Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 100 2002

DIVERGENCE TERMS IN THE SUPERTRACE HEAT ASYMPTOTICS FOR THE DE RHAM COMPLEX ON A MANIFOLD WITH BOUNDARY

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Abstract. We use invariance theory to determine the coefficient $a^{d+\delta}_{m+1,m}$ in the super trace for the twisted de Rham complex with absolute boundary conditions.

1. Introduction

Let (M,g) be a compact m dimensional Riemannian manifold with smooth, non-empty boundary ∂M. Let $D_{\mathcal{B}}$ be the realization of an operator of Laplace type on M with respect to a suitable local boundary condition $\mathcal{B}$ and let $f\in C^\infty(M)$ be a smooth smearing function. The Greiner-Seeley calculus [14, 19] can be used to show that the fundamental solution of the heat equation $e^{-tD_{\mathcal{B}}}$ is an infinitely smoothing operator and that the smeared heat trace has a complete asymptotic expansion as $t\downarrow 0$ of the form:

$$\mathrm{Tr}_{L^2}(fe^{-tD_{\mathcal{B}}})\sim\textstyle\sum_{n\geq 0}a_{n,m}(f,D,\mathcal{B})t^{(n-m)/2}.$$

The *heat trace invariants* $a_{n,m}(f,D,\mathcal{B})$ are locally computable. Let $\nabla_m^k f$ be the k^{th} covariant derivative of f with respect to the inward unit normal on ∂M. Let dx and dy be the Riemannian elements of volume on M and on ∂M, respectively. There exist local invariants $a_{n,m}(x,D)$ and $a_{n,m,k}(y,D,\mathcal{B})$ which are defined on M and on ∂M, respectively, so:

$$\begin{array}{rcl}a_{n,m}(f,D,\mathcal{B})&=&\int_M f(x)a_{n,m}(x,D)dx\\ &+&\sum_k\int_{\partial M}\left(\nabla_m^k f(y)\right)a_{n,m,k}(y,D,\mathcal{B})dy.\end{array}$$

The interior invariants $a_{n,m}(x,D)$ vanish if n is odd; the boundary invariants $a_{n,m,k}(y,D,\mathcal{B})$ are generically non-zero for all $n\geq 1$. The presence of the smearing function f localizes the problem and permits the recovery of divergence terms which would otherwise be lost. The presence of terms involving $\nabla_m^k f$ indicates that the kernel function for $e^{-tD_{\mathcal{B}}}$ behaves asymptotically as $t\downarrow 0$ like a distribution near the boundary. We refer to the discussion in Section 2 for further details.

This formalism can be applied to index theory. Let $\phi\in C^\infty(M)$ be an auxiliary smooth function called the dilaton. Let $d_\phi:=e^{-\phi}de^{\phi}$ and let $\delta_{\phi,g}:=e^{\phi}\delta_g e^{-\phi}$ be the twisted exterior derivative and the co-derivative, respectively, on the space of smooth differential forms. The *twisted* or *Witten* Laplacian is given by:

$$\Delta^p_{\phi,g}:=d_\phi\delta_{\phi,g}+\delta_{\phi,g}d_\phi\quad\text{on}\quad C^\infty(\Lambda^p(M)).$$

Key words and phrases. Heat trace asymptotics, twisted de Rham complex, Witten Laplacian, invariants of the orthogonal group.

2000 *Mathematics Subject Classification:* 58J50.

[1]Research partially supported by the NSF (USA) and Mittag-Leffler (Stockholm, Sweden).

[2]Research partially supported by the MPI (Leipzig, Germany).

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This operator appears in the study of quantum p form fields interacting with a background dilaton [13, 21]. It has also been used in supersymmetric quantum mechanics [5] and in Morse theory [23].

We impose *absolute boundary conditions* $\mathcal{B}_a$, see [12] for details, motivated by the Hodge-de Rham theorem:

$$\ker(\Delta^p_{0,g,\mathcal{B}_a}) = H^p(M).$$

We shall not consider relative boundary conditions $\mathcal{B}_r$ since [13]

$$a_{n,m}(f, \Delta^p_{\phi,g,\mathcal{B}_a}) = a_{n,m}(f, \Delta^{m-p}_{-\phi,g,\mathcal{B}_r}).$$

We define the local *supertrace heat asymptotics* by setting:

$$\begin{aligned} a^{d+\delta}_{n,m}(\phi, g)(x) &:= \textstyle\sum_p (-1)^p a_{n,m}(x, \Delta^p_{\phi,g}), \\ a^{d+\delta}_{n,m,k}(\phi, g)(y) &:= \textstyle\sum_p (-1)^p a_{n,m,k}(y, \Delta^p_{\phi,g}, \mathcal{B}_a). \end{aligned}$$

