inverse image $\gamma^{-1}(s) \subset Y_h$ is the point $0 \in (K_X \otimes \mathcal{O}_X(S))_s$ (with multiplicity two).

Given a parabolic Higgs bundle, there is a corresponding spectral curve and a torsion-free sheaf on it [7], [4]. To describe the construction, take a semistable parabolic Higgs bundle $(E_*\,,\theta) \in \mathcal{M}_H$ on X. The spectral curve associated to it is the one defined by $\psi((E_*\,,\theta)) \in \mathcal{H}$ by the above construction, where ψ is the Hitchin map defined in (2.7). Denote the point $\psi((E_*\,,\theta))$ by h.

There is a torsionfree sheaf $\mathcal{L}$ of rank one on the spectral curve Y_h such that

$$(2.10) \qquad\qquad\qquad\qquad \gamma_*\mathcal{L} \cong E$$

(the underlying vector bundle of the parabolic bundle), where γ as in (2.9). The spectral curve can be thought of as the eigenvalues of the endomorphism θ. The sheaf $\mathcal{L}$ is defined by the corresponding eigenvectors (see [7] for the details). Since the map γ is ramified over any point of $s \in S$, the direct image $\gamma_*\mathcal{L}$ has a filtration over s. This filtration is defined by the order of the vanishing (at $\gamma^{-1}(s)$) of a (locally defined) section of $\mathcal{L}$. In the isomorphism of $\gamma_*\mathcal{L}$ with E, The filtration of $\gamma_*\mathcal{L}$ at any $s \in S$ coincides with the filtration $F_s \subset E_s$ for the parabolic structure.

In the next section we will define parabolic triples.

3. Parabolic triples and Hilbert scheme

In Section 2.1 we fixed the degree of a parabolic vector bundle to be d. Now, obviously what is important is the parity (even or odd) of the integer d, and its exact value is not important. More precisely, if we fix a line bundle ξ of degree d', then by tensoring with ξ the parabolic vector bundles of degree d get identified with parabolic vector bundles of degree $d + 2d'$ keeping rest of the parabolic data unchanged.

Henceforth, we will assume that the integer d, which is the degree of a parabolic vector bundle, satisfies the condition

$$d > 6g - 6 + n$$

where $n = \#S$. (The above remark justifies this assumption in the sense that the assumption does not cause any loss of generality.)

Lemma 3.1. *For any semistable parabolic Higgs bundle $(E_*, \theta) \in \mathcal{M}_H$ over X, we have $H^1(X, E) = 0$, where E is the underlying vector bundle. Consequently, $\dim H^0(X, E) = d + 2(1 - g)$.*

Proof. Take any $(E_*, \theta) \in \mathcal{M}_H$. Let E be the underlying vector bundle for the parabolic vector bundle E_*. Since $H^1(X, E) \cong H^0(X, E^* \otimes K_X)^*$, it suffices to show $H^0(X, E^* \otimes K_X) = 0$. Assume that $\tau \in H^0(X, E^* \otimes K_X) \setminus \{0\}$ is nonzero section.

So τ defines a nonzero homomorphism from E to K_X, which will be denoted by $\overline{\tau}$. The kernel of $\overline{\tau}$ is a torsionfree coherent subsheaf of E. Therefore, it defines a line bundle over X, which will be denoted by L. Now we have a diagram

$$
\begin{array}{ccccc}
L & \stackrel{\iota}{\longrightarrow} & E & \stackrel{\overline{\tau}}{\longrightarrow} & K_X \\
 & & \downarrow{\theta} & & \\
L \otimes K_X \otimes \mathcal{O}_X(S) & \stackrel{\iota \otimes \mathrm{Id}}{\longrightarrow} & E \otimes K_X \otimes \mathcal{O}_X(S) & \stackrel{\overline{\tau} \otimes \mathrm{Id}}{\longrightarrow} & K_X^{\otimes 2} \otimes \mathcal{O}_X(S)
\end{array}
$$

(3.1)

We will show that the composition $(\overline{\tau} \otimes \mathrm{Id}) \circ \theta \circ \iota = 0$. To prove this, first note that the top sequence in (3.1) shows that

$$(3.2) \qquad \mathrm{degree}(L) \geq \mathrm{degree}(E) - \mathrm{degree}(K_X) = d - 2g + 2.$$

On the other hand, $\mathrm{degree}(K_X^{\otimes 2} \otimes \mathcal{O}_X(S)) = 2g - 4 + n$, where n is the cardinality of the set S. Since d is assumed to be at least $6g - 5 + n$, we have

$$\mathrm{degree}(L) > \mathrm{degree}(K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).$$

Consequently, there is no nonzero homomorphism from L to $K_X^{\otimes 2} \otimes \mathcal{O}_X(S))$. In particular, the composition $(\overline{\tau} \otimes \mathrm{Id}) \circ \theta \circ \iota = 0$.

