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Covariant theory of asymptotic symmetries, conservation laws and central charges

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Abstract

Under suitable assumptions on the boundary conditions, it is shown that there is a bijective correspondence between equivalence classes of asymptotic reducibility parameters and asymptotically conserved $n-2$ forms in the context of Lagrangian gauge theories. The asymptotic reducibility parameters can be interpreted as asymptotic Killing vector fields of the background, with asymptotic behaviour determined by a new dynamical condition. A universal formula for asymptotically conserved $n-2$ forms in terms of the reducibility parameters is derived. Sufficient conditions for finiteness of the charges built out of the asymptotically conserved $n-2$ forms and for the existence of a Lie algebra $\mathfrak{g}$ among equivalence classes of asymptotic reducibility parameters are given. The representation of $\mathfrak{g}$ in terms of the charges may be centrally extended. An explicit and covariant formula for the central charges is constructed. They are shown to be 2-cocycles on the Lie algebra $\mathfrak{g}$. The general considerations and formulas are applied to electrodynamics, Yang-Mills theory and Einstein gravity.

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Contents

1 Introduction and summary

1.1 The Noether-charge puzzle for gauge symmetries

A physical motivation for studying asymptotic symmetries and conservation laws is the problem of a systematic construction of meaningful charges related to gauge symmetries, such as the electric charge in electrodynamics or energy and momentum in general relativity. This problem is closely related to the famous Noether-charge puzzle for gauge symmetries. It is encountered when one tries to define the charge related to a gauge symmetry "in the usual manner", by applying Noether's first theorem [1] on the relation of symmetries and conserved currents. The problem of such an approach is that a Noether current associated to a gauge symmetry necessarily vanishes on-shell (i.e., for *every* solution of the Euler-Lagrange equations of motion), up to the divergence of an arbitrary superpotential. This is a direct consequence of Noether's second theorem [1] and was already pointed out by Noether herself. Let us first review this problem and then indicate how it can be resolved through asymptotic symmetries and conservation laws.

We denote the fields of the theory by ϕ^i, the Lagrangian by $L[\phi]$ and a generating set of non trivial gauge symmetries by $\delta_f\phi^i = R^i_\alpha(f^\alpha) = R^i_f$ for some operators $R^i_\alpha =$

$\sum_{k=0} R^{i(\mu_1 \dots \mu_k)}_\alpha \partial_{\mu_1} \dots \partial_{\mu_k}$ and arbitrary local functions f^α (which should be understood as possibly field dependent gauge parameters). By definition of a gauge symmetry, the $R^i_\alpha(f^\alpha)$ satisfy

$$R^i_\alpha(f^\alpha)\frac{\delta L}{\delta \phi^i} = \partial_\mu j^\mu_f, \tag{1.1}$$

for some set of local functions j^μ_f, where $\delta L/\delta\phi^i$ is the Euler-Lagrange derivative of $L[\phi]$ with respect to ϕ^i. The sets of local functions j^μ_f are the Noether currents associated to the gauge symmetry δ_f. Noether's second theorem states that there exist associated identities among the Euler-Lagrange equations of motions,[1]

$$R^{+i}_\alpha \frac{\delta L}{\delta \phi^i} = 0, \tag{1.2}$$

where the operators R^{+i}_α are obtained from the operators R^i_α defining the gauge transformations through "integrations by parts" and "forgetting about the boundary term". Explicitly, they are defined for all collections of local functions Q_i by $R^{+i}_\alpha(Q_i) = \sum_{k=0}(-)^k \partial_{\mu_1} \dots \partial_{\mu_k}[R^{i(\mu_1 \dots \mu_k)}_\alpha Q_i]$.

Doing these integrations by parts without forgetting any boundary terms, one obtains

$$\forall Q_i, f^\alpha : \quad Q_i R^i_\alpha(f^\alpha) = f^\alpha R^{+i}_\alpha(Q_i) + \partial_\mu S^{\mu i}_\alpha(Q_i, f^\alpha), \tag{1.3}$$

where the $S^{\mu i}_\alpha(\cdot,\cdot)$ are some bidifferential operators. Choosing now $Q_i = \delta L/\delta\phi^i$ and using (1.2), one obtains

$$\frac{\delta L}{\delta \phi^i} R^i_\alpha(f^\alpha) = \partial_\mu S^{\mu i}_\alpha(\frac{\delta L}{\delta \phi^i}, f^\alpha). \tag{1.4}$$

Hence, $S^{\mu i}_\alpha(\delta L/\delta\phi^i, f^\alpha)$ is a particular Noether current satisfying (1.1). This current vanishes on-shell because it is a linear combination (with field dependent coefficients) of the Euler-Lagrange derivatives $\delta L/\delta\phi^i$ and their derivatives. For any other current j^μ_f satisfying (1.1) one obtains from (1.1) and (1.4):

$$\partial_\mu \Big(j^\mu_f - S^{\mu i}_\alpha(\frac{\delta L}{\delta \phi^i}, f^\alpha) \Big) = 0. \tag{1.5}$$

Using the algebraic Poincaré lemma (see e.g. [6, 7, 8, 9, 10, 11, 2, 12, 13, 14, 15]), one concludes from (1.5) that j^μ_f is given on-shell by the divergence of a superpotential $k^{[\nu\mu]}_f$,

$$n > 1 : \quad j^\mu_f = S^{\mu i}_\alpha(\frac{\delta L}{\delta \phi^i}, f^\alpha) - \partial_\nu k^{[\nu\mu]}_f, \tag{1.6}$$

[1] A particularly simple derivation of (1.2) is to take the Euler-Lagrange derivative of (1.1) with respect to f^α which is possible because (1.1) holds for all functions f^α; the result is (1.2). Other derivations can be found for instance in [1, 2, 3, 4, 5].

where n is the spacetime dimension[2] (in one dimension one has instead $j_f = S^i_\alpha(\delta L/\delta\phi^i, f^\alpha) + C$ where C is an arbitrary constant). (1.6) is the general solution of (1.1) for given L and δ_f. Note that the superpotential is completely *arbitrary* because it drops out of (1.1) owing to $\partial_\mu\partial_\nu k_f^{[\nu\mu]} = 0$. This implies in particular that the Noether charge corresponding to δ_f is undefined because it is given by the surface integral of an arbitrary $n-2$ form,

$$n > 1: \quad Q[\phi(x)] = \int_\Sigma j_f|_{\phi(x)} = \int_{\partial\Sigma} k_f|_{\phi(x)}, \tag{1.7}$$

where $\phi(x)$ is a solution of the Euler-Lagrange equations of motion, Σ is an $n-1$ dimensional spacelike surface of spacetime with boundary $\partial\Sigma$, j_f is the current $n-1$ form and k_f is the $n-2$ form associated to the superpotential defined according to

$$j_f = j^\mu_f (d^{n-1}x)_\mu\,, \quad k_f = k_f^{[\mu\nu]}(d^{n-2}x)_{\mu\nu}\,,$$
$$(d^{n-p}x)_{\mu_1\dots\mu_p} := \tfrac{1}{p!(n-p)!}\,\epsilon_{\mu_1\dots\mu_n} dx^{\mu_{p+1}}\dots dx^{\mu_n}, \quad \epsilon_{0\dots(n-1)} = 1.$$

Equation (1.7) expresses the problem described in the beginning, but at the same time it hints at a resolution of this problem. Since (1.7) is the flux of the superpotential through the boundary $\partial\Sigma$, it depends solely on the properties of the superpotential near the boundary. The situation is familiar from electrodynamics where the electric charge reduces to the flux of the electric field and the superpotential is the field strength $F^{\mu\nu}$ itself. This suggests to define charges of gauge symmetries through corresponding superpotentials rather than through currents, and calls for an appropriate criterion which allows one to single out these superpotentials, such as $F^{\mu\nu}$ in electrodynamics. Such a criterion is the requirement that the superpotentials be asymptotically conserved $n-2$ forms, for specific boundary conditions imposed on the fields.

This indicates that there may exist a general relation between asymptotically conserved $n-2$ forms and gauge symmetries in Lagrangian gauge theories. The formulation of such a relation is one of our central results. It states that, under quite generic conditions, every nontrivial asymptotically conserved $n-2$ form is related to nontrivial asymptotic reducibility parameters, and vice versa. Asymptotic reducibility parameters are the parameters f^α of gauge transformations that vanish sufficiently fast when evaluated at a background field configuration characterizing (partly) the boundary conditions for the fields. Therefore asymptotic reducibility parameters may be interpreted as asymptotic Killing vectors of the background. Asymptotic symmetries are generated by gauge transformations whose parameters are asymptotic reducibility parameters.

1.2 Various approaches

The link between asymptotically conserved charges and asymptotic reducibility parameters had been known for a long time in general relativity (see e.g. [16, 17, 18] and

[2]For simplicity and clarity we asssume throughout the paper that the topologies of spacetime and of the field space are trivial, and that so are all bundles possibly associated with the fields. If one drops this assumption, (1.6) still holds locally, but globally its right hand side may contain in addition a topological conserved current, and other equations of the paper may get analogously modified.

references therein), where the gauge symmetry is diffeomorphism invariance and the asymptotic reducibility parameters are asymptotic Killing vectors of the background metric. Later it was realized that it also applies to other gauge theories, such as Yang-Mills theory [19].

From a general point of view, a criterion for the construction of asymptotic charges and their relation to asymptotic symmetries was given in [17] in the context of the Hamiltonian formalism. This criterion was subsequently used in [20, 21] to develop the canonical theory of the central charges that appear in the representation of the Lie algebra of asymptotic symmetries in terms of the Poisson brackets of the canonical generators. The problem of defining and constructing asymptotically conserved currents and charges and of establishing their correspondence with asymptotic symmetries in a manifestly covariant way has received of lot of attention for quite some time. Recent approaches are often referred to as the Lagrangian Noether method [3, 4, 22, 23, 24, 25, 26] or the covariant phase space approach [27, 28] (see also [29, 30]). Let us also mention quasi-local techniques [31] and conformal methods (see e.g. [32, 33] and references therein).

From a more technical point of view, the starting point for the present paper is the recent investigation by Anderson and Torre [34] (see also [35]) who have shown that lower degree asymptotic conservation laws should be understood as suitable asymptotic cohomology groups of the variational bicomplex pulled back to the surface defined by the equations of motion.

Our approach is guided by the results derived in [36] for exact global reducibility identities and conserved $n-2$ forms. For instance, in the case of pure Maxwell theory, the superpotential $F^{\mu\nu}$ is related in a precise way (through descent equations) to the global reducibility of the electromagnetic gauge symmetry which constrains the gauge parameter f through $\partial_\mu f = 0$, and such a relation between global reducibility identities and conserved $n-2$ form holds generally for gauge theories [36]. For interacting gauge theories, such as Yang-Mills theory with a semi-simple gauge group or general relativity, there are no non trivial reducibility identities and consequently no non trivial conserved $n-2$ forms. However, nontrivial *asymptotic* reducibility identities and *asymptotically* conserved $n-2$ forms may well exist in interacting gauge theories, if the theory near the boundary becomes asymptotically linear when expanded around a suitable background. This is the basic idea of [34, 35] that also underlies our paper. It allows us to apply and extend the methods used in [36] to asymptotic quantities.

1.3 Summary of results

1.3.1 Asymptotic reducibility parameters and asymptotic symmetries

Suppose that $\bar{\phi}(x)$ is a background solution of the Euler Lagrange equations describing the theory near the boundary. Let us decompose the fields according to $\phi = \bar{\phi}(x) + \varphi$ and suppose that there are some functions χ^i that specify the boundary conditions imposed on the field ϕ^i through $\varphi^i \longrightarrow O(\chi^i)$. Typically one has $\chi^i = 1/r^{m^i}$ for some number m^i that may depend on the specific field.

We denote by $L^{\rm free}$ the Lagrangian of the theory linearized around the background, and by χ_i the asymptotic behaviour of the corresponding field equations times the volume form, when evaluated for generic fields φ^i that satisfy the boundary conditions,

$$\forall \varphi^i(x) \longrightarrow O(\chi^i):\ \frac{\delta L^{\rm free}}{\delta \varphi^i}|_{\varphi(x)} d^n x \longrightarrow O(\chi_i). \tag{1.8}$$

Let ψ_i denote a field that behaves asymptotically as the linearized field equations times the volume form. Asymptotic reducibility parameters are functions $\tilde{f}^\alpha(x)$ which satisfy

$$\forall \psi_i \longrightarrow O(\chi_i):\ \psi_i R^i_\alpha|_{\bar\phi(x)}(\tilde{f}^\alpha) \longrightarrow 0. \tag{1.9}$$

Furthermore let χ_α denote the asymptotic behaviour of the Noether operators evaluated at the background when acting on a field ψ_i,

$$\forall \psi_i \longrightarrow O(\chi_i):\ R^{+i}_\alpha|_{\bar\phi(x)}(\psi_i) \longrightarrow O(\chi_\alpha). \tag{1.10}$$

Then functions $\tilde{f}^\alpha$ that fall-off as $\tilde{f}^\alpha \longrightarrow o(1/\chi_\alpha)$ are automatically asymptotic reducibility parameters if we assume that integrations by parts do not increase the asymptotic degree. Such asymptotic reducibility parameters are considered as trivial, and two sets of asymptotic reducibility parameters are called equivalent ($\sim$) if they differ by a trivial one.

By definition, asymptotic symmetries are gauge transformations $R^i_\alpha(\tilde{f}^\alpha)$ with asymptotic reducibility parameters.

Notice that (1.9) is *not* the condition that asymptotic reducibility parameters yield gauge transformations which preserve the boundary conditions satisfied by the fields. Indeed, such a condition, which one encounters in many articles on asymptotic symmetries and conservation laws, would read $R^i_\alpha|_{\bar\phi(x)}(\tilde{f}^\alpha) \longrightarrow O(\chi^i)$, whereas (1.9) requires $R^i_\alpha|_{\bar\phi(x)}(\tilde{f}^\alpha) \longrightarrow o(1/\chi_i)$ which involves the boundary conditions of the asymptotic equations of motion rather than those of the fields. However, notice also that the two conditions have important solutions in common, namely those which satisfy $R^i_\alpha|_{\bar\phi(x)}(\tilde{f}^\alpha) = 0$ (with exact rather than asymptotic equality). Adopting the terminology of general relativity, these solutions may be called "Killing vectors of the background". Asymptotic reducibility parameters may thus be interpreted as "asymptotic Killing vectors of the background" but one should bear in mind that their asymptotic behaviour is determined by the asymptotic behaviour of the linearized equations of motion and not directly by the boundary conditions for the fields themselves.

1.3.2 Asymptotically conserved n–2 forms

An $n-2$ form constructed of the fields and their derivatives is called asymptotically conserved if its exterior derivative vanishes asymptotically for the solutions of the field equations satisfying the boundary conditions. When the theory is asymptotically linear, asymptotically conserved $n-2$ forms can be constructed as follows. Let $\delta L^{\rm free}/\delta\varphi^i$ denote the "left hand sides" of the field equations linearized around the background and

$$s^\mu_{\tilde{f}}[\varphi; \bar\phi(x)] = S^{\mu i}_\alpha|_{\bar\phi(x)}(\frac{\delta L^{\rm free}}{\delta \varphi^i}, \tilde{f}^\alpha) \tag{1.11}$$

the current defined by equation (1.3), evaluated at the background solution, for the linearized field equations and for the asymptotic reducibility parameters. The divergence of this current (multiplied by the volume form) vanishes asymptotically, $\partial_\mu s^\mu_{\tilde f} d^n x \longrightarrow 0$, when the $\tilde f^\alpha$ are asymptotic reducibility parameters as one deduces from (1.4). This implies, under suitable assumptions on the boundary conditions, that the $n-1$ form $s^\mu_{\tilde f}(d^{n-1}x)_\mu$ is asymptotically the exterior derivative of an $n-2$ form that is a local function in the fields φ^i,

$$s^\mu_{\tilde f}[\varphi;\bar\phi(x)](d^{n-1}x)_\mu \longrightarrow -d_H \tilde k^{[\nu\mu]}_{\tilde f}[\varphi;\bar\phi(x)](d^{n-2}x)_{\nu\mu} \tag{1.12}$$

with $d_H = dx^\mu \partial_\mu$ and ∂_μ the total derivative, so that $d_H \tilde k_{\tilde f} = \partial_\nu \tilde k^{[\mu\nu]}_{\tilde f}(d^{n-1}x)_\mu$. The explicit expression of $\tilde k^{[\nu\mu]}_{\tilde f}$ is obtained by applying the contracting homotopy of the algebraic Poincaré lemma to $s^\mu_{\tilde f}$. In the case of equations of motion that are at most of second order in derivatives, this expression reduces to

$$\tilde k^{[\mu\nu]}_{\tilde f} = \frac{1}{2}\varphi^i \frac{\partial^S s^\nu_{\tilde f}}{\partial \varphi^i_\mu} + \Big(\frac{2}{3}\varphi^i_\lambda - \frac{1}{3}\varphi^i \partial_\lambda\Big)\frac{\partial^S s^\nu_{\tilde f}}{\partial \varphi^i_{\lambda\mu}} - (\mu \leftrightarrow \nu) \tag{1.13}$$

where lower indices of φ^i represent derivatives, i.e., φ^i_μ and $\varphi^i_{\mu\nu}$ represent the first and second order derivatives of φ^i, respectively, and the operation $\partial^S/\partial\varphi^i_{\mu_1\dots\mu_k}$ is defined according to $\partial^S \varphi^i_{\mu_1\dots\mu_k}/\partial\varphi^j_{\nu_1\dots\nu_k} = \delta^i_j \delta^{\nu_1}_{(\mu_1}\dots\delta^{\nu_k}_{\mu_k)}$. Here, the round parentheses denote symmetrization with weight one. For example, $\partial^S\varphi^i_{\mu\nu}/\partial\varphi^j_{\rho\lambda} = 1/2\ \delta^i_j(\delta^\rho_\mu\delta^\lambda_\nu + \delta^\rho_\nu\delta^\lambda_\mu)$.

For the remainder of the introduction, we restrict ourselves to the case where the asymptotic behaviour of the asymptotic reducibility parameters is given by

$$\tilde f^\alpha \longrightarrow O(1/\chi_\alpha). \tag{1.14}$$

An asymptotic solution $\varphi_s(x)$ of the linearized equations of motion near the boundary is defined by

$$\frac{\delta L^{\text{free}}}{\delta\varphi^i}\Big|_{\varphi_s(x)} d^n x \longrightarrow o(\chi_i). \tag{1.15}$$

In this case, the associated $n-2$ forms $\tilde k_{\tilde f}$ satisfy the asymptotic conservation law

$$d_H \tilde k_{\tilde f}|_{\varphi_s(x)} \longrightarrow 0. \tag{1.16}$$

Defining a corresponding $n-2$ form of the full theory by

$$k_{\tilde f}[\phi;\bar\phi(x)] = \tilde k_{\tilde f}[\phi - \bar\phi(x);\bar\phi(x)] \tag{1.17}$$

and considering an infinitesimal field variation,

$$d_V = \sum_{k=0} d_V \phi^i_{\mu_1\dots\mu_k} \frac{\partial^S}{\partial\phi^i_{\mu_1\dots\mu_k}}\ , \tag{1.18}$$

one has by construction $(d_V k_{\tilde f})|_{\bar\phi(x),\varphi} = \tilde k_{\tilde f}[\varphi;\bar\phi(x)]$, and the asymptotic conservation law can be written from the point of view of the full theory as

$$d_H(d_V k_{\tilde f})|_{\bar\phi(x),\varphi_s(x)} \longrightarrow 0. \tag{1.19}$$

When the theory is asymptotically linear, we *define* an asymptotically conserved $n-2$ form k to be an $n-2$ form for which this equation holds. Two asymptotically conserved $n-2$ forms should be considered equivalent if their linearization around the given background agree asymptotically up to the horizontal differential of an $n-3$ form when evaluated for asymptotic solutions,

$$k_1 \sim k_2 \iff \big(d_V(k_1-k_2)\big)|_{\bar\phi(x),\varphi_s(x)} \longrightarrow d_H \tilde l^{n-3} \tag{1.20}$$

where $\tilde l^{n-3}$ depends linearly on the fields φ and their derivatives.

1.3.3 Bijective correspondence

With these definitions and assumptions, we will show that every asymptotically conserved $n-2$ form is equivalent to an $n-2$ form $k_{\tilde f}$ obtained from (1.12) for some asymptotic reducibility parameters $\tilde f^\alpha$, and that there is a one-to-one correspondence between equivalence classes $[\tilde f^\alpha]$ of asymptotic reducibility parameters and equivalence classes $[k]$ of asymptotically conserved $n-2$ forms. This is the analog of (the complete version of) Noether's first theorem for the case of asymptotic symmetries. Furthermore, for irreducible gauge theories, there are no nontrivial asymptotically conserved forms in degrees strictly smaller than $n-2$.

1.3.4 Charges

Consider an $n-2$ dimensional compact manifold $\mathcal{C}^{n-2}$ without boundary, $\partial\mathcal{C}^{n-2}=\emptyset$, that lies in the asymptotic region and an asymptotically conserved $n-2$ form $\tilde k_{\tilde f}$. The charge in the full theory is defined by

$$Q_{\tilde f}[\phi;\bar\phi(x)] = \int_{\mathcal{C}^{n-2}} \tilde k_{\tilde f}[\phi-\bar\phi(x);\bar\phi(x)] + N_{\tilde f}, \tag{1.21}$$

where the normalization $N_{\tilde f}$ is the charge of the background $\bar\phi(x)$. If we evaluate this integral for an asymptotic solution $\phi(x)=\bar\phi(x)+\varphi_s(x)$, the charges are finite when (1.14) holds, and we can then apply Stokes theorem because of the conservation law (1.16) to prove asymptotic independence on the choice of representatives for the homology class $[\mathcal{C}^{n-2}]$ and for the equivalence class $[\tilde k_{\tilde f}]$.

1.3.5 Algebra of asymptotic reducibility parameters

Because we have assumed that $\delta_f\phi^i = R^i_\alpha(f^\alpha)$ provide a generating set of non trivial gauge symmetries, the commutator algebra of the non trivial gauge symmetries closes on-shell in the sense

$$\delta_{f_1}R^i_\alpha(f^\alpha_2) - (1\longleftrightarrow 2) \approx R^i_\gamma(C^\gamma_{\alpha\beta}(f^\alpha_1,f^\beta_2)+\delta_{f_1}f^\gamma_2-\delta_{f_2}f^\gamma_1), \tag{1.22}$$

for some bidifferential operators $C^{\gamma}_{\alpha\beta}$ ($\approx$ denotes equality on-shell). Additional (sufficient) contraints on the asymptotic reducibility parameters, on the cubic vertex of the theory and on the gauge symmetries of the linearized theory will be given that guarantee that asymptotic reducibility parameters $\tilde{f}^\alpha$ form a Lie algebra, with bracket determined by the structure operators $C^{\gamma}_{\alpha\beta}$ evaluated at the background,

$$[\tilde{f}_1, \tilde{f}_2]^{\gamma}_M = C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha_1, \tilde{f}^\beta_2). \tag{1.23}$$

with $C^{\gamma 0}_{\alpha\beta} \equiv C^{\gamma}_{\alpha\beta}|_{\bar{\phi}(x)}$. This Lie algebra induces a well defined Lie algebra $\mathfrak{g}$ for the equivalence classes with bracket denoted by $[\ ,\ ]_G$:

$$[[\tilde{f}_1], [\tilde{f}_2]]_G = [[\tilde{f}_1, \tilde{f}_2]_M]. \tag{1.24}$$

1.3.6 Induced global symmetries

Under approriate assumptions, the asymptotic reducibility parameters can be shown to determine asymptotic linear global symmetries

$$\delta^g_{\tilde{f}} \varphi^i = (d_V R^i_\alpha)|_{\bar{\phi}(x),\varphi}(\tilde{f}^\alpha) \equiv R^{i1}_\alpha(\tilde{f}^\alpha), \tag{1.25}$$

for the linearized theory, in the sense that $\delta^g_{\tilde{f}} L^{\text{free}} \longrightarrow d_H(\ )$.

1.3.7 Algebra of asymptotically conserved n–2 forms

On the level of the equivalence classes of asymptotically conserved $(n-2)$-forms of the linearized theory near the boundary, the Lie algebra $\mathfrak{g}$ of the equivalence classes of asymptotic reducibility parameters can be represented asymptotically by a covariant Poisson bracket, which is defined through the action of the associated global symmetry,

$$\{[\tilde{k}_{\tilde{f}_1}], [\tilde{k}_{\tilde{f}_2}]\}_F := [\delta^g_{\tilde{f}_1} \tilde{k}_{\tilde{f}_2}] = [\tilde{k}_{[\tilde{f}_1, \tilde{f}_2]_M}]. \tag{1.26}$$

The property $-[\delta^g_{\tilde{f}_2} \tilde{k}_{\tilde{f}_1}] = [\tilde{k}_{[\tilde{f}_1, \tilde{f}_2]_M}]$ implies that alternative equivalent expressions for the covariant Poisson bracket are $-[\delta^g_{\tilde{f}_2} \tilde{k}_{\tilde{f}_1}]$ or $\frac{1}{2}([\delta^g_{\tilde{f}_1} \tilde{k}_{\tilde{f}_2}] - [\delta^g_{\tilde{f}_2} \tilde{k}_{\tilde{f}_1}])$.

1.3.8 Algebra of charges and central extensions

On the level of the charges of the full theory, the Lie algebra $\mathfrak{g}$ can be represented by a covariant Poisson bracket that is defined by applying an asymptotic symmetry $\delta_{\tilde{f}} \phi^i = R^i_\alpha(\tilde{f}^\alpha) = R^i_{\tilde{f}}$. This representation may contain non trivial central extensions. Explicitly,

$$\{Q_{\tilde{f}_1}, Q_{\tilde{f}_2}\}_{CF} := \delta_{\tilde{f}_1} Q_{\tilde{f}_2} \sim Q_{[\tilde{f}_1, \tilde{f}_2]_M} - N_{[\tilde{f}_1, \tilde{f}_2]_M} + K_{\tilde{f}_1, \tilde{f}_2}, \tag{1.27}$$

where $\sim$ is asymptotic equality when the charges are evaluated for asymptotic solutions, the N's are normalization constants, and the $K_{\tilde{f}_1, \tilde{f}_2}$ are central charges given by

$$K_{\tilde{f}_1, \tilde{f}_2} = Q_{\tilde{f}_2}[R_{\tilde{f}_1}|_{\bar{\phi}(x)}; \bar{\phi}(x)] = -Q_{\tilde{f}_1}[R_{\tilde{f}_2}|_{\bar{\phi}(x)}; \bar{\phi}(x)]. \tag{1.28}$$

These K's are Chevalley-Eilenberg 2-cocycles on the Lie algebra $\mathfrak{g}$.

1.4 Organization of the paper

Our analysis of asymptotic symmetries and conservation laws is guided by methods and results known in the context of exact symmetries and conservation laws. Therefore we first summarize in sections 2 through 4 facts about exact symmetries and conservation laws. The results on asymptotic symmetries and conservation laws are collected in section 5 and illustrated in section 6 for the most prominent gauge theories. The details of the analysis are presented in section 7 and in the appendix. Section 8 describes briefly the relation to other approaches to asymptotic symmetries and conservation laws.

1.4.1 Section 2

We review, besides well known facts on global symmetries and conserved currents, the results of [36, 5] on the bijective correspondence between suitably defined equivalence classes of global symmetries and conserved currents in the context of gauge theories, without using the cohomological tools related to the BRST formalism.

1.4.2 Section 3

The bijective correspondence between equivalence classes of reducibility parameters of gauge symmetries and conserved $n-2$ forms [36, 5] is reviewed, independently of BRST cohomological arguments. Universal formulas for the conserved $n-2$ forms associated to reducibility parameters are given and the definition and properties of corresponding charges are recalled. It is shown that there is a well defined Lie action of equivalence classes of global symmetries on equivalence classes of reducibility parameters and thus also on equivalence classes of conserved $n-2$ forms.

1.4.3 Section 4

It is shown how the expansion of an interacting gauge theory around a solution allows one to associate global symmetries to the (field independent) reducibility parameters of the linearized theory. The Lie action of these symmetries is then used to define a Lie algebra of equivalence classes of field independent reducibility parameters.

1.4.4 Section 5

The section begins with a general discussion of the boundary conditions followed by a discussion of asymptotic reducibility parameters and asymptotically conserved $n-2$ forms and their algebra from the point of view of the linearized theory near the boundary. The assumptions that allow one to use and reexpress these results from the point of view of the bulk theory are discussed next. Finally, some remarks on the associated boundary theory are given.

1.4.5 Section 6

It is shown how the familiar expression for the electric charge in electrodynamics arises from an asymptotically conserved $n-2$ form related to an exact Killing vector of the background. Non abelian Yang-Mills theories are discussed next and it is shown that and how our results reproduce those of [19]. Then we discuss in more detail Einstein gravity in spacetime dimensions larger than 2, with or without cosmological constant. We derive a general expression for the gravitational asymptotically conserved $n-2$ forms which reproduces, in the particular case that the reducibility parameters are exact Killing vectors of the background, the expressions given in [18] and [34]. We also derive an explicit general expression for the potential gravitational central charges which, to our knowledge, is completely new, and illustrate the covariant theory of central charges in the case of 3-dimensional asymptotically anti-de Sitter gravity, where the results of [20] obtained in the canonical framework are recovered.

1.4.6 Section 7

BRST cohomological methods are used to reformulate, prove and partly generalize the statements of the previous sections, with subsection $7.x$ corresponding to section x for $x = 2, 3, 4, 5$, while in subsection 7.1 we first recall the basic ingredients of the BRST approach (antifields, ghost fields, Batalin-Vilkovisky master equation, antibracket, BRST differential). The cohomological formulation of Noether's first theorem and of the relation between reducibility parameters and conserved $n-2$ forms via descent equations [36, 5] is briefly reviewed. The induced global symmetries, and the Lie algebras discussed in the previous sections are derived from the antibracket map. In order to discuss asymptotic symmetries and conservation laws, we define linear and exact linear characteristc cohomology. When evaluated at a background this latter cohomology is shown to provide the right framework to discuss asymptotic reducibility parameters, asymptotically conserved $n-2$ forms and to prove the bijective correspondence between the appropriate equivalence classes.

1.4.7 Section 8

The covariant theory of asymptotic symmetries and conservation laws derived in the previous sections is related to the original Regge-Teitelboim canonical approach [17]. The comparison with the covariantized Regge-Teitelboim formalism [22] recently proposed in the context of the Lagrangian Noether method is direct. The main formulas that allow one to connect our results to the covariant phase space method are given, and finally we briefly compare the assumptions, techniques and results of our investigation to those of the original cohomological analysis in the context of the variational bicomplex [34].

1.4.8 Appendix

In the appendix we first collect conventions and notation, especially those concerning multiindices, then we give compact expressions for higher order Lie-Euler operators and for the contracting homotopy of the horizontal complex, and finally the proof of a central theorem of the paper.

2 Global symmetries and conserved currents

2.1 Definitions

Global symmetries are evolutionary vector fields with characteristic X^i such that their prolongation leaves the Lagrangian L invariant up to a total divergence,

$$\delta_X L = \partial_\mu k^\mu. \tag{2.1}$$

Conserved currents j^μ are currents whose divergence vanishes when the Euler-Lagrange equations of motion hold,

$$\partial_\mu j^\mu = Y^{i(\nu)}\partial_{(\nu)}\frac{\delta L}{\delta\phi^i}. \tag{2.2}$$

Note that in this context, the Lagrangian only serves to define the dynamics through its Euler-Lagrange derivatives and is defined up to a total divergence because

$$\frac{\delta f}{\delta\phi^i} = 0 \Longleftrightarrow f = \partial_\mu m^\mu, \tag{2.3}$$

for some local functions m^μ. This ambiguity does not affect the definition of the global symmetries because $[\delta_X, \partial_\mu] = 0$. Note also that the spacetime points x^μ are not transformed so that $\delta_X\phi^i$ corresponds to the "variations of the fields at the same point".

2.2 From symmetries to conserved currents

The current $V_i^\mu(Q^i, f)$ is defined through the equation

$$\delta_Q f = Q^i\frac{\delta f}{\delta\phi^i} + \partial_\mu V_i^\mu(Q^i, f), \tag{2.4}$$

for all Q^i. It then follows from (2.1) that $j^\mu = k^\mu - V_i^\mu(X^i, L)$ is a conserved current because it satisfies

$$\partial_\mu j^\mu = X^i\frac{\delta L}{\delta\phi^i}. \tag{2.5}$$

2.3 From conserved currents to symmetries

Using on the right hand side of equation (2.2) repeatedly Leibniz' rule under the form $f\partial_\mu g = \partial_\mu(fg) - (\partial_\mu f)g$, and bringing the terms $\partial_\mu(fg)$ to the left hand side, one obtains

$$\partial_\mu j'^\mu = (-\partial)_{(\nu)} Y^{i(\nu)} \frac{\delta L}{\delta \phi^i} \; . \tag{2.6}$$

Using the same notation as in (2.4), one has

$$(-\partial)_{(\nu)} Y^{i(\nu)} \frac{\delta L}{\delta \phi^i} = \delta_X L - \partial_\mu V_i^\mu(X^i, L),$$

with

$$X^i = (-\partial)_{(\nu)} Y^{i(\nu)}.$$

Combining these equations, one obtains that equation (2.2) implies (2.1), with $X^i = (-\partial)_{(\nu)} Y^{i(\nu)}$.

2.4 Bijective correspondence between equivalence classes

In order to understand a complete version of Noether's first theorem on the correspondence between global symmetries and conserved currents in the context of gauge theories, it is crucial to define what are trivial symmetries and currents and to consider equivalence classes of symmetries respectively currents up to trivial ones.

Indeed, both correspondences between symmetries and currents are not uniquely defined because of the existence of identically conserved currents $\partial_\mu j^\mu = 0$, and the existence of Noether identities, $N^{i(\mu)} \partial_{(\mu)} \delta L / \delta \phi^i = 0$. For identically conserved currents, the algebraic Poincaré lemma mentioned already in the introduction guarantees that, at least locally,

$$\partial_\mu j^\mu = 0 \iff j^\mu = \partial_\nu k^{[\nu\mu]} + \delta_1^n C, \tag{2.7}$$

for some local functions $k^{[\nu\mu]}$ and some constant $C \in \mathbb{R}$. This is equivalent to the statement that the cohomology of d_H vanishes in form degree $n-1$ (locally) except for the constants in spacetime dimension $n = 1$.