We set $f = 1$. We also assume ϕ satisfies Neumann boundary conditions, i.e. the normal derivative of ϕ vanishes on ∂M. This ensures that d_ϕ and $\delta_{\phi,g}$ interact properly with absolute boundary conditions. The cancellation argument of McKean and Singer [16] then generalizes to this setting to yield a formula for the Euler-Poincaré characteristic $\chi(M)$:

$$\textstyle\sum_p (-1)^p \mathrm{Tr}_{L^2}(e^{-t\Delta^p_{\phi,g,\mathcal{B}_a}}) = \chi(M);$$

see [13] for details. Equating terms in the asymptotic series then shows:

Lemma 1.1. *Let ϕ satisfy Neumann boundary conditions. Then*

(1) *If $n = m$, $\int_M a^{d+\delta}_{n,m}(\phi, g)(x)dx + \int_{\partial M} a^{d+\delta}_{n,m,0}(\phi, g)(y)dy = \chi(M)$.*

(2) *If $n \neq m$, $\int_M a^{d+\delta}_{n,m}(\phi, g)(x)dx + \int_{\partial M} a^{d+\delta}_{n,m,0}(\phi, g)(y)dy = 0$.*

The local index density has been computed in this setting [13]. Let indices $i, j, ...$ range from 1 to m and index a local orthonormal frame for the tangent bundle of M; let R_{ijkl} be the associated components of the Riemann curvature tensor with the sign convention that $R_{1221} = +1$ on the unit sphere $S^2 \subset \mathbb{R}^3$. Near the boundary, normalize the choice of the orthonormal frame so e_m is the inward unit geodesic normal. Let indices $a, b, ...$ range from 1 to $m-1$ and index the induced orthonormal frame for the tangent bundle of the boundary; let L_{ab} be the components of the second fundamental form.

We proved the following vanishing theorem [13] generalizing previous results of [1, 10, 11, 18] to the twisted setting.

Theorem 1.2.

(1) *If n is odd, then $a^{d+\delta}_{n,m}(\phi, g) = 0$.*

(2) *If m is odd, then $a^{d+\delta}_{n,m}(0, g) = 0$.*

(3) *If $n < m$, then $a^{d+\delta}_{n,m}(\phi, g) = 0$.*

(4) *If $n + k < m$, then $a^{d+\delta}_{n,m,k}(\phi, g) = 0$.*

Let $\varepsilon^V_U := g(e_{u_1} \wedge ... \wedge e_{u_\mu}, e_{v_1} \wedge ... \wedge e_{v_\mu})$ be the totally anti-symmetric tensor. Let I and J be m tuples of indices indexing an orthonormal frame for $T(M)$. Let A and B be $m-1$ tuples of indices indexing an orthonormal frame for $T(\partial M)$. Set:

$$\begin{aligned} \mathcal{R}^{I,t}_{J,s} &:= R_{i_s i_{s+1} j_{s+1} j_s} ... R_{i_{t-1} i_t j_t j_{t-1}}, \\ \mathcal{R}^{A,t}_{B,s} &:= R_{a_s a_{s+1} b_{s+1} b_s} ... R_{a_{t-1} a_t b_t b_{t-1}}, \\ \mathcal{L}^{A,t}_{B,s} &:= L_{a_s b_s} ... L_{a_t b_t}. \end{aligned}$$

We set $\mathcal{R}^{I,t}_{J,s} = 1$, $\mathcal{R}^{A,t}_{B,s} = 1$, and $\mathcal{L}^{A,t}_{B,s} = 1$ if $t < s$.

We adopt the Einstein convention and sum over repeated indices and refer to [13] for the proof of the following result which identifies the local index density in the twisted setting:

Theorem 1.3.

(1) *If $m=2\bar m$ is even,* $a^{d+\delta}_{m,m}(\phi,g)=\frac{1}{\pi^{\bar m}8^{\bar m}\bar m!}\varepsilon^I_J\mathcal{R}^{I,m}_{J,1}$.

(2) $a^{d+\delta}_{m,m,0}(\phi,g)=\sum_k\frac{1}{\pi^k8^kk!(m-1-2k)!\mathrm{vol}(\mathrm{S}^{m-1-2k})}\varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}$.