Let L' denote the line subbundle of E generated by L (note that ι may not be fiberwise injective). Since $(\overline{\tau} \otimes \mathrm{Id}) \circ \theta \circ \iota = 0$, it follows immediately, that $\theta(L') \subseteq L' \otimes K_X \otimes \mathcal{O}_X(S)$.

Finally, we have

$$\text{degree}(L') \geq \text{degree}(L) \geq d - 2g + 2 = \frac{d}{2} + \frac{1}{2}(d - 4g + 4) > \frac{d}{2} + \frac{n}{2} = \frac{\text{par-deg}(E_*)}{2}$$

where the second inequality was obtained in (3.2) and the third one follows from the assumption $d > 6g - 6 + n$; the first inequality follows from the fact that there is a nonzero homomorphism from L to L'. The above inequality shows that the line subbundle L' of E contradicts the semistability condition of the parabolic Higgs bundle (E_*, θ).

Therefore, we have $H^1(X, E) = 0$. Now the Riemann–Roch says that $\dim H^0(X, E) = d + 2(1 - g)$. This completes the proof of the lemma. $\qquad\square$

The above lemma says that $\dim H^0(X, E)$ remains constant over $\mathcal{M}_H^s$. Therefore, there is a natural projective bundle

$$(3.3) \qquad\qquad \phi : \mathbb{P}_H \longrightarrow \mathcal{M}_H^s$$

of relative dimension $d - 2g + 1$ such that the fiber over any point $(E_*, \theta) \in \mathcal{M}_H$ is the projective space $\mathbb{P}H^0(X, E)$ consisting of all lines in $H^0(X, E)$; as before, E denotes the underlying vector bundle for the parabolic bundle E_*.

Therefore, $\mathbb{P}_H$ is the moduli space of triples of the form (E_*, θ, σ), where (E_*, θ) is a stable parabolic Higgs bundle and $\sigma \in H^0(X, E) \setminus \{0\}$ a nonzero section.

By a *parabolic triple* we will mean a triple (E_*, θ, σ) of the above type. Consequently, $\mathbb{P}_H$ is the moduli space of all parabolic triples.

In Section 2.3 we defined a symplectic structure $d\Omega$ on $\mathcal{M}_H^s$. Define the algebraic one–form

$$(3.4) \qquad\qquad \Omega' := \phi^*\Omega$$

on $\mathbb{P}_H$, where ϕ is defined in (3.3). So,

$$\mathrm{d}\Omega' = \phi^* d\Omega$$

is the pullback to $\mathbb{P}_H$ of the symplectic form Ω on $\mathcal{M}_H^s$.

3.1. Parabolic triples and spectral data.
Take a parabolic triple $(E_*, \theta, \sigma) \in \mathbb{P}_H$. Its image $\psi((E_*, \theta)) \in \mathcal{H}$ will be denoted by h, where ψ is the Hitchin map defined in (2.7).

As we saw in Section 2.4, the parabolic Higgs bundle (E_*, θ) gives a spectral curve Y_h and a torsionfree sheaf $\mathcal{L}$ of rank one on Y_h. It was noted in (2.10) that $\gamma_*\mathcal{L} \cong E$, where γ is the projection of Y_h to X. Now, since γ is a finite map, the natural homomorphism

$$H^i(Y_h, \mathcal{L}) \longrightarrow H^i(X, \gamma_*\mathcal{L}) = H^i(X, E)$$

is an isomorphism for all $i \geq 0$. Therefore, $\mathbb{P}H^0(Y_h, \mathcal{L}) \cong \mathbb{P}H^0(X, E)$. The point in $\mathbb{P}H^0(Y_h, \mathcal{L})$ corresponding the point $\sigma \in \mathbb{P}H^0(X, E)$ will be denoted by σ'. In particular, σ' is a divisor on Y_h.

We will calculate the degree of the divisor defined by σ'. The Hitchin map ψ in (2.7) is a complete integrable system for the symplectic structure $d\Omega$ on $\mathcal{M}_H$ and the fiber of ψ over any $h' \in \mathcal{H}$ is the Jacobian of the corresponding spectral curve $Y_{h'}$. Consequently, the genus of $Y_{h'}$ coincides with $\dim \mathcal{M}_H / 2 = 4g - 3 + n$. Since

$$\text{degree}(\mathcal{L}) - (4g - 3 + n) + 1 = \chi(\mathcal{L}) = \chi(E) = d + 2(1 - g)$$

we conclude that $\text{degree}(\mathcal{L}) = d + n + 2(g - 1)$. Hence the degree of the divisor defined by the section σ' of $\mathcal{L}$, which coincides with the degree of $\mathcal{L}$, is $d + n + 2(g - 1)$, where $n = \#S$.