The requirement that the operators R^i_α be a generating set of gauge symmetries means that every gauge symmetry, i.e., every symmetry of the Lagrangian whose characteristic G^i depends linearly and homogeneously on an arbitrary local function f and its derivatives, $G^i(f) = G^{i(\mu)} \partial_{(\mu)} f$ with $\delta_{G(f)} L = \partial_\mu k^\mu(f)$, can be written as

$$G^i(f) = R^i_\alpha(Z^\alpha(f)) + M^{+ji}(\frac{\delta L}{\delta \phi^j}, f), \tag{2.8}$$

for some operators $Z^\alpha = Z^{\alpha(\mu)} \partial_{(\mu)}$. Here and in the following, $M^{ji}(Q_j, g)$ and $M^{+ji}(Q_j, g)$ (with two arguments) are regarded as differential operators acting on their second argument (g), and $M^{+ji}(Q_j)$ (with only one argument) as local functions equal to $M^{+ji}(Q_j,1)$,

$$\begin{aligned} M^{ji}(Q_j,\ \cdot\ ) &= \partial_{(\nu)}Q_j\, M^{[j(\nu)i(\mu)]}\,\partial_{(\mu)}\,\cdot\ , \\ M^{+ji}(Q_j,\ \cdot\ ) &= (-\partial)_{(\mu)}(\ \cdot\ M^{[j(\nu)i(\mu)]}\,\partial_{(\nu)}Q_j), \\ M^{+ji}(Q_j) &= (-\partial)_{(\mu)}(M^{[j(\nu)i(\mu)]}\,\partial_{(\nu)}Q_j), \end{aligned} \tag{2.9}$$

where

$$M^{[j(\nu)i(\mu)]} = -M^{[i(\mu)j(\nu)]}. \tag{2.10}$$

Equivalently, every operator $N^{i(\mu)}\partial_{(\mu)}$ defining a Noether identity can be written as

$$N^{i(\mu)}\partial_{(\mu)}\delta L/\delta\phi^i = 0 \iff N^{i(\mu)}\partial_{(\mu)} = Z^{+\alpha}\circ R^{+i}_{\alpha} + M^{[j(\nu)i(\mu)]}\partial_{(\nu)}\frac{\delta L}{\delta\phi^j}\partial_{(\mu)}, \tag{2.11}$$

for some operators $Z^{+\alpha}$ and some $M^{[j(\nu)i(\mu)]}$.

Armed with these definitions, one can prove that there is a bijective correspondence $[X^i] \longleftrightarrow [j^\mu]$ between equivalence classes $[X^i]$ of global symmetries X^i satisfying (2.1), with equivalence ($\sim$) defined by

$$X^i \sim X^i + R^i_\alpha(f^\alpha) + M^{+ji}(\frac{\delta L}{\delta\phi^j}), \tag{2.12}$$

and equivalence classes $[j^\mu]$ of conserved currents j^μ satisfying (2.2), with equivalence defined by

$$j^\mu \sim j^\mu + \partial_\nu k^{[\nu\mu]} + l^{\mu i(\nu)}\partial_{(\nu)}\frac{\delta L}{\delta\phi^i} + \delta^n_1 C, \quad C\in\mathbb{R}. \tag{2.13}$$

The proof using the Koszul-Tate resolution of the stationary surface and descent equations techniques, originally given in [36], is briefly reviewed in section 7.

Explicity, the correspondence is given by

$$\begin{aligned} [X^i] &\longrightarrow [k^\mu - V^\mu_i(X^i,L)], \\ [j^\mu] &\longrightarrow [(-\partial)_{(\nu)}Y^{i(\nu)}]. \end{aligned} \tag{2.14}$$

From the point of view of equivalence classes of global symmetries, symmetries of the form $\delta_f\phi^i = R^i_\alpha(f^\alpha)$ (non trivial gauge symmetries) and of the form $\delta_M\phi^i = M^{+ji}(\frac{\delta L}{\delta\phi^j})$ (trivial gauge symmetries) should thus be considered as trivial, while from the point of view of equivalence classes of conserved currents, trivial currents are on-shell equal to the divergence of an arbitrary superpotential.

One can furthermore show under fairly general assumptions (linearizable, normal theories; see theorems 6.5 and 6.6 in [5], and especially the remark after theorem 6.6 there) that global symmetries whose characteristic X^i vanishes when the Euler-Lagrange equations of motion hold, can be assumed to be trivial

$$\delta_X L = \partial_\mu k^\mu,\ X^i \approx 0 \implies X^i = R^i_\alpha(f^\alpha) + M^{+ji}(\frac{\delta L}{\delta\phi^j}), \tag{2.15}$$

for some f^α and some $M^{[j(\nu)i(\mu)]}$.

In the language of differential forms, the equivalence class $[j]$ is an element of the characteristic cohomology in form degree $n-1$ associated with the surface defined by the Euler-Lagrange equations of motion [37, 38, 39, 40, 41],

$$H^{n-1}_{\text{char}} = \{[j]; d_H j \approx 0, j \sim j + d_H k + t + \delta^n_1 C, t \approx 0\}, \tag{2.16}$$

with j, t in form degree $n-1$ and k in form degree $n-2$.

2.5 Application of Stokes theorem

Stokes theorem implies that for a compact $n-1$ dimensional manifold $\mathcal{C}^{n-1}$ without boundary, $\partial\mathcal{C}^{n-1} = \emptyset$, and a solution $\phi(x)$ of the equations of motion, the charge

$$Q([\mathcal{C}^{n-1}], [j])[\phi(x)] = \int_{\mathcal{C}^{n-1}} j|_{\phi(x)} \tag{2.17}$$

does not dependent on the choice of representatives for the homology class $[\mathcal{C}^{n-1}]$ or for the equivalence class $[j]$.

2.6 Algebra

The vector space of evolutionary vector fields is an infinite dimensional Lie algebra for the bracket defined by

$$[Q_1, Q_2]^i_L = \delta_{Q_1} Q^i_2 - \delta_{Q_2} Q^i_1. \tag{2.18}$$

The vector space of global symmetries is an infinite dimensional Lie sub-algebra for this bracket.

Furthermore, the bracket of a trivial global symmetry with any global symmetry is again trivial. Indeed, $[R_\alpha(g^\alpha), X]^i_L$ defines a family of symmetries depending on the arbitrary local functions g^α and, by the definition of a generating set of non trivial gauge symmetries, it can thus be written as in (2.8). Similarly, $[M^{+j}(\delta L/\delta\phi^j), X]^i_L$ defines a global symmetry for any choice of functions $M^{[j(\nu)i(\mu)]}$, so that it can again be written as in (2.8). (Moreover, $[M^{+j}(\delta L/\delta\phi^j), X]^i$ vanishes on-shell because $\delta_X(\delta L/\delta\phi^i) \approx 0$, which implies by corollary 6.3 of [5] that it can be assumed to be of the form of the second term on the right hand side of (2.8).)

Hence, there is a well defined induced bracket for the equivalence classes,

$$\big[[X_1], [X_2]\big]^i_L = \big[[X_1, X_2]^i_L\big]. \tag{2.19}$$

The induced Lie algebra for the equivalence classes of global symmetries is the relevant algebra from a physical point of view.

There is also a well defined Lie bracket for equivalence classes of conserved currents, the Dickey bracket [42]. This bracket has the following equivalent expressions:

$$\big[[j_1], [j_2]\big]^\mu_D = [\delta_{X_1} j^\mu_2] = [-\delta_{X_2} j^\mu_1] = [\frac{1}{2}(\delta_{X_1} j^\mu_2 - \delta_{X_2} j^\mu_1)] = -[\omega^\mu(X_1, X_2)], \tag{2.20}$$

where the presymplectic current 2-form is defined by $\omega^\mu = d_V(V_i^\mu(d_V\phi^i, L))$. One can show [42, 43] that the Lie algebras of equivalence classes of global symmetries and equivalence classes of conserved currents are isomorphic,

$$\big[[X_1],[X_2]\big]^i_L \longleftrightarrow \big[[j_1],[j_2]\big]^\mu_D. \tag{2.21}$$

3 Reducibility parameters and conserved n–2 forms

3.1 Reducibility parameters

(Global) reducibility parameters are a collection of local functions f^α that satisfy the global reducibility identity

$$R^i_\alpha(f^\alpha) = M^{+ji}(\frac{\delta L}{\delta\phi^j}), \tag{3.1}$$

for some skew symmetric functions $M^{[j(\nu)i(\mu)]}$, cf. (2.9) and (2.10). The corresponding gauge transformations leave thus all solutions of the equations of motion invariant ("ineffective gauge transformations"). Trivial reducibility parameters are given by functions f^α that vanish on any solution of the equations of motion, $f^\alpha \approx 0$. Indeed, such functions always define a global reducibility identity, because if $f^\alpha = k^{\alpha j(\nu)}\partial_{(\nu)}\delta L/\delta\phi^j$, then (3.1) holds with $M^{[j(\nu)i(\mu)]} = -R^{+j(\nu)}_\alpha k^{\alpha i(\mu)} + R^{+i(\mu)}_\alpha k^{\alpha j(\nu)}$, by using the Noether identity (1.2). We define equivalence classes $[f^\alpha]$ of reducibility parameters by identifying parameters which coincide on-shell,

$$f^\alpha \sim f^{\alpha\prime} \quad\Longleftrightarrow\quad f^\alpha \approx f^{\alpha\prime} \quad\Longleftrightarrow\quad f^\alpha - f^{\alpha\prime} = k^{\alpha j(\nu)}\partial_{(\nu)}\delta L/\delta\phi^j. \tag{3.2}$$

3.2 Conserved n–2 forms

Conserved $n-2$ forms are defined by superpotentials whose divergence vanishes when the Euler-Lagrange equations of motion hold,

$$\partial_\nu k^{[\nu\mu]} = J^{i\mu(\lambda)}\partial_{(\lambda)}\frac{\delta L}{\delta\phi^i}. \tag{3.3}$$

Equivalence classes of conserved $n{-}2$ forms are defined by identifying the superpotentials of $n-2$ forms that differ by the sum of a weakly vanishing superpotential and the divergence of an antisymmetric tensor with three indices,

$$k^{[\mu\nu]} \sim k^{[\mu\nu]} + \partial_\sigma l^{[\sigma\mu\nu]} + t^{[\mu\nu]i(\lambda)}\partial_{(\lambda)}\frac{\delta L}{\delta\phi^i} + \delta^n_2\epsilon^{\mu\nu}C. \tag{3.4}$$

In the language of differential forms, the equivalence classe $[k]$ is an element of the characteristic cohomology in form degree $n-2$ associated with the Euler-Lagrange equations of motion,

$$H^{n-2}_{\text{char}} = \{[k]; d_Hk \approx 0, k \sim k + d_Hl + t + \delta^n_2 C, t \approx 0\}, \tag{3.5}$$

with k,t in form degree $n-2$ and l in form degree $n-3$.

3.3 From reducibility parameters to conserved n–2 forms

Contracting the reducibility identity (3.1) with $\delta L/\delta\phi^i$ and using (1.4), we obtain

$$\partial_\mu[S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha)] = M^{+ji}(\frac{\delta L}{\delta\phi^j})\,\frac{\delta L}{\delta\phi^i}\ . \tag{3.6}$$

By definition (2.9) of M^{+ji} and the skew-symmetry (2.10) of $M^{[j(\nu)i(\mu)]}$, one has

$$\begin{aligned}
M^{+ji}(\frac{\delta L}{\delta\phi^j})\,\frac{\delta L}{\delta\phi^i} &= (-\partial)_{(\mu)}\Big[M^{[j(\nu)i(\mu)]}\,\partial_{(\nu)}\frac{\delta L}{\delta\phi^j}\Big]\frac{\delta L}{\delta\phi^i} \\
&= -\partial_\mu M^{\mu ji}(\frac{\delta L}{\delta\phi^j}, \frac{\delta L}{\delta\phi^i}) + \underbrace{M^{[j(\nu)i(\mu)]}\,\partial_{(\nu)}\frac{\delta L}{\delta\phi^j}\,\partial_{(\mu)}\frac{\delta L}{\delta\phi^i}}_{0} \\
&= -\partial_\mu M^{\mu ji}(\frac{\delta L}{\delta\phi^j}, \frac{\delta L}{\delta\phi^i})
\end{aligned} \tag{3.7}$$

where the second equality follows by using repeatedly Leibniz' rule for the derivative ∂_μ. Hence, the right hand side of (3.6) is the divergence of a "doubly" weakly vanishing Noether current $-M^{\mu ji}(\frac{\delta L}{\delta\phi^j}, \frac{\delta L}{\delta\phi^i})$ associated to the trivial global symmetry $\delta_{-M}\phi^i = -M^{+ji}(\frac{\delta L}{\delta\phi^j})$ and (3.6) reduces to the divergence identity

$$\partial_\mu J^\mu_f = 0, \quad J^\mu_f = S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha) + M^{\mu ji}(\frac{\delta L}{\delta\phi^j}, \frac{\delta L}{\delta\phi^i}). \tag{3.8}$$

Because of (2.7), it follows that in dimensions $n \geq 2$, there exists a superpotential $k^{[\mu\nu]}$ whose divergence is conserved on-shell,

$$\partial_\nu k^{[\nu\mu]}_f = J^\mu_f \approx 0. \tag{3.9}$$

3.4 From conserved n–2 forms to reducibility parameters

Given an $n-2$ form k with associated superpotential satisfying (3.3), it follows, by contracting with ∂_μ and using skew-symmetry of the indices of the superpotential, that the operators $\partial_\mu \circ J^{i\mu}$, with $J^{i\mu} = J^{i\mu(\lambda)}\partial_{(\lambda)}$ define a Noether identity. Because we assume that the R^{+i}_α define a generating set of non trivial Noether identities (2.11), these operators can be expressed as

$$\partial_\mu \circ J^{i\mu} = Z^{+\alpha} \circ R^{i+}_\alpha - M^{[j(\nu)i(\mu)]}\partial_{(\nu)}\frac{\delta L}{\delta\phi^j}\,\partial_{(\mu)}. \tag{3.10}$$

The adjoint operator relations are

$$-J^{+i\mu} \circ \partial_\mu = R^i_\alpha \circ Z^\alpha - M^{+ji}(\frac{\delta L}{\delta\phi^j}, \ \cdot\ ) \tag{3.11}$$

with $M^{+ji}(\delta L/\delta\phi^j, \ \cdot\ )$ as in (2.9). Applying (3.11) to 1, we find the relation (3.1), with the reducibility parameters given by

$$f^\alpha = Z^\alpha(1) = (-\partial)_{(\mu)} Z^{+\alpha(\mu)}. \tag{3.12}$$

3.5 Operator currents

So far, we have considered the problem from the point of view of the reducibility parameters or the conserved $n-2$ forms. One can also make the discussion from the point of view of the current that is involved. Consider currents $J^{\mu i}(Q_i) = J^{\mu i(\lambda)}\partial_{(\lambda)}Q_i$ that depend linearly and homogeneously on arbitrary local functions Q_i. We call such currents operator currents. Let us consider operator currents that either satisfy the condition that their divergence gives a Noether operator,

$$\partial_\mu J^{\mu i}(Q_i) = N^i(Q_i), \quad N^i(\frac{\delta L}{\delta \phi^i}) = 0, \tag{3.13}$$

or the condition that they are given by the divergence of a superpotential upon replacing Q_i by the Euler-Lagrange derivatives of the Lagrangian,

$$J^{\mu i}(\frac{\delta L}{\delta \phi^i}) = \partial_\nu k^{[\nu\mu]}. \tag{3.14}$$

Equivalence classes of currents $J^{\mu i}(Q_i)$ satisfying either condition (3.13) or (3.14) are obtained by identifying currents that differ by a current defining itself a Noether identity or the divergence of a superpotential operator,

$$J^{\mu i}(Q_i) \sim J^{\mu i}(Q_i) + N^{i\mu}(Q_i) + \partial_\nu[k^{[\nu\mu]i}(Q_i)], \; N^{i\mu}(\frac{\delta L}{\delta \phi^i}) = 0. \tag{3.15}$$

For every operator current satisfying (3.13), one writes the Noether operator N^i as in the right hand side of (3.10) and finds again that one can associate reducibility parameters through $f^\alpha = (-\partial)_{(\mu)}Z^{+(\alpha)}$, while $\partial_\mu J^{\mu i}(\delta L/\delta\phi^i) = 0$ implies that equation (3.14) holds for some local $k^{[\nu\mu]}$ (owing to (2.7)) which gives thus a conserved $n-2$ form k. Similarly, if the operator current satisfies (3.14), the associated conserved $n-2$ form is again k and associated operator currents satisfying (3.13) as well as associated reducibility parameters are constructed as in subsection 3.4.

3.6 Bijective correspondence between equivalence classes

In order to prove directly that there is a bijective correspondence between equivalence classes of reducibility parameters, equivalence classes of conserved $n-2$ forms and equivalence classes of operator currents satisfying (3.13) or (3.14), one has to use, in addition to (2.7) and (2.11), that, by the algebraic Poincaré lemma,

$$\partial_\nu k^{[\nu\mu]} = 0 \iff k^{[\mu\nu]} = \partial_\sigma l^{[\sigma\mu\nu]} + \delta^n_2 \epsilon^{\mu\nu} C, \tag{3.16}$$

for some local functions $l^{[\sigma\mu\nu]}$ or (if $n=2$) some constant $C \in \mathbb{R}$, and the assumption that the generating set of non trivial gauge symmetries is irreducible in the sense that

$$Z^{+\alpha} \circ R^{+i}_\alpha \approx 0 \quad \Longrightarrow \quad Z^{+\alpha} \approx 0. \tag{3.17}$$

Because this equation has to hold as an operator equation, it is equivalent to the adjoint operator equation, which can be written as

$$R^i_\alpha(Z^\alpha(f)) \approx 0 \quad \forall f \quad \Longrightarrow \quad Z^\alpha \approx 0. \tag{3.18}$$

Irreducible gauge theories, to which the present investigation is limited, are thus characterized by the absence of "local" reducibility identities, i.e., reducibility identities defined by operators Z^α satisfying the left hand side of (3.18) without being weakly zero themselves. We stress the difference from *global* reducibility identities which involve particular local functions, the reducibility parameters, and which may very well exist in irreducible gauge theories. This difference is analogous to the difference between global and gauge symmetries.

The direct proof of the bijective correpondences will not be given here, but we will briefly review in section 7 the proof given in [36] based on the Koszul-Tate resolution.

3.7 Explicit computation of associated conserved n–2 form

3.7.1 General expression

In order to explicitly construct an expression for the superpotential $k^{[\mu\nu]}_f$ in (3.9), associated to the divergence free current J^μ_f of (3.8) built out of the data of the reducibility identity defined by f^α, one needs to use the contracting homotopy that is involved in the algebraic Poincaré lemma.

For instance, one can use the formula for the homotopy operator given in [44] (see also [2], chapter 5 for an expression with a different summation convention). The part concerning the fields of this homotopy operator is given in appendix A, together with the definition and relevant properties of the higher order Lie-Euler operators $\delta/\delta\phi^i_{(\mu)}$. Explicitly, when applied to an $n-1$ form, one finds from $d_H J_f = 0$ that $J_f = -d_H k_f$, respectively $J^\mu_f = \partial_\nu k^{[\nu\mu]}_f$, with

$$k^{[\nu\mu]}_f = \int_0^1 dt\ \Big(\partial_{(\lambda)}\big[\phi^i(\frac{|\lambda|+1}{|\lambda|+2}\frac{\delta J^\mu_f}{\delta\phi^i_{(\lambda)\nu}}[x,t\phi])\big] + x^\nu J^\mu[tx,0] - \mu \longleftrightarrow \nu\Big). \tag{3.19}$$

3.7.2 Simplifications in the computation

- If the current J^μ_f vanishes when the fields and their derivatives are set to zero, one has $J^\mu_f[tx,0] = 0$ in (3.19). This is for instance the case when the Lagrangian contains only terms of degree $k \geq 2$ in the fields and their derivatives.

- If the Euler-Lagrange equations are linear and homogeneous in the fields and the generating set of non trivial gauge transformations contains only field independent operators, any reducibility identity with field independent gauge parameters $\bar f^\alpha = \bar f^\alpha(x)$ holds off-shell, because the left hand side of (3.1) does not depend on the fields, so that one can take $M^{[j(\nu)i(\mu)]} = 0$. It follows that $J^\mu_{\bar f}$ reduces to

$$J^\mu_{\bar f} = S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, \bar f^\alpha) \equiv S^\mu_{\bar f}, \tag{3.20}$$

which is then also linear and homogeneous in the fields (the field dependence coming only from the Euler-Lagrange derivatives of L) and the integration over t can be evaluated trivially. Because in this case one has $J^\mu[x,0] = 0$, (3.19) becomes

$$k^{[\nu\mu]}_{\bar f} = \partial_{(\lambda)}\big[\frac{|\lambda|+1}{|\lambda|+2}\phi^i\frac{\delta S^\mu_{\bar f}}{\delta\phi^i_{(\lambda)\nu}} - (\mu\leftrightarrow\nu)\big]. \tag{3.21}$$

- If the current $S^\mu_{\bar f}$ contains at most second order derivatives of the fields, (3.21) reduces to

$$k^{[\nu\mu]}_{\bar f} = \frac{1}{2}\phi^i\frac{\delta S^\mu_{\bar f}}{\delta\phi^i_\nu} + \frac{2}{3}\partial_\lambda(\phi^i\frac{\delta S^\mu_{\bar f}}{\delta\phi^i_{\lambda\nu}}) - (\mu\leftrightarrow\nu). \tag{3.22}$$

The higher order Euler-operators are then explicitly given by

$$\frac{\delta f}{\delta\phi^i} = \frac{\partial^S f}{\partial\phi^i} - \partial_\lambda\frac{\partial^S f}{\partial\phi^i_\lambda} + \partial_\lambda\partial_\rho\frac{\partial^S f}{\partial\phi^i_{\lambda\rho}}\ , \tag{3.23}$$

$$\frac{\delta f}{\delta\phi^i_\nu} = \frac{\partial^S f}{\partial\phi^i_\nu} - 2\partial_\lambda\frac{\partial^S f}{\partial\phi^i_{\lambda\nu}}\ , \tag{3.24}$$

$$\frac{\delta f}{\delta\phi^i_{\lambda\nu}} = \frac{\partial^S f}{\partial\phi^i_{\lambda\nu}}\ , \tag{3.25}$$

so that (3.22) becomes

$$k^{[\nu\mu]}_{\bar f} = \frac{1}{2}\phi^i\frac{\partial^S S^\mu_{\bar f}}{\partial\phi^i_\nu} + [\frac{2}{3}\phi^i_\lambda - \frac{1}{3}\phi^i\partial_\lambda]\frac{\partial^S S^\mu_{\bar f}}{\partial\phi^i_{\lambda\nu}} - (\mu\leftrightarrow\nu). \tag{3.26}$$

3.8 Application of Stokes theorem

Stokes theorem implies that for an $n-2$ dimensional compact manifold $\mathcal{C}^{n-2}$ without boundary, $\partial\mathcal{C}^{n-2} = \emptyset$, and a solution $\phi(x)$ of the equations of motion, the charge

$$Q([\mathcal{C}^{n-2}],[k])[\phi(x)] = \int_{\mathcal{C}^{n-2}} k|_{\phi(x)} \tag{3.27}$$

is independent of the choice of representatives both for the homology class $[\mathcal{C}^{n-2}]$ and for the equivalence class $[k]$.

3.9 Algebra

One can define a bilinear skew-symmetric operation among the parameters of the non trivial gauge symmetries, i.e., the collection of local functions f^α according to

$$[f_1,f_2]^\gamma_P = C^\gamma_{\alpha\beta}(f^\alpha_1,f^\beta_2), \tag{3.28}$$

where the structure operators $C^{\gamma}_{\alpha\beta}(f_1^\alpha, f_2^\beta) \equiv C^{\gamma(\mu)(\nu)}_{\alpha\beta}\partial_{(\mu)} f_1^\alpha \partial_{(\nu)} f_2^\beta$ depend on the choice of the generating set according to (1.22).

The Jacobi identity for the Lie bracket $[\cdot,\cdot]_L$ of evolutionary vector fields then implies a relation for the bracket of gauge parameters:

$$\begin{gathered}[R_{f_3},[R_{f_1},R_{f_2}]_L]^i_L + \text{cyclic}(1,2,3) = 0 \quad \Longrightarrow \\ R^i_\rho([f_3,[f_1,f_2]_P + \delta_{f_1} f_2 - \delta_{f_2} f_1]^\rho_P + \delta_{f_3}[f_1,f_2]^\rho_P) + \text{cyclic}(1,2,3) \approx 0.\end{gathered} \tag{3.29}$$

For irreducible gauge theories one deduces because of (3.18) that

$$[f_3,[f_1,f_2]_P]^\rho_P + [f_3,\delta_{f_1} f_2 - \delta_{f_2} f_1]^\rho_P + \delta_{f_3}[f_1,f_2]^\rho_P + \text{cyclic}(1,2,3) \approx 0. \tag{3.30}$$

We also note the commutation relation between a global symmetry X^i and a gauge symmetry $R^i_\alpha(f^\alpha)$:

$$[X,R_f]^i_L = R^i_\alpha(X^\alpha_\beta(f^\beta) + \delta_X f^\alpha) + M^{+ji}_\alpha(\frac{\delta L}{\delta\phi^j}, f^\alpha), \tag{3.31}$$

for some operators $X^\alpha_\beta = X^{\beta(\mu)}_\alpha \partial_{(\mu)}$, and $M^{+ji}_\alpha(\delta L/\delta\phi^j,\ \cdot\ )$ as in (2.9).

There is a well defined Lie action from equivalence classes of global symmetries on equivalence classes of reducibility parameters. Indeed, by acting with a global symmetry δ_X on $R^i_\alpha(f^\alpha) \approx 0$ and using the algebra (3.31) together with $\delta_f \approx 0$, we deduce that $R^i_\alpha(X^\alpha_\beta(f^\beta) + \delta_X f^\alpha) \approx 0$, so that the global symmetry defines a mapping $(X^i, f^\alpha) \mapsto X^\alpha_\beta(f^\beta) + \delta_X f^\alpha$ which maps a collection of reducibility parameters to another collection of reducibility parameters. Furthermore, if $f^\alpha \approx 0$, the result vanishes weakly, so that the mapping gives trivial reducibility parameters and induces a well defined map for equivalence classes of reducibility parameters. Similarly, if $X^i \approx 0$ the left hand side of (3.31) defines, for all f^α, a weakly vanishing, and thus a trivial gauge symmetry, and the irreducibility of the generating set then implies that $X^\alpha_\beta(f^\beta) + \delta_X f^\alpha \approx 0$ for all f^α and thus also in particular for reducibility parameters f^α. Finally, if we choose $X^i = R^i_\alpha(f_1^\alpha)$ with arbitrary local functions f_1^α, and $f^\alpha \equiv f_2^\alpha$ reducibility parameters, the parameters in the reducibility identity $R^i_\alpha(X^\alpha_\beta(f^\beta) + \delta_X f^\alpha) \approx 0$ depend on the arbitrary local function f_1. This implies by the irreducibility of the generating set that $X^\alpha_\beta(f^\beta) + \delta_X f^\alpha \approx 0$.

Hence, the mapping

$$\big([X^i],[f^\alpha]\big) \mapsto \big[X^\alpha_\beta(f^\beta) + \delta_X f^\alpha\big] \tag{3.32}$$

is well defined.

By the isomorphism of equivalence classes of reducibility parameters and equivalence classes of non constant conserved $n-2$ forms, it follows that this last space is an isomorphic Lie module. Explicitly, the associated module action is defined by

$$\big([X^i],[k_f]\big) \mapsto [\delta_X k_f]. \tag{3.33}$$

This will be proved by cohomological methods in section 7.

Remark:
Suppose that f_1^α and f_2^α are reducibility parameters, $R^i_\alpha(f_1^\alpha) \approx 0$ and $R^i_\alpha(f_2^\alpha) \approx 0$. Equation (1.22) defining the algebra of the gauge transformations then implies that $[f_1, f_2]^\alpha_P$ are also reducibility parameters (because of $\delta_{f_1} \approx 0$, $\delta_{f_2} \approx 0$):

$$R^i_\gamma([f_1, f_2]^\gamma_P) \approx 0. \tag{3.34}$$

Furthermore, the bracket of weakly vanishing gauge parameters with arbitrary gauge parameters is again weakly vanishing (if, say, $f_1^\alpha \approx 0$, then $[f_1, f_2]^\alpha_P \approx 0$, for all f_2^α), and (3.30) implies that the Jacobi identity holds for equivalence classes of reducibility parameters. Hence, the bracket $[\cdot,\cdot]_P$ induces a well defined Lie bracket among equivalence classes of reducibility parameters:

$$\big[[f_1],[f_2]\big]^\gamma_P = \big[[f_1, f_2]^\gamma_P\big]. \tag{3.35}$$

However, this algebra is always Abelian. Indeed, suppose that f_2^α are reducibility parameters, while f^1_α are arbitary local functions. Then (1.22) implies

$$R^i_\gamma([f_1, f_2]^\gamma_P + \delta_{f_1} f_2^\gamma) \approx 0 \tag{3.36}$$

for arbitary f_1^α, which implies by irreducibility

$$[f_1, f_2]^\gamma_P + \delta_{f_1} f_2^\gamma \approx 0. \tag{3.37}$$

In the case where f_1^α are reducibility parameters, $\delta_{f_1} \approx 0$, which gives the result.

4 Induced symmetries of the linearized theory

4.1 Gauge and global symmetries of the linearized theory

Consider the change of variables $\phi = \bar\phi(x) + \varphi$, where $\bar\phi(x)$ is a solution of the equations of motion,

$$\frac{\delta L}{\delta \phi^i}\Big|_{\bar\phi(x)} = 0. \tag{4.1}$$

We have

$$L[\bar\phi(x) + \varphi] = L[\bar\phi(x)] + \partial_\mu V^\mu_i(\varphi^i, L^1) + \sum_{n=2} L^n, \tag{4.2}$$

$$L^n = \frac{1}{n!}\frac{\partial^{S^n} L[\phi]}{\partial\phi^{i_1}_{(\mu_1)}\dots\partial\phi^{i_n}_{(\mu_n)}}\Big|_{\bar\phi(x)} \varphi^{i_1}_{(\mu_1)}\dots\varphi^{i_n}_{(\mu_n)}. \tag{4.3}$$

[See Eq. (2.4) for the notation $V^\mu_i(\varphi^i, L^1)$.] In the following, we will assume that the operators $R^{+i}_\alpha|_{\bar\phi(x)}$ provide an (irreducible) generating set for the Noether identities of the "free theory" defined by $L^{\rm free}[\varphi] \equiv L^2$, i.e., that the theory is "linearizable" [5] around the solution $\bar\phi(x)$.

Consider the equation expressing the fact that $R^i_\alpha(f^\alpha)$ are symmetries of the Lagrangian for all local functions f^α,

$$R^i_\alpha(f^\alpha)\frac{\delta L}{\delta\phi^i} = \partial_\mu S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha). \tag{4.4}$$

Expanding in powers of the fields φ^i, we obtain to lowest non trivial order,

$$R^{i0}_\alpha(f^{\alpha 0})\frac{\delta L^2}{\delta\varphi^i} = \partial_\mu S^{\mu i0}_\alpha(\frac{\delta L^2}{\delta\varphi^i}, f^{\alpha 0}), \tag{4.5}$$

which reflects the gauge invariance of L^2 under the transformations $\delta^0_f\varphi^i = R^{i0}_\alpha(f^\alpha)$. The next order gives

$$\begin{aligned} &R^{i0}_\alpha(f^{\alpha 0})\frac{\delta L^3}{\delta\varphi^i} + \big(R^{i1}_\alpha(f^{\alpha 0}) + R^{i0}_\alpha(f^{\alpha 1})\big)\frac{\delta L^2}{\delta\varphi^i} \\ = \;& \partial_\mu\Big(S^{\mu i0}_\alpha(\frac{\delta L^3}{\delta\varphi^i}, f^{\alpha 0}) + S^{\mu i1}_\alpha(\frac{\delta L^2}{\delta\varphi^i}, f^{\alpha 0}) + S^{\mu i0}_\alpha(\frac{\delta L^2}{\delta\varphi^i}, f^{\alpha 1})\Big). \end{aligned} \tag{4.6}$$

Suppose that $\tilde{f}^\alpha$ is the parameter of a field independent reducibility identity of the free theory,

$$R^{i0}_\alpha(\tilde{f}^\alpha) = 0, \quad \tilde{f}^\alpha = \tilde{f}^\alpha(x). \tag{4.7}$$

[There can be no equations of motion terms here because these are at least linear in the fields. If one would however consider reducibility parameters of the free theory that depend on the fields, equations of motion terms are relevant in general. The analysis can be extended to cover this case. This is done in section 7 using cohomological methods.] Specializing (4.6) by choosing parameters $f^{\alpha 0} = \tilde{f}^\alpha$ and $f^{\alpha 1} = 0$, it reads

$$R^{i1}_\alpha(\tilde{f}^\alpha)\frac{\delta L^2}{\delta\varphi^i} = \partial_\mu\Big(S^{\mu i0}_\alpha(\frac{\delta L^3}{\delta\varphi^i}, \tilde{f}^\alpha) + S^{\mu i1}_\alpha(\frac{\delta L^2}{\delta\varphi^i}, \tilde{f}^\alpha)\Big), \tag{4.8}$$

which means that $R^{i1}_\alpha(\tilde{f}^\alpha)$ defines a global symmetry of the free theory.

Hence, the parameters $\tilde{f}^\alpha$ of a field independent reducibility identity of the free theory determine a global, linear, symmetry of the free theory given by $\delta^1_{\tilde{f}}\varphi^i = R^{i1}_\alpha(\tilde{f}^\alpha)$.