The fact that the local index density was not dependent on the dilaton field had important physical consequences [13]. One can also combine Lemma 1.1 and Theorem 1.3 to obtain a heat equation proof of the Chern-Gauss-Bonnet theorem [8, 9] for manifolds with boundary:

$$\begin{aligned}\chi(M^{2\bar m}) &= \int_M\frac{1}{\pi^{\bar m}8^{\bar m}\bar m!}\varepsilon^I_J\mathcal{R}^{I,m}_{J,1}dx\\ &+ \sum_k\int_{\partial M}\frac{1}{\pi^k8^kk!(m-1-2k)!\mathrm{vol}(S^{m-1-2k})}\varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}dy,\\ \chi(M^{2\bar m+1}) &= \sum_k\int_{\partial M}\frac{1}{\pi^k8^kk!(m-1-2k)!\mathrm{vol}(S^{m-1-2k})}\varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}dy.\end{aligned}$$

By Theorem 1.2, the first non-trivial 'divergence' terms can first arise in the supertrace when $n=m+1$. Let ';' denote multiple covariant differentiation with respect to the Levi-Civita connection on M. By Theorem 1.2, $a^{d+\delta}_{m+1,m}(\phi,g)=0$ if m is even. Furthermore $a^{d+\delta}_{m+1,m,k}(\phi,g)=0$ if $k\ge 2$. The following is the main result of this paper:

Theorem 1.4.

(1) *If $m=2\bar m+1$ is odd, then* $a^{d+\delta}_{m+1,m}(\phi,g)=\frac{1}{\sqrt{\pi}}\frac{1}{\pi^{\bar m}8^{\bar m}\bar m!}\varepsilon^I_J\phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}$.

(2) $a^{d+\delta}_{m+1,m,0}=\sum_k\frac{1}{\sqrt{\pi}}\frac{1}{\pi^k8^kk!(m-2k-2)!\mathrm{vol}(\mathrm{S}^{m-2k-2})}\varepsilon^A_B\phi_{;a_1b_1}\mathcal{R}^{A,2k+1}_{B,2}\mathcal{L}^{A,m-1}_{B,2k+2}$.

(3) $a^{d+\delta}_{m+1,m,1}=\sum_k\frac{1}{8^k\pi^kk!}\frac{\sqrt{\pi}}{\mathrm{vol}(\mathrm{S}^{m-2k})(\mathrm{m}-2k)!}\varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}$.

Let M be a closed manifold. The local index density for the untwisted de Rham complex was identified in dimension 2 by McKean and Singer [16] and in arbitrary dimensions by Atiyah, Bott, and Patodi [1], by Gilkey [10], and by Patodi [18]. The case of manifolds with boundary was studied in [11]. We also refer to [2, 3, 17] for other treatments of the local index theorem.

Patodi's approach involved a direct calculation analyzing cancellation formulas for the fundamental solution of the heat equation. Atiyah, Bott, and Patodi used invariance theory to identify the local index density for the twisted signature and twisted spin complexes. They then expressed the de Rham complex locally in terms of the spin complex twisted by a suitable coefficient bundle. Neither of these approaches seemed particularly well adapted to the twisted setting. In particular, since the operator d_ϕ relies on the $\mathbb{Z}$ grading of the de Rham complex, it is not described in terms of an operator on the twisted signature or spin complexes. Thus we choose in [13] to generalize the approach of [10] to determine the local index density for the twisted de Rham complex.

There are explicit combinatorial formulas [6, 7, 15] for the invariants $a_{n,m}(f,D,\mathcal{B})$ for $n\le 5$, see the discussion in Section 2 for further details. However, these formulas become very complicated and it seems hopeless to prove Theorem 1.4 by an explicit computation. Instead, we proceed indirectly. In Section 3, we establish some functorial properties of these invariants. In Section 4, we discuss H. Weyl's results on the invariants of the orthogonal group. We introduce the following notation:

Definition 1.5. (1) $\mathcal{F}^{1,k}_{m,m}:=\varepsilon^A_B\phi_{;a_1b_1}\mathcal{R}^{A,2k+1}_{B,2}\mathcal{L}^{A,m-1}_{B,2k+2}$,

(2) $\mathcal{F}^{2,k}_{m,m}:=\varepsilon^A_B\phi_{;a_1}\phi_{;b_1}\mathcal{R}^{A,2k+1}_{B,2}\mathcal{L}^{A,m-1}_{B,2k+2}$,

(3) $\mathcal{F}^{3,k}_{m,m}:=\varepsilon^A_BR_{a_1a_2b_2b_1;m}\mathcal{R}^{A,2k}_{B,3}\mathcal{L}^{A,m-1}_{B,2k+1}$,

(4) $\mathcal{F}^{4,k}_{m,m}:=\varepsilon^A_BR_{a_1mmb_1}\mathcal{R}^{A,2k+1}_{B,2}\mathcal{L}^{A,m-1}_{B,2k+2}$,