Set $\delta = d + n + 2(g - 1)$. Let $\text{Hilb}^\delta(Z)$ denote the Hilbert scheme of zero dimensional subschemes of the surface Z (the total space of $K_X \otimes \mathcal{O}_X(S)$) of length δ.

The divisor of σ' is a zero dimensional subscheme of Y_h of length δ. Taking the image of σ' by the inclusion map of the spectral curve Y_h in Z, we get a zero dimensional subscheme of Z of length δ. Therefore, we have an element of $\text{Hilb}^\delta(Z)$. Let

$$(3.5) \qquad\qquad f \;:\; \mathbb{P}_H \;\longrightarrow\; \text{Hilb}^\delta(Z)$$

be the map that sends any parabolic triple to the zero dimensional subscheme of Z defined by the divisor of the corresponding section on the spectral curve for the underlying parabolic Higgs bundle. In other words, f sends any $(E_*, \theta, \sigma) \in \mathbb{P}_H$ to the element of $\text{Hilb}^\delta(Z)$ defined by the divisor of σ' on Y_h.

3.2. **Forms on moduli of triples.** Using the map f defined in (3.5) we will construct an algebraic one–form on $\mathbb{P}_H$, and for that we will first show the existence of a natural meromorphic one–form on $\text{Hilb}^\delta(Z)$.

We start by observing that the variety Z has a natural meromorphic one–form with pole, of order at most one, along the divisor $\gamma^{-1}(S)$, where γ, as before is the projection of Z to X. The $\gamma^* \mathcal{O}_X(S)$ valued one–form sends any tangent vector $v \in T_z Z$ to $z \otimes d\gamma(z)(v)$, where $d\gamma(z) : T_z Z \longrightarrow T_{\gamma(z)}X$ is the differential of γ at z. Note that since z is an element of the fiber $(K_X \otimes \mathcal{O}_X(S))_{\gamma(z)}$, the tensor product $z \otimes d\gamma(z)(v)$ gives an element of the fiber $(\mathcal{O}_X(S))_{\gamma(z)}$ after contracting $(K_X)_{\gamma(z)}$ with $T_{\gamma(z)}X$. Since S is an effective reduced divisor, a $\gamma^* \mathcal{O}_X(S)$ valued one–form on Z is a meromorphic one–form on Z with a pole along $\gamma^{-1}(S)$ of order at most one.

The meromorphic one–form on Z defined above will be denoted by Ω_Z. Using Ω_Z, a meromorphic one–form will be constructed on Hilbert scheme $\text{Hilb}^k(Z)$ of zero dimensional subschemes of Z of length k, where $k \geq 1$.

Consider the Zariski open dense subset $U_k \subset \text{Hilb}^k(Z)$ consisting of *distinct* k points of Z. Let

$$\underline{z} = \{z_1, z_2, \cdots, z_k\} \in \text{Hilb}^k(Z)$$

be a point of U_k, that is, all z_i are distinct. Then we have

$$T_{\underline{z}}\mathrm{Hilb}^k(Z) \;=\; \bigoplus_{i=1}^{k} T_{z_i}Z\,.$$

Therefore, the meromorphic one–form Ω_Z on Z defines a meromorphic one–form on the Zariski open subset U_k of $\mathrm{Hilb}^k(Z)$. In other words, this form sends any tangent vector

$$\{v_1, v_2, \cdots, v_k\} \;\in\; T_{\underline{z}}\mathrm{Hilb}^k(Z)$$

where $v_i \in T_{z_i}Z$, to

$$\sum_{i=1}^{k}\Omega_Z(z_i)(v_i)$$

whenever the sum makes sense. Evidently, this one–form is regular on the complement of the divisor on U_k defined by all points $\{z_1, z_2, \cdots, z_k\}$ such that

$$\{\gamma(z_1), \gamma(z_2), \cdots, \gamma(z_k)\} \cap S \;\neq\; \emptyset$$

as the above sum makes sense on the complement. More precisely, the pole of the one–form on U_k is over this divisor, and the order of the pole is one. Since U_k is a Zariski open dense subset of $\mathrm{Hilb}^k(Z)$, the meromorphic one–form on U_k defines a meromorphic one–form on $\mathrm{Hilb}^k(Z)$.