4.2 Algebra

By expanding (1.22) in terms of φ around $\bar{\phi}(x)$, with $f^\alpha_1 = \tilde{f}^\alpha$ field independent reducibility parameters satisfying (4.7) and $f^\alpha_2 = g^\alpha$ arbitrary field independent parameters, one finds, to zeroth order, the action of a gauge symmetry of the linearized theory on the global symmetry associated to the reducibility parameters:

$$[R^1_{\tilde{f}}, R^0_g]_L = -\delta^0_g R^{i1}_\beta(\tilde{f}^\beta) = R^{i0}_\gamma(C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha, g^\beta)). \tag{4.9}$$

If one chooses $g^\alpha = \tilde{g}^\alpha$ where $\tilde{g}^\alpha$ are also field independent reducibility parameters of the free theory ($R^{i0}_{\tilde{g}} = 0$), this gives

$$R^{i0}_\gamma(C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha, \tilde{g}^\beta)) = 0. \tag{4.10}$$

The latter equation shows that the bracket

$$[\tilde{f}, \tilde{g}]^\gamma_M := C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha, \tilde{g}^\beta) \tag{4.11}$$

gives again parameters of a field independent reducibility identity, whenever $\tilde{f}^\alpha$ and $\tilde{g}^\alpha$ are such parameters. Furthermore, (3.30) then guarantees that this bracket satisfies the Jacobi identity so that the vector space of field independent reducibility parameters equipped with this bracket is a Lie algebra. Note that this bracket may be non trivial and that there is no contradiction with the analysis of section 3.9. Indeed, the commutators $[\delta^0_{f_1}, \delta^0_{f_2}]$ vanish for all field independent parameters f^α_1 and f^α_2 (since R^{i0}_α is field independent), i.e., the structure operators arising in these commutators vanish and are not given by $C^{\gamma 0}_{\alpha\beta}$; hence the bracket (4.11) is *not* the counterpart in the linearized theory of the bracket (3.28) in the full theory.

The Lie algebra with bracket (4.11) can also be expressed in terms of a basis $\{\tilde{f}_A\}$ for the field independent reducibility parameters. Such a basis is defined analogously to a basis for Killing vector fields of a Riemannian metric: each $\tilde{f}_A$ is a "vector field" with components $\tilde{f}^\alpha_A$ such that (i) the vector fields $\tilde{f}_A$ are linearly independent, and (ii) every vector field $\tilde{f}$ of field independent reducibility parameters $\tilde{f}^\alpha$ is a linear combination $C^A \tilde{f}_A$ of the $\tilde{f}_A$ with constant coefficients C^A. In particular, one thus has

$$C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha_A, \tilde{f}^\beta_B) = C_{AB}{}^C \tilde{f}^\gamma_C \tag{4.12}$$

for some constant coefficients $C_{AB}{}^C$, which are the structure constants of the Lie algebra defined by $[\ ,\ ]_M$ in the basis $\{\tilde{f}_A\}$,

$$[\tilde{f}_A, \tilde{f}_B]^\alpha_M = C_{AB}{}^C \tilde{f}^\alpha_C\ . \tag{4.13}$$

Let us denote by δ_A the induced global symmetry of the free theory associated to $\tilde{f}_A$,

$$\delta_A \varphi^i = R^{i1}_\alpha(\tilde{f}^\alpha_A).$$

Note that some linear combinations of the subset of global symmetries $\{\delta_A\}$ might be trivial global symmetries. For $g^\alpha = \tilde{g}^\alpha$, (4.9) defines a Lie action of these global symmetries on the field independent reducibility parameters. Owing to (4.12), this Lie module action is

$$(\delta_A, [\tilde{f}_B]) \mapsto C_{AB}{}^C [\tilde{f}_C]. \tag{4.14}$$

To first order in φ, Eq. (1.22) gives the commutator algebra of the induced global symmetries,

$$[R^1_{\tilde{f}}, R^1_{\tilde{g}}]^i_L \approx^{\text{free}} R^{i1}_\gamma(C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha, \tilde{g}^\beta)) + R^{i0}_\gamma(C^{\gamma 1}_{\alpha\beta}(\tilde{f}^\alpha, \tilde{g}^\beta)), \tag{4.15}$$

where $\approx^{\text{free}}$ means an equality when the equations of motion of the free theory hold. As the second term on the right hand side is a trivial symmetry of the free theory (it is a gauge transformation), one obtains, using (4.12),

$$[\delta_A, \delta_B] \sim C_{AB}{}^C \delta_C \qquad (4.16)$$

where $\sim$ denotes equivalence in the free theory. Hence, the commutator algebra of the induced global symmetries reflects, modulo trivial global symmetries, the Lie algebra (4.13) associated to the reducibility parameters.

5 Asymptotic symmetries and conservation laws

5.1 Boundary conditions

Our aim is to capture general properties of symmetries, conservation laws and their algebra in Lagrangian field theories, for different models and various choices of boundary conditions. Therefore we try to avoid, as much as possible, too specific assumptions on the boundary conditions. In fact, a detailed specification of the boundary conditions cannot be done in a model independent manner. For instance, a basic physical requirement on the boundary conditions could be that they contain certain solutions of the full equations of motion that are of physical interest and such a requirement depends on the model and the particluar solutions under investigation.

What we want here are generic assumptions in connection with the boundary conditions that allow us to extend the bijective correspondence between equivalence classes of exact reducibility parameters and conserved $n-2$ forms described in sections 3 to the asymptotic counterparts of these quantities. Nevertheless we find it useful to describe in the following a certain type of boundary conditions and related assumptions that are sufficient for this bijective correspondence[3]. These conditions are adapted from the Hamiltonian analysis of asymptotically anti-de Sitter gravity in 3 and 4 dimensions in [45, 20]. Three-dimensional asymptotically anti-de Sitter gravity will be discussed in some detail in section 6.3.4, to which we refer for a concrete example.

The conditions are formulated in terms of Landau's O-notation and a corresponding "asymptotic degree" which characterize the behaviour of the functions of interest near the boundary [i.e., the behaviour of the fields, gauge parameters and local forms constructed of them; the boundary need not be at (spatial) infinity]. We denote the asymptotic degree of a function f by $|f|$. The notation $f \longrightarrow O(g)$ and $f \longrightarrow o(g)$ mean $|f| \leq |g|$ and $|f| < |g|$, respectively. For instance, in three-dimensional asymptotically anti-de Sitter gravity, the asymptotic degree of a function is its leading power in the radial coordinate r for $r \longrightarrow \infty$ so that a function $f \longrightarrow r^m h(t, \theta)$ has asymptotic degree $|f| = m$.

Moreover we assume that we can also assign a definite asymptotic degree to each of the relevant differential operators, independently of its arguments. For definiteness and

[3]They are not necessary, i.e., one may relax them; a central requirement for our purpose is the validity of the asymptotic acyclicity properties described in section 5.4.

simplicity, let us assume in particular that the derivatives ∂_μ have asymptotic degree opposite to the corresponding coordinates and differentials,

$$|x^\mu| = |dx^\mu| = -|\partial_\mu|. \tag{5.1}$$

This implicitly is an assumption on properties of the space of functions in which the fields are assumed to live. For instance, in three-dimensional asymptotically anti-de Sitter gravity, we assign asymptotic degree -1 to the derivative ∂_r with respect to the radial coordinate meaning that $f \longrightarrow O(r^m) \Longrightarrow \partial_r f \longrightarrow O(r^{m-1})$ for $r \longrightarrow \infty$; this excludes in particular functions with an oscillating dependence on the radial coordinate near the boundary, such as outgoing or incoming waves. Restrictions of this kind on the space of allowed functions are quite commonly used in studies of asymptotic quantities; for example, they were already used in [46] and also in [45, 20].

The boundary conditions for the fields refer to a background $\bar{\phi}^i(x)$ and the fields $\phi^i(x)$ are assumed to approach the background fields near the boundary, i.e., $\phi^i(x)/\bar{\phi}^i(x) \longrightarrow 1$ at some rate. Accordingly, the deviations $\varphi^i = \phi^i - \bar{\phi}^i(x)$ of the fields from the background, which are used as basic field variables near the boundary, satisfy $\varphi^i(x)/\bar{\phi}^i(x) \longrightarrow 0$. We denote the resulting asymptotic behaviour of the fields $\varphi^i(x)$ by

$$\varphi^i(x) \equiv \phi^i(x) - \bar{\phi}^i(x) \longrightarrow O(\chi^i). \tag{5.2}$$

In general, the fields φ^i are not "small". Near the boundary, however, they are small as compared to the corresponding background fields so that $\varphi^i/\bar{\phi}^i \longrightarrow 0$. Nevertheless it may happen that φ^i does not approach zero at the boundary if $\bar{\phi}^i$ does not do so. In the following, we assume that $\varphi^i(x)$ are generic fields that satisfy the boundary conditions (5.2).

In the case of three dimensional anti-de Sitter gravity for example, the metric deviations $h_{\mu\nu}$ satisfying the boundary conditions are required to be of the form

$$h_{\mu\nu}(x) \longrightarrow r^{m_{\mu\nu}} \tilde{h}_{\mu\nu}(t,\theta) + o(r^{m_{\mu\nu}}), \tag{5.3}$$

with $\tilde{h}_{\mu\nu}(t,\theta)$ arbitrary functions of t, θ.

When discussing the asymptotic behaviour of a local form ω^p, we will consider the asymptotic degree of the differential form obtained after evaluating the form ω^p for generic fields that satisfy the boundary conditions.

Equation (5.1) implies that the differential d_H and the associated contracting homotopy $\rho_{H,\varphi}$ (A.9) have vanishing asymptotic degree,

$$\omega^p\Big|_{\varphi(x)} \longrightarrow O(\chi^p) \quad \Longrightarrow \quad (d_H\omega^p)\Big|_{\varphi(x)} \longrightarrow O(\chi^p), \quad (\rho^p_{H,\varphi}\omega^p)\Big|_{\varphi(x)} \longrightarrow O(\chi^p). \tag{5.4}$$

Furthermore, (5.1) implies that a field independent differential operator $Z = Z^{(\mu)}(x)\partial_{(\mu)}$ has the same asymptotic degree as its adjoint $Z^+ = (-\partial)_{(\mu)}[Z^{(\mu)}(x)\cdot]$,

$$Z = Z^{(\mu)}(x)\partial_{(\mu)} \quad \Longrightarrow \quad |Z| = |Z^+|. \tag{5.5}$$

Now, let $L[\phi^i; j_a(x)]$ be the Lagrangian of the model under study. It may involve external sources $j_a(x)$ but these and their derivatives are supposed to vanish in a neighborhood of the boundary, so that near the boundary, the theory is described by the source free Lagrangian $L[\phi^i; 0]$. We shall assume here that the background is an exact solution of the field equations derived from $L[\phi^i; 0]$.

The boundary conditions may allow one to completely neglect a subset of the fields φ^i near the boundary (possibly after a field redefinition), so that one may use a simplified Lagrangian there (with less fields and less terms). A typical example is the case where "matter fields" in general relativity, electrodynamics or Yang-Mills theory decrease sufficiently fast near the boundary so that there one may use the pure Einstein, Maxwell or Yang-Mills Lagrangian, respectively.

We denote by $O(\chi_i)$ the behaviour of the linearized (source free) field equations evaluated at $\varphi^i(x)$ and multiplied by the volume element,

$$\forall \varphi^i(x) \longrightarrow O(\chi^i): \quad d^n x \, \frac{\delta L^{\text{free}}}{\delta \varphi^i}\Big|_{\varphi(x)} \longrightarrow O(\chi_i), \tag{5.6}$$

with a χ_i of minimal asymptotic degree. To determine that degree, we use that the linearized field equations take the form

$$\frac{\delta L^{\text{free}}}{\delta \varphi^i} = D_{ij} \varphi^j$$

where $D_{ij} = d_{ij}^{(\mu)}(x) \partial_{(\mu)}$ are differential operators involving the background fields and their derivatives. The asymptotic degree of the function χ_i is thus

$$|\chi_i| = \max_j \{|D_{ij}| + |\chi^j| + |d^n x|\}, \tag{5.7}$$

where we use the convention $|0| = -\infty$, i.e., the vanishing of an operator D_{ij} does not affect $|\chi_i|$. Furthermore, let us denote by $O(\chi_\alpha)$ the behaviour of the linearized Noether identities when evaluated at a field that behaves like the linearized equations of motion times the volume form,

$$\forall \psi_i \longrightarrow O(\chi_i): \quad R_\alpha^{+i0}(\psi_i) \longrightarrow O(\chi_\alpha), \tag{5.8}$$

where

$$|\chi_\alpha| = \max_i \{|R_\alpha^{i0}| + |\chi_i|\}. \tag{5.9}$$

The left hand sides of the (Euler-Lagrange) equations of motion of the full and the free theory and their total derivatives, $\partial_{(\mu)} \delta L / \delta \phi^i$ and $\partial_{(\mu)} \delta L^{\text{free}} / \delta \varphi^i$, have been assumed to satisfy important regularity conditions described for instance in [47, 48] and spelled out in detail in the context of Yang-Mills theories in [5]. We assume that these regularity conditions also hold asymptotically. By this we mean that the leading order terms of $\partial_{(\mu)} \delta L^{\text{free}} / \delta \varphi^i d^n x$, after substitution of generic fields that saturate the boundary conditions, satisfy the mentioned regularity conditions.

The Noether operators R^{+i}_α of the full theory were assumed to form an irreducible generating set of Noether identitites, as expressed by (2.11) and (3.17). Similarly, these Noether operators evaluated at the background R^{+i0}_α were assumed to form an irreducible generating set of Noether operators for the linearized theory near the boundary. Now, we require in addition that these properties also hold asymptotically. More precisley,

$$\forall\varphi^i(x):\ N^i(\frac{\delta L^{\text{free}}}{\delta\varphi^i})\Big|_{\varphi(x)} d^nx \longrightarrow 0 \implies \exists\{Z^\alpha\}\ \forall\psi_i:\ N^i(\psi_i)\longrightarrow Z^{+\alpha}(R^{+i0}_\alpha(\psi_i)), \tag{5.10}$$

for field independent differential operators N^i, and

$$\forall\psi_i:\ Z^{+\alpha}(R^{+i0}_\alpha(\psi_i))\longrightarrow 0 \implies \forall\psi_\alpha:\ Z^{+\alpha}(\psi_\alpha)\longrightarrow 0, \tag{5.11}$$

for field independent differential operators $Z^{+\alpha}$. Here and in the following, ψ_i and ψ_α are generic fields satisfying the boundary conditions

$$\psi_i \longrightarrow O(\chi_i), \quad \psi_\alpha \longrightarrow O(\chi_\alpha). \tag{5.12}$$

5.2 Analysis from the viewpoint of the linearized theory

5.2.1 Asymptotic solutions

Asymptotic solutions are particular fields $\varphi_s(x)$ satisfying the boundary conditions (5.2) together with the condition

$$\frac{\delta L^{\text{free}}}{\delta\varphi^i}\Big|_{\varphi_s(x)} d^nx \longrightarrow o(\chi_i). \tag{5.13}$$

5.2.2 Asymptotic reducibility parameters

Definition: Asymptotic reducibility parameters are field independent gauge parameters $\tilde{f}^\alpha$ satisfying the condition

$$\forall\psi_i:\quad \psi_i R^{i0}_\alpha(\tilde{f}^\alpha)\longrightarrow 0. \tag{5.14}$$

Because of (5.8), this condition is automatically satisfied for parameters with asymptotic degrees smaller than $-|\chi_\alpha|$. Such parameters will thus be considered as trivial and called "pure gauge". Equivalence classes of asymptotic reducibility parameters are defined by asymptotic reducibility parameters up to parameters that are pure gauge. In particular, parameters that are pure gauge are thus equivalent to zero (~ 0),

$$\tilde{f}^\alpha \sim 0 \quad\iff\quad \tilde{f}^\alpha \longrightarrow o(\chi^\alpha), \tag{5.15}$$

where χ^α is a function with asymptotic degree equal to $-|\chi_\alpha|$,

$$|\chi^\alpha| = -|\chi_\alpha|. \tag{5.16}$$

5.2.3 Asymptotically conserved n–2 forms

Definition: An asymptotically conserved $n-2$ form is an $n-2$ form $\tilde{k}[\varphi]$ that depends linearly and homogeneously on $\varphi^i_{(\mu)}$ such that

$$\forall \varphi^i(x): \quad d_H \tilde{k}|_{\varphi(x)} \longrightarrow \tilde{s}^i(\frac{\delta L^{\text{free}}}{\delta \varphi^i})|_{\varphi(x)}, \tag{5.17}$$

with $\tilde{s}^i(Q_i)$ an $n-1$ form that depends linearly and homogeneously on Q_i and its derivatives.

An asymptotically conserved $n-2$ form $\tilde{k}$ is trivial if

$$\forall \varphi^i(x): \quad \tilde{k}|_{\varphi(x)} \longrightarrow \tilde{t}^i(\frac{\delta L^{\text{free}}}{\delta \varphi^i})|_{\varphi(x)} + d_H \tilde{l}|_{\varphi(x)}, \tag{5.18}$$

with $\tilde{t}^i(Q_i)$ an $n-2$ form that depends linearly and homogeneously on Q_i and its derivatives.

5.2.4 Bijective correspondence

The $n-1$ form $s^i_\alpha(\psi_i, \tilde{f}^\alpha)$ is defined by

$$\forall Q_i: \quad d^n x\, Q_i R^{i0}_\alpha(\tilde{f}^\alpha) = d^n x\, R^{+i0}_\alpha(Q_i)\tilde{f}^\alpha + d_H s^i_\alpha(Q_i, \tilde{f}^\alpha). \tag{5.19}$$

For $Q_i = \delta L^{\text{free}}/\delta\varphi^i$, this relation reduces to

$$d_H \tilde{s}_{\tilde{f}} = d^n x\, \frac{\delta L^{\text{free}}}{\delta \varphi^i}\, R^{i0}_\alpha(\tilde{f}^\alpha), \tag{5.20}$$

with $\tilde{s}_{\tilde{f}} = s^i_\alpha(\delta L^{\text{free}}/\delta\varphi^i, \tilde{f}^\alpha)$. Suppose that $\tilde{f}^\alpha$ are asymptotic reducibility parameters so that (5.14) holds. This implies

$$\forall \varphi^i(x): \quad d_H \tilde{s}_{\tilde{f}}|_{\varphi(x)} \longrightarrow 0. \tag{5.21}$$

Applying the contracting homotopy $\rho_{H,\varphi}$ to $\tilde{s}_{\tilde{f}}$ and using (5.21) together with (5.4), it follows that

$$\forall \varphi^i(x): \quad \tilde{s}_{\tilde{f}}|_{\varphi(x)} \longrightarrow -d_H \tilde{k}_{\tilde{f}}|_{\varphi(x)}, \tag{5.22}$$

with $\tilde{k}_{\tilde{f}} = -\rho^{n-1}_{H,\varphi}\tilde{s}_{\tilde{f}}$. Since $\tilde{s}_{\tilde{f}}$ depends linearly and homogeneously on the "left hand sides" of the linearized field equations, the $n-2$ form $\tilde{k}_{\tilde{f}}$ is thus an asymptotically conserved $n-2$ form.

Conversely, for an asymptotically conserved $n-2$ form $\tilde{k}$, application of d_H to (5.17) implies

$$\forall \varphi^i(x): \quad d_H \tilde{s}^i(\frac{\delta L^{\text{free}}}{\delta \varphi^i})|_{\varphi(x)} \longrightarrow 0. \tag{5.23}$$

Hence $d_H\tilde{s}^i(\cdot) \equiv d^n x N^i$ defines an asymptotic Noether operator as in (5.10) which implies that there are operators Z^α such that $N^i(\psi_i) \longrightarrow Z^{+\alpha}(R^{+i0}_\alpha(\psi_i))$. Setting $\psi_i = Q_i d^n x$, we obtain $d^n x Z^{+\alpha}(R^{+i0}_\alpha(Q_i)) = d_H(\dots) + d^n x Q_i R^{i0}_\alpha(\tilde{f}^\alpha)$ with $\tilde{f}^\alpha = Z^\alpha(1)$. Furthermore we have $d^n x N^i(Q_i) = d_H\tilde{s}^i(Q_i)$ by definition of N^i. We thus obtain $\psi_i R^{i0}_\alpha(\tilde{f}^\alpha) = d^n x Q_i R^{i0}_\alpha(\tilde{f}^\alpha) \longrightarrow d_H\omega$ for some $(n-1)$-form ω. Recall that this holds for *all* ψ_i with $\psi_i \longrightarrow O(\chi_i)$. This is only possible if both $\psi_i R^{i0}_\alpha(\tilde{f}^\alpha) \longrightarrow 0$ and $d_H\omega \longrightarrow 0$. It follows that the $\tilde{f}^\alpha = Z^\alpha(1)$ satisfy (5.14) and are thus asymptotic reducibility parameters.

We have thus shown that asymptotic reducibility parameters correspond to asymptotically conserved $n-2$ forms and vice versa. This correspondence extends to the equivalence classes associated with these quantities. This will be proved in section 7 using cohomological methods and is summarized by the following theorem.

Theorem 1. *There is a bijective correspondence between the quotient space of asymptotic reducibility parameters factored by pure gauge parameters on the one hand, and equivalence classes of asymptotically conserved $n-2$ forms on the other hand.*

Remark:

Because of (5.4), the asymptotic behaviour of the forms $\tilde{s}_{\tilde{f}}$ and $\tilde{k}_{\tilde{f}}$ is determined by the asymptotic behaviour of the asymptotic reducibility parameters according to

$$
\begin{aligned}
|\tilde{s}_{\tilde{f}}|_{\varphi(x)}| &\le \max_\alpha\{|\tilde{f}^\alpha| + |\chi_\alpha|\}, \\
|\tilde{k}_{\tilde{f}}|_{\varphi(x)}| &\le \max_\alpha\{|\tilde{f}^\alpha| + |\chi_\alpha|\}.
\end{aligned} \tag{5.24}
$$

In particular, these forms vanish asymptotically for trivial asymptotic reducibility parameters because then one obtains $|\tilde{f}^\alpha| + |\chi_\alpha| < -|\chi_\alpha| + |\chi_\alpha| = 0$, see (5.15) and (5.16), while they are asymptotically finite for asymptotic reducibility parameters that satisfy

$$
\forall\alpha: \quad |\tilde{f}^\alpha| \le -|\chi_\alpha|. \tag{5.25}
$$

In this latter case, the horizontal differential of the asymptotically conserved $n-2$ form $\tilde{k}_{\tilde{f}}$ vanishes asymptotically when evaluated at an arbitrary asymptotic solution $\varphi_s(x)$,

$$
d_H\tilde{k}_{\tilde{f}}|_{\varphi_s(x)} \longrightarrow 0. \tag{5.26}
$$

Similarly, a trivial asymptotically conserved $n-2$ form, evaluated at an arbitrary asymptotic solution $\varphi_s(x)$, is asymptotically given by the horizontal differential of an $n-3$ form,

$$
\tilde{k}_{\tilde{f}} \sim 0 \quad \Longrightarrow \quad \tilde{k}_{\tilde{f}}|_{\varphi_s(x)} \longrightarrow d_H l|_{\varphi_s(x)}. \tag{5.27}
$$

5.2.5 Asymptotic charges

Consider an $n-2$ dimensional compact manifold $\mathcal{C}^{n-2}$ without boundary, $\partial\mathcal{C}^{n-2} = \emptyset$, that lies in the asymptotic region and an asymptotically conserved $n-2$ form $\tilde{k}_{\tilde{f}}$. The associated charge in the linearized theory is defined by

$$
\tilde{Q}_{\tilde{f}}[\varphi; \bar{\phi}(x)] = \int_{\mathcal{C}^{n-2}} \tilde{k}_{\tilde{f}}[\varphi; \bar{\phi}(x)]. \tag{5.28}
$$

If the condition (5.25) holds, the charges are finite when evaluated at a field $\varphi(x)$ that satisfies the boundary conditions (5.2). If furthermore we evaluate the charge for a solution $\varphi_s(x)$ of the linearized equations of motion, we can apply Stokes theorem because of the conservation law (5.26) to prove asymptotic independence of $\tilde{Q}_{\tilde{f}}$ on the choice of representatives for the homology class $[\mathcal{C}^{n-2}]$ and for the equivalence class $[\tilde{k}_{\tilde{f}}]$.

5.2.6 Asymptotic algebra

Let us suppose now, and in the following, that the reducibility parameters $\tilde{f}^\alpha$ defined by (5.14) satisfy condition (5.25), i.e., that $\tilde{f}^\alpha \longrightarrow O(\chi^\alpha)$, which guarantees that the associated $n-1$ and $n-2$ forms are asymptotically finite. Consider fields ψ^α satisfying the boundary conditions $\psi^\alpha \longrightarrow O(\chi^\alpha)$. Suppose now that the additional constraints

$$\forall \psi_i, \varphi^i(x): \qquad \psi_i R^{i1}_\alpha(\tilde{f}^\alpha)|_{\varphi(x)} \longrightarrow O(1), \tag{5.29}$$

$$\forall \psi_\alpha, \psi^\alpha: \qquad \psi_\alpha C^{\alpha 0}_{\beta\gamma}(\psi^\beta, \tilde{f}^\gamma) \longrightarrow O(1), \tag{5.30}$$

$$\forall \psi_i, \psi^\alpha: \qquad \frac{\delta}{\delta\varphi^j}[\psi_i R^{i1}_\alpha(\psi^\alpha)] \longrightarrow O(\chi_j), \tag{5.31}$$

$$\forall \varphi^i(x): \qquad \frac{\delta L^3}{\delta\varphi^i}\Big|_{\varphi(x)} d^n x \longrightarrow O(\chi_i) \tag{5.32}$$

hold for asymptotic reducibility parameters $\tilde{f}^\alpha$ that satisfy (5.25). Under these conditions, we have

Theorem 2. *The vector space of asymptotic reducibility parameters forms a Lie algebra for the bracket* (4.11). *Furthermore, the bracket induced among equivalence classes of asymptotic reducibility parameters is well defined,*

$$[[\tilde{f}], [\tilde{g}]]^\gamma_G = [[\tilde{f}, \tilde{g}]_M]^\gamma. \tag{5.33}$$

The space of equivalence classes of asymptotic reducibility parameters equipped with the bracket $[\cdot,\cdot]_G$ defines the physically relevant Lie algebra $\mathfrak{g}$. Again, the theorem will be proved in section 7 by cohomological means.

If the additional constraints

$$\forall \varphi^i(x): \qquad R^{i1}_\alpha(\tilde{f}^\alpha) \longrightarrow O(\chi^i), \tag{5.34}$$

$$\forall \varphi^i(x): \qquad \Big[\frac{\delta}{\delta\varphi^j}[R^{i0}_\alpha(\tilde{f}^\alpha)\frac{\delta L^3}{\delta\varphi^i}]\Big]_{\varphi(x)} d^n x \longrightarrow O(\chi_j), \tag{5.35}$$

$$\forall \psi^\alpha: \qquad R^{j0}_\beta(\tilde{f}^\beta)\frac{\delta R^{i1}_\alpha(\psi^\alpha)}{\delta\varphi^j} \longrightarrow O(\chi^i) \tag{5.36}$$

hold for asymptotic reducibility parameters $\tilde{f}^\alpha$ that satisfy (5.25), the Lie algebra $\mathfrak{g}$ of equivalence classes of asymptotic reducibility parameters can be represented on the level of the equivalence classes of asymptotically conserved $(n-2)$-forms of the linearized theory near the boundary by a covariant Poisson bracket, which is defined through the action of the associated "global symmetry"[4]:

$$\{[\tilde{k}_{\tilde{f}_1}],[\tilde{k}_{\tilde{f}_2}]\}_F := [\delta^g_{\tilde{f}_1}\tilde{k}_{\tilde{f}_2}] = [\tilde{k}_{[\tilde{f}_1,\tilde{f}_2]_M}]. \tag{5.37}$$

The property $-[\delta^g_{\tilde{f}_2}\tilde{k}_{\tilde{f}_1}] = [\tilde{k}_{[\tilde{f}_1,\tilde{f}_2]_M}]$ implies that alternative equivalent expressions for the covariant Poisson bracket are $-[\delta^g_{\tilde{f}_2}\tilde{k}_{\tilde{f}_1}]$ or $\frac{1}{2}([\delta^g_{\tilde{f}_1}\tilde{k}_{\tilde{f}_2}] - [\delta^g_{\tilde{f}_2}\tilde{k}_{\tilde{f}_1}])$.

When evaluated for solutions of the linearized equations of motion, the Lie algebra $\mathfrak{g}$ can also be represented by a covariant Poisson bracket of the charges $\tilde{Q}_{\tilde{f}}$ of the free theory, defined in the same way:

$$\{\tilde{Q}_{\tilde{f}_1},\tilde{Q}_{\tilde{f}_2}\}_{CL} := \delta^g_{\tilde{f}_1}\tilde{Q}_{\tilde{f}_2} \overset{\approx^{\text{free}}}{\longrightarrow} \tilde{Q}_{[\tilde{f}_1,\tilde{f}_2]_M}. \tag{5.38}$$

That both of these representations also provide representations of the Lie algebra $\mathfrak{g}$ follows from the fact that the asymptotically conserved $n-2$ forms $\tilde{k}_{\tilde{f}}$ and the associated charges $\tilde{Q}_{\tilde{f}}$ vanish asymptotically whenever the $\tilde{f}^\alpha$ are pure gauge.

The proof of these statements is postponed until section 7.

5.3 Analysis from the viewpoint of the bulk theory

5.3.1 Asymptotic linearity

In order for the previous discussion of asymptotic reducibility parameters and asymptotically conserved $n-2$ forms to correctly describe these quantities from the point of view of the bulk theory, additional assumptions on the Lagrangian, the gauge transformations and the boundary conditions are needed. They state that the theory is "asymptotically linear". By that we mean that in the vicinity of the boundary, the full theory can be approximated by the linearized theory with Lagrangian L^{free}.

More precisely, this translates into the following requirements:

(i) the only terms of the equations of motion that are relevant near the boundary are the equations of motion of the linear theory,

$$[\frac{\delta L}{\delta\phi^i} - \frac{\delta L^{\text{free}}}{\delta\varphi^i}]|_{\varphi(x)} d^n x \longrightarrow o(\chi_i), \tag{5.39}$$

(ii) the generating set of non trivial Noether operators are appropriately described by the Noether operators of the linearized theory,

$$\forall\psi_i \longrightarrow O(\chi_i):\ [R^{+i}_\alpha(\psi_i) - R^{+i0}_\alpha(\psi_i)]|_{\varphi(x)} \longrightarrow o(\chi_\alpha). \tag{5.40}$$

[4]Strictly speaking, when $\tilde{f}^\alpha$ are asymptotic reducibility parameters, the variations $\delta^g_{\tilde{f}}\varphi^i = R^{i1}_\alpha(\tilde{f}^\alpha)$ are not global symmetries of the linearized theory, but it will be shown below that they induce symmetries of the equations of motion of the boundary theory.

(iii) the gauge transformation associated to asymptotic reducibility parameters $\tilde{f}^\alpha$ are appropriately described by the sum of the corresponding gauge transformation of the linearized theory and the associated "global" symmetry,

$$\forall \tilde{f}^\alpha \text{ satisfying (5.14)}: \ [R^i_\alpha(\tilde{f}^\alpha) - R^{i0}_\alpha(\tilde{f}^\alpha) - R^{i1}_\alpha(\tilde{f}^\alpha)]|_{\varphi(x)} \longrightarrow o(\chi^i). \tag{5.41}$$

We shall also use the fact that the Euler-Lagrange derivatives $\delta L^{\text{free}}/\delta\varphi^i$ of the linearized Lagrangian are the linearization of the Euler-Lagrange derivatives $\delta L/\delta\phi^i$ of the full Lagrangian,

$$(d_V \frac{\delta L}{\delta \phi^i})|_{\bar{\phi}(x),\varphi} = \frac{\delta L^{\text{free}}}{\delta \varphi^i} \ , \tag{5.42}$$

which holds for all $\bar{\phi}(x)$ and not only for $\bar{\phi}(x)$ that are solutions of the equations of motion. In (5.42) and throughout this paper, evaluation at $\bar{\phi}(x), \varphi$ is obtained by replacing $\phi^i_{\mu_1\dots\mu_k}$ by $\partial^k \bar{\phi}^i(x)/\partial x^{\mu_1}\dots\partial x^{\mu_k}$ and the Grassmann odd variables $d_V \phi^i_{\mu_1\dots\mu_k}$ by $\varphi^i_{\mu_1\dots\mu_k}$.

5.3.2 Asymptotic solutions

On account of (5.39), from the point of view of the full theory, asymptotic solutions can equivalently be defined by fields $\phi_s(x)$ that satisfy the boundary conditions (5.2) together with the condition

$$\frac{\delta L}{\delta \phi^i}\Big|_{\phi_s(x)} d^n x \longrightarrow o(\chi_i). \tag{5.43}$$

5.3.3 Asymptotic reducibility parameters

From the point of view of the full theory, one can allow for possibly field dependent gauge parameters f^α. The condition for asymptotic reducibility parameters then becomes

$$\forall \psi_i \longrightarrow O(\chi_i): \ \psi_i R^i_\alpha(f^\alpha)|_{\bar{\phi}(x)} \longrightarrow 0, \tag{5.44}$$

while trivial asymptotic reducibility parameters correspond to reducibility parameters f^α that fall off fast enough when evaluated at the background,

$$f^\alpha|_{\bar{\phi}(x)} \longrightarrow o(\chi^\alpha). \tag{5.45}$$

The identification $f^\alpha|_{\bar{\phi}(x)} = \tilde{f}^\alpha$ shows that there is no difference between the two points of view.

5.3.4 Asymptotic symmetries

One can define asymptotic symmetries to be gauge transformations $\delta_f \phi^i = R^i_\alpha(f^\alpha)$ of the full theory with gauge parameters that are asymptotic reducibility parameters. Trivial asymptotic symmetries are defined as asymptotic symmetries that involve trivial reducibility parameters and equivalence classes of asymptotic symmetries can as usual be defined by asymptotic symmetries up to trivial ones. According to assumption (5.41), the action of asymptotic symmetries near the boundary is determined by the action of the first two terms in their expansion:

$$\delta_{\tilde{f}}\phi^i = R^i_\alpha(\tilde{f}^\alpha) \longrightarrow R^{i0}_\alpha(\tilde{f}^\alpha) + R^{i1}_\alpha(\tilde{f}^\alpha) + o(\chi^i). \tag{5.46}$$

There is *no* bijective correspondence between equivalence classes of asymptotic symmetries and asymptotic reducibility parameters. Indeed, for instance in the case of pure Maxwell theory, there is one exact reducibility parameter given by a constant gauge parameter, but the associated gauge transformation vanishes. The reason why we will focus our attention on equivalence classes of reducibility parameters and not on equivalence classes of asymptotic symmetries, is that the former and not the latter are in bijective correspondence with equivalence classes of asymptotically conserved $n-2$ forms.

5.3.5 Asymptotically conserved n–2 forms

From the point of view of the full theory, an asymptotically conserved $n-2$ form k is defined as an $n-2$ form whose linearization at the background $\tilde{k} = (d_V k)_{\bar{\phi}(x),\varphi}$ satisfies (5.17). Such a form is trivial if its linearization is, i.e., if it satisfies (5.18).

An equivalent characterization of asymptotically conserved $n-2$ forms and their relation to asymptotic reducibility parameters is the following.