(5) $\mathcal{F}^{5,k}_{m,m}:=\varepsilon^A_BR_{a_1mb_2b_1}R_{a_2a_3mb_3}\mathcal{R}^{A,2k+1}_{B,4}\mathcal{L}^{A,m-1}_{B,2k+2}$,

(6) $\mathcal{F}^{6,k}_{m,m}:=\varepsilon^A_BL_{a_1c}L_{cb_1}\mathcal{R}^{A,2k+1}_{B,2}\mathcal{L}^{A,m-1}_{B,2k+2}$,

(7) $\mathcal{F}^{7,k}_{m,m}:=\varepsilon^A_BL_{a_1c}R_{ca_2b_2b_1}\mathcal{R}^{A,2k}_{B,3}\mathcal{L}^{A,m-1}_{B,2k+1}$,

(8) $\mathcal{F}^{8,k}_{m,m} := \varepsilon^A_B R_{a_1cb_2b_1} R_{a_2a_3cb_3} \mathcal{R}^{A,2k+1}_{B,4} \mathcal{L}^{A,m-1}_{B,2k+2}$.

In Section 5, we use the results of Sections 3 and 4 to prove:

Lemma 1.6. *There are universal constants so:*

(1) $a^{d+\delta}_{m+1,m}(\phi,g) = c_{m+1,m}\varepsilon^I_J \phi_{;i_1j_1}\mathcal{R}^{I,m}_{J,2}$ *for* m *odd.*
(2) $a^{d+\delta}_{m+1,m,0}(\phi,g) = \sum_{i,k} c^{i,k}_{m,m}\mathcal{F}^{i,k}_{m,m}$.
(3) $a^{d+\delta}_{m+1,m,1}(\phi,g) = \sum_k c^k_{m+1,m,1}\varepsilon^A_B \mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}$.

This is an enormous simplification in that it reduces the proof of Theorem 1.4 to the evaluation of the relatively small number of undetermined coefficients $\{c_{m+1,m}, c^{i,k}_{m,m}, c^k_{m+1,m,1}\}$. In Section 6 we evaluate the constants $c_{m+1,m}$ and $c^{i,k}_{m,m}$ and in Section 7 we evaluate the constants $c^k_{m+1,m,1}$.

The approach taken by Gilkey in [10] suffered from the disadvantage that the techniques involved were rather ad hoc and cumbersome as they did not make full use of the machinery of invariance theory developed by H. Weyl [22]. In the present paper, we use both the first and second main theorems of invariance theory; this enters in a crucial way in the proof of Lemma 1.6.

2. Formulas for the heat trace asymptotics

We summarize some properties of the heat trace invariants which we shall need and refer to [6] for the proof:

Lemma 2.1.

(1) *Let* $(M,D,\mathcal{B}) = (M_1\times M_2, D_1\otimes \mathrm{Id} + \mathrm{Id}\otimes D_2, \mathcal{B})$ *where* $\partial M_1 = \emptyset$, *where* D_i *are operators of Laplace type on* M_i, *and where* $\mathcal{B}$ *arises from structures on* M_2. *Then*

$$a_{n,m}((x_1,x_2),D) = \textstyle\sum_{n_1+n_2=n} a_{n_1,m_1}(x_1,D_1)\cdot a_{n_2,m_2}(x_2,D_2),\ \textit{and}$$
$$a_{n,m,k}((x_1,y_2),D,\mathcal{B}) = \textstyle\sum_{n_1+n_2=n} a_{n_1,m_1}(x_1,D_1)\cdot a_{n_2,m_2,k}(y_2,D_2,\mathcal{B}).$$

(2) *Let* $D_\varepsilon := e^{-2\varepsilon f}D_0$ *and let* $\mathcal{B}_\varepsilon := e^{-\varepsilon f}\mathcal{B}_0$. *Then*

$$\partial_\varepsilon|_{\varepsilon=0} a_n(1,D_\varepsilon,\mathcal{B}_\varepsilon) = (m-n)a_n(f,D_0,\mathcal{B}_0).$$

Let D be an arbitrary operator of Laplace type on a vector bundle V. There is a canonical connection ∇ on V which we use to differentiate tensors of all types and a canonical endomorphism E of V so that

$$Du = -(u_{;ii} + Eu);$$

see, for example, the discussion in [12]. Let Ω_{ij} be the components of the curvature endomorphism defined by ∇.

Let χ be an endomorphism of $V|_{\partial M}$ so $\chi^2 = 1$. Decompose $\chi = \Pi_+ - \Pi_-$ where $\Pi_\pm := \frac{1}{2}(\mathrm{Id}\pm\chi)$ are the projections on the ± 1 eigenspaces of χ. We extend χ and $\Pi_\pm$ to be parallel with respect to the geodesic normal vector field