The meromorphic one–form on $\mathrm{Hilb}^k(Z)$ defined above will be denoted by Ω_k. Note that $\mathrm{Hilb}^1(Z) = Z$ and $\Omega_1 = \Omega_Z$.

Consider the meromorphic one–form $f^*\Omega_\delta$ on $\mathbb{P}_H$, where the map f is defined in (3.5). It was noted in Section 2.4 that for any spectral curve Y_h and any $s \in S$, we have $\gamma^{-1}(s) \cap Y_h = \{0\}$ (that is, the spectral curve is totally ramified over s and passes through 0). From this it follows immediately that $f^*\Omega_\delta$ is a holomorphic one–form on $\mathbb{P}_H$. Indeed, for the origin $0 \in (K_X \otimes \mathcal{O}_X(S))_x$, where $x \in X$, the form Ω_Z vanishes at 0. Therefore, $f^*\Omega_\delta$ is a holomorphic one–form on $\mathbb{P}_H$.

Recall the one–form $\Omega' = \phi^*\Omega$ on $\mathbb{P}_H$ constructed in (3.4).

Theorem 3.2. *The one–form $f^*\Omega_\delta$ on $\mathbb{P}_H$ coincides with the one–form Ω'. In particular, $\mathrm{d}f^*\Omega_\delta$ coincides with the pullback $\phi^*\mathrm{d}\Omega$ of the symplectic form $\mathrm{d}\Omega$ on $\mathcal{M}_H^s$.*

This theorem will be proved in the next section.

4. Identification of one–forms

We start with the following lemma.

Lemma 4.1. *There is a holomorphic one–form ω on $\mathcal{M}_H^s$ such that $\phi^*\omega$ coincides with $f^*\Omega_\delta$, where ϕ is the projection defined in (3.3).*

Proof. Recall that $\mathcal{P}_H$ is a projective bundle over $\mathcal{M}_H^s$ with ϕ being the projection map. Since there is no nonzero holomorphic one–form on a projective space, the pullback of $f^*\Omega_\delta$ by the inclusion map of a fiber of ϕ must vanish identically.

Let $T^\phi \subset T\mathcal{P}_H$ be the relative tangent bundle. In other words, T^ϕ is the kernel of the differential $\mathrm{d}\phi : T\mathcal{P}_H \longrightarrow \phi^*T\mathcal{M}_H^s$. Since the evaluation of $f^*\Omega_\delta$ on T^ϕ vanishes, there is a homomorphism

$$\omega' : T\mathcal{P}_H/T^\phi \longrightarrow \mathcal{O}_{\mathcal{P}_H}$$

such that $f^*\Omega_\delta$ coincides with the composition of the natural projection of $T\mathcal{P}_H$ to $T\mathcal{P}_H/T^\phi$ with the homomorphism ω'; here $\mathcal{O}_{\mathcal{P}_H}$ is the structure sheaf of $\mathcal{P}_H$, or equivalently, the sheaf defined by the trivial line bundle.

For any point $\zeta \in \mathcal{M}_H^s$, the restriction of $T\mathcal{P}_H/T^\phi$ to the fiber $\phi^{-1}(\zeta)$ is a trivial vector bundle. In fact, $T\mathcal{P}_H/T^\phi$ is identified with the pullback $\phi^*T\mathcal{M}_H^s$. Since $\phi^{-1}(\zeta)$ is a compact and connected, the homomorphism $\omega'(p)$ is independent of $p \in \phi^{-1}(\zeta)$ (with ζ fixed). In other words, there is a holomorphic one–form ω on $\mathcal{M}_H^s$ such that ω' is the pullback of ω. This completes the proof of the lemma. $\square$

Recall the one–form Ω on $\mathcal{M}_H^s$ constructed in Section 2.3. Since $\Omega' = \phi^*\Omega$ (see (3.4)), in view of Lemma 4.1, to prove Theorem 3.2 it suffices to establish that the two one–forms ω and Ω on $\mathcal{M}_H^s$ coincide.

Consider the Hitchin map ψ defined in (2.7). We want to show that there is a holomorphic one–form Ω_H on the Hitchin space $\mathcal{H}$ such that

$$(4.1) \qquad\qquad\qquad \Omega - \omega = \psi^*\Omega_H .$$

To prove the existence of such a form Ω_H, take a point $h \in \mathcal{H}$ such that the corresponding spectral curve Y_h is smooth. We recall that there is a nonempty Zariski open subset U of $\mathcal{H}$ such that for any point $h' \in U$ the corresponding spectral curve $Y_{h'}$ is smooth.