Theorem 3. *Let Σ be any $n-1$ dimensional hypersurface with boundary $\partial\Sigma$ and $\delta\phi^i = R^i_\alpha(f^\alpha)$ be a non trivial gauge symmetry. The associated weakly vanishing "Noether charge"*

$$\int_\Sigma S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha)(d^{n-1}x)_\mu \tag{5.47}$$

can be improved through the addition of a surface integral

$$\oint_{\partial\Sigma} k^{\mu\nu}_\alpha(f^\alpha)(d^{n-2}x)_{\mu\nu} \tag{5.48}$$

to a charge that is asymptotically extremal at $\bar{\phi}(x)$ for arbitrary variations $d_V\phi^i$ (not restricted by any boundary conditions) if and only if the f^α are asymptotic reducibility parameters. For solutions of the equations of motions, the improved Noether charge reduces to the surface integral whose integrand is the associated asymptotically conserved $n-2$ form.

The proof of this theorem is given in section 7.

5.3.6 Algebra and central extensions for the full theory

Let $Q_{\tilde{f}}$ be the charge associated to a given collection of asymptotic reducibility parameters $\tilde{f}^\alpha$,

$$Q_{\tilde{f}}[\phi;\bar{\phi}(x)] = \int_{\mathcal{C}^{n-2}} \tilde{k}_{\tilde{f}}[\phi-\bar{\phi}(x);\bar{\phi}(x)] + N_{\tilde{f}}, \tag{5.49}$$

where the field independent normalization "constant" $N_{\tilde{f}}$ is the arbitrarily chosen charge of the background and $\mathcal{C}^{n-2}$ denotes an $n-2$ dimensional compact and closed manifold that lies in the asymptotic region.

On the level of the charges of the full theory, the Lie algebra $\mathfrak{g}$ of equivalence classes of asymptotic reducibility parameters is represented by acting with an asymptotic symmetry associated to one collection of reducibility parameters on the charge associated to another such collection,

$$\{Q_{\tilde{f}_1}, Q_{\tilde{f}_2}\}_{CF} := \delta_{\tilde{f}_1} Q_{\tilde{f}_2} = \int_{\mathcal{C}^{n-2}} \tilde{k}_{\tilde{f}_2}[R_{\tilde{f}_1}; \bar{\phi}(x)]. \tag{5.50}$$

Because of (5.46), only the first two terms in the expansion of the asymptotic symmetries contribute near the boundary [since we assume that (5.25) holds]. Central charges are contributions to $\delta_{\tilde{f}_1} Q_{\tilde{f}_2}$ which have no counterpart in the Lie algebra $\mathfrak{g}$ associated with the asymptotic reducibility parameters and with the charges in the linearized theory. They arise from the first term on the right hand side of (5.46), while the "regular" terms arise from the second term. (We assume of course the validity of the assumptions of section 5.2.6, that guarantee that the algebra of equivalence classes of asymptotic reducibility parameters is well defined and can be represented by a Poisson algebra of the conserved charges for the free theory.)

Theorem 4. *The covariant Poisson algebra of the charges defined by* (5.50) *is given by*

$$\{Q_{\tilde{f}_1}, Q_{\tilde{f}_2}\}_{CF} \sim Q_{[\tilde{f}_1,\tilde{f}_2]_M} - N_{[\tilde{f}_1,\tilde{f}_2]_M} + K_{\tilde{f}_1,\tilde{f}_2} , \tag{5.51}$$

$$K_{\tilde{f}_1,\tilde{f}_2} = \int_{\mathcal{C}^{n-2}} \tilde{k}_{\tilde{f}_2}[R^0_{\tilde{f}_1}; \bar{\phi}(x)], \tag{5.52}$$

where $\sim$ is asymptotic equality when the charges are evaluated for asymptotic solutions.

The $n-2$ forms $\tilde{k}_{f'}[R^0_f; \bar{\phi}(x)]$ are skew-symmetric, up to a d_H-exact $n-2$ form, under the exchange of arbitrary field independent gauge parameters f, f',

$$\forall f^\alpha, f^{\alpha\prime}: \quad k_f[R^0_{f'}; \bar{\phi}(x)] = -k_{f'}[R^0_f; \bar{\phi}(x)] + d_H(\dots). \tag{5.53}$$

This implies the skew-symmetry of $K_{\tilde{f}_1,\tilde{f}_2}$ under exchange of $\tilde{f}^\alpha_1$ and $\tilde{f}^\alpha_2$, and that $K_{\tilde{f}_1,\tilde{f}_2}$ are 2-cocycles on the Lie algebra of all asymptotic reducibility parameters,

$$K_{\tilde{f}_1,\tilde{f}_2} = -K_{\tilde{f}_2,\tilde{f}_1} , \tag{5.54}$$

$$K_{[\tilde{f}_1,\tilde{f}_2]_M,\tilde{f}_3} + K_{[\tilde{f}_3,\tilde{f}_1]_M,\tilde{f}_2} + K_{[\tilde{f}_2,\tilde{f}_3]_M,\tilde{f}_1} = 0. \tag{5.55}$$

The proof of this theorem is given in appendix A.4. We add a few comments:

- In general, the finiteness of the charges (5.49) does not imply the finiteness of the central charges (5.52). In particular, condition (5.25) which guarantees the existence of the charges $Q_{\tilde{f}}$ does not guarantee the existence of the central charges, unless the additional conditions

$$R^{i0}_\alpha(\tilde{f}^\alpha) \longrightarrow O(\chi^i) \tag{5.56}$$

on the asymptotic reducibility parameters are satisfied. The reason is that $K_{\tilde{f}_1,\tilde{f}_2}$ arises from $Q_{\tilde{f}_2}$ by substituting $R^{i0}_\alpha(\tilde{f}_1^\alpha)$ for φ^i. Furthermore, when (5.25) holds, parameters which satisfy

$$R^{i0}_\alpha(\tilde{f}^\alpha) \longrightarrow o(\chi^i) \tag{5.57}$$

do not contribute to central charges. (5.56) was the starting point of the analysis of [20, 45]. In the case of asymptotically adS$_3$ gravity, it implies the conditions (5.14), (5.25), (5.29), (5.30).

- Let N and K be the alternating linear maps on the Lie algebra of all asymptotic reducibility parameters defined by $N(\tilde{f}) = N_{\tilde{f}}$ and $K(\tilde{f}_1,\tilde{f}_2) = K_{\tilde{f}_1,\tilde{f}_2}$, respectively. The consistency condition (5.55) can be written in terms of the Chevalley-Eilenberg differential δ^{CE} [49] as $\delta^{CE}K = 0$, while the term involving the normalization on the right hand side of the covariant Poisson bracket can be written as the coboundary $(\delta^{CE}N)(\tilde{f}_1,\tilde{f}_2)$. The central charge $K_{\tilde{f}_1,\tilde{f}_2}$ can be removed from (5.51) by a choice of normalization $N_{\tilde{f}}$ if there exists a normalization $N_{\tilde{f}}$ such that $K_{\tilde{f}_1,\tilde{f}_2} = N_{[\tilde{f}_1,\tilde{f}_2]_M}$, i.e., if the 2 cocycle K is a coboundary, $K = \delta^{CE}N$.

- The Lie algebra of physical interest is not the Lie algebra of all asymptotic reducibility parameters, but the Lie algebra $\mathfrak{g}$ of equivalence classes of asymptotic reducibility parameters. Hence, nontrivial central charges are to be viewed as (representatives of) cohomology classes in degree 2 of the Lie algebra $\mathfrak{g}$. This can be done consistently if conditions (5.25) and (5.56) are satisfied, provided trivial asymptotic reducibility parameters that satisfy (5.56) automatically also satisfy (5.57). Then the finite charges $\int_{\mathcal{C}^{n-2}} k_{\tilde{f}_2}[R^0_{\tilde{f}_1}]$ vanish whenever $\tilde{f}_1$ are trivial asymptotic reducibility parameters, so that $K_{\tilde{f}_1,\tilde{f}_2}$ really only depends on the equivalence classes $[\tilde{f}_1]$, $[\tilde{f}_2]$. Similarly, because the boundary conditions guarantee that the charges $\int k_{\tilde{f}}[\phi - \bar{\phi}(x); \bar{\phi}(x)]$ vanish (asymptotically) for trivial asymptotic reducibility parameters (when evaluated at a solution satisfying the boundary condition), the charge $Q_{\tilde{f}}$ only depends on the equivalence class $[\tilde{f}]$ of the asymptotic reducibility parameters.

- Two particular important cases where the central charges are necessarily trivial and can be absorbed by an appropriate choice of normalization are the case that the Lie algebra cohomology of equivalence classes of asymptotic reducibility parameters in degree 2 is trivial ("safe algebras", e.g., semi-simple finite dimensional algebras), and the case that all asymptotic reducibility parameters are equivalent to exact Killing vectors of the background, because $R^{i0}_{\tilde{f}} = 0$ implies $K_{\tilde{f}_1,\tilde{f}_2} = 0$. In the latter case, an appropriate choice is to normalize the charges of the backgound to zero, $N_f = 0$, whereas for a semi-simple $\mathfrak{g}$, the existence of an appropriate normalization follows from $H^2(\mathfrak{g}) = 0$. Furthermore, because $H^1(\mathfrak{g}) = 0$, the requirement that there should be no central extension then completely fixes the normalization of the background.

5.3.7 Effective sources

Usually, the charges (5.49) are integrals over boundaries, i.e., $\mathcal{C}^{n-2} = \partial\Sigma$ is the boundary of an $(n-1)$-dimensional region Σ of spacetime. One may then try to identify source terms in Σ and represent the charges as $(n-1)$-dimensional integrals over Σ of the source terms. For example, one may define "source currents"

$$j^\mu_{\rm eff} := S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i},\tilde{f}^\alpha) - \partial_\nu \tilde{k}^{[\nu\mu]}_{\tilde{f}}[\phi-\bar{\phi}(x);\bar{\phi}(x)]. \tag{5.58}$$

This implies

$$\begin{aligned} Q_{\tilde{f}} - N_{\tilde{f}} &\overset{(5.49)}{=} \int_{\partial\Sigma}\tilde{k}_{\tilde{f}}[\phi-\bar{\phi}(x);\bar{\phi}(x)] \\ &\overset{\text{Stokes}}{=} \int_\Sigma d\tilde{k}_{\tilde{f}}[\phi-\bar{\phi}(x);\bar{\phi}(x)] \overset{(5.58)}{\approx} \int_\Sigma (d^{n-1}x)_\mu\, j^\mu_{\rm eff}\ , \end{aligned} \tag{5.59}$$

where $\approx$ denotes weak equality in the full theory. That is, one has $Q_{\tilde{f}} = N_{\tilde{f}} + \int_\Sigma (d^{n-1}x)_\mu\, j^\mu_{\rm eff}$ for solutions of the field equations satisfying the respective boundary conditions. By construction, the currents $j^\mu_{\rm eff}$ thus yield the same value for the charges upon integration and they are conserved,

$$\partial_\mu j^\mu_{\rm eff} \approx -\partial_\mu\partial_\nu\tilde{k}^{[\nu\mu]}_{\tilde{f}}[\phi-\bar{\phi}(x);\bar{\phi}(x)] = 0.$$

The motivation for the definition (5.58) is that the $j^\mu_{\rm eff}$ contain the terms in the field equations that depend on external sources (if any), or, as in [18], terms that are at least quadratic in the fields $\varphi = \phi - \bar{\phi}$.

The difference between our approach here and the one in [18] is that we concentrate first on the $n-2$ forms and then consider the effective sources as derived quantities, instead of the other way around. The advantage is that the procedure becomes constructive and ambiguities or equivalences for various expressions of the charges can be controlled.

5.4 Remarks on the boundary theory

Suppose for definiteness that, in addition to the assumptions of section 5.1, we are in the situation where we have coordinates r, s^a, (with s^a denoting for instance coordinates such as time or some angles) and the boundary is at $r \longrightarrow \infty$ with boundary conditions

$$\varphi^i(x) = r^{m^i}\tilde{\varphi}^i(s) + o(r^{m^i}), \tag{5.60}$$

i.e., $|\chi^i| = m^i$. This means that, when evaluated at fields that satisfy the boundary conditions, a linear local form is to leading order a form that lives on the jet-bundle with base space coordinates s^a and fiber coordinates $\tilde{\varphi}^i$ and their derivatives with respect to s^a with a parametrical dependence on r.

If we define

$$\frac{\delta L^{\text{free}}}{\delta\varphi^i}\Big|_{\varphi(x)} d^n x = L^{as}_i + o(\chi_i), \tag{5.61}$$

so that $|L^{as}_i| = |\chi_i|$, the boundary theory that controls the leading order contributions of asymptotic solutions of the bulk theory can be defined to be the linear theory for the fields $\tilde{\varphi}^i$ with dynamics determined by the equations $\partial_{(a)} L^{as}_i = 0$ [a priori, it is not guaranteed that the equations $L^{as}_i = 0$ derive from a variational principle].

Asymptotic solutions are determined by exact solutions $\tilde{\varphi}^i(s)$ of the boundary theory,

$$L^{as}_i\Big|_{\tilde{\varphi}^i(s)} = 0. \tag{5.62}$$

Denoting $m_i := |\chi_i|$ and $m_\alpha := |\chi_\alpha|$, we have $\psi_i = r^{m_i}\tilde{\psi}_i(s) + o(r^{m_i})$ and $\psi_\alpha = r^{m_\alpha}\tilde{\psi}_\alpha(s) + o(r^{m_\alpha})$. One can decompose the generating set of Noether operators of the linearized theory according to

$$R^{+i0}_\alpha \psi_i = r^{m_\alpha}\tilde{R}^{+i0}_\alpha \tilde{\psi}_i(s) + o(r^{m_\alpha}) \tag{5.63}$$

with $\tilde{R}^{+i0}_\alpha = \tilde{R}^{+i0(a)}_\alpha(s)\partial_{(a)}$. Generically $\{\tilde{R}^{+i0}_\alpha\}$ will be a generating set of Noether operators of the boundary theory and the asymptotic regularity conditions of section 5.1 will imply standard regularity conditions for the boundary theory, at least when the latter can be traced to identities involving only field independent operators (as one would expect for a linear theory). Indeed, suppose that $\tilde{N}^i = r^{M-m_i}\tilde{N}^{i(a)}(s)\partial_{(a)}$ is a field independent Noether operator of the boundary theory, $\tilde{N}^i L^{as}_i = 0$, for some M. Defining $N^i := r^{-n'-M}\tilde{N}^i$ where $n' = |d^n x|$, we obtain $d^n x N^i \delta L^{\text{free}}/\delta\varphi^i \longrightarrow 0$. (5.10) implies now $r^{m_i}\tilde{N}^i\tilde{\psi}_i = r^M \tilde{Z}^{+\alpha}\tilde{R}^{+i0}_\alpha\tilde{\psi}_i$ for some operators $\tilde{Z}^\alpha = \tilde{Z}^{\alpha(a)}(s)\partial_{(a)}$ and all $\tilde{\psi}_i$, i.e., $\tilde{N}^i = r^{M-m_i}\tilde{Z}^{+\alpha}\tilde{R}^{+i0}_\alpha$.

Suppose the functions $\tilde{f}^\alpha$ are asymptotic reducibility parameters that satisfy condition (5.25) for finite charges, i.e.,

$$\tilde{f}^\alpha = \tilde{f}^\alpha_m + o(r^{-m_\alpha}), \quad \tilde{f}^\alpha_m = r^{-m_\alpha}h^\alpha(s). \tag{5.64}$$

The leading order of the definition (5.14) of asymptotic reducibility parameters then requires $\tilde{f}^\alpha_m$ to be exact reducibility parameters for the operators $\tilde{R}^{i0}_\alpha$ associated to the boundary theory,

$$\tilde{R}^{i0}_\alpha(\tilde{f}^\alpha_m) = 0. \tag{5.65}$$

If (5.29)-(5.34) hold we obtain from (4.6):

$$R^{i1}_\alpha(\tilde{f}^\alpha)\,\frac{\delta L^{\text{free}}}{\delta\varphi^i}\,d^n x \longrightarrow d_H(\cdot), \quad \Longrightarrow \quad \varphi^i\,\frac{\delta(\delta^1_{\tilde{f}} L^{\text{free}})}{\delta\varphi^i}\,d^n x \longrightarrow 0. \tag{5.66}$$

Commuting the Euler-Lagrange derivative with the vector field $\delta^1_{\tilde{f}} = \partial_{(\mu)}[R^{i1}_\alpha(\tilde{f}^\alpha)]\partial/\partial\varphi^i_{(\mu)}$ then implies

$$\varphi^j\Big[\delta^1_{\tilde{f}}\,\frac{\delta L^{\text{free}}}{\delta\varphi^j} + (-\partial)_{(\mu)}\Big[\frac{\partial R^{i1}_\alpha(\tilde{f}^\alpha)}{\partial\varphi^j_{(\mu)}}\,\frac{\delta L^{\text{free}}}{\delta\varphi^i}\Big]\Big]\,d^n x \longrightarrow 0. \tag{5.67}$$

Assuming that (5.34) holds, we may write

$$R^{i1}_\alpha(\tilde{f}^\alpha) = \tilde{R}^{i1}_\alpha(\tilde{f}^\alpha_m) + o(\chi^i), \tag{5.68}$$

with $|\tilde{R}^{i1}_\alpha(\tilde{f}^\alpha_m)| = |\chi^i| = m^i$.

By choosing $\varphi^i = r^{m^i}\tilde{\varphi}^i(s)$ and considering only the leading order in (5.67), we obtain

$$\tilde{\delta}^1_{\tilde{f}} L^{as}_j \approx^{\text{bd}} 0, \tag{5.69}$$

where $\approx^{\text{bd}}$ means equality when the equations of motions of the boundary theory hold, and

$$\tilde{\delta}^1_{\tilde{f}} = \partial_{(a)}[\tilde{R}^{i1}_\alpha(\tilde{f}^\alpha_m)]\frac{\partial}{\partial\tilde{\varphi}^i_{(a)}}. \tag{5.70}$$

Hence, $\tilde{\delta}^1_{\tilde{f}}$ defines a symmetry of the equations of motions of the boundary theory.

For field independent gauge parameters, the contribution linear in the fields in the expansion of equation (1.22) gives

$$\partial_{(\mu)}R^{j1}_\alpha(\tilde{f}^\alpha_1)\frac{\partial R^{i1}_\beta(\tilde{f}^\beta_2)}{\partial\varphi^j_{(\mu)}} + \partial_{(\mu)}R^{j0}_\alpha(\tilde{f}^\alpha_1)\frac{\partial R^{i2}_\beta(\tilde{f}^\beta_2)}{\partial\varphi^j_{(\mu)}} - (1 \longleftrightarrow 2) \approx^{\text{free}} R^{i1}_\gamma(C^{\gamma 0}_{\alpha\beta}(\tilde{f}^\alpha_1, \tilde{f}^\beta_2). \tag{5.71}$$

Under the assumptions (5.34), (5.41) and (5.56), the leading order contribution to this equation gives

$$\tilde{\delta}^1_{\tilde{f}_1}\tilde{R}^{i1}_\alpha(\tilde{f}^\alpha_{2m}) - (1 \longrightarrow 2) \approx^{\text{bd}} \tilde{R}^{i1}_\alpha([\tilde{f}_{1m}, \tilde{f}_{2m}]_{\text{bd}}), \tag{5.72}$$

while

$$[\tilde{f}_1, \tilde{f}_2]^\alpha_M = [\tilde{f}_{1m}, \tilde{f}_{2m}]^\alpha_{\text{bd}} + o(1/\chi_\alpha), \tag{5.73}$$

with $|[\tilde{f}_{1m}, \tilde{f}_{2m}]^\alpha| = |1/\chi_\alpha| = -m_\alpha$. Hence, on-shell for the boundary theory, the commutator algebra of the equations of motion symmetries $\tilde{\delta}^1_{\tilde{f}_m}$ represents the Lie algebra $\mathfrak{g}$ of equivalence classes of asymptotic reducibility parameters:

$$[\tilde{\delta}^1_{\tilde{f}_{1m}}, \tilde{\delta}^1_{\tilde{f}_{2m}}] \approx^{\text{bd}} \tilde{\delta}^1_{[\tilde{f}_{1m}, \tilde{f}_{2m}]_{\text{bd}}}. \tag{5.74}$$

6 Standard applications

In this section, we illustrate and test the general results by applying them to the well studied cases of electrodynamics, Yang-Mills theory and Einstein gravity. We shall specify in each case the superpotential of Eq. (1.13) and related quantities, such as asymptotic reducibility parameters and conserved charges. In all cases treated here, the boundary conditions are imposed at the boundary $\partial\Sigma$ of a spatial $(n-1)$-dimensional volume Σ (not necessarily at spatial infinity). For simplicity we shall assume that all "matter fields" fall off sufficiently fast to be negligible near $\partial\Sigma$ and that external sources vanish there. Accordingly, all background matter fields vanish and the background gauge or metric fields solve the source-free Maxwell, Yang-Mills and Einstein equations, respectively, possibly with a cosmological constant in the gravitational case. Furthermore, we shall mostly discuss the particular case of asymptotic reducibility parameters that are exact Killing vectors of the background ($R^{i0}_\alpha(\tilde{f}^\alpha) = 0$) because for these parameters the discussion can be made without more specific assumptions on the boundary conditions. The only exception where we consider precise boundary conditions and determine all the associated asymptotic reducibility parameters is the well-known example of three-dimensional asymptotically anti-de Sitter gravity. The reason is of course that this example gives rise to central extensions in the algebra of conserved charges, and thus provides a particularly nontrivial illustration of our general framework.

6.1 Electrodynamics

As a warm-up, we briefly discuss electrodynamics with Lagrangian $L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + L_{\text{matter}}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the electromagnetic field strengths and L_{matter} is a "matter field Lagrangian" of the standard type (such as Dirac spinor fields minimally coupled to the gauge fields via covariant derivatives), or contains terms with external sources (such as $A_\mu j^\mu(x)$, with $\partial_\mu j^\mu(x) = 0$). For the standard cases that L_{matter} contains only terms that are at least quadratic in the matter fields and that gauge transformations of the matter fields do not contain derivatives of the gauge parameter, the current in Eq. (1.11) is $s^\mu_{\tilde{f}} = \tilde{f}\partial_\nu f^{\nu\mu}$. Here $\tilde{f}$ is an asymptotic reducibility parameter and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength of $a_\mu = A_\mu - \bar{A}_\mu(x)$, with $\bar{A}_\mu(x)$ the background gauge fields. The only asymptotic reducibility parameters that are exact Killing vectors of the background are constants, $\tilde{f} = c = constant$. They yield $s^\mu_c = \partial_\nu(cf^{\nu\mu})$ and the corresponding superpotential (1.13) is simply $\tilde{k}^{[\nu\mu]}_c = cf^{\nu\mu}$. An asymptotically conserved $(n-2)$-form is thus $c(d^{n-2}x)_{\nu\mu}(F^{\nu\mu} - \bar{F}^{\nu\mu})$. Owing to $d_V(F^{\nu\mu} - \bar{F}^{\nu\mu}) = d_V F^{\nu\mu}$, a simpler (equivalent) choice is the $(n-2)$-form

$$k_c[A] = c\,(d^{n-2}x)_{\nu\mu}F^{\nu\mu}.$$

By integrating $k_c[A]$ over $\partial\Sigma$, one gets the corresponding conserved charge. Notice that, actually, there is a one-parameter family of conserved charges Q_c parametrized by c. The charge is the "generator" $Q := \partial Q_c/\partial c$ of this family,[5]

$$Q = \int_{\partial\Sigma} d\sigma_i\, F^{0i}.$$

(5.58) gives here $j^\mu_{\text{eff}} = \delta L_{\text{matter}}/\delta A_\mu$, and (5.59) then implies

$$Q = -\int_{\partial\Sigma} d\sigma_i\, F^{i0} = -\int_\Sigma d\sigma\, \partial_i F^{i0} \approx \int_\Sigma d\sigma\, j^0.$$

[5]Here and in the following we use the notation $d\sigma_i \equiv 2(d^{n-2}x)_{0i}$

Hence, when evaluated for solutions to the equations of motion, Q agrees with $\int_\Sigma d\sigma j^0$ where $j^0 = \delta L_{\rm matter}/\delta A_0$ is the charge density appearing in the Maxwell equation $\partial_i F^{i0} = -j^0$, so that the standard textbook expression for the electric charge is recovered.

6.2 Yang-Mills theory

We consider a Lagrangian

$$L = \frac{1}{4}\,Tr(F^{\mu\nu}F_{\mu\nu}) + L_{\rm matter}, \tag{6.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ are the nonabelian field strengths of the gauge fields $A_\mu = A^a_\mu T_a$. We use here matrix notation and the conventions that T_a are antihermitian representation matrices normalized according to $Tr(T_aT_b) = -\delta_{ab}$. Analogously to electrodynamics discussed before, $L_{\rm matter}$ may contain matter fields or external sources coupled to the gauge fields. Again, we assume that all matter fields are negligible and all external sources vanish near $\partial\Sigma$. In particular, all background matter fields vanish.

6.2.1 Superpotentials

Assuming a standard Lagrangian which contains only terms that are at least quadratic in the matter fields, one obtains

$$\frac{\delta L^{\rm free}}{\delta a^a_\mu}\,\delta^{ab}T_b = \bar D_\nu f^{\nu\mu} + [a_\nu, \bar F^{\nu\mu}], \tag{6.2}$$

where

$$a_\mu = A_\mu - \bar A_\mu, \quad f_{\mu\nu} = \bar D_\mu a_\nu - \bar D_\nu a_\mu, \quad \bar D_\mu\,\cdot\, = \partial_\mu + [\bar A_\mu, \cdot\,].$$

The currents $s^\mu_{\tilde f}$ of Eq. (1.11) are

$$s^\mu_{\tilde f}[a;\bar A] = -\,Tr(\tilde f \bar D_\nu f^{\nu\mu} + \tilde f [a_\nu, \bar F^{\nu\mu}]), \tag{6.3}$$

where $\tilde f = \tilde f^a(x) T_a$ involves the asymptotic reducibility parameters $\tilde f^a(x)$. The latter are subject to (5.14) which requires in this case

$$\forall a^a_\mu \longrightarrow O(\chi^a_\mu): \quad d^nx\; Tr(\bar D_\nu f^{\nu\mu}\bar D_\mu \tilde f + [a_\nu, \bar F^{\nu\mu}]\bar D_\mu \tilde f) \longrightarrow 0, \tag{6.4}$$

where χ^a_μ characterizes the boundary condition for a^a_μ. According to (1.13), the associated superpotentials are given by

$$\tilde k^{[\mu\nu]}_{\tilde f}[a;\bar A] = -\,Tr\Big(\frac{3}{2}[\bar A^\mu, a^\nu]\tilde f + \frac{1}{2}a^\mu\partial^\nu\tilde f + \tilde f\partial^\mu a^\nu - (\mu \leftrightarrow \nu)\Big). \tag{6.5}$$

Let us now discuss asymptotic reducibility parameters that are exact Killing vectors of the background,

$$\bar D_\mu \tilde f = 0. \tag{6.6}$$

For parameters satisfying (6.6), we can simplify (6.5) by substituting $-[\bar{A}^\nu, \tilde{f}]$ for $\partial^\nu \tilde{f}$. This yields

$$\tilde{k}_{\tilde{f}}^{[\mu\nu]}[a;\bar{A}] \stackrel{(6.6)}{=} -Tr(\tilde{f} f^{\mu\nu}), \tag{6.7}$$

which agrees with equation (5) of [19]. Equivalently, we can use (6.6) to substitute $-\partial^\nu \tilde{f}$ for $[\bar{A}^\nu, \tilde{f}]$ in (6.5). Then we obtain, using once again (6.6),

$$\tilde{k}_{\tilde{f}}^{[\mu\nu]}[a;\bar{A}] \stackrel{(6.6)}{=} \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu, \quad \mathcal{A}^\mu = -Tr(\tilde{f} a^\mu). \tag{6.8}$$

Remark. Actually (6.5) is not restricted to the case that matter fields can be neglected near $\partial\Sigma$. Rather, it even holds for solutions with possibly non-negligible matter fields near $\partial\Sigma$, assuming a Yang-Mills-matter Lagrangian of the standard type (with matter fields that are fermions or scalar fields). The reason is that, for a standard Yang-Mills-matter system, the current $s^\mu_{\tilde{f}}$ involves only the linearized field equations for the gauge fields but not those for the matter fields because the gauge transformations of standard matter fields do not involve derivatives of the gauge parameters. The matter field dependent terms in $s^\mu_{\tilde{f}}$ then either do not contain derivatives at all (in the case of fermions) or they contain precisely one derivative whose index coincides with the index μ of $s^\mu_{\tilde{f}}$ (in the case of scalar fields). As a consequence, they give no contributions to $\tilde{k}_{\tilde{f}}^{[\mu\nu]}$ at all, as one easily reads off from (1.13). The only possible effect that the matter fields may then have are extra conditions on the parameters $\tilde{f}$, but (6.5) does not change.

6.2.2 Asymptotically conserved n–2 forms

Equation (6.5) yields directly asymptotically conserved $(n-2)$-forms given by

$$k_{\tilde{f}}[A;\bar{A}(x)] = (d^{n-2}x)_{\mu\nu} \tilde{k}_{\tilde{f}}^{[\mu\nu]}[A-\bar{A};\bar{A}]. \tag{6.9}$$

When (6.6) holds, there are somewhat more elegant, equivalent expressions for $k_{\tilde{f}}$ which do not explicitly depend on the background fields. The first one corresponds to (6.8) and reads

$$k'_{\tilde{f}}[A] \stackrel{(6.6)}{=} -2(d^{n-2}x)_{\mu\nu} \partial^\mu\, Tr(\tilde{f} A^\nu). \tag{6.10}$$

Another one corresponds to (6.7) and reads

$$k''_{\tilde{f}}[A] \stackrel{(6.6)}{=} -(d^{n-2}x)_{\mu\nu}\, Tr(\tilde{f} F^{\mu\nu}). \tag{6.11}$$

The equivalence of all these expressions is due to

$$\Big[d_V k''_{\tilde{f}}[A]\Big]_{\bar{A}(x)} \stackrel{(6.6)}{=} d_V k_{\tilde{f}}[A;\bar{A}(x)] \stackrel{(6.6)}{=} d_V k'_{\tilde{f}}[A].$$

6.2.3 Example: asymptotically flat connections

Let us finally consider asymptotically flat connections as in [19], using a background $\bar{A}_\mu = g^{-1}(x)\partial_\mu g(x)$. The exact Killing vectors of such a background are easily found: multiplying (6.6) from the left with $g(x)$ and from the right with $g^{-1}(x)$ gives $\partial_\mu[g(x)\tilde{f}g^{-1}(x)] = 0$ and thus $g(x)\tilde{f}g^{-1}(x) = c^a T_a$ with constant parameters c^a. Hence, let us consider

$$\tilde{f} = c^a g^{-1}(x) T_a g(x).$$

We define corresponding "color" charges by $Q_a := \partial Q_{\tilde{f}}/\partial c^a$ where $Q_{\tilde{f}}$ is the integral over $\partial\Sigma$ of the asymptotically conserved $(n-2)$-forms. Using (6.11), one obtains

$$Q_a = -\int_{\partial\Sigma} d\sigma_i \; Tr\Big[g^{-1}(x)T_a g(x) F^{0i}\Big].$$

Because we have considered exact Killing vectors, there is no central extension in the covariant Poisson algebra of the corresponding color charges:

$$\{Q_a, Q_b\}_{CF} = f_{ab}{}^c Q_c\ .$$

This can be easily verified using $\delta_{\tilde{f}} F^{0i} = [F^{0i}, \tilde{f}]$. Here $f_{ab}{}^c$ are the structure constants of the Lie algebra of the gauge group in the basis associated to $\{T_a\}$, $[T_a, T_b] = f_{ab}{}^c T_c$.

Of course, whether or not these charges vanish depends on the behaviour of the gauge fields near the boundary. For instance, the BPST instanton solution [50] of Yang-Mills theory in four-dimensional Euclidean space satisfies $A_\mu \longrightarrow g^{-1}(x)\partial_\mu g(x)$ at infinity but yields only vanishing charges Q_a because the field strengths fall off too fast. The same is actually true for all Yang-Mills instantons and related to the fact that they have finite action as this requires that the field strengths fall off faster than $1/r^2$.

6.3 Einstein gravity

We finally discuss Einstein gravity (without or with cosmological constant Λ) in spacetime dimensions $n \geq 3$ with Lagrangian

$$L = \frac{1}{16\pi}\sqrt{-g}(R - 2\Lambda) + L_{\text{matter}},$$

where $R = g^{\mu\nu} R_{\rho\mu\nu}{}^\rho$, $R_{\rho\mu\nu}{}^\lambda = \partial_\rho \Gamma_{\mu\nu}{}^\lambda + \Gamma_{\rho\sigma}{}^\lambda \Gamma_{\mu\nu}{}^\sigma - (\rho \leftrightarrow \mu)$. As before in the cases of electrodynamics and Yang-Mills theory, L_{matter} may contain matter fields which are assumed to be negligible near $\partial\Sigma$, or external sources which vanish near $\partial\Sigma$. We introduce the standard notation $h_{\mu\nu} = h_{\nu\mu}$ for the deviation of the metric fields from the background metric $\bar{g}_{\mu\nu}(x)$ ($g_{\mu\nu} = h_{\mu\nu} + \bar{g}_{\mu\nu}(x)$). The background metric and its inverse are used to lower and raise world indices. In particular we thus use the notation $h_\mu{}^\nu = \bar{g}^{\nu\rho} h_{\mu\rho}$ and $h^{\mu\nu} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} h_{\rho\sigma}$. Furthermore, h denotes the trace of $h_\mu{}^\nu$, i.e., $h = \bar{g}^{\mu\nu} h_{\mu\nu}$, and

$\bar{D}_\mu$ are background covariant derivatives, such as $\bar{D}_\mu h^{\nu\rho} = \partial_\mu h^{\nu\rho} + \bar{\Gamma}_{\mu\lambda}{}^{\nu} h^{\lambda\rho} + \bar{\Gamma}_{\mu\lambda}{}^{\rho} h^{\nu\lambda}$, $\bar{D}^\mu h = \partial^\mu h = \bar{g}^{\mu\nu}\partial_\nu h$. We write the (full) equations of motion for the metric as

$$\mathcal{H}^{\mu\nu} + \frac{\sqrt{-\bar{g}}}{2} T^{\mu\nu}_{\text{eff}} = 0, \tag{6.12}$$

where $\mathcal{H}^{\mu\nu}[h;\bar{g}]$ is the linear part of $\delta[(1/16\pi)\sqrt{-g}(R-2\Lambda)]/\delta g_{\mu\nu}$,

$$\begin{aligned}
\mathcal{H}^{\mu\nu}[h;\bar{g}] &:= -\frac{1}{16\pi}\Big[d_V(\sqrt{-g}R^{\mu\nu} - \frac{1}{2}\sqrt{-g}g^{\mu\nu}R + \sqrt{-g}g^{\mu\nu}\Lambda)\Big]_{\bar{g},h} \\
&= \frac{\sqrt{-\bar{g}}}{32\pi}\Big(- h\bar{R}^{\mu\nu} + \frac{1}{2}h\bar{R}\bar{g}^{\mu\nu} + 2h^{\mu\alpha}\bar{R}_\alpha{}^\nu + 2h^{\nu\beta}\bar{R}_\beta{}^\mu - h^{\mu\nu}\bar{R} - h^{\alpha\beta}\bar{R}_{\alpha\beta}\bar{g}^{\mu\nu} \\
&\quad + \bar{D}^\mu\bar{D}^\nu h + \bar{D}^\lambda\bar{D}_\lambda h^{\mu\nu} - 2\bar{D}_\lambda\bar{D}^{(\mu}h^{\nu)\lambda} - \bar{g}^{\mu\nu}(\bar{D}^\lambda\bar{D}_\lambda h - \bar{D}_\lambda\bar{D}_\rho h^{\rho\lambda}) \\
&\quad + 2\Lambda h^{\mu\nu} - \Lambda\bar{g}^{\mu\nu}h\Big),
\end{aligned} \tag{6.13}$$

and $(1/2)\sqrt{-\bar{g}}T^{\mu\nu}_{\text{eff}}$ simply collects all terms of $\delta L/\delta g_{\mu\nu}$ not contained in $\mathcal{H}^{\mu\nu}$,

$$T^{\mu\nu}_{\text{eff}} := \frac{2}{\sqrt{-\bar{g}}}\Big(\frac{\delta L}{\delta g_{\mu\nu}} - \mathcal{H}^{\mu\nu}\Big).$$

We also note that (6.13) reduces to the following expression when $\bar{g}_{\mu\nu}$ solves the Einstein equations $\bar{R}_{\mu\nu} = 2(n-2)^{-1}\Lambda\bar{g}_{\mu\nu}$:

$$\begin{aligned}
\mathcal{H}^{\mu\nu}[h;\bar{g}] = \frac{\sqrt{-\bar{g}}}{32\pi}\Big(&\frac{2\Lambda}{n-2}(2h^{\mu\nu} - \bar{g}^{\mu\nu}h) + \bar{D}^\mu\bar{D}^\nu h + \bar{D}^\lambda\bar{D}_\lambda h^{\mu\nu} \\
&- 2\bar{D}_\lambda\bar{D}^{(\mu}h^{\nu)\lambda} - \bar{g}^{\mu\nu}(\bar{D}^\lambda\bar{D}_\lambda h - \bar{D}_\lambda\bar{D}_\rho h^{\rho\lambda})\Big).
\end{aligned} \tag{6.14}$$

6.3.1 Superpotentials and asymptotically conserved n–2 forms

The assumption that the matter fields are negligible near the boundary means more precisely that near the boundary they give no contribution to the $(n-1)$-form constructed of the current (1.11). Then this $(n-1)$-form is

$$(d^{n-1}x)_\mu\, s^\mu_\xi[h;\bar{g}] \longrightarrow 2(d^{n-1}x)_\mu\mathcal{H}^{\mu\nu}\xi_\nu\ , \tag{6.15}$$

where the use of $\longrightarrow$ (instead of $=$) indicates that terms with matter fields (if any) have been neglected [in general, s^μ_ξ may contain terms with the linearized equations of motion of matter fields, as the gauge transformations of matter fields may contain derivatives of the gauge parameters]. $\xi_\nu = \xi_\nu(x)$ are asymptotic reducibility parameters satisfying (5.14) which requires in this case

$$\forall h_{\mu\nu} \longrightarrow O(\chi_{\mu\nu}): \quad d^nx\,\mathcal{H}^{\mu\nu}\bar{D}_\mu\xi_\nu \longrightarrow 0, \tag{6.16}$$

where $\chi_{\mu\nu}$ characterizes the boundary condition for $h_{\mu\nu}$. Applying (1.13) to $2\mathcal{H}^{\mu\nu}\xi_\nu$, we find for the superpotential $\tilde{k}^{[\nu\mu]}_\xi$:

$$\begin{aligned}
\tilde{k}^{[\nu\mu]}_\xi[h;\bar{g}] \;=\; -\frac{\sqrt{-\bar{g}}}{16\pi}\Big[&\bar{D}^\nu(h\xi^\mu) + \bar{D}_\sigma(h^{\mu\sigma}\xi^\nu) + \bar{D}^\mu(h^{\nu\sigma}\xi_\sigma) \\
&+ \frac{3}{2}h\bar{D}^\mu\xi^\nu + \frac{3}{2}h^{\sigma\mu}\bar{D}^\nu\xi_\sigma + \frac{3}{2}h^{\nu\sigma}\bar{D}_\sigma\xi^\mu - (\mu\leftrightarrow\nu)\Big].
\end{aligned} \tag{6.17}$$

This expression can be more compactly written as

$$\tilde{k}_\xi^{[\nu\mu]}[h;\bar{g}] = \frac{\sqrt{-\bar{g}}}{16\pi}\Big(\xi_\rho \bar{D}_\sigma H^{\rho\sigma\nu\mu} + \frac{1}{2}H^{\rho\sigma\nu\mu}\partial_\rho\xi_\sigma\Big), \tag{6.18}$$

where $H^{\rho\sigma\mu\nu}[h;\bar{g}]$ is the following background tensor with the symmetries of the Riemann tensor:

$$H^{\mu\alpha\nu\beta}[h;\bar{g}] = -\hat{h}^{\alpha\beta}\bar{g}^{\mu\nu} - \hat{h}^{\mu\nu}\bar{g}^{\alpha\beta} + \hat{h}^{\alpha\nu}\bar{g}^{\mu\beta} + \hat{h}^{\mu\beta}\bar{g}^{\alpha\nu}, \tag{6.19}$$

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h. \tag{6.20}$$

The first term on the right hand side of (6.18) just collects all terms in (6.17) with background covariant derivatives of $h_{\mu\nu}$, the second one the terms with background covariant derivatives of ξ (we used that $\bar{D}_{[\mu}\xi_{\nu]} = \partial_{[\mu}\xi_{\nu]}$). Equation (6.18) generalizes superpotentials that are familiar in the particular case of asymptotically flat spacetimes to more general asymptotics (asymptotically flat spacetimes will be briefly discussed in the next subsection). It had been originally obtained for exact Killing vectors of the background metric in [18], equation (2.17).

Note that neither the terms in $\mathcal{H}^{\mu\nu}$ with the background curvatures nor those with the cosmological constant contribute to $\tilde{k}_\xi^{[\nu\mu]}$ because they do not contain derivatives of $h_{\mu\nu}$. Hence, a cosmological constant affects the superpotential only indirectly via its influence on the background.

From (6.17) or (6.18) one obtains the asymptotically conserved $(n-2)$-form associated to the asymptotic reducibility parameters $\xi_\mu(x)$:

$$k_\xi[g;\bar{g}] = (d^{n-2}x)_{\nu\mu}\,\tilde{k}_\xi^{[\nu\mu]}[g-\bar{g};\bar{g}]. \tag{6.21}$$

Suppose now that the ξ_μ are exact Killing vectors of the background metric,

$$\bar{D}_\mu\xi_\nu + \bar{D}_\nu\xi_\mu = 0. \tag{6.22}$$

In this case, (6.21) reduces to the $(n-2)$-form given in equation (11) of [34]:

$$k_\xi[h;\bar{g}] \overset{(6.22)}{=} \frac{\sqrt{-\bar{g}}}{16\pi}(d^{n-2}x)_{\nu\mu}\Big[\xi^\nu\bar{D}^\mu h - \xi^\nu\bar{D}_\sigma h^{\mu\sigma} + \xi_\sigma\bar{D}^\nu h^{\mu\sigma} + \frac{1}{2}h\bar{D}^\nu\xi^\mu - h^{\nu\sigma}\bar{D}_\sigma\xi^\mu - (\mu\leftrightarrow\nu)\Big]. \tag{6.23}$$

This is so because (6.21) is given by (6.23) plus the term

$$\frac{\sqrt{-\bar{g}}}{16\pi}(d^{n-2}x)_{\nu\mu}\,h^{\rho\nu}(\bar{D}^\mu\xi_\rho + \bar{D}_\rho\xi^\mu). \tag{6.24}$$

More generally, one may use the somewhat simpler $(n-2)$-form (6.23) instead of (6.21) whenever (6.24) vanishes asymptotically.

6.3.2 Central charges

According to section 5.3.6, the central extensions $K_{\xi',\xi}$ which can occur in the algebra of the gravitational conserved charges arise from (6.21) by substituting there $\bar{D}_\mu\xi'_\nu + \bar{D}_\nu\xi'_\mu$ for $h_{\mu\nu}$ and then integrating over $\partial\Sigma$. Performing this substitution in (6.17) we obtain

$$\begin{aligned}
&\tilde{k}^{[\nu\mu]}_\xi[\bar{D}_\lambda\xi'_\rho + \bar{D}_\rho\xi'_\lambda;\bar{g}_{\lambda\rho}] \\
&= \frac{\sqrt{-\bar{g}}}{16\pi}\Big[\bar{D}_\rho(\xi^\nu\bar{D}^\mu\xi'^\rho + \xi^\mu\bar{D}^\rho\xi'^\nu + \xi^\rho\bar{D}^\nu\xi'^\mu) \\
&\quad -\bar{D}_\rho\xi^\rho\,\bar{D}^\nu\xi'^\mu + \bar{D}_\rho\xi'^\rho\,\bar{D}^\nu\xi^\mu + 2\bar{D}_\rho\xi^\nu\,\bar{D}^\rho\xi'^\mu \\
&\quad +\frac{1}{2}(\bar{D}^\rho\xi'^\nu + \bar{D}^\nu\xi'^\rho)(\bar{D}^\mu\xi_\rho + \bar{D}_\rho\xi^\mu) - 2\bar{R}^{\mu\rho}\xi^\nu\xi'_\rho + \bar{R}^{\mu\nu\rho\sigma}\xi_\rho\xi'_\sigma\Big] - (\mu\leftrightarrow\nu). \quad (6.25)
\end{aligned}$$

The terms in the first line on the right hand side do not contribute to $K_{\xi',\xi}$ because they only contribute an exact form to its integrand,

$$\begin{aligned}
(d^{n-2}x)_{\nu\mu}\sqrt{-\bar{g}}\,\bar{D}_\rho(\xi^{[\nu}\bar{D}^\mu\xi'^{\rho]}) &= (d^{n-2}x)_{\nu\mu}\partial_\rho(\sqrt{-\bar{g}}\,\xi^{[\nu}\bar{D}^\mu\xi'^{\rho]}) \\
&= d_H\Big[(d^{n-3}x)_{\nu\mu\rho}\sqrt{-\bar{g}}\,\xi^\nu\bar{D}^\mu\xi'^\rho\Big].
\end{aligned}$$

When the background metric satisfies the Einstein equations $\bar{R}_{\mu\nu} = 2(n-2)^{-1}\Lambda\bar{g}_{\mu\nu}$, we obtain for the gravitational central charges:

$$\begin{aligned}
K_{\xi',\xi} = \frac{1}{16\pi}\int_{\partial\Sigma}(d^{n-2}x)_{\nu\mu}\sqrt{-\bar{g}}\Big[&-2\bar{D}_\rho\xi^\rho\,\bar{D}^\nu\xi'^\mu + 2\bar{D}_\rho\xi'^\rho\,\bar{D}^\nu\xi^\mu \\
&+4\bar{D}_\rho\xi^\nu\,\bar{D}^\rho\xi'^\mu + (\bar{D}^\rho\xi'^\nu + \bar{D}^\nu\xi'^\rho)(\bar{D}^\mu\xi_\rho + \bar{D}_\rho\xi^\mu) \\
&+\frac{8\Lambda}{2-n}\,\xi^\nu\xi'^\mu + 2\bar{R}^{\mu\nu\rho\sigma}\xi_\rho\xi'_\sigma\Big]. \quad (6.26)
\end{aligned}$$

This expression is manifestly skew symmetric under exchange of ξ and ξ', owing to $(d^{n-2}x)_{\nu\mu} = -(d^{n-2}x)_{\mu\nu}$. Remember, however, that it is not guaranteed to be finite unless the charges $\int_{\partial\Sigma} k_\xi[h;\bar{g}]$ themselves are finite and $\bar{D}_\mu\xi_\nu + \bar{D}_\nu\xi_\mu = O(\chi_{\mu\nu}) = \bar{D}_\mu\xi'_\nu + \bar{D}_\nu\xi'_\mu$ holds.

6.3.3 Asymptotically flat spacetimes

We shall now briefly discuss the important case of asymptotically flat spacetimes (with $\Lambda = 0$). In particular we shall show that the superpotentials (6.17) or, equivalently, (6.18) reproduce standard expressions for conserved quantities in asymptotically flat spacetimes. We shall thus use as background metric the Minkowski metric $\eta_{\mu\nu}$, so that $h_{\mu\nu}$ is the deviation of the $g_{\mu\nu}$ from the Minkowski metric,

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}\ , \quad \bar{g}_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,+1,\dots,+1).$$

The Einstein equations in the form (6.12) read now

$$\partial_\rho\partial_\sigma H^{\mu\rho\nu\sigma} = 16\pi T^{\mu\nu}_{\mathrm{eff}}, \quad H^{\mu\rho\nu\sigma} = H^{\mu\rho\nu\sigma}[h;\eta], \quad (6.27)$$

with $H^{\mu\rho\nu\sigma}[h;\bar{g}]$ as in (6.19) [one has $32\pi\mathcal{H}^{\mu\nu}[h;\eta] = -\partial_\rho\partial_\sigma H^{\mu\rho\nu\sigma}$]. The exact isometries of the flat background are given by the Killing vector fields $\xi_\lambda = c_\lambda$ and $\xi_\lambda = x^\rho(c_{\rho\lambda}-c_{\lambda\rho})$ where c_λ and $c_{\lambda\rho}$ are constant parameters. For these ξ's, (6.18) reads, respectively,

$$\tilde{k}^{[\nu\mu]}_{\xi_\lambda=c_\lambda}[h;\eta] = \frac{1}{16\pi}\, c_\rho\partial_\sigma H^{\rho\sigma\nu\mu}\ , \tag{6.28}$$

$$\tilde{k}^{[\nu\mu]}_{\xi_\lambda=x^\rho(c_{\rho\lambda}-c_{\lambda\rho})}[h;\eta] = \frac{1}{16\pi}\, c_{\rho\sigma}[x^\rho\partial_\lambda H^{\sigma\lambda\nu\mu} - x^\sigma\partial_\lambda H^{\rho\lambda\nu\mu} + H^{\rho\sigma\nu\mu}]. \tag{6.29}$$

Analogously to the procedure in electrodynamics and Yang-Mills theory, we define the associated charges through derivatives with respect to the parameters c_μ and $c_{\mu\nu}$ of the integrated $(n-2)$-form (6.21). Denoting these charges by P^μ and $M^{\mu\nu}$, we obtain

$$P^\mu \ := \ \frac{\partial}{\partial c_\mu}\int_{\partial\Sigma} d\sigma_i\, \tilde{k}^{[0i]}_{\xi_\mu=c_\mu}[h;\eta] = \frac{1}{16\pi}\int_{\partial\Sigma} d\sigma_i\, \partial_\lambda H^{\mu\lambda 0i}, \tag{6.30}$$

$$\begin{aligned} M^{\mu\nu} \ &:= \ \frac{\partial}{\partial c_{\mu\nu}}\int_{\partial\Sigma} d\sigma_i\, \tilde{k}^{[0i]}_{\xi_\mu=x^\nu(c_{\nu\mu}-c_{\mu\nu})}[h;\eta] \\ &= \ \frac{1}{16\pi}\int_{\partial\Sigma} d\sigma_i\, [x^\mu\partial_\lambda H^{\nu\lambda 0i} - x^\nu\partial_\lambda H^{\mu\lambda 0i} + H^{\mu\nu 0i}]. \end{aligned} \tag{6.31}$$

Owing to $H^{\mu\nu 0i} = H^{\mu i0\nu} - H^{\nu i0\mu}$, these are precisely the expressions derived in chapter 20 of [51] (but note that they are not restricted to four dimensions). P^0 gives the ADM mass formula [52]

$$P^0 = \frac{1}{16\pi}\int_{\partial\Sigma} d\sigma_i\, \eta^{ik}\eta^{jl}(\partial_j g_{kl} - \partial_k g_{jl}). \tag{6.32}$$

Equation (6.18) can also be used to establish the relation to the Landau-Lifshitz expressions [53] for the total momentum and angular momentum in asymptotically flat spacetimes. Let us denote by $H^{\mu\rho\nu\sigma}_{\rm LL}$ the Landau-Lifshitz weight-2-tensor density,

$$H^{\mu\rho\nu\sigma}_{\rm LL} = -g(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\nu}g^{\mu\sigma}). \tag{6.33}$$

One has

$$\Big[d_V H^{\mu\rho\nu\sigma}_{\rm LL}\Big]_{\eta,h} = H^{\mu\rho\nu\sigma}.$$

This implies that

$$k'_\xi[g] = \frac{1}{16\pi}\,(d^{n-2}x)_{\nu\mu}\Big(\xi_\rho\partial_\sigma H^{\rho\sigma\nu\mu}_{\rm LL} + \frac{1}{2}H^{\rho\sigma\nu\mu}_{\rm LL}\partial_\rho\xi_\sigma\Big) \tag{6.34}$$

is equivalent to (6.21) in asymptotically flat spacetimes because of

$$\Big[d_V k'_\xi[g]\Big]_{\eta,h} = k_\xi[h;\eta].$$

Note, however, that (6.34) does in general not vanish when evaluated for $g_{\mu\nu} = \eta_{\mu\nu}$, in contrast to (6.21); rather, for $g_{\mu\nu} = \eta_{\mu\nu}$ it equals $(1/16\pi)(d^{n-2}x)_{\mu\nu}\partial^\mu\xi^\nu$. To obtain from

(6.34) equivalent asymptotically conserved $(n-2)$-forms that vanish for $g_{\mu\nu} = \eta_{\mu\nu}$, one may simply subtract $(1/16\pi)(d^{n-2}x)_{\mu\nu}\partial^\mu\xi^\nu$ from (6.34).

Integrated over $\partial\Sigma$, (6.34) reproduces the expressions for the total momentum and angular momentum in §96 of [53]. These expressions arise from (6.30) and (6.31) by substituting $H^{\mu\nu 0i}_{\rm LL}$ for $H^{\mu\nu 0i}$ everywhere in the integrands.

Analogously (6.21) yields the charges for the asymptotic isometries of flat spacetimes found in [54, 55, 46], when the parameters of these isometries satisfy equation (6.16).

6.3.4 Asymptotically 3d anti-de Sitter spacetimes with central charges

The formulas derived in sections 6.3.1 and 6.3.2 are valid in the presence of a non vanishing cosmological constant. That is why they can be used to rediscuss, from a covariant point of view, asymptotically anti-de Sitter spacetimes in 3-dimensional gravity. The original Hamiltonian analysis in [20], in addition to its considerable intrinsic interest, was to illustrate that non trivial central extensions may occur in the classical algebra of the canonical generators. In the same spirit, this model serves here as an example for the covariant theory of such central extensions proposed in section 5.

The background metric is represented in coordinates $\{x^\mu\} = \{t, r, \theta\}$ as in section 4 of [20] by

$$(ds^2)_{\rm background} = -\frac{r^2}{\ell^2}\,dt^2 + \frac{\ell^2}{r^2}\,dr^2 + r^2 d\theta^2, \tag{6.35}$$

where ℓ is a constant and θ has periodicity 2π. The nonvanishing components of the background Christoffel connection are

$$\bar\Gamma_{tt}{}^r = \frac{r^3}{\ell^4}\ ,\ \bar\Gamma_{rr}{}^r = -\frac{1}{r}\ ,\ \bar\Gamma_{\theta\theta}{}^r = -\frac{r^3}{\ell^2}\ ,\ \bar\Gamma_{tr}{}^t = \bar\Gamma_{rt}{}^t = \bar\Gamma_{\theta r}{}^\theta = \bar\Gamma_{r\theta}{}^\theta = \frac{1}{r}\ . \tag{6.36}$$

The background Ricci tensor and the cosmological constant are

$$\bar R_{\mu\nu} = 2\Lambda\,\bar g_{\mu\nu} \quad , \quad \Lambda = -1/\ell^2. \tag{6.37}$$

As in section 4 of [20], we study spacetimes which are asymptotically anti-de Sitter in the sense that the metric is $g_{\mu\nu} = \bar g_{\mu\nu} + h_{\mu\nu}$ with boundary conditions

$$\begin{array}{lll} h_{tt} \longrightarrow O(1), & h_{rr} \longrightarrow O(r^{-4}), & h_{\theta\theta} \longrightarrow O(1), \\ h_{tr} \longrightarrow O(r^{-3}), & h_{t\theta} \longrightarrow O(1), & h_{r\theta} \longrightarrow O(r^{-3}). \end{array} \tag{6.38}$$

The asymptotic behaviour is determined only by the dependence on r because the boundary conditions are imposed at $r \longrightarrow \infty$. Using Eq. (6.13), one obtains that the boundary conditions (6.38) imply

$$\begin{array}{lll} \mathcal{H}^{tt} \longrightarrow O(r^{-3}), & \mathcal{H}^{rr} \longrightarrow O(r), & \mathcal{H}^{\theta\theta} \longrightarrow O(r^{-3}), \\ \mathcal{H}^{tr} \longrightarrow O(r^{-2}), & \mathcal{H}^{t\theta} \longrightarrow O(r^{-3}), & \mathcal{H}^{r\theta} \longrightarrow O(r^{-2}), \end{array} \tag{6.39}$$

where we restricted the space of allowed functions to those which satisfy $h_{\mu\nu} \longrightarrow O(r^m) \Rightarrow \partial_r h_{\mu\nu} \longrightarrow O(r^{m-1})$ (and analogously for the derivatives of $h_{\mu\nu}$). In particular we thus exclude oscillating functions in the coordinate r, such as $r^m \sin(r)$. As we must assign $O(r)$ to d^3x, (6.16) imposes in this case

$$\begin{gathered}\bar{D}_t\xi_t \longrightarrow o(r^2), \quad \bar{D}_r\xi_r \longrightarrow o(r^{-2}), \quad \bar{D}_\theta\xi_\theta \longrightarrow o(r^2), \\ \bar{D}_t\xi_r + \bar{D}_r\xi_t \longrightarrow o(r), \quad \bar{D}_t\xi_\theta + \bar{D}_\theta\xi_t \longrightarrow o(r^2), \quad \bar{D}_r\xi_\theta + \bar{D}_\theta\xi_r \longrightarrow o(r). \end{gathered} \tag{6.40}$$

The functions χ_α in equation (5.8) are in this case $\chi_t = \chi_\theta = 1$, $\chi_r = 1/r$. Hence, trivial solutions to (6.40) are:

$$\xi^\mu \sim 0 \iff \xi^t \longrightarrow 0,\ \xi^r \longrightarrow o(r),\ \xi^\theta \longrightarrow 0. \tag{6.41}$$

The general solution to the conditions (6.40) in the space of functions satisfying $\xi^\mu \longrightarrow O(r^m) \Rightarrow \partial_r\xi^\mu \longrightarrow O(r^{m-1})$ is

$$\begin{aligned}\xi^t &\longrightarrow \ell T(t,\theta), \\ \xi^r &\longrightarrow -r\partial_\theta\Phi(t,\theta) + o(r), \\ \xi^\theta &\longrightarrow \Phi(t,\theta),\end{aligned} \tag{6.42}$$

where $T(t,\theta)$ and $\Phi(t,\theta)$ are functions of t and θ which are 2π-periodic in θ and satisfy

$$\ell\partial_t T(t,\theta) = \partial_\theta\Phi(t,\theta), \quad \ell\partial_t\Phi(t,\theta) = \partial_\theta T(t,\theta). \tag{6.43}$$

The general solution of these equations are functions $T(t,\theta)$ and $\Phi(t,\theta)$ which are superpositions of modes $f(nt/\ell)g(n\theta)$ with $f, g \in \{\sin,\cos\}$, $n \in \mathbb{Z}$, see [20] for details. We note that (6.42) agrees to leading order with the asymptotic Killing vector fields determined in [20] from the conditions $\mathcal{L}_\xi\,\bar{g}_{\mu\nu} \longrightarrow O(\chi_{\mu\nu})$. The latter conditions are stronger than (6.40) and impose also constraints on contributions to the ξ's at subleading order (see remark at the end of this section). However, contributions to the ξ's of subleading order do not contribute to the charges obtained from (6.21) because they are trivial, see (6.41). Furthermore, condition (5.25) is satisfied in this case and guarantees that the charges corresponding to (6.42) are finite. We choose $\partial\Sigma$ the circle of radius r for $r \to \infty$ (so that $dr = 0$, $dt = 0$ on $\partial\Sigma$). The conserved charges are then

$$Q_\xi = \frac{1}{2}\int_{\partial\Sigma} dx^\rho \epsilon_{\mu\nu\rho}\,\tilde{k}^{[\mu\nu]}_\xi[h,\bar{g}] = \lim_{r\to\infty}\int_0^{2\pi} d\theta\,\tilde{k}^{[tr]}_\xi[h,\bar{g}], \tag{6.44}$$

where $\tilde{k}^{[tr]}_\xi[h,\bar{g}]$ is the $[tr]$-component of the superpotential (6.17) evaluated for $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}(x)$ with the background metric (6.35). Explicitly one obtains

$$16\pi\tilde{k}^{[tr]}_\xi[h,\bar{g}] \longrightarrow -\xi^t(\frac{r^4}{\ell^4}\,h_{rr} + \frac{2}{\ell^2}\,h_{\theta\theta} - \frac{r}{\ell^2}\,\partial_r h_{\theta\theta}) - \xi^\theta(2h_{t\theta} - r\partial_r h_{t\theta}). \tag{6.45}$$

Notice that, indeed, the charges are finite and only the leading order terms in (6.42) contribute to them. Equations (6.44) and (6.45) may now be used to compute explicitly the values of the charges for a given metric satisfying the boundary conditions (6.38). For example, let us consider the metric given in equation (4.2) of [20]:

$$ds^2 = -(\frac{r^2}{\ell^2} + \alpha^2)\,dt^2 + 2\alpha A\,dtd\theta + (\frac{r^2 - A^2}{\ell^2} + \alpha^2)^{-1}dr^2 + (r^2 - A^2)\,d\theta^2,$$

where A and α are constant parameters. Evaluating (6.44) for $\xi = (\ell, 0, 0)$ (i.e., $T = 1$, $\Phi = 0$) and for $\xi = (0, 0, -1)$ (i.e., $T = 0$, $\Phi = -1$), respectively, one obtains

$$\begin{aligned} 16\pi Q_{(\ell,0,0)} &= 2\pi\ell(\alpha^2 + A^2/\ell^2), \\ 16\pi Q_{(0,0,-1)} &= 4\pi\alpha A, \end{aligned}$$

in agreement with Eq. (4.12) of [20] (modulo conventions).

Let us finally discuss the algebra of the charges. As we have pointed out, the existence of a well-defined algebra generally may impose additional conditions on the asymptotic reducibility parameters. Conditions which are sufficient for the existence of the algebra when (5.25) holds, are given in equations (5.29)–(5.32), (5.34)–(5.36) and (5.56). In the present case, it turns out that actually one only needs (5.34) in order to get a well-defined algebra; (5.34) reads in this case $\mathcal{L}_\xi h_{\mu\nu} \longrightarrow O(\chi_{\mu\nu})$ and imposes

$$\begin{aligned} \xi^t &\longrightarrow \ell\, T(t,\theta) + O(r^{-2}), \\ \xi^r &\longrightarrow -r\partial_\theta\Phi(t,\theta) + o(r), \\ \xi^\theta &\longrightarrow \Phi(t,\theta) + O(r^{-2}), \end{aligned} \tag{6.46}$$

where the functions $T(t,\theta)$ and $\Phi(t,\theta)$ are still only subject to (6.43). (6.46) especially implies the existence (finiteness) of the central charges (6.26); one obtains

$$\begin{aligned} K_{\xi_1,\xi_2} &= \frac{1}{16\pi}\lim_{r\to\infty}\int_0^{2\pi} d\theta\,\frac{2}{r}\,(\partial_\theta\xi_1^r\,\partial_\theta\xi_2^t - \partial_\theta\xi_2^r\,\partial_\theta\xi_1^t) \\ &= \frac{2\ell}{16\pi}\int_0^{2\pi} d\theta\Big[\partial_\theta T_1(t,\theta)\,\partial_\theta^2\Phi_2(t,\theta) - \partial_\theta T_2(t,\theta)\,\partial_\theta^2\Phi_1(t,\theta)\Big], \end{aligned} \tag{6.47}$$

which is the covariant expression for the central charge derived previously by different means in [56], equation (13).

Using a mode expansion of $\Phi(t,\theta)$ and $T(t,\theta)$ as in [20], it can be explicitly verified that the Poisson algebra (5.51) of the conserved charges for parameters (6.46) coincides with the algebra of canonical generators found in [20]. As shown there, this algebra can be written as the direct sum of two copies of the Virasoro algebra.

Remarks:

- The final expression (6.47) for the central charges involves solely the leading order terms in (6.46) which agree with those in (6.42). Nevertheless, (6.46) was used in the computation, as we dropped terms which vanish for $r \longrightarrow \infty$ on account of (6.46), but which would in general diverge in this limit for parameters that satisfy only the boundary conditions (6.42). These terms are

$$\frac{1}{16\pi}\lim_{r\to\infty}\int_0^{2\pi} d\theta\,\frac{r^3}{\ell^2}\Big[\partial_r\xi_1^t(\partial_\theta\xi_2^\theta + \frac{1}{r}\xi_2^r) + \frac{1}{2}(\partial_\theta\xi_1^t - \ell^2\partial_t\xi_1^\theta)\partial_r\xi_2^\theta - (1 \leftrightarrow 2)\Big].$$

- (6.46) is a weaker condition than (5.56) applied in the present case, as the latter imposes

$$\bar{D}_t\xi_t \longrightarrow O(1),\ \bar{D}_r\xi_r \longrightarrow O(r^{-4}),\ \bar{D}_\theta\xi_\theta \longrightarrow O(1),$$
$$\bar{D}_{(t}\xi_{r)} \longrightarrow O(r^{-3}),\ \bar{D}_{(t}\xi_{\theta)} \longrightarrow O(1),\ \bar{D}_{(r}\xi_{\theta)} \longrightarrow O(r^{-3}). \tag{6.48}$$

  The general solution of these conditions in the same space of functions as above is

$$\begin{aligned} \xi^t &\longrightarrow \ell T(t,\theta) + \frac{\ell^3}{2r^2}\partial_\theta^2 T(t,\theta) + O(1/r^4), \\ \xi^r &\longrightarrow -r\partial_\theta \Phi(t,\theta) + O(1/r), \\ \xi^\theta &\longrightarrow \Phi(t,\theta) - \frac{\ell^2}{2r^2}\partial_\theta^2\Phi(t,\theta) + O(1/r^4). \end{aligned} \tag{6.49}$$

  (6.48) are the conditions imposed in [20]. The fact that (6.46) leads to the same conclusions shows that these conditions can be relaxed. This demonstrates that (5.56) is only a sufficient but not a necessary condition for finiteness of the central charges as given in (5.52).

7 Cohomological approach

7.1 Antifield BRST formalism

7.1.1 Koszul-Tate resolution

The cohomological set-up used so far was given by the free variational bicomplex (see e.g. [2, 40, 42]), i.e., horizontal and vertical form valued local functions with horizontal differential $d_H = dx^\mu\partial_\mu$ and vertical differential $d_V = d_V\phi^i_{(\mu)}\partial^S/\partial\phi^i_{(\mu)}$ and by the variational bicomplex pulled back to the surface defined by the Euler-Lagrange equations of motion (and their total derivatives).

In the absence of vertical generators, one constructs a homological resolution of the horizontal complex associated with the equations of motion by introducing "antifields" [57, 58]: for irreducible gauge theories, the antifields are given by Grassmann odd generators $\phi^*_{i(\mu)}$ of antifield number 1 and Grassmann even generators $C^*_{\alpha(\mu)}$ of antifield number 2. The so-called Koszul-Tate differential [47, 48] is defined by

$$\delta = \partial_{(\mu)}\frac{\delta L}{\delta\phi^i}\frac{\partial^S}{\partial\phi^*_{i(\mu)}} + \partial_{(\mu)}[R^{+i}_\alpha(\phi^*_i)]\frac{\partial^S}{\partial C^*_{\alpha(\mu)}}. \tag{7.1}$$

The cohomology of δ can then be shown to be trivial in the space of horizontal forms in the fields and the antifields with strictly positive antifield number, $H_k(\delta) = 0$ for $k \geq 1$, while $H_0(\delta)$ is given by equivalence classes of forms $[\omega_0]$ in the original fields alone, where two such forms have to be identified if they agree when evaluated on every solution of the equations of motion, $\omega_0 \sim \omega_0'$ if $\omega_0 - \omega_0' \approx 0$.

7.1.2 Antibracket, master equation and BRST differential

In many problems involving the gauge symmetries of a classical Lagrangian, and in particular for the discussion of the Lie algebra associated with global reducibility identities, it is most convenient to extend the Koszul-Tate differential to the full BRST differential of the antifield formalism. This differential is canonically generated in the antibracket by the so-called minimal solution of the master equation [57, 59, 58, 60, 61, 62, 63, 47, 48] (for reviews, see e.g. [64, 65]). This formulation is crucial for the quantum theory, because canonical transformations in the antibracket are used on the one hand to fix the gauge while retaining the original gauge invariance in the form of the gauge fixed BRST invariance, and, on the other hand, to absorb trivial, BRST exact divergences. At the classical level, a great advantage of the formalism is for instance that different choices of field parametrizations or of generating sets of gauge transformations are again related by canonical transformations.