The fiber $\psi^{-1}(h)$ is identified with the Picard variety $J_h := \mathrm{Pic}^{d+2(1-g)}(Y_h)$ of degree $d + 2(1 - g)$ line bundles on Y_h. (The degree $d + 2(1 - g)$ was computed in the proof of Lemma 3.1.) Let $j_h : J_h \longrightarrow \mathcal{M}_H^s$ be the inclusion map of the fiber of ψ. From the constructions of Ω and ω it follows that that $j_h^*\Omega = j_h^*\omega$.

Therefore, exactly as in the proof of Lemma 4.1, we conclude that for any point $z \in \psi^{-1}(h)$, the homomorphism

$$(\Omega - \omega)(z) : T_z\mathcal{M}_H^s \longrightarrow \mathbb{C}$$

factors through the projection $\mathrm{d}\psi(z) : T_z\mathcal{M}_H^s \longrightarrow T_{\psi(z)}\mathcal{H}$ defined the differential of ψ at the point h. Consequently, there is a holomorphic one–form Ω_H on U such that $\Omega - \omega = \psi^*\Omega_H$ on $\psi^{-1}(U)$, where U, as before, is the open subset of $\mathcal{H}$ defined by the points corresponding to smooth spectral curves. Since $\Omega - \omega$ on $\psi^{-1}(U)$ extends to $\mathcal{M}_H^s$,

U is a Zariski open dense subset of $\mathcal{H}$ and the map ψ is a submersion everywhere, it follows immediately that Ω_H extends to $\mathcal{H}$ and the equality in (4.1) is valid on $\mathcal{M}_H^s$.

Lemma 4.2. *The one–form Ω_H on $\mathcal{H}$ vanishes identically.*

Proof. Recall that $\mathcal{H} = H^0(X, K_X) \oplus H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S))$. Set $\mathcal{H}' := \mathcal{H} \setminus \{0\}$, the nonzero vectors. For any nonzero complex number c, consider the automorphism of $\mathcal{H}'$ that sends any point (α, β) to $(c\alpha, c^2\beta)$, where $\alpha \in H^0(X, K_X)$ and

$$\beta \in H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).$$

So we have a free action of $\mathbb{C}^*$ on $\mathcal{H}'$ defined this way. The quotient space

$$Q := \mathcal{H}'/\mathbb{C}^*$$

is a weighted projective space. Let

(4.2) $\rho : \mathcal{H}' \longrightarrow Q$

be the quotient map.

In Section 2.4, given a point of $\mathcal{H}$ we constructed a spectral curve, which is a divisor on Z, the total space of $K_X \otimes \mathcal{O}_X(S)$. We want to describe the above action of $\mathbb{C}^*$ on $\mathcal{H}'$ in terms of spectral curves. On Z there is an action of $\mathbb{C}^*$ defined by the condition that the action of any $c \in \mathbb{C}^*$ sends a point z to cz, where the scalar multiplication is defined by the vector space structure of the fibers of the line bundle $K_X \otimes \mathcal{O}_X(S)$. It is easy to see that for any $h \in \mathcal{H}'$ and $c \in \mathbb{C}^*$, the spectral curve corresponding to the point ch coincides with the image of the spectral curve corresponding to the point h by the automorphism of Z defined action of c.

On the other hand, the meromorphic one–form Ω_Z on Z (constructed in Section 3.2) evidently has the property that it vanishes along the orbits of $\mathbb{C}^*$ on Z. In other words, for the projection γ of Z to X, the pullback of Ω_Z by the inclusion map of a fiber of γ vanishes identically. Furthermore, for any $c \in \mathbb{C}^*$, if T_c denotes the automorphism of Z defined by the multiplication by c, then $T_c^*\Omega_Z = c\Omega_Z$. From these it follows immediately that there is a one–form Ω_Q on the weighted projective space Q such that $\rho^*\Omega_Q = \Omega_H$ on $\mathcal{H}'$, where ρ is the projection in (4.2).

A weighted projective space does not admit any nonzero holomorphic one–form. Hence we have $\Omega_Q = 0$. Since $\rho^*\Omega_Q = \Omega_H$, it follows immediately that $\Omega_H = 0$, and the proof of the lemma is complete. $\square$

The above results clearly combine together in imply Theorem 3.2.

Proof of Theorem 3.2. Lemma 4.2 (4.1) together imply that $\Omega = \omega$ on $\mathcal{M}_H^s$. So,

$$\Omega' := \phi^*\Omega = \phi^*\omega.$$

Now Lemma 4.1, which says that $\phi^*\omega = f^*\Omega_\delta$, completes the proof of the theorem.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`

MAX-PLANCK-INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, D-04103 LEIPZIG, GERMANY

E-mail address: `avijit@mis.mpg.de`