The full antifield BRST formalism involves as additional fields not only the antifields ϕ^*_i and C^*_α, but also the ghosts C^α and their derivatives. They can be understood as Grassmann odd gauge parameters. There is a well defined graded Lie bracket in $H^n(d_H)$, which is induced by the local antibracket. With $\{\phi^A\}=\{\phi^i, C^\alpha\}$, $\{\phi^*_A\}=\{\phi^*_i, C^*_\alpha\}$, it is defined in terms of Euler-Lagrange left and right derivatives (indicated by superscripts L and R, respectively) through

$$(f d^n x, g d^n x) = \Big[\frac{\delta^R f}{\delta\phi^A}\frac{\delta^L g}{\delta\phi^*_A} - \frac{\delta^R f}{\delta\phi^*_A}\frac{\delta^L g}{\delta\phi^A}\Big] d^n x. \tag{7.2}$$

To each equivalence class $[f d^n x]\in H^n(d_H)$, one can associate a "Hamiltonian" vector field defined by

$$\delta_{f d^n x} = \partial_{(\mu)}\frac{\delta^R f}{\delta\phi^A}\frac{\partial^S}{\partial\phi^*_{A(\mu)}} - \partial_{(\mu)}\frac{\delta^R f}{\delta\phi^*_A}\frac{\partial^S}{\partial\phi^A_{(\mu)}}. \tag{7.3}$$

[The operators $\partial^S/\partial(\dots)$ are left derivatives.] If $[\cdot,\cdot]$ denotes the graded commutator of vector fields, these vector fields satisfy

$$\begin{aligned}[][\delta_{f d^n x}, \partial_\mu] &= 0,\\ [\delta_{f d^n x}, d_H] &= 0,\\ [\delta_{f d^n x}, \delta_{g d^n x}] &= \delta_{(f d^n x, g d^n x)}.\end{aligned} \tag{7.4}$$

An algebraic proof of the last identity can be found for instance in [43].

The classical Lagrangian L is extended to the Lagrangian L_M that solves the (classical) master equation

$$(L_M\ d^n x, L_M\ d^n x) = d_H(\ ). \tag{7.5}$$

In an expansion according to the antifield number, the Lagrangian L_M reads

$$L_M = L + \phi^*_i R^i_\alpha(C^\alpha) + \frac{1}{2}C^*_\gamma C^\gamma_{\alpha\beta}(C^\alpha, C^\beta) + \frac{1}{4}M^{ij}_{\alpha\beta}(\phi^*_i,\phi^*_j, C^\alpha, C^\beta) + \text{terms of antifield number} \geq 3, \tag{7.6}$$

where the structure operators $M^{ij}_{\alpha\beta}$ describe the weakly vanishing terms in the commutators of gauge transformations. The full BRST differential involving the antifields is generated from the solution of the master equation according to

$$s = \delta_{L_M d^n x}. \tag{7.7}$$

In an expansion according to the antifield number, $s = \delta + \gamma + s_1 + \ldots$, it starts at antifield number -1 with the Koszul-Tate differential δ. The component γ at antifield number 0 is the so-called longitudinal differential along the gauge orbits [64].

The cohomology groups $H^n_k(\delta|d_H)$ can be shown [36, 5] to be isomorphic to the local BRST cohomological groups $H^{-k,n}(s|d_H)$, where the first superscript denotes the ghost number obtained by assigning 1 to C^α, 0 to $\phi^i, \partial_\mu, x^\mu, dx^\mu$, -1 to ϕ^*_i, and -2 to C^*_α,

$$H^n_k(\delta|d_H) \simeq H^{-k,n}(s|d_H) \quad \text{for} \quad k > 0. \tag{7.8}$$

Owing to the properties of $H^n_k(\delta|d_H)$ summarized below, this implies that equivalence classes of global symmetries and of reducibility parameters, and at the same time the associated characteristic cohomology, can be described as local BRST cohomology classes. Explicitly, representatives of $H^n_k(\delta|d_H)$ are completed by terms of higher antifield number containing the ghosts C^α and their derivatives to representatives of $H^{-k,n}(s|d_H)$, while conversely, representatives of $H^{-k,n}(s|d_H)$ determine representatives of $H^n_k(\delta|d_H)$ by setting to zero the ghosts C^α and their derivatives.

One of the advantages of this description is that the antibracket map induces a graded Lie bracket (with grading +1) in local BRST cohomology,

$$(\cdot,\cdot)_M : H^{g_1,n}(s|d_H) \otimes H^{g_2,n}(s|d_H) \longrightarrow H^{g_1+g_2+1,n}(s|d_H),$$
$$([\omega^{g_1,n}], [\eta^{g_2,n}])_M = [(\omega^{g_1,n}, \eta^{g_2,n})].$$

Alternative equivalent expressions for the antibracket map are[6]

$$([\omega^{g_1,n}], [\eta^{g_2,n}])_M = [\delta_{\omega^{g_1,n}} \eta^{g_2,n}] = -(-)^{(g_1+1)(g_2+1)} [\delta_{\eta^{g_2,n}} \omega^{g_1,n}]. \tag{7.9}$$

This map will be used below to describe the Lie algebra of equivalence classes of global symmetries, the Lie action of equivalence classes of global symmetries on equivalence classes of reducibility parameters, the global symmetries induced from reducibility parameters and the Lie algebra of equivalence classes of reducibility parameters.

7.2 Global symmetries and conserved currents

The equivalence classes of global symmetries can be identified with the cohomology classes of the group $H^n_1(\delta|d_H)$, which admit canonical representatives of the form

[6]For simplicity we assume throughout this paper that all fields ϕ^i are bosonic (Grassmann even). Then the Grassmann parity of a local function of the fields, ghosts and antifields equals its ghost number (modulo 2). This allows us to write Grassmann parity dependent signs as in (7.9) in terms of ghost numbers.

$\omega_1^n = \phi_i^* X^i d^n x$, while equivalence classes of conserved currents correspond to cohomology classes of the group $H_0^{n-1}(d_H|\delta)$ with representatives $\omega_0^{n-1} = j^\mu\, (d^{n-1}x)_\mu$. In this set-up, Noether's first theorem, in its complete(d) formulation as in section 2.4, is precisely the cohomological relation

$$H_1^n(\delta|d_H) \simeq H_0^{n-1}(d_H|\delta)/\delta_0^{n-1}\mathbb{R}, \tag{7.10}$$

which is a rather direct consequence of the properties of the cohomology of δ and the fact that the cohomology of the horizontal differential in the space of horizontal form valued local functions in the fields and antifields is given (locally) by $H^k(d_H) = \delta_0^k\mathbb{R}$ for $k \leq n-1$ (algebraic Poincaré lemma).

For completeness, let us note that because of (2.3), $H^n(d_H)$ is given by the equivalence classes $[\omega^n]$ of n-forms having the same Euler-Lagrange derivatives with respect to the fields and the antifields, $\omega^n \sim \tilde{\omega}^n$ iff $\frac{\delta}{\delta Z^A}(\omega^n - \tilde{\omega}^n) = 0$, for all $Z^A \in \{\phi^i, \phi_i^*, C_\alpha^*\}$.

We also note that Noether's first theorem holds in exactly the same form for reducible gauge theories, although one needs to introduce additional antifields of antifield number higher than 2 and extend the definition of δ on these additional antifields in such a way that the cohomology of δ remains trivial in strictly positive antifield number and unchanged in antifield number 0.

7.3 Reducibility parameters and conserved n–2 forms

7.3.1 Characteristic cohomology and $H^n(\delta|d_H)$.

The cohomology group $H_0^{n-k}(d_H|\delta)$ is also called the characteristic cohomology in form degree $n-k$ and is represented by conserved $(n-k)$ forms. The characteristic cohomology is the cohomology of the horizontal complex associated to the (Euler-Lagrange) equations of motion [37, 38, 39, 40, 41]. Its representatives are local forms which are d_H-closed on-shell modulo local forms which are d_H-exact on-shell; in other words: the representatives are conserved local forms. Using the cohomology of d_H and of δ, one can prove [36, 5]:

$$H_0^{n-k}(d_H|\delta)/\delta_k^n\mathbb{R} \simeq H_k^n(\delta|d_H) \quad \text{for} \quad 1 \leq k \leq n-1. \tag{7.11}$$

Note that this generalizes (7.10) to $k \geq 1$ and might therefore be regarded as a generalization of Noethers first theorem. For $k = 2$, it encodes the bijective correspondence between conserved $(n-2)$ forms and global reducibility identities as we shall explain in more detail below. For irreducible gauge theories, one can show under fairly general assumptions (linearizable, normal) [36, 5] that there is no characteristic cohomology in form degree strictly smaller than $n-2$ except for the constant 0-forms, i.e., that $H_0^{n-k}(d_H|\delta) = \delta_k^n\mathbb{R}$ and $H_k^n(\delta|d_H) = 0$ for $k > 2$. More generally, one can show for reducible (linearizable, normal) gauge theories of reducibility order r ($r = -1$ for models without nontrivial gauge symmetry, $r = 0$ for irreducible gauge theories, etc): $H_0^{n-k}(d_H|\delta) = \delta_k^n\mathbb{R}$ and $H_k^n(\delta|d_H) = 0$ for $k > r + 2$ [36].

7.3.2 Descent equations

Let us now explain in more detail that and how the isomorphism (7.11) yields the bijective correspondence between conserved $(n-2)$ forms and global reducibility identities. The isomorphism (7.11) is based on so-called descent equations for δ and d_H. For $k=2$, these descent equations relate, in intermediate steps, $H^{n-2}_0(d_H|\delta)$ to $H^{n-1}_1(\delta|d_H)$ and $H^{n-1}_1(\delta|d_H)$ to $H^n_2(\delta|d_H)$,

$$H^{n-2}_0(d_H|\delta)/\delta^{n-2}_0\mathbb{R} \simeq H^{n-1}_1(\delta|d_H) \simeq H^n_2(\delta|d_H). \tag{7.12}$$

[Analogous intermediate steps are behind (7.11) for $k>2$ [36, 5].]

$H^{n-2}_0(d_H|\delta)/\delta^{n-2}_0\mathbb{R} \simeq H^{n-1}_1(\delta|d_H)$ is explicitly given by associating to any class $[\omega^{n-2}_0] \in H^{n-2}_0(d_H|\delta)$ (except for the constants in 2 spacetime dimensions), a class $[\omega^{n-1}_1] \in H^{n-1}_1(\delta|d_H)$. $H^{n-1}_1(\delta|d_H) \simeq H^n_2(\delta|d_H)$ is explicitly given by associating to any class $[\omega^{n-1}_1] \in H^{n-1}_1(\delta|d_H)$ a class $[\omega^n_2] \in H^n_2(\delta|d_H)$. The representatives satisfy the chain of descent equations

$$\delta\omega^n_2 + d_H\omega^{n-1}_1 = 0, \tag{7.13}$$
$$\delta\omega^{n-1}_1 + d_H\omega^{n-2}_0 = 0. \tag{7.14}$$

Under the same general assumptions as above, one can show [36, 5] that $H^n_2(\delta|d_H)$ is isomorphic to the space of equivalence classes of global reducibility identities up to trivial ones.

For a collection of functions f^α and $M^{[j(\nu)i(\mu)]}$ that satisfy (3.1), the forms $\omega^n_2, \omega^{n-1}_1, \omega^{n-2}_0$ that satisfy the descent equations (7.13) and (7.14) can be constructed as follows. The form ω^n_2 is given by

$$\omega^n_2 = [f^\alpha C^*_\alpha - \frac{1}{2}\phi^*_{j(\nu)}\phi^*_{i(\mu)}M^{[j(\nu)i(\mu)]}]d^nx. \tag{7.15}$$

The form

$$\omega^{n-1}_1 = -[S^{\mu i}_\alpha(\phi^*_i, f^\alpha) + M^{\mu ji}(\frac{\delta L}{\delta\phi^j}, \phi^*_i)](d^{n-1}x)_\mu, \tag{7.16}$$

is a particular solution to the first of the descent equations (7.13) because of (3.1). Finally, a particular solution to the second of the descent equations (7.14) is given by

$$\omega^{n-2}_0 = -k^{[\mu\nu]}_f (d^{n-2}x)_{\mu\nu}\ , \tag{7.17}$$

with $k^{[\mu\nu]}_f$ given by (3.19).

The advantage of this cohomological formulation is that the ambiguities in the solutions of the descent equations are automatically taken care of by the triviality of the cohomology of d_H and δ in the appropriate degrees: the various forms are all defined only up to the addition of d_H and δ exact terms. This leads to the isomorphisms (7.12), which states the bijective correspondence between the equivalence classes of reducibility parameters, conserved $n-2$ forms (up to the constant form in $n=2$) and operator currents satisfying (3.13) or (3.14).

7.3.3 Lie algebra and action of global symetries from antibracket map

In ghost number $g=-1$, the antibracket map describes the Lie algebra of equivalence classes of global symmetries, up to a shift in the grading, and an overall minus sign. Indeed, if the cocycle

$$\omega_X^{-1,n}=(\phi^*_i X^i - C^*_\alpha X^\alpha_\beta(C^\beta)+\dots)d^nx \tag{7.18}$$

describes the global symmetry with characteristic X^i, we have

$$([\omega_{X_1}^{-1,n}],[\omega_{X_2}^{-1,n}])_M=[\omega^{-1,n}_{[X_2,X_1]_L}]. \tag{7.19}$$

Furthermore, due to the graded Jacobi identity for $(\cdot,\cdot)_M$,

$$\begin{aligned}([\omega_{X_1}^{-1,n}],([\omega_{X_2}^{-1,n}],[\omega^{*,n}])_M)_M-([\omega_{X_2}^{-1,n}],([\omega_{X_1}^{-1,n}],[\omega^{*,n}])_M)_M\\ =([\omega^{-1,n}_{[X_2,X_1]_L}],[\omega^{*,n}])_M,\end{aligned} \tag{7.20}$$

there is a well defined Lie action of equivalence classes of global symmetries on local BRST cohomology classes. In particular, if the cocycle

$$\omega_f^{-2,n}=(C^*_\alpha f^\alpha-\frac{1}{2}\phi^*_{j(\nu)}\phi^*_{i(\mu)}M^{[j(\nu)i(\mu)]}+C^*_\alpha k^{\alpha i}_\beta(\phi^*_i,C^\beta)+\dots)d^nx, \tag{7.21}$$

describes the reducibility parameters f^α, we can choose

$$([\omega_X^{-1,n}],[\omega_f^{-2,n}])_M=[\omega^{-2,n}_{-(X,f)}], \tag{7.22}$$

with $(X,f)^\alpha=\delta_X f^\alpha+X^\alpha_\beta(f^\beta)$, in agreement with (3.32).

Owing to

$$H^{n-2}_{\rm char}/\delta^n_2\mathbb{R}\simeq H^n_2(\delta|d_H)\simeq H^{-2,n}(s|d_H), \tag{7.23}$$

there is an isomorphic Lie action of equivalence classes of global symmetries on equivalence classes of conserved $n-2$ forms.

The proof that this Lie action is given by (3.33) proceeds as follows. The vector field $\delta_{\omega_X^{-1,n}}$ anticommutes not only with d_H but also with s. Indeed, (7.4) implies that $[\delta_{\omega^{*,n}},s]=0$ for every s modulo d_H cocycle $\omega^{*,n}$. Hence, by appying $\delta_{\omega_X^{-1,n}}$ to the descent equations

$$\begin{aligned}s\omega_f^{-2,n}+d_H\omega_f^{-1,n-1}=0,\\ s\omega_f^{-1,n-1}+d_H\omega_f^{0,n-2}=0,\end{aligned} \tag{7.24}$$

one can move $\delta_{\omega_X^{-1,n}}$ past s and d_H. The result then follows from the fact that $[\delta_{\omega_X^{-1,n}}\omega_f^{-2,n}]=([\omega_X^{-1,n}],[\omega_f^{-2,n}])_M$ and that $[\delta_{\omega_X^{-1,n}}\omega_f^{0,n-2}]=[-\delta_X k^{\mu\nu}(d^{n-2}x)_{\mu\nu}+\dots]$.

Let us also show by cohomological means that the bracket $[\cdot,\cdot]_P$ induced by (3.28) in the space of equivalence classes of reducibility parameters is trivial. Indeed, consider the trivial global symmetry $X^i = R^i_\alpha(f_1^\alpha)$ for an arbitrary local function f_1^α described by the cocycle

$$\begin{aligned}\omega^{-1,n}_{R_{f_1}} &= s(C^*_\alpha f_1^\alpha d^n x) + d_H(\ )\\ &= [\phi^*_i R^i_\alpha(f_1^\alpha) + C^*_\alpha \partial_{(\mu)} R^i_\alpha(C^\alpha)\frac{\partial^S f_1^\alpha}{\partial \phi^i_{(\mu)}} + C^*_\alpha C^\alpha_{\beta\gamma}(C^\beta, f_1^\gamma) + \dots]d^n x.\end{aligned} \tag{7.25}$$

For a given cocycle $\omega^{-2,n}_{f_2}$, we have

$$([\omega^{-1,n}_{R_{f_1}}], [\omega^{-2,n}_{f_2}])_M = [0] = [\omega^{-2,n}_{-(R_{f_1},f_2)}], \tag{7.26}$$

which implies the triviality of the resulting reducibility parameters,

$$(R_{f_1}, f_2)^\alpha = \delta_{f_1} f_2^\alpha - \delta_{f_2} f_1^\alpha + C^\alpha_{\beta\gamma}(f_1^\beta, f_2^\gamma) \approx 0, \tag{7.27}$$

and also the triviality of the conserved $n-2$ form obtained by applying δ_{f_1} to a given conserved $n-2$ form. In particular, if f_1^α are reducibility parameters, we get

$$C^\alpha_{\beta\gamma}(f_1^\beta, f_2^\gamma) \approx 0, \tag{7.28}$$

which proves the triviality of the bracket induced by $[\cdot,\cdot]_P$ among equivalence classes of reducibility parameters.

7.4 Induced symmetries and associated algebra

7.4.1 Global symmetries out of reducibility parameters

Let $[\omega^{0,n}] \in H^{0,n}(s|d_H)$. Because $(\cdot,\cdot)_M : H^{0,n}(s|d) \otimes H^{-2,n}(s|d) \longrightarrow H^{-1,n}(s|d_H)$, the antibracket map with a given $[\omega^{0,n}]$ provides a way to induce a possibly non trivial global symmetry from a set of reducibility parameters,

$$([\omega^{0,n}], [\omega^{-2,n}_f])_M = [\omega^{-1,n}_f]. \tag{7.29}$$

Because $(\cdot,\cdot)_M : H^{-1,n}(s|d) \otimes H^{-2,n}(s|d) \longrightarrow H^{-2,n}(s|d_H)$, there is an action of global symmetries, and in particular of induced global symmetries, on reducibility parameters. Using the graded Jacobi identity for the antibracket $(\cdot,\cdot)$, it follows that

$$(\omega^{-1,n}_{f_1}, \omega^{-2,n}_{f_2}) = -(\omega^{-1,n}_{f_2}, \omega^{-2,n}_{f_1}) + (\omega^{0,n}, (\omega^{-2,n}_{f_1}, \omega^{-2,n}_{f_2})) + d_H(\ ). \tag{7.30}$$

By assumption, we are dealing with irreducible gauge theories which do not admit non trivial characteristic cohomology in ghost number -3,

$$H^{n-3}_{\text{char}}/\delta^n_3\mathbb{R} \simeq H^n_3(\delta|d_H) \simeq H^{-3,n}(s|d_H) \simeq 0, \tag{7.31}$$

so that the s modulo d_H cocycle $(\omega^{-2,n}_{f_1}, \omega^{-2,n}_{f_2})$ is trivial, $(\omega^{-2,n}_{f_1}, \omega^{-2,n}_{f_2}) = s(\ ) + d_H(\ )$. It follows that the action of induced global symmetries on reducibility parameters is skew-symmetric (modulo trivial terms),

$$([\omega^{-1,n}_{f_1}], [\omega^{-2,n}_{f_2}])_M = -([\omega^{-1,n}_{f_2}], [\omega^{-2,n}_{f_1}])_M. \tag{7.32}$$

Explicitly, for

$$\omega^{0,n} = (v_0 + \phi_i^* v_\alpha^i(C^\alpha) + \frac{1}{2} C_\gamma^* v_{\alpha\beta}^\gamma(C^\alpha, C^\beta) + \dots) d^n x, \tag{7.33}$$

with v_0 a local ϕ-dependent function and v_α^i, $v_{\alpha\beta}^\gamma$ local ϕ-dependent operators, (7.29) holds with

$$\begin{aligned} \omega_f^{-1,n} &= \Big(\phi_i^*[v_\alpha^i(f^\alpha) - M^{+ji}(\frac{\delta^R v_0}{\delta\phi^j})] \\ &+ C_\alpha^*[k_\beta^{\alpha i}(\frac{\delta^R v_0}{\delta\phi^j}, C^\beta) + \partial_{(\mu)} v_\beta^i(C^\beta)\frac{\partial^L f^\alpha}{\partial\phi^i_{(\mu)}} + v_{\beta\gamma}^\alpha(C^\beta, f^\gamma)] + \dots\Big) d^n x. \end{aligned} \tag{7.34}$$

Remark: We note that if the reducibility parameters give rise to a reducibility identity off-shell, $M^{[j(\nu)i(\mu)]} = 0$, the induced global symmetry can only be non trivial if the dependence on the antifields of antifield number 1 of $\omega^{0,n}$ is non trivial.

From the point of view of a free theory, the existence of such an element $\omega^{0,n}$ is a necessary condition for the existence of an interacting theory with a non trivial deformation R_α^{i1} of the generating set of gauge symmetries. More precisely, it is a necessary condition for the existence of a first order deformation.

Starting from an interacting gauge theory that deforms the gauge transformations of the linearized theory in a non trivial way through terms linear in the fields, the cubic part $\omega_0^n = L_M^3\, d^n x$ that arises in an expansion in the number of fields and antifields of the solution L_M of the master equation around a solution of the classical equations of motion, $L_M\, d^n x = L_M^2\, d^n x + d_H(\ ) + L_M^3\, d^n x + \dots$, is automatically a cocycle (modulo d_H) for the BRST differential of the linearized theory s^{free} generated by L_M^2, with a non trivial dependence on the antifields.

Explicitly, for $L_M^3\, d^n x = (L^3 + \varphi_i^* R_\alpha^{i1}(C^\alpha) + \frac{1}{2} C_\gamma^* v_{\alpha\beta}^{\gamma 0}(C^\alpha, C^\beta)) d^n x$, the induced symmetry for field independent reducibility parameters f^α is given by $\omega_f^{-1,n} = (\varphi_i^* R_\alpha^{i1}(f^\alpha) + C_\alpha^* v_{\beta\gamma}^{\alpha 0}(C^\beta, f^\gamma)) d^n x$.

7.4.2 Lie algebra of reducibility parameters

According to the previous section, for a given element $[\omega^{0,n}] \in H^{0,n}(s|d_H)$, there exists a bilinear operation $[f_1, f_2]^\alpha$ between reducibility parameters f_1^α, f_2^α defined so that

$$[\omega^{-2,n}_{[f_2,f_1]}] = ([\omega_{f_1}^{-1,n}], [\omega_{f_2}^{-2,n}])_M, \tag{7.35}$$

which induces a skew-symmetric bracket among equivalence classes of reducibility parameters,

$$[[f_1],[f_2]]_M = [[f_1, f_2]]. \tag{7.36}$$

Explicitly, one can choose

$$\begin{aligned} [f_1, f_2]^\alpha = v_{\beta\gamma}^\alpha(f_1^\beta, f_2^\gamma) + \partial_{(\mu)} v_\beta^i(f_1^\beta)\frac{\partial^S f_2^\alpha}{\partial\phi^i_{(\mu)}} - \partial_{(\mu)} v_\beta^i(f_2^\beta)\frac{\partial^S f_1^\alpha}{\partial\phi^i_{(\mu)}} \\ - k_{1\beta}^{\alpha i}(\frac{\delta^R v_0}{\delta\phi^i}, f_2^\beta) - \partial_{(\mu)} M_1^{+ji}(\frac{\delta^R v_0}{\delta\phi^i})\frac{\partial^S f_2^\alpha}{\partial\phi^i_{(\mu)}}. \end{aligned} \tag{7.37}$$

The graded Jacobi identity for $(\cdot,\cdot)$ implies that

$$(\omega^{-1,n}_{f_1},(\omega^{-1,n}_{f_2},\omega^{-2,n}_{f_3}))=(\omega^{-1,n}_{f_2},(\omega^{-1,n}_{f_1},\omega^{-2,n}_{f_3})+((\omega^{-1,n}_{f_1},\omega^{-1,n}_{f_2}),\omega^{-2,n}_{f_3})+d_H(\,).\quad(7.38)$$

Suppose the element $[\omega^{0,n}]$ satisfies the condition

$$([\omega^{0,n}],[\omega^{0,n}])_M=[0]\iff(\omega^{0,n},\omega^{0,n})+s(\,)+d_H(\,)=0.\qquad(7.39)$$

The graded Jacobi identity for $(\cdot,\cdot)$ then implies

$$(([\omega^{-1,n}_{f_1}],[\omega^{-1,n}_{f_2}])_M,[\omega^{-2,n}_{f_3}])_M=([\omega^{-1,n}_{f_3}],([\omega^{-1,n}_{f_2}],[\omega^{-2,n}_{f_1}])_M)_M.\qquad(7.40)$$

By using (7.38), (7.32) and (7.40) to transform the first term, it follows that

$$([\omega^{-1,n}_{f_1}],([\omega^{-1,n}_{f_2}],[\omega^{-2,n}_{f_3}])_M)_M+\text{cyclic }(1,2,3)=0.\qquad(7.41)$$

This implies that the bracket $[\cdot,\cdot]_M$ among equivalence classes of reducibility parameters satisfies the Jacobi identity. We denote the Lie algebra of equivalence classes of reducibility parameters equipped with the bracket $[\cdot,\cdot]_M$ by $\mathfrak{g}$.

One can introduce a basis $\{f^\alpha_A\}$ in the space of equivalence classes of reducibility parameters. Such a basis has the property that all reducibility parameters can be expressed as a linear combination of the basis, up to trivial reducibility parameters,

$$f^\alpha\approx k^Af^\alpha_A,\qquad(7.42)$$

and that the basis vectors are independent in the sense that

$$k^Af^\alpha_A\approx 0\Longrightarrow k^A=0.\qquad(7.43)$$

A corresponding basis of $H^{-2,n}(s|d_H)$ is then given by $\{\omega^{-2,n}_{f_A}\}$ and satisfies

$$\omega^{-2,n}=k^A\omega^{-2,n}_{f_A}+s(\,)+d_H(\,),\qquad(7.44)$$

for any s modulo d_H cocycle $\omega^{-2,n}$ and

$$k^A\omega^{-2,n}_{f_A}=s(\,)+d_H(\,)\Longrightarrow k^A=0.\qquad(7.45)$$

Induced global symmetries associated to the basis can be defined by

$$\omega^{-1,n}_{f_A}=(\omega^{0,n},\omega^{-2,n}_{f_A})+s(\,)+d_H(\,).\qquad(7.46)$$

We note that, in general, the set $\{\omega^{-1,n}_{f_A}\}$ is not a basis of $H^{-1,n}(s|d_H)$ because the induced global symmetries do not necessarily span all the non trivial global symmetries and some linear combinations of the induced global symmetries can be trivial global symmetries.

By reasonings similar to the above using in addition (7.44) and (7.45), it follows from these definitions that

- the module action of the induced global symmetries can be described by skew-symmetric structure constants C^C_{AB},

$$(\omega^{-1,n}_{f_A}, \omega^{-2,n}_{f_B}) = C^C_{AB}\omega^{-2,n}_{f_C} + s(\ ) + d_H(\ ); \tag{7.47}$$

- the algebra of the induced global symmetries involves the same structure constants,

$$(\omega^{-1,n}_{f_A}, \omega^{-1,n}_{f_B}) = C^C_{AB}\omega^{-1,n}_{f_C} + s(\ ) + d_H(\ ); \tag{7.48}$$

- the structure constants satisfy the Jacobi identity provided $\omega^{0,n}$ satisfies (7.39),

$$C^D_{AE}C^E_{BC} + \text{cyclic }(A, B, C) = 0. \tag{7.49}$$

Remarks: (a) From the point of view of a free theory, the condition (7.39) is a necessary condition for the first order deformation to be extendable to a second order deformation.

(b) The cocycle $\omega^n_0 = L^3_M\, d^nx$ of $H^{0,n}(s^{\text{free}}|d_H)$, obtained from the interacting theory by expanding L_M, automatically satisfies (7.39) in the linearized theory, because the expansion of the master equation to order 4 reads

$$\frac{1}{2}(L^3_M d^nx, L^3_M d^nx) + s^{\text{free}} L^4_M d^nx + d_H(\ ) = 0. \tag{7.50}$$

7.5 Asymptotic symmetries and conservation laws

7.5.1 Linear characteristic cohomology

Our starting point for understanding asymptotic symmetries, conservation laws and their interplay is the approach of reference [34] where asymptotic conservation laws have been studied in the context of the variational bicomplex, including the vertical generators $d_V\phi^i_{(\mu)} = d\phi^i_{(\mu)} - \phi^i_{(\mu)\nu}dx^\nu$ and the vertical differential $d_V = d_V\phi^i_{(\mu)}\partial^S/\partial\phi^i_{(\mu)}$. The vertical differential corresponds to an "infinitesimal field variation" (independent of the variation of the base space) with these variations and their derivatives being Grassmann odd. The concept "vanish on all solutions of the equations of motion" (≈ 0) includes the equations $\delta L/\delta\phi^i = 0$ and $d_V(\delta L/\delta\phi^i) = 0$ for the fields $\phi^i, d_V\phi^i$.

Linear characteristic cohomology is defined in terms of vertical 1-forms and horizontal $n-k$ forms through the cocycle condition

$$d_H\omega^{n-k,1}_0 + \omega^{n-k+1,1i(\mu)}_0\partial_{(\mu)}\frac{\delta L}{\delta\phi^i} + \omega^{n-k+1,0i(\mu)}_0\partial_{(\mu)}d_V\frac{\delta L}{\delta\phi^i} = 0, \tag{7.51}$$

and the coboundary condition

$$\omega^{n-k,1}_0 = d_H\eta^{n-k-1,1}_0 + \eta^{n-k,1i(\mu)}_0\partial_{(\mu)}\frac{\delta L}{\delta\phi^i} + \eta^{n-k,0i(\mu)}_0\partial_{(\mu)}d_V\frac{\delta L}{\delta\phi^i}. \tag{7.52}$$

The Koszul-Tate resolution is extended to the full variational bicomplex associated with all the equations of motion [36, 43] through the addition of the additional vertical generators $d_V\phi^*_{i(\mu)}, d_V C^*_{\alpha(\mu)}$ and the definition $\delta_T = \delta + \delta_V$, where

$$\delta_V = -\partial_{(\mu)} d_V \frac{\delta L}{\delta\phi^i} \frac{\partial^S}{\partial d_V \phi^*_{i(\mu)}} - \partial_{(\mu)} d_V \left[R^{+i}_\alpha(\phi^*_i)\right] \frac{\partial^S}{\partial d_V C^*_{\alpha(\mu)}}. \tag{7.53}$$

The cocycle and coboundary conditions (7.51) and (7.52) can then be written as

$$d_H\omega_0^{n-k,1} + \delta_T\omega_1^{n-k+1,1} = 0, \tag{7.54}$$

$$\omega_0^{n-k,1} = d_H\eta_0^{n-k-1,1} + \delta_T\eta_1^{n-k,1}. \tag{7.55}$$

As in the case without vertical generators, one uses descent equation techniques to show for instance the isomorphism $H_0^{n-k,1}(d_H|\delta_T) \simeq H_k^{n,1}(\delta_T|d_H)$. Note that in this case, the isomorphism holds exactly in all spacetime dimensions and not only up to constants, because of the presence of the vertical generators.

The following technical lemma is a direct generalization of theorems 6.5 and 6.6 of [5].

Lemma 1 (Trivial linear characteristic cohomology). *For linearizable, normal gauge theories,*

(i) if $k \geq 3$ and the theory is irreducible, or

(ii) if $k = 2$ and $N_{d_V C^*_\alpha}(\omega_2^{n,1}) = 0 = N_{C^*_\alpha}(\omega_2^{n,1})$, or

(iii) if $k = 1$ and $\omega_1^{n,1} \approx 0$,

then

$$\delta_T\omega_k^{n,1} + d_H\omega_k^{n-1,1} = 0 \Longrightarrow \omega_k^{n,1} = \delta_T\eta_{k+1}^{n,1} + d_H\eta_k^{n-1,1}. \tag{7.56}$$

7.5.2 Exact linear characteristic cohomology

Exact linear characteristic cohomology is defined through elements $\omega_0^{n-k,0}$ such that $\omega_0^{n-k,1} = d_V\omega_0^{n-k,0}$ satisfies the cocycle condition (7.54). For such a representative, the second term of the cocycle condition (7.54) can also be assumed to be d_V-exact. This can be understood as a consequence of the fact that

$$H_k^*(\delta_T|d_V) = 0, \text{ for } k > 0, \tag{7.57}$$

which itself follows from the fact that the contracting homotopy, which allows one to prove that $H_k^*(\delta_T) = 0$ for $k > 0$, anticommutes with d_V. Hence, exact linear characteristic cohomology is defined through the cocycle condition

$$d_H d_V\omega_0^{n-k,0} + \delta_T d_V\omega_1^{n-k+1,0} = 0 \tag{7.58}$$

for the form $\omega_0^{n-k,0}$. The cocycle $\omega_0^{n-k,0}$ is trivial as an element of exact linear characteristic cohomology if

$$d_V\omega_0^{n-k,0} = d_H\eta_0^{n-k-1,1} + \delta_T\eta_1^{n-k,1}. \tag{7.59}$$

We shall now show that standard characteristic cohomology and exact linear characteristic cohomology are isomorphic, except for the presence of the constants in the former. Because $\{d_V, d_H\} = 0 = \{d_V, \delta_T\}$, there is a well defined map from standard characteristic cohomology to exact linear characteristic cohomology: $[\omega_0^{n-k,0}] \longrightarrow [\omega_0^{n-k,0}]$. The kernel of this map is given by $\mathbb{R}$ in form degree 0. Indeed, the kernel is defined by a cocycle $\omega_0^{n-k,0}$ of standard characteristic cohomology such that (7.59) holds. Let $\omega^{k,l} = b^{k,l} + \tilde{\omega}^{k,l}$, where $b^{k,l}$ is obtained from $\omega^{k,l}$ by setting to zero all the fields, antifields, their derivatives and their vertical derivatives. By applying the contracting homotopy $\rho_V(\cdot) = \int_0^1 dt/t\ [\partial_{(\mu)} Z^a \partial/\partial d_V Z^a_{(\mu)}(\cdot)][tZ, t d_V Z, x, dx]$ of d_V to (7.59), we get

$$\tilde{\omega}_0^{n-k,0} = -d_H \rho_V \eta_0^{n-k-1,1} - \delta_T \rho_V \eta_1^{n-k,1} + \{\rho_V, \delta_T\} \eta_1^{n-k,1}. \tag{7.60}$$

Furthermore,

$$\{\rho_V, \delta_T\} = \partial_{(\mu)}[2\frac{\delta L}{\delta \phi^i} - \frac{\delta N_\phi(L)}{\delta \phi^i}]\frac{\partial}{\partial d_V \phi^*_{i(\mu)}} - \partial_{(\mu)}[N_\phi(R^{+j}_\alpha)(\phi^*_j)]\frac{\partial}{\partial d_V C^*_{\alpha(\mu)}}. \tag{7.61}$$

This shows that for linear theories one has $\{\rho_V, \delta_T\} = 0$, so that $\tilde{\omega}_0^{n-k,0}$ is a trivial characteristic cohomology class. For linearizable theories, an induction on the homogeneity in the fields $Z, d_V Z$ with $\{Z\} = \{\phi, \phi^*, C^*\}$, allows one to prove the same result in the space of formal power series in $Z, d_V Z$, which is extended to the case of linearizable, normal theories, to spaces involving a finite number of derivatives as in section 6 of [5]. The part $b^{n-k,0}$ of $\omega_0^{n-k,0}$ satisfies $d_H b^{n-k,0} = 0$, implying $b^{n-k,0} = d_H b^{n-k-1,0} + \delta_0^{n-k} k$, $k \in \mathbb{R}$, which gives the result.

The map from standard to exact linear characteristic cohomology is surjective. Indeed, if $\omega_0^{n-k,0} = b^{n-k,0} + \tilde{\omega}_0^{n-k,0}$, the part $b^{n-k,0}$ is always trivial in exact linear characteristic cohomology because $d_V b^{n-k,0} = 0$ satisfies (7.59) with $\eta_0^{n-k-1,0} = 0 = \eta_1^{n-k,1}$, while $\tilde{\omega}_0^{n-k,0}$ corresponds to a cocycle of standard characteristic cohomology. This follows by using the free cohomology of d_V for the cocycle condition (7.58) with $\omega_0^{n-k,0}$ replaced by $\tilde{\omega}_0^{n-k,0}$ and $\omega_1^{n-k+1,0}$ by $\tilde{\omega}_0^{n-k+1,1}$.

Hence, except for the constants, exact and standard characteristic cohomology are indeed isomorphic, and nothing is gained by considering exact linear characteristic cohomology. However, this changes if one evaluates at a fixed background.

7.5.3 Koszul-Tate resolution of the linearized theory

In this and the following subsubsections, the Lagrangian L is the source free Lagrangian relevant near the boundary and $\bar{\phi}(x)$ is a solution of the associated field equations. The differential $\delta_T|_{\bar{\phi}(x)}$ is nilpotent because for a solution $\bar{\phi}(x)$ of the field equations relevant near the boundary, $R^{+i}_\alpha(\delta L/\delta\phi^i) = 0$ implies $R^{+i}_\alpha|_{\bar{\phi}(x)}((d_V \delta L/\delta \phi^i)|_{\bar{\phi}(x)}) = 0$, while $\delta_T|_{\bar{\phi}(x), \phi^*=0} \equiv \delta^{\rm free}$ with

$$\delta^{\rm free} = -\partial_{(\mu)}(d_V \frac{\delta L}{\delta \phi^i})|_{\bar{\phi}(x)} \frac{\partial^S}{\partial d_V \phi^*_{i(\mu)}} - \partial_{(\mu)} \left[R^{+i}_\alpha|_{\bar{\phi}(x)}(d_V \phi^*_i)\right] \frac{\partial^S}{\partial d_V C^*_{\alpha(\mu)}}, \tag{7.62}$$

is acyclic in positive vertical antifield number because, up to an overall shift of grading, it is the Koszul-Tate differential associated with the free theory valid near the boundary. Furthermore, the identity

$$\delta^{\text{free}}(d_V\omega)|_{\bar{\phi}(x),\phi^*=0} = -(d_V\delta_T\omega)|_{\bar{\phi}(x),\phi^*=0}, \tag{7.63}$$

for all ω, will allow us to relate expressions constructed in the free theory near the boundary to expressions constructed in the full theory.

7.5.4 Boundary conditions and asymptotic acyclicity

In the following, the asymptotic behaviour of forms is understood after evaluation for fields and antifields that satisfy the asymptotic behaviour $\varphi^i(x) \longrightarrow O(\chi^i)$, $d^nx\phi^*_i \longrightarrow O(\chi_i)$, $d^nxC^*_\alpha \longrightarrow O(\chi_\alpha)$ with the χ_i, χ_α as defined in section 5.1.

Our aim is to extend the results derived in [36, 5] and reviewed in sections 3 and 7.3 on exact reducibility parameters and conserved $n-2$ forms to their asymptotic counterparts. The analysis in [36, 5] is based on acyclicity properties of d_H and δ. Therefore these results can be extended to the asymptotic context when the differentials d_H and δ^{free} have analogous "*asymptotic acyclicity properties*". More precisely, what one needs is

$$\begin{cases} d_H\omega^k \longrightarrow 0 & \Longleftrightarrow \quad \omega^k \longrightarrow d_H\eta^{k-1} \quad \text{for } 0<k<n, \\ \omega^n \longrightarrow d_H\eta^{n-1} & \Longleftrightarrow \quad \forall d_VZ^A:\ d_VZ^A\dfrac{\delta\omega^n}{\delta d_VZ^A} \longrightarrow 0, \end{cases} \tag{7.64}$$

and

$$\begin{cases} \delta^{\text{free}}\omega_k \longrightarrow 0 \quad \Longleftrightarrow \quad \omega_k \longrightarrow \delta^{\text{free}}\eta_{k+1} \quad \text{for } k\geq 1, \\ \forall\omega_0 \longrightarrow O(1):\ \ \omega_0 \longrightarrow \delta^{\text{free}}\eta_1 \quad \Longleftrightarrow \quad \forall\varphi_s(x):\ \omega_0|_{\varphi_s(x)} \longrightarrow 0, \end{cases} \tag{7.65}$$

on forms which are homogeneous and linear in the variables $\{d_VZ^A\} = \{d_V\phi^i, d_V\phi^*_i, d_VC^*_\alpha\}$ and their derivatives, with coefficients that are ordinary differential forms made up of x^μ and dx^μ [$\varphi_s(x)$ are asymptotic solutions as in (5.13)][7]. Therefore, we shall assume that the boundary conditions are such that (7.64) and (7.65) hold. At first glance, this may appear to be a strong assumption. However, as we tried to show by a detailed analysis of exact reducibility parameters and conserved $n-2$ forms, it is actually quite natural.

The validity of the first part of (7.64) is related to properties of the contracting homotopy associated to d_H: equation (A.10) gives, for a k-form $\omega^k[x, d_VZ]$ depending linearly on the d_VZ^A and their derivatives:

$$\omega^k[x, d_VZ] = \rho^{k+1}_{H,d_VZ}(d_H\omega^k) + d_H(\rho^k_{H,d_VZ}\omega^k), \tag{7.66}$$

[7]As the last conditions in (7.64) and (7.65) show, what we call acyclicity properties includes not only absence of non trivial cohomology in appropriate degrees. In addition it requires that the horizontal complex provides an algebraic resolution of equivalence classes of local n forms with asymptotically identical Euler-Lagrange derivatives, while the Koszul-Tate complex provides asymptotically a resolution of the horizontal forms pulled back to the surface defined by the linearized equations of motion.

with homotopy operators $\rho_{H,d_V Z}$ as in (A.9). The homotopy operators remove a differential dx^ν and one derivative ∂_ν of one of the fields, and redistribute the other derivatives over the fields and the coefficient functions. Hence, whenever the fields, as functions of x^μ, and the coefficient functions are sufficiently well-behaved (as discussed and illustrated in more details in sections 5.1, 5.4 and 6.3.4), asymptotic acyclicity of d_H will indeed hold. Similarly, the second part of (7.64) is related to the identity

$$\omega^n[x, d_V Z] = d_V Z^A \frac{\delta \omega^n}{\delta d_V Z^A} + d_H(\rho^n_{H,d_V Z}\omega^n). \tag{7.67}$$

This identity evidently provides the implication $\Longleftarrow$ of the second part of (7.64); it also gives $\Longrightarrow$ whenever the following (very reasonable) implication holds

$$\forall d_V Z^A :\ d_V Z^A \frac{\delta \omega^n}{\delta d_V Z^A} \longrightarrow d_H \eta'^{n-1} \Longrightarrow d_V Z^A \frac{\delta \omega^n}{\delta d_V Z^A} \longrightarrow 0. \tag{7.68}$$

The first condition in (7.65) is trivially satisfied for $k \geq 3$ because we consider irreducible gauge theories and forms that are linear and homogeneous in the fields and antifields [in irreducible gauge theories, there are no such forms because there are no antifields with antifield number ≥ 3]. For $k = 1, 2$, this condition is equivalent to (5.10) and (5.11), respectively. The second condition in (7.65) is a consequence of the asymptotic regularity conditions discussed in sections 5.1 and 5.4.

7.5.5 Definitions and bijective correspondence

Let $1 \leq k \leq n$. The form $\omega_0^{n-k,0}$ is an asymptotic conservation law of order $n-k$, relative to the fixed background $\bar{\phi}^i(x)$, if the cocycle condition (7.58) holds asymptotically when evaluated at $\bar{\phi}(x)$,

$$d_H(d_V\omega_0^{n-k,0})|_{\bar{\phi}(x)} + \delta^{\text{free}}(d_V\omega_1^{n-k+1,0})|_{\bar{\phi}(x)} \longrightarrow 0. \tag{7.69}$$

The asymptotic conservation law $\omega_0^{n-k,1}$ at $\bar{\phi}(x)$ is trivial if

$$(d_V\omega_0^{n-k,0})|_{\bar{\phi}(x)} \longrightarrow d_H\tilde{\eta}_0^{n-k-1,1} + \delta^{\text{free}}(d_V\eta_1^{n-k,0})|_{\bar{\phi}(x)}, \tag{7.70}$$

where $\tilde{\eta}_0^{n-k-1,1}$ is a local form involving linearly only the fields $d_V\phi^i$. Asymptotic characteristic cohomology is defined as the set of equivalence classes of asymptotic conservation laws up to trivial ones.

The form $\omega_k^{n,0}$ is an asymptotic degree k symmetry for the fixed background $\bar{\phi}(x)$ if

$$\delta^{\text{free}}(d_V\omega_k^{n,0})|_{\bar{\phi}(x),\phi^*=0} + d_H(d_V\eta_{k-1}^{n-1,0})|_{\bar{\phi}(x),\phi^*=0} \longrightarrow 0. \tag{7.71}$$

The degree k asymptotic symmetry at $\bar{\phi}(x)$ is trivial if

$$(d_V\omega_k^{n,0})|_{\bar{\phi}(x),\phi^*=0} \longrightarrow \delta^{\text{free}}\tilde{\eta}_{k+1}^{n,1} + d_H(d_V\eta_k^{n-1,0})|_{\bar{\phi}(x),\phi^*=0}, \tag{7.72}$$

where $\tilde{\eta}^{n,1}_{k+1}$ involves linearly only the vertical derivatives of the antifields. Equivalence classes of asymptotic degree k symmetries are defined as asymptotic degree k symmetries modulo trivial ones.

Because the forms are linear and homogeneous in the fields and antifields and the theory is irreducible, only the cases $k = 1, 2$ can give non trivial cohomology. Furthermore, the asymptotic acyclicity properties assumed in subsubsection 7.5.4 allow one to prove the bijective correspondence between equivalence classes of asymptotic degree k symmetries and of degree $n-k$ conservation laws exactly as done in [36, 5] in the exact case.

7.5.6 Asymptotic global symmetries and asymptotically conserved currents

The cocycle condition both for the asymptotic global (i.e., degree 1) symmetries and the asymptotically conserved $n-1$ forms at $\bar{\phi}(x)$ reads:

$$\delta^{\text{free}}(d_V\omega^{n,0}_1)|_{\bar{\phi}(x),\phi^*=0} + d_H(d_V\omega^{n-1,0}_0)|_{\bar{\phi}(x),\phi^*=0} \longrightarrow 0. \tag{7.73}$$

The general form of an asymptotic global symmetry is

$$\omega^{n,0}_1 = \phi^*_{i(\mu)}Q^{+i(\mu)}d^nx = (\phi^*_iQ^i + \partial_\mu T^{i\mu}(\phi^*_i))d^nx, \tag{7.74}$$

where we used "integrations by parts": we applied repeatedly Leibniz' rule $(\partial f)g = -f(\partial g) + \partial(fg)$ and collected the terms $\partial(fg)$ in $\partial_\mu T^{i\mu}(\phi^*_i)$. The second term in $\omega^{n,0}_1$ corresponds to the trivial asymptotic symmetry $d_Hd_V(T^{i\mu}(\phi^*_i))|_{\bar{\phi}(x),\phi^*=0}(d^{n-1}x)_\mu$ and can be absorbed by the trivial asymptotically conserved $n-1$ form $\delta^{\text{free}}d_V(T^{i\mu}(\phi^*_i))|_{\bar{\phi}(x),\phi^*=0}(d^{n-1}x)_\mu$. Accordingly, we can assume, $\omega^{n,0}_1 = \phi^*_iQ^id^nx$. With $\omega^{n-1,0}_0 = j^\mu(d^{n-1}x)_\mu$, the cocycle condition (7.73) is explicitly given by

$$-(d_V\frac{\delta L}{\delta\phi^i})|_{\bar{\phi}(x)}Q^i|_{\bar{\phi}(x)}d^nx + \partial_\mu(d_Vj^\mu)|_{\bar{\phi}(x)}d^nx \longrightarrow 0, \tag{7.75}$$

which means

$$\frac{\delta L^{\text{free}}}{\delta\varphi^i}Q^i|_{\bar{\phi}(x)}d^nx \longrightarrow \partial_\mu(d_Vj^\mu)|_{\bar{\phi}(x),\varphi}d^nx. \tag{7.76}$$

The coboundary condition (7.72) for $k=1$, with

$$\tilde{\eta}^{n,1}_2 = -d_VC^*_{\alpha(\mu)}\tilde{f}^{\alpha(\mu)}d^nx, \tag{7.77}$$

and $\tilde{f}^{\alpha(\mu)}$ depending on x alone, implies that the asymptotic symmetry defined by Q^i is trivial at $\bar{\phi}(x)$ if and only if

$$\forall\psi_i \longrightarrow O(\chi_i):\ Q^i|_{\bar{\phi}(x)}\psi_i \longrightarrow \psi_iR^i_\alpha|_{\bar{\phi}(x)}(\tilde{f}^\alpha), \tag{7.78}$$

with $\tilde{f}^\alpha = (-\partial)_{(\mu)}\tilde{f}^{\alpha(\mu)}$. This follows by "integrations by parts" and using the fact that the coboundary condition holds for all $d_V\phi^*_i$ satisfying the boundary conditions. The coboundary condition for asymptotically conserved $n-1$ forms states that such a form is trivial if it is asymptotically d_H-exact up to a form that is proportional to the field equations of the linearized theory.

In order to be able to interpret asymptotic global symmetries and asymptotically conserved $n-1$ forms from the point of view of the bulk theory, we use relation (7.63) to rewrite the cocycle condition (7.73) as

$$-d_V(\delta_T\omega_1^{n,0})|_{\bar{\phi}(x),\phi^*=0}+d_H(d_V\omega_0^{n-1,0})|_{\bar{\phi}(x),\phi^*=0}\longrightarrow 0, \tag{7.79}$$

or explicitly,

$$-d_V(\frac{\delta L}{\delta\phi^i}Q^i)|_{\bar{\phi}(x)}d^nx+\partial_\mu(d_Vj^\mu)|_{\bar{\phi}(x)}d^nx\longrightarrow 0. \tag{7.80}$$

In other words, consider the weakly vanishing charge $\int_M \delta L/\delta\phi^i Q^i$ associated to the transformation $\delta\phi^i=Q^i$. The necessary and sufficient condition that allows one to improve this charge by the substraction of a surface integral $\oint_{\partial M} j$ to a charge that is asymptotically extremal at $\bar{\phi}(x)$ for arbitary variations $d_V\phi^i$ not restricted by any boundary conditions is the requirement that Q^i defines an asymptotic global symmetry. For solutions of the equations of motion, the improved charge reduces to the surface integral whose integrand is the asymptotically conserved $n-1$ form.

7.5.7 Asymptotic reducibility parameters and conserved n–2 forms

The cocycle condition for an asymptotic degree 2 symmetry at $\bar{\phi}(x)$ is

$$\delta^{\rm free}(d_V\omega_2^{n,0})|_{\bar{\phi}(x),\phi^*=0}+d_H(d_V\omega_1^{n-1,0})|_{\bar{\phi}(x),\phi^*=0}\longrightarrow 0. \tag{7.81}$$

Applying $\delta^{\rm free}$ and using asymptotic acyclicity of d_H gives

$$\delta^{\rm free}(d_V\omega_1^{n-1,0})|_{\bar{\phi}(x),\phi^*=0}+d_H\tilde{\omega}_0^{n-2,1}\longrightarrow 0. \tag{7.82}$$

The general form of an asymptotic degree 2 symmetry is

$$\omega_2^{n,0}=[f^\alpha C^*_\alpha-\frac{1}{2}\phi^*_{j(\nu)}\phi^*_{i(\mu)}M^{[j(\nu)i(\mu)]}]d^nx, \tag{7.83}$$

where again "integrations by parts" have been done to reduce the first term to one not involving the derivatives of C^*_α. With $\omega_1^{n-1,0}=T^{\mu i}(\phi^*_i)(d^{n-1}x)_\mu$, the cocycle condition gives

$$-R^{+i}_\alpha|_{\bar{\phi}(x)}(d_V\phi^*_i)f^\alpha|_{\bar{\phi}(x)}d^nx+\partial_\mu T^{\mu i}|_{\bar{\phi}(x)}(d_V\phi^*_i)d^nx\longrightarrow 0. \tag{7.84}$$

By using the fact that this equation has to hold for arbitrary $d_V\phi^*_i$ satisfying the boundary conditions, one finds that asymptotic degree 2 symmetries are determined by asymptotic reducibility parameters $\tilde{f}^\alpha=f^\alpha|_{\bar{\phi}(x)}$ satisfying (5.14). According to (1.3), $\omega_1^{n-1,0}$ can be taken to be $S^{\mu i}_\alpha(\phi^*_i,\tilde{f}^\alpha)(d^{n-1}x)_\mu$.

The equation defining the corresponding asymptotically conserved $n-2$ form $k^{[\nu\mu],0}$ can be taken to be

$$\delta^{\text{free}}(d_V S^{\mu i}_\alpha(\phi^*_i, f^\alpha))|_{\bar\phi(x),\phi^*=0}(d^{n-1}x)_\mu + \partial_\nu(d_V k^{[\nu\mu],0}_\alpha(f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu \longrightarrow 0, \quad (7.85)$$

which gives explicitly

$$S^{\mu i}_\alpha(\frac{\delta L^{\text{free}}}{\delta\varphi^i}, f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu + \partial_\nu(d_V k^{[\nu\mu],0}_\alpha(f^\alpha))|_{\bar\phi(x),\varphi}(d^{n-1}x)_\mu \longrightarrow 0. \quad (7.86)$$

How the expression $d_V k^{[\nu\mu],0}_\alpha(f^\alpha))$ can be explicitly constructed out of $s^\mu_f = S^{\mu i}_\alpha(\delta L^{\text{free}}/\delta\varphi^i, f^\alpha))|_{\bar\phi(x)}$ has been explained in section 3.7.

The coboundary condition for the asymptotic symmetry at $\bar\phi(x)$ is $(d_V\omega^{n,0}_2)|_{\bar\phi(x),\phi^*=0} \longrightarrow \delta^{\text{free}}\tilde\eta^{n,1}_3 + d_H(d_V\eta^{n-1,0}_2))|_{\bar\phi(x),\phi^*=0}$. Because there is no $\tilde\eta^{n,1}_3$ in irreducible gauge theories, and integrations by parts have already been used to remove all derivatives from C^*_α, the asymptotic degree 2 symmetries are trivial iff the associated asymptotic reducibility parameters are trivial as defined in (5.15).

Hence, equivalence classes of asymptotic degree 2 symmetries are in bijective correspondence to equivalence classes of asymptotic reducibility parameters and theorem 1 follows from the bijective correspondence between asymptotic degree 2 symmetries and degree $n-2$ conservation laws.

According to (7.63), equation (7.85) can also be written as

$$-(d_V\delta_T S^{\mu i}_\alpha(\phi^*_i, f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu + \partial_\nu(d_V k^{[\nu\mu],0}_\alpha(f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu \longrightarrow 0, \quad (7.87)$$

which gives

$$-(d_V S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu + \partial_\nu(d_V k^{[\nu\mu],0}_\alpha(f^\alpha))|_{\bar\phi(x)}(d^{n-1}x)_\mu \longrightarrow 0. \quad (7.88)$$

This equivalent formulation of the cocycle condition for asymptotic degree 2 symmeties and degree $n-2$ conservation laws proves theorem 3 of subsection 5.3.

7.5.8 Asymptotic algebra

As we have seen above, the asymptotic degree 2 symmetry can be identified with the n form $\omega^{-2,n}_{\tilde f} = \tilde f^\alpha C^*_\alpha d^n x$ of the free theory, with $\tilde f^\alpha$ asymptotic reducibility parameters. Assuming here and everywhere below that condition (5.25) holds, the asymptotic behaviour of this n-form is $\omega^{-2,n}_{\tilde f} \longrightarrow O(1)$. Furthermore we have $s^{\text{free}}\omega^{-2,n}_{\tilde f} = \delta^{\text{free}}\omega^{-2,n}_{\tilde f} \longrightarrow d_H(\;)$.

Let us define the asymptotic behaviour of the ghosts C^α to be the same as that of the asymptotic reducibility parameters: $C^\alpha \longrightarrow O(\chi^\alpha)$. The cubic vertex $\omega^{0,n} = [L^3 + \phi^*_i R^{i1}_\alpha(C^\alpha) + 1/2 C^*_\alpha C^{\alpha 0}_{\beta\gamma}(C^\beta, C^\gamma)]d^n x$ induced by the full theory satisfies $s^{\text{free}}\omega^{0,n} = d_H(\;)$. Furthermore, assumptions (5.29) and (5.30) then guarantee that the form representing the induced "global symmetry" is at most of order 1

$$\omega^{-1,n}_{\tilde f} = \delta_{\omega^{0,n}}\omega^{-2,n}_{\tilde f} + d_H(\cdot) = [\phi^*_i R^{i1}_\alpha(\tilde f^\alpha) + C^*_\alpha C^{\alpha 0}_{\beta\gamma}(C^\beta, \tilde f^\gamma)]d^n x \longrightarrow O(1),$$

with

$$\delta_{\omega^{0,n}} = \partial_{(\mu)}[R^{+i1}_\alpha(\phi^*_i) + C^{+\alpha 0}_{\beta\gamma}(C^*_\alpha, C^\beta)]\frac{\partial^S}{\partial C^*_{\gamma(\mu)}} - \partial_{(\mu)}[\frac{1}{2}C^{\alpha 0}_{\beta\gamma}(C^\beta, C^\gamma)]\frac{\partial^S}{\partial C^\alpha_{(\mu)}}$$
$$+\partial_{(\mu)}[\frac{\delta}{\delta\varphi^j}(L^3 + \phi^*_i R^{i1}_\alpha(C^\alpha))]\frac{\partial^S}{\partial\phi^*_{j(\mu)}} + \partial_{(\mu)}[R^{i1}_\alpha(C^\alpha)]\frac{\partial^S}{\partial\varphi^i_{(\mu)}}, \qquad (7.89)$$

and $[s^{\text{free}}, \delta_{\omega^{0,n}}] = 0$. When $\delta_{\omega^{0,n}}$ is applied to $s^{\text{free}}\omega^{-2,n}_{\tilde{f}}$, only the part that acts on the antifields ϕ^*_i is involved, and conditions (5.31) and (5.32) guarantee that this action does not increase the asymptotic degree. It follows that $\omega^{-1,n}_{\tilde{f}}$ is asymptotically a BRST cocycle modulo d_H

$$s^{\text{free}}\omega^{-1,n}_{\tilde{f}} = \delta_{\omega^{0,n}} \underbrace{s^{\text{free}}\omega^{-2,n}_{\tilde{f}}}_{\longrightarrow d_H()} \longrightarrow d_H(\cdot). \qquad (7.90)$$

Similarly,

$$\omega^{-2,n}_{[\tilde{f}_2,\tilde{f}_1]_M} = \delta_{\omega^{-1,n}_{\tilde{f}_1}}\omega^{-2,n}_{\tilde{f}_2} + d_H(\cdot) = d^n x C^*_\alpha[\tilde{f}_2, \tilde{f}_1]^\alpha_M \longrightarrow O(1),$$

so that $[\tilde{f}_1, \tilde{f}_2]^\alpha_M$ satisfies condition (5.25). Explicitly,

$$\delta_{\omega^{-1,n}_{\tilde{f}_1}} = -\partial_{(\mu)}[C^{+\alpha 0}_{\beta\gamma}(C^*_\alpha, \tilde{f}^\beta_1)]\frac{\partial^S}{\partial C^*_{\gamma(\mu)}} - \partial_{(\mu)}[C^{\alpha 0}_{\beta\gamma}(C^\beta, \tilde{f}^\gamma_1)]\frac{\partial^S}{\partial C^\alpha_{(\mu)}}$$
$$+\partial_{(\mu)}[\frac{\delta}{\delta\varphi^j}(\phi^*_i R^{i1}_\alpha(\tilde{f}^\alpha_1))]\frac{\partial^S}{\partial\phi^*_{j(\mu)}} - \partial_{(\mu)}[R^{i1}_\alpha(\tilde{f}^\alpha_1)]\frac{\partial^S}{\partial\varphi^i_{(\mu)}}. \qquad (7.91)$$

Now, $[s^{\text{free}}, \delta_{\omega^{-1,n}_{\tilde{f}_1}}] = \delta_{s^{\text{free}}\omega^{-1,n}_{\tilde{f}_1}}$ with

$$s^{\text{free}}\omega^{-1,n}_{\tilde{f}_1} = [\frac{\delta L^2}{\delta\varphi^i}R^{i1}_\alpha(\tilde{f}^\alpha_1) - \phi^*_i\partial_{(\mu)}R^{j0}_\beta(C^\beta)\frac{\partial R^{i1}_\alpha(\tilde{f}^\alpha_1)}{\partial\varphi^j_{(\mu)}} + R^{+i0}_\alpha(\phi^*_i)C^{\alpha 0}_{\beta\gamma}(C^\beta, \tilde{f}^\gamma_1)]d^n x.$$

In the action of $\delta_{s^{\text{free}}\omega^{-1,n}_{\tilde{f}_1}}$ on $\omega^{-2,n}_{\tilde{f}_2}$ only the ghost dependent part of $s^{\text{free}}\omega^{-1,n}_{\tilde{f}_1}$ is involved, and this part does not change the asymptotic behaviour because the ghost dependent terms are at most of order 1. It follows from (7.90) that $\delta_{s^{\text{free}}\omega^{-1,n}_{\tilde{f}_1}}\omega^{-2,n}_{\tilde{f}_2} \longrightarrow 0$. Since again the part of $\delta_{\omega^{-1,n}_{\tilde{f}_1}}$ that acts on $s^{\text{free}}\omega^{-2,n}_{\tilde{f}_2}$ only involves the antifields ϕ^*_i and is at most of order 1,

$$s^{\text{free}}\omega^{-2,n}_{[\tilde{f}_2,\tilde{f}_1]_M} \longrightarrow d_H(\cdot).$$

This implies that $\psi_i R^{i0}_\alpha([\tilde{f}_1, \tilde{f}_2]^\alpha_M) \longrightarrow 0$ for all ψ_i, meaning that $[\tilde{f}_1, \tilde{f}_2]^\alpha_M$ are asymptotic reducibility parameters. Hence, asymptotic reducibility parameters form a Lie algebra for the bracket $[\cdot,\cdot]_M$. Finally, when the $\tilde{f}^\alpha_2$ are pure gauge, the associated form $\omega^{-2,n}_{\tilde{f}_2}$

vanishes asymptotically and so do the forms $\omega^{-1,n}_{\tilde f_2}$ and $\omega^{-2,n}_{[\tilde f_2,\tilde f_1]_M}$, which implies that $[\tilde f_2,\tilde f_1]^\alpha_M$ are pure gauge. The skew symmetry (7.32) of the bracket induced in cohomology shows that the same conclusion holds if the $\tilde f^\alpha_1$ are pure gauge. Hence, there is a well defined induced Lie algebra for the quotient space of asymptotic reducibility parameters modulo pure gauge parameters. This completes the proof of theorem 2.

In subsubsection 7.3.3, we have applied $\delta_{\omega^{-1,n}_X}$ to the set of descent equations that relates the n forms representing the (exact) reducibility parameters to the associated conserved $n-2$ forms in order to investigate the action of the associated global symmetry on equivalence classes of conserved $n-2$ forms. In order to use the same reasoning here, we use the fact that the asymptotic behaviour of the operator $\delta_{\omega^{-1,n}_{\tilde f_1}}$ is at most of order 1 due to the additional assumption (5.34). Furthermore, since (4.6) and (1.22) imply

$$s^{\rm free}\omega^{-1,n}_{\tilde f_1} = [-\frac{\delta L^3}{\delta\varphi^i}R^{i0}_\alpha(\tilde f^\alpha_1) + \phi^*_i\partial_{(\mu)}R^{j0}_\beta(\tilde f^\beta_1)\frac{\partial R^{i1}_\alpha(C^\alpha)}{\partial\varphi^j_{(\mu)}}]d^nx + d_H(\cdot), \tag{7.92}$$

the additional assumptions (5.35) and (5.36) guarantee that $[s^{\rm free},\delta_{\omega^{-1,n}_{\tilde f_1}}] = \delta_{s^{\rm free}\omega^{-1,n}_{\tilde f_1}} \longrightarrow 0$. This implies that $\delta_{\omega^{-1,n}_{\tilde f_1}}$ can be applied to the asymptotic descent equations (7.81) and (7.82) and allows one to prove the statements on the representation of the Lie algebra of equivalence classes of asymptotic reducibility parameters in exactly the same way as in the case of equivalence classes of exact reducibility parameters.

8 Relation to other approaches

8.1 Hamiltonian approach

A systematic approach to asymptotic conservation laws, especially in the context of general relativity, was given in [17] in the context of the Hamiltonian formalism. In order to make contact with this approach, we will apply the covariant Lagrangian results obtained in the previous sections to the action in first order Hamiltonian form,

$$S = \int dt\int d^{n-1}x\ (\frac{1}{2}\dot z^A\sigma_{AB}z^B - h - \gamma_a\lambda^a), \tag{8.1}$$

where we assume for simplicity that the constraints γ_a are first class, irreducible and time independent and that σ_{AB} is the symplectic matrix with $\sigma^{AB}\sigma_{BC} = \delta^A_C$. In the following, we will use a local Poisson bracket with spatial Euler-Lagrange derivatives for spatial $n-1$ forms,

$$\{f\, d^{n-1}x, g\, d^{n-1}x\} = \frac{\tilde\delta f}{\delta z^A}\sigma^{AB}\frac{\tilde\delta g}{\delta z^B}\, d^{n-1}x. \tag{8.2}$$

If $\tilde d_H$ denotes the spatial exterior derivative, this Poisson bracket is well defined in the space $H^{n-1}(\tilde d_H)$, and thus does not depend on "boundary terms" that are added to improve spatial functionals. Similiarily, the Hamiltonian vector field associated to an $n-1$ form

$$\tilde{\delta}_{fd^{n-1}x} = \partial_{(i)} \frac{\tilde{\delta} f}{\delta z^A} \sigma^{AB} \frac{\partial^S}{\partial z^B_{(i)}} \tag{8.3}$$

only depends on the class $[f d^{n-1}x] \in H^{n-1}(\tilde{d}_H)$. If we denote

$$\hat{\gamma}_a = \gamma_a \, d^{n-1}x, \quad \hat{h}_E = h \, d^{n-1}x + \lambda^a \hat{\gamma}_a, \tag{8.4}$$

an irreducible generating set of gauge transformations for (8.1) (see e.g. [64] chapter 3) is given by

$$\delta_f z^A = \tilde{\delta}_{\hat{\gamma}_a f^a} z^A = \frac{\tilde{\delta}(\gamma_a f^a)}{\delta z^B} \sigma^{BA}, \tag{8.5}$$

$$\delta_f \lambda^a = \frac{D}{Dt} f^a + \tilde{\delta}_{\hat{h}_E} f^a + C^a_{bc}(\lambda^b, f^c) + V^a_b(f^b), \tag{8.6}$$

where the gauge parameters f^a may depend on x^μ, the Lagrange multipliers and their derivatives as well as the canonical variables and their spatial derivatives and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\lambda}^a \frac{\partial}{\partial \lambda^a} + \ddot{\lambda}^a \frac{\partial}{\partial \dot{\lambda}^a} + \ldots, \tag{8.7}$$

$$\{\hat{\gamma}_a \xi^a_1, \hat{\gamma}_b \xi^b_2\} = \hat{\gamma}_c C^c_{ab}(\xi^a_1, \xi^b_2) + \tilde{d}_H(\ ), \tag{8.8}$$

$$\{h \, d^{n-1}x \ , \hat{\gamma}_a \xi^a\} = \hat{\gamma}_b V^b_a(\xi^a) + \tilde{d}_H(\ ), \tag{8.9}$$

with $\xi^a = \xi^a(x)$.

According to section 3, if one is interested in equivalence classes of exact reducibility parameters, the remaining ambiguity in the f^a is that two f^a that agree on the constraint surface defined by γ_a and their spatial derivatives have to be identified. Let $\approx' 0$ denote terms that vanish on this surface. The condition for exact reducibility parameters then reads

$$\frac{\tilde{\delta}(\gamma_a f^a)}{\delta z^A} \approx' 0, \tag{8.10}$$

$$\frac{D}{Dt} f^a + \tilde{\delta}_{\hat{h}_E} f^a + C^a_{bc}(\lambda^b, f^c) + V^a_b(f^b) \approx' 0. \tag{8.11}$$

This last condition means that one can assume that f^a is independent of the Lagrange multipliers or any of its derivatives. This is so because the highest order time derivative of a Lagrange multiplier needs to be multiplied by a term that vanishes weakly, so it can be be absorbed. This can be repeated until all the dependence on the Lagrange multipliers has been absorbed, so that $f^a = f^a[x, z]$ and D/Dt reduces to $\partial/\partial t$ in the above condition.

If ψ_A are arbitrary fields with the asymptotic behaviour of the Hamiltonian evolution equations times the volume form and ψ_a arbitrary fields that behave like the constraints times the volume form, the conditions that determine the asymptotic reducibility parameters $\tilde{f}^a(x)$ for a given background solution $\bar{z}^A(x)$ are

$$\psi_A \frac{\tilde{\delta}(\gamma_a \tilde{f}^a)}{\delta z^A}\Big|_{\bar{z}(x)} \longrightarrow 0, \tag{8.12}$$

$$\psi_a \Big[\frac{\partial \tilde{f}^a}{\partial t} + V^{a0}_b(\tilde{f}^b)\Big] \longrightarrow 0. \tag{8.13}$$

The first condition implies that the Euler-Lagrange derivatives with respect to the canonical coordinates of the constraints contracted with the asymptotic reducibility parameters have to vanish asymptotically when evaluated at the background, while the second condition fixes the asymptotic time behaviour of the asymptotic reducibility parameters.

In order to construct the associated conservation laws, note that in the first order case

$$\begin{aligned} R^i_\alpha(f^\alpha)\frac{\delta^L L}{\delta \phi^i} &= \frac{\tilde{\delta}(\gamma_c f^c)}{\delta z^C}\sigma^{CA}(-\sigma_{AB}\dot{z}^B - \frac{\delta^L h}{\delta z^A} - \frac{\delta^L \lambda^a \gamma_a}{\delta z^A}) \\ &\quad -(\frac{D}{Dt}f^a + \tilde{\delta}_{\hat{h}_E} f^a + C^a_{bc}(\lambda^b, f^c) + V^a_b(f^b))\gamma_a \\ &= -\frac{d}{dt}(f^a\gamma_a) - \partial_k\Big(V^k_A(\frac{\delta h_E}{\delta z^C}\sigma^{CA}, \gamma_c f^c) + j^{kb}_c(\gamma_b, f^c)\Big), \end{aligned} \tag{8.14}$$

where the current $j^{kb}_c(Q_b, f^c)$ is determined in terms of the Hamiltonian structure operators through the formula

$$\begin{aligned} \partial_k j^{kb}_c(Q_b, f^c) &= Q_a V^a_b(f^b) - f^b V^{+a}_b(Q_a) \\ &\quad + Q_a C^a_{b,c}(\lambda^b, f^c) - C^{+a}_{bc}(\gamma_a, \lambda^b) f^c \end{aligned} \tag{8.15}$$

for all Q_b, f^c. Hence

$$S^{0i}_\alpha(\frac{\delta L}{\delta \phi^i}, f^\alpha) \equiv -\gamma_a f^a, \tag{8.16}$$

$$S^{ki}_\alpha(\frac{\delta L}{\delta \phi^i}, f^\alpha) \equiv -V^k_A(\frac{\delta h_E}{\delta z^C}\sigma^{CA}, \gamma_c f^c) - j^{kb}_c(\gamma_b, f^c). \tag{8.17}$$

By applying theorem 3 for the surface Σ defined by $t = \text{cste}$, we have *proved* that, if the gauge parameters f^a are asymptotic reducibility parameters, the integrated constraints

$$G_a[f^a] = \int d^{n-1}x \ \gamma_a f^a \tag{8.18}$$

can be be improved through the subtraction of a surface integral

$$\oint_{\partial\Sigma} k^{0i}_\alpha(f^\alpha)(d^{n-2}x)_i \tag{8.19}$$

to a charge that is asymptotically extremal at the background $\bar{z}(x)$ for arbitrary variations of the canonical variables not restricted by any boundary conditions. The integrand of the surface integral is determined the time components of the associated conserved $n-2$ form $k^{\mu\nu}_\alpha(f^\alpha)(d^{n-2}x)_{\mu\nu}$. This provides an a posteriori justification of the Hamiltonian procedure of [17].

8.2 Lagrangian Noether method

Theorem 3 or its non integrated formulation in (7.88) can be understood either as a precise formulation, a generalization or a justification of the Lagrangian Noether method of references [4, 22, 66, 24, 25].

8.3 Covariant phase space approach

Let us first recall the two main formulas from the calculus of variations. The first variational formula is simply

$$d_V L = d_V \phi^i \frac{\delta L}{\delta \phi^i} + \partial_\mu V_i^\mu (d_V \phi^i, L) \tag{8.20}$$

The second variational formula is obtained by applying d_V (which is equivalent to taking two different variations and skew-symmetrizing):

$$0 = -d_V \phi^i d_V \frac{\delta L}{\delta \phi^i} + \partial_\mu \omega^\mu, \tag{8.21}$$

where the presymplectic current is defined by

$$\omega^\mu = d_V V_i^\mu (d_V \phi^i, L). \tag{8.22}$$

This formula is contracted with the evolutionary vector field $R^j_f \frac{\partial}{\partial \phi^j}$, i.e., a gauge transformation

$$0 = -R^i_f d_V \frac{\delta L}{\delta \phi^i} + d_V \phi^i \partial_{(\mu)} R^j_f \frac{\partial^S}{\partial \phi^j_{(\mu)}} \frac{\delta L}{\delta \phi^i} + \partial_\mu i_{R_f} \omega^\mu, \tag{8.23}$$

where $i_{R_f}\omega^\mu$ denotes the contraction of ω^μ with the evolutionary vector field $R^j_f \frac{\partial}{\partial \phi^j}$. To the second term, we apply the formula for the commutator of an Euler-Lagrange derivative and an evolutionary vector field (see e.g. [5] equation (6.43)). Using the fact that $R^j_f \frac{\delta L}{\delta \phi^j}$ is a total divergence and is thus annihilitated by the Euler-Lagrange derivative, we get

$$0 = -R^i_f d_V \frac{\delta L}{\delta \phi^i} - d_V \phi^i (-\partial)_{(\lambda)} [\frac{\partial^S R^j_f}{\partial \phi^i_{(\lambda)}} \frac{\delta L}{\delta \phi^j}] + \partial_\mu i_{R_f} \omega^\mu. \tag{8.24}$$

Applying repeatedly Leibniz' rule to the second term, we obtain

$$0 = -d_V (R^i_f \frac{\delta L}{\delta \phi^i}) + \partial_\mu [i_{R_f} \omega^\mu + t^{\mu,1}], \tag{8.25}$$

where $t^{\mu,1}$ is in vertical degree 1 and vanishes weakly as it is linear in the Euler-Lagrange derivatives $\delta L/\delta \phi^i$ and their derivatives. Finally, using (1.4), we deduce

$$\partial_\mu (i_{R_f} \omega^\mu + t^{\mu,1} - d_V (S^{\mu i}_\alpha (\frac{\delta L}{\delta \phi^i}, f^\alpha)) = 0, \tag{8.26}$$

Hence, there exists an $n-2$ form $r^{[\nu\mu],1}$ in vertical degree 1 such that

$$d_V(S^{\mu i}_\alpha(\frac{\delta L}{\delta\phi^i}, f^\alpha)) = i_{R_f}\omega^\mu + t^{\mu,1} + \partial_\nu r^{[\nu\mu],1}, \tag{8.27}$$

Evaluating at a solution $\bar\phi(x)$ of the Euler-Lagrange equations of motion, this gives

$$S^{\mu i}_\alpha|_{\bar\phi(x)}(d_V\frac{\delta L}{\delta\phi^i}|_{\bar\phi(x)}, f^\alpha|_{\bar\phi(x)})(d^{n-1}x)_\mu = i_{R_f}\omega^\mu|_{\bar\phi(x)}(d^{n-1}x)_\mu + \partial_\nu r^{[\nu\mu],1}|_{\bar\phi(x)}(d^{n-1}x)_\mu, \tag{8.28}$$

Since our results imply that the the left hand side of this equation reduces to the exterior derivative of a conserved $n-2$ form if and only if the $f^\alpha|_{\bar\phi(x)}$ are asymptotic reducibility parameters, we get in this case,

$$-d_H(d_V k^{[\nu\mu]}_f(d^{n-2}x)_{\nu\mu} \longrightarrow i_{R_f}\omega^\mu|_{\bar\phi(x)}(d^{n-1}x)_\mu - d_H(r^{[\nu\mu],1}|_{\bar\phi(x)}(d^{n-2}x)_{\nu\mu}) \tag{8.29}$$

In other words, it is possible to subtract the exterior derivative of an $n-2$ form $r^{[\nu\mu],1}|_{\bar\phi(x)}(d^{n-2}x)_{\nu\mu}$ from the presymplectic $n-1$ form contracted with a gauge transformations in order to get asymptotically the exterior derivative of a conserved $n-2$ form if and only if the gauge parameters define asymptotic reducibility parameters.

8.4 Characteristic cohomology and generalized symmetries

Algebraic and differential-geometric techniques have been used for quite some time for the symmetry analysis of standard partial differential equations (see e.g. [2, 42, 40, 44] and references therein). The application of these techniques in the context of gauge theories, i.e., possibly degenerate Lagrangian field theories, is more involved because of the presence of gauge symmetries [67, 41, 36, 68]. Recent results show that in interacting gauge theories, there are only few non trivial global symmetries involving the gauge fields alone. For instance, there are none in the case of pure four dimensional Einstein gravity [69, 70]. (See also [71] for a recent classification of generalized symmetries of semi-simple Yang-Mills theory and compare to free electromagnetism treated in [72, 73, 74, 75, 76]). Similarly, there are only few exact lower degree conservation laws. For instance, there are none in semi-simple Yang-Mills theory or in Einstein gravity (see e.g. [36]).

In [34], lower degree linear characteristic cohomology for generic second-order field equations have been classified directly. In particular, the equations are not assumed to derive from a Lagrangian. The technical starting point for the current investigation has been the "central premise" of [34] (see also [35]) that cohomological techniques "can be successfully adapted to the analysis of asymptotic conservation laws". We have made more restrictive assumptions, namely, that the field equations are Lagrangian and that a generating set of gauge symmetries is known, and have suitably extended the approach of [34] by introducing additional cohomological tools of the BRST-antifield formalism. This has allowed us to flesh out the general classification theorem of [34].

Conclusion

Using cohomological tools as a guiding line, we have investigated in detail asymptotic symmetries and conservation laws, their relation and their algebra.

For simplicity and clarity, we have restricted the present investigation to the case of irreducible gauge theories. The extension to reducible gauge theories or non trivial topology along these lines is straightforward, because the associated cohomological techniques are well under control (see e.g. [44, 77]).

We stress that the cohomological methods used in this paper are technical tools which are not necessary but rather convenient to derive the results. In particular they motivate the definitions of equivalence classes of asymptotic symmetries and conservation laws and facilitate the proof of the various statements. This is because they take into account in a natural way the ambiguities inherent in the definitions. The various definitions and results are stated in this paper both in terms of equivalence classes of quantities that involve only the original fields and also in terms of local BRST cohomology classes. The latter have the advantage that they are manifestly invariant under field redefinitions, changes in the description of the generating set of gauge transformations and elimination of auxiliary and generalized auxiliary fields, because these operations do not modify the local BRST cohomology.

The most essential prerequisites for the validity of the results presented in this work are that the theory is asymptotically linear and the "asymptotic acyclicity properties", as they play a central role in the cohomological analysis.

Sufficient conditions on the asymptotic behaviour of fields and gauge parameters have been given that guarantee bijective correspondence between equivalence classes of asymptotic reducibility parameters and asymptotically conserved $n-2$ forms, finite charges, a well defined algebra and finite central charges. It would be of interest to investigate in how far these desirable properties still hold for more relaxed assumptions, or for different formulations of boundary conditions.

We have tested in this paper our general approach in the non trivial case of three dimensional anti-de Sitter gravity and have been able to reproduce in a straightforward way the results originally obtained by Hamiltonian methods in [21]. In the future, we hope to report on new applications of the present analysis, in particular for models that have not been previously treated in the canonical framework.

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A Appendix

A.1 Conventions and notation

We assume for notational simplicity that all fields ϕ^i are (Grassmann) even.

Consider k-th order derivatives $\frac{\partial^k \phi^i(x)}{\partial x^{\mu_1} \ldots \partial x^{\mu_k}}$ of a field $\phi^i(x)$. The corresponding jet-coordinate is denoted by $\phi^i_{\mu_1 \ldots \mu_k}$. Because the derivatives are symmetric under permutations of the derivative indices $\mu_1, \ldots, \mu_k$, these jet-coordinates are not independent, but one has $\phi^i_{\mu\nu} = \phi^i_{\nu\mu}$ etc. Local functions are smooth functions depending on the coordinates x^μ of the base space M, the fields ϕ^i, and a finite number of the jet-coordinates $\phi^i_{\mu_1 \ldots \mu_k}$. Local horizontal forms involve in addition the differentials dx^μ which we treat as anticommuting (Grassmann odd) variables, $dx^\mu dx^\nu = -dx^\nu dx^\mu$.

As in [78, 44], we define derivatives $\partial^S / \partial \phi^i_{\mu_1 \ldots \mu_k}$ that act on the basic variables through

$$\frac{\partial^S \phi^j_{\nu_1 \ldots \nu_k}}{\partial \phi^i_{\mu_1 \ldots \mu_k}} = \delta^j_i \, \delta^{\mu_1}_{(\nu_1} \ldots \delta^{\mu_k}_{\nu_k)} \, , \quad \frac{\partial^S \phi^j_{\nu_1 \ldots \nu_m}}{\partial \phi^i_{\mu_1 \ldots \mu_k}} = 0 \quad \text{for } m \neq k,$$
$$\frac{\partial^S x^\mu}{\partial \phi^i_{\mu_1 \ldots \mu_k}} = 0, \quad \frac{\partial^S dx^\mu}{\partial \phi^i_{\mu_1 \ldots \mu_k}} = 0, \tag{A.1}$$

where the round parantheses denote symmetrization with weight one,

$$\delta^{\mu_1}_{(\nu_1} \delta^{\mu_2}_{\nu_2)} = \frac{1}{2} (\delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} + \delta^{\mu_1}_{\nu_2} \delta^{\mu_2}_{\nu_1}) \quad \text{etc.}$$

For instance, the definition gives explicitly (with ϕ any of the ϕ^i):

$$\frac{\partial^S \phi_{11}}{\partial \phi_{11}} = 1 \, , \quad \frac{\partial^S \phi_{12}}{\partial \phi_{12}} = \frac{\partial^S \phi_{21}}{\partial \phi_{12}} = \frac{1}{2} \, , \quad \frac{\partial^S \phi_{112}}{\partial \phi_{112}} = \frac{1}{3} \, , \quad \frac{\partial^S \phi_{123}}{\partial \phi_{123}} = \frac{1}{6} \, .$$

We note that the use of these operators takes automatically care of many combinatorical factors which arise in other conventions, such as those used in [2].

The vertical differential is defined by (1.18) with Grassmann odd generators $d_V \phi^i_{\mu_1 \ldots \mu_k}$, so that $d_V^2 = 0$. The total derivative is the vector field denoted by ∂_ν and acts on local functions according to

$$\partial_\nu = \frac{\partial}{\partial x^\nu} + \sum_{k=0} \phi^i_{\mu_1 \ldots \mu_k \nu} \frac{\partial^S}{\partial \phi^i_{\mu_1 \ldots \mu_k}} \, . \tag{A.2}$$

Here $\sum_{k=0}$ means the sum over all k, from $k = 0$ to infinity, with the summand for $k = 0$ given by $\phi^i_\nu \partial/\partial\phi^i$, i.e., by definition $k = 0$ means "no indices μ_i". Furthermore we are using Einstein's summation convention over repeated indices, i.e., for each k there is a summation over all tupels $(\mu_1, \ldots, \mu_k)$. Hence, for $k = 2$, the sum over μ_1 and μ_2 contains both the tupel $(\mu_1, \mu_2) = (1, 2)$ and the tupel $(\mu_1, \mu_2) = (2, 1)$. These conventions extend to all other sums of similar type.

The horizontal differential on horizontal forms is defined by $d_H = dx^\nu \partial_\nu$. It is extended to the vertical generators in such a way that $\{d_H, d_V\} = 0$.

A vector field of the form $Q^i \partial/\partial\phi^i$, for Q^i a set of local functions, is called an evolutionary vector field with characteristic Q^i. Its prolongation which acts on local functions is

$$\delta_Q = \sum_{k=0} (\partial_{\mu_1} \ldots \partial_{\mu_k} Q^i) \frac{\partial^S}{\partial\phi^i_{\mu_1 \ldots \mu_k}} . \tag{A.3}$$

More details on the variational bicomplex can be found for instance in the textbooks [2, 79, 42, 44].

The set of multiindices is simply the set of all tupels $(\mu_1, \ldots, \mu_k)$, including (for $k = 0$) the empty tupel. The tuple with one element is denoted by μ_1 without round parentheses, while a generic tuple is denoted by (μ). The length, i.e., the number of individual indices, of a multiindex (μ) is denoted by $|\mu|$. We use Einstein's summation convention also for repeated multiindices as in [44]. For instance, an expression of the type $(-\partial)_{(\mu)} K^{(\mu)}$ stands for a free sum over all tupels $(\mu_1, \ldots, \mu_k)$ analogous to the one in (A.2),

$$(-\partial)_{(\mu)} K^{(\mu)} = \sum_{k=0} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} K^{\mu_1 \ldots \mu_k}.$$

If $Z = Z^{(\mu)} \partial_{(\mu)}$ is a differential operator, its adjoint is defined by $Z^+ = (-\partial)_{(\nu)} [Z^{(\nu)} \cdot]$ and its 'components' are denoted by $Z^{+(\mu)}$, i.e., $Z^+ = Z^{+(\mu)} \partial_{(\mu)}$. Furthermore, we assume that the Euler-Lagrange equations of motion and their total derivatives $\partial_{(\mu)} \delta L/\delta\phi^i$ satisfy suitable regularity conditions, such that a local function f vanishes when evaluated on *every* solution $\phi^i(x)$ of the Euler-Lagrange equations of motion ($f|_{\phi(x)} = 0$ whenever $(\delta L/\delta\phi^i)|_{\phi(x)} = 0$) if and only if $f = G^i \delta L/\delta\phi^i$ for some differential operators $G^i = G^{i(\mu)} \partial_{(\mu)}$.

A.2 Higher order Lie-Euler operators

Except for a different notation, we follow in this and the next subsection [44]. The higher order Lie-Euler operators $\delta/\delta\phi^i_{\mu_1 \ldots \mu_k}$ can be defined through the formula

$$\forall Q^i : \quad \delta_Q f = \partial_{(\mu)} \Big[Q^i \frac{\delta f}{\delta\phi^i_{(\mu)}} \Big]. \tag{A.4}$$

Explicitly,

$$\frac{\delta f}{\delta\phi^i_{(\mu)}} = \begin{pmatrix} |\mu| + |\nu| \\ |\mu| \end{pmatrix} (-\partial)_{(\nu)} \frac{\partial^S f}{\partial\phi^i_{((\mu)(\nu))}} , \tag{A.5}$$

or, equivalently,

$$\frac{\delta f}{\delta \phi^i_{\mu_1\ldots\mu_k}} = \sum_{l=0} \begin{pmatrix} k+l \\ k \end{pmatrix} (-)^l \partial_{\nu_1}\ldots\partial_{\nu_l} \frac{\partial^S f}{\partial \phi^i_{\mu_1\ldots\mu_k\nu_1\ldots\nu_l}} \ , \tag{A.6}$$

i.e., there is a summation over (ν) in (A.5) by Einstein's summation convention for repeated multiindices, and the multiindex $((\mu)(\nu))$ is the tupel $(\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_l)$ when (μ) and (ν) are the tupels $(\mu_1,\ldots,\mu_k)$ and $(\nu_1,\ldots,\nu_l)$, respectively. Note that the sum contains only finitely many nonvanishing terms whenever f is a local function: if f depends only on variables with at most M "derivatives", i.e., on the $\phi^i_{(\rho)}$ with $|\rho|\leq M$, the only possibly nonvanishing summands are those with $|\nu|\leq M-|\mu|$ $(l\leq M-k)$. Note also that $\delta/\delta\phi^i$ is the Euler-Lagrange derivative.

The crucial property of these operators is that they "absorb total derivatives",

$$|\mu|=0: \quad \frac{\delta(\partial_\nu f)}{\delta\phi^i} = 0, \tag{A.7}$$

$$|\mu|>0: \quad \frac{\delta(\partial_\nu f)}{\delta\phi^i_{(\mu)}} = \delta_\nu^{(\mu}\frac{\delta f}{\delta\phi^i_{(\mu'))}} \ , \quad (\mu)=(\mu(\mu')), \tag{A.8}$$

where, e.g.,

$$\delta_\nu^{(\mu}\frac{\delta f}{\delta\phi^i_{\lambda)}} = \frac{1}{2}\big(\delta_\nu^\mu\frac{\delta f}{\delta\phi^i_\lambda}+\delta_\nu^\lambda\frac{\delta f}{\delta\phi^i_\mu}\big).$$

A.3 Contracting homotopy for the horizontal complex

Define

$$\rho^p_{H,\phi}(\omega^p) = \int_0^1 dt\ \frac{|\mu|+1}{n-p+|\mu|+1}\ \partial_{(\mu)}\left(\phi^i\Big[\frac{\delta}{\delta\phi^i_{((\mu)\nu)}}\frac{\partial\omega^p}{\partial dx^\nu}\Big][x,t\phi]\right) \tag{A.9}$$

for ω^p a horizontal p-form (note that there is a summation over (μ) by Einstein's summation convention). Then:

$$0\leq p<n: \quad \omega^p[x,\phi]-\omega^p[x,0] = \rho^{p+1}_{H,\phi}(d_H\omega)+d_H(\rho^p_{H,\phi}\omega); \tag{A.10}$$

$$p=n: \quad \omega^n[x,\phi]-\omega^n[x,0] = \int_0^1 dt\ \phi^i\Big[\frac{\delta\omega^n}{\delta\phi^i}\Big][x,t\phi]+d_H(\rho^n_{H,\phi}\omega^n). \tag{A.11}$$

A.4 Proof of theorem 4

Proof of equations (5.51) and (5.52). Using (5.46) and (5.25), we have $\delta_{\tilde f_1}\tilde k_{\tilde f_2} \longrightarrow \delta^0_{\tilde f_1}\tilde k_{\tilde f_2}+\delta^1_{\tilde f_1}\tilde k_{\tilde f_2}$. According to (1.26), $\delta^1_{\tilde f_1}\tilde k_{\tilde f_2} \stackrel{\approx^{\rm free}}{\longrightarrow} \tilde k_{[\tilde f_1,\tilde f_2]_M}+d_H(\ )$. Integrating over the boundary $\mathcal{C}^{n-2}$ and using the definition (5.49) togther with (5.39) then gives directly (5.51) and (5.52).

Proof of equations (5.53) and (5.54). Consider the $(n-1)$-form

$$s_f[\varphi,\bar{\phi}(x)] = s_f^\mu[\varphi,\bar{\phi}(x)](d^{n-1}x)_\mu$$

where $s_f^\mu[\varphi,\bar{\phi}(x)]$ is the weakly vanishing Noether current of the free theory (1.11) for arbitrary field independent gauge parameters f^α (rather than for asymptotic reducibility parameters). We apply formula (A.10) to this $(n-1)$-form, using the contracting homotopy (A.9) for the φ. Since each term in $s_f[\varphi,\bar{\phi}(x)]$ is linear and homogeneous in the φ and their derivatives, one has $s_f[0,\bar{\phi}(x)] = 0$ (and the integral over t can be evaluated trivially). Furthermore, $\rho^{n-1}_{H,\varphi} s_f[\varphi,\bar{\phi}(x)]$ is nothing but $-\tilde{k}_f[\varphi,\bar{\phi}(x)]$. Hence, we obtain from (A.10):

$$s_f[\varphi,\bar{\phi}(x)] = -d_H\tilde{k}_f[\varphi,\bar{\phi}(x)] + \rho^n_{H,\varphi} d_H s_f[\varphi,\bar{\phi}(x)]. \tag{A.12}$$

We now apply a transformation $\delta^0_{f'}$ ($\delta^0_{f'}\varphi^i = R^{i0}_\alpha f^{\alpha\prime}$) with arbitrary field independent gauge parameters $f^{\alpha\prime}$ to (A.12). The facts that $\delta^0_{f'}$ is a gauge symmetry of $L^{\rm free}$ and that $\delta^0_{f'}\varphi^i$ does not depend on the φ or their derivatives implies

$$\delta^0_{f'}\,\frac{\delta L^{\rm free}}{\delta\varphi^i} = 0. \tag{A.13}$$

[This can be verified using formula (6.43) in [5], for instance.] As $s_f[\varphi,\bar{\phi}(x)]$ depends on the φ and their derivatives only via the $\delta L^{\rm free}/\delta\varphi^i$, (A.13) implies $\delta^0_{f'} s_f[\varphi,\bar{\phi}(x)] = 0$. Hence, applying $\delta^0_{f'}$ to (A.12), one obtains

$$d_H\tilde{k}_f[R^0_{f'},\bar{\phi}(x)] = \delta^0_{f'}\rho^n_{H,\varphi} d_H s_f[\varphi,\bar{\phi}(x)] = \delta^0_{f'}\rho^n_{H,\varphi}[\delta^0_f\varphi^i L^{\rm free}_i d^n x], \tag{A.14}$$

where $L^{\rm free}_i \equiv \delta L^{\rm free}/\delta\varphi^i$. The point is now that the last expression on the right hand side of (A.14) is skew symmetric under the exchange of f and f',

$$\delta^0_{f'}\rho^n_{H,\varphi}[\delta^0_f\varphi^i L^{\rm free}_i d^n x] = -\delta^0_f\rho^n_{H,\varphi}[\delta^0_{f'}\varphi^i L^{\rm free}_i d^n x] \tag{A.15}$$

and thus also

$$\delta^0_{f'}\rho^n_{H,\varphi} d_H s_f[\varphi,\bar{\phi}(x)] = -\delta^0_f\rho^n_{H,\varphi} d_H s_{f'}[\varphi,\bar{\phi}(x)], \tag{A.16}$$

owing to the properties of $\rho^n_{H,\varphi}$. Let us postpone the demonstration of (A.15) and first complete the proof of the theorem, assuming that (A.15) holds. Equations (A.14) and (A.16) imply

$$d_H(\tilde{k}_f[R^0_{f'},\bar{\phi}(x)] + \tilde{k}_{f'}[R^0_f,\bar{\phi}(x)]) = 0. \tag{A.17}$$

As f^α (and $f^{\alpha\prime}$) are arbitrary functions, one can apply the contracting homotopy $\rho^{n-1}_{H,f}$. Using (A.10) and $\tilde{k}_0[R^0_{f'},\bar{\phi}(x)] = 0 = \tilde{k}_{f'}[0,\bar{\phi}(x)]$ gives

$$\tilde{k}_f[R^0_{f'},\bar{\phi}(x)] + \tilde{k}_{f'}[R^0_f,\bar{\phi}(x)] = d_H\rho^{n-2}_{H,f}\Big(\tilde{k}_f[R^0_{f'},\bar{\phi}(x)] + \tilde{k}_{f'}[R^0_f,\bar{\phi}(x)]\Big), \tag{A.18}$$

proving (5.53). Integration of (A.18) over $\mathcal{C}^{n-2}$ yields

$$K_{f',f} + K_{f,f'} = 0, \tag{A.19}$$

for any $f^{\alpha}(x)$ and $f^{\alpha\prime}(x)$. This implies (5.54) and shows that the skew symmetry of K is actually not restricted to asymptotic reducibility parameters but holds for to general gauge parameters. Notice also that this proof of (5.54) uses only the standard algebraic Poincaré lemma rather than its asymptotic version (7.64), i.e., (A.19) holds independently of assumptions on the boundary conditions.

Proof of equation (5.55). The proof of (5.55) is now very easy. (1.22) implies

$$[\delta_{\tilde{f}_1}, \delta_{\tilde{f}_2}]Q_{\tilde{f}_3} \approx \delta_{[\tilde{f}_1,\tilde{f}_2]_P} Q_{\tilde{f}_3} \ , \tag{A.20}$$

We first evaluate the commutator on the left hand side of (A.20) using (5.51), and then extract from the result the φ-independent part. We obtain from (5.51):

$$\delta_{\tilde{f}_1}(\delta_{\tilde{f}_2} Q_{\tilde{f}_3}) \sim \delta_{\tilde{f}_1} Q_{[\tilde{f}_2,\tilde{f}_3]_M} \sim Q_{[\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M]_M} + K_{\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M} - N_{[\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M]_M}.$$

Hence, the φ-independent part of the left hand side of (A.20) is

$$K_{\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M} - K_{\tilde{f}_2,[\tilde{f}_1,\tilde{f}_3]_M} - N_{[\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M]_M} + N_{[\tilde{f}_2,[\tilde{f}_1,\tilde{f}_3]_M]_M}.$$

The φ-independent part of the right hand side is $\sim (K_{[\tilde{f}_1,\tilde{f}_2]_M,\tilde{f}_3} - N_{[[\tilde{f}_1,\tilde{f}_2]_M,\tilde{f}_3]_M})$, where we used again (5.51). Since (A.20) gives an exact equality for the φ-independent part (the equation-of-motion-terms do not contribute to this part since they are at least linear in the φ), and since the terms involving the normalization vanish separately because of the Jacobi identity for $[\cdot,\cdot]_M$, (which corresponds to $(\delta^{CE})^2 N = 0$), we obtain

$$K_{\tilde{f}_1,[\tilde{f}_2,\tilde{f}_3]_M} - K_{\tilde{f}_2,[\tilde{f}_1,\tilde{f}_3]_M} = K_{[\tilde{f}_1,\tilde{f}_2]_M,\tilde{f}_3} \ . \tag{A.21}$$

Equation (5.55) follows immediately from (A.21) and (A.19).

Direct demonstration of equation (A.15). Writing out $\rho^n_{H,\varphi}$ in the left hand side of (A.15) gives

$$\begin{aligned} &\rho^n_{H,\varphi}(\delta^0_f \varphi^i L^{\mathrm{free}}_i d^n x) = \omega^\nu_f (d^{n-1}x)_\nu, \\ &\omega^\nu_f = \begin{pmatrix} |\mu| + |\rho| + 1 \\ |\mu| + 1 \end{pmatrix} \partial_{(\mu)} \Big[\varphi^i (-\partial)_{(\rho)} \frac{\partial^S \delta^0_f \varphi^j L^{\mathrm{free}}_j}{\partial \varphi^i_{((\mu)(\rho)\nu)}} \Big]. \end{aligned} \tag{A.22}$$

Equation (A.15) can be directly verified by evaluation of (A.22) for a general quadratic Lagrangian L^{free}. Up to a total divergence which can be neglected because it gives no contribution to the Euler-Lagrange derivatives, every Lagrangian L^{free} takes the form

$$L^{\mathrm{free}} = \varphi^i a^{(\mu)}_{ij} \varphi^j_{(\mu)} \ , \tag{A.23}$$

where $a_{ij}^{(\mu)}$ are x-dependent coefficient functions (of the background fields and their derivatives). The Euler-Lagrange derivatives of $L^{\rm free}$ are

$$L_i^{\rm free} = a_{ij}^{(\mu)}\varphi^j_{(\mu)} + (-)^{|\mu|+|\rho|}\begin{pmatrix}|\mu|+|\rho|\\ |\rho|\end{pmatrix}\varphi^j_{(\mu)}\partial_{(\rho)}a_{ji}^{((\mu)(\rho))}. \tag{A.24}$$

One now inserts (A.24) in (A.22) and verifies (A.15) by direct computation. This reduces to an exercise in binomial coefficients. The binomial coefficients which occur are those in (A.22), those in (A.24) and those coming from distributing the derivatives in $\partial_{(\mu)}$ and $(-\partial)_{(\rho)}$ occurring in (A.22). The computation is straightforward but the formulas become involved. Let us explicitly demonstrate it for a Lagrangian with two derivatives because it involves all characteristic features (the cases without or with only one derivative are rather trivial),

$$L^{\rm free} = \varphi^i a_{ij}^{\mu\nu}\varphi^j_{\mu\nu}\ .$$

Its Euler-Lagrange derivatives are

$$L_i^{\rm free} = \varphi^j_{\mu\nu}(a_{ij}^{\mu\nu} + a_{ji}^{\mu\nu}) + 2\varphi^j_\mu\partial_\nu a_{ji}^{\mu\nu} + \varphi^j\partial_\mu\partial_\nu a_{ji}^{\mu\nu}.$$

Since these contain no third or higher order derivatives of the φ, the only nonvanishing contributions to the sums in (A.22) are those with $(|\mu|,|\rho|)\in\{(0,0),(1,0),(0,1)\}$,

$$\begin{aligned}\omega_f^\nu &= \varphi^i\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_\nu} + \partial_\mu\Big[\varphi^i\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_{\mu\nu}}\Big] - 2\varphi^i\partial_\rho\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_{\rho\nu}}\\ &= \varphi^i\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_\nu} + \varphi^i_\mu\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_{\mu\nu}} - \varphi^i\partial_\mu\,\frac{\partial^S\delta_f^0\varphi^j L_j^{\rm free}}{\partial\varphi^i_{\mu\nu}}\ .\end{aligned}$$

One explicitly obtains

$$\begin{aligned}&\omega_f^\nu = 2\varphi^i\delta_f^0\varphi^j\partial_\mu a_{ij}^{\mu\nu} + \varphi^i_\mu\delta_f^0\varphi^j(a_{ij}^{\mu\nu}+a_{ji}^{\mu\nu}) - \varphi^i\partial_\mu[(a_{ij}^{\mu\nu}+a_{ji}^{\mu\nu})\delta_f^0\varphi^j]\\ \Longrightarrow\quad &\delta_{f'}^0\omega_f^\nu = \delta_{f'}^0\varphi^i\delta_f^0\varphi^j\partial_\mu a_{ij}^{\mu\nu} + \delta_{f'}^0\varphi^i_\mu\delta_f^0\varphi^j(a_{ij}^{\mu\nu}+a_{ji}^{\mu\nu}) - (f\leftrightarrow f'),\end{aligned}$$

which demonstrates (A.15) for this case. Analogously one can verify (A.15) for a Lagrangian with any other fixed number of derivatives which then implies (A.15) because every Lagrangian $L^{\rm free}$ is a linear combination of such particular Lagrangians.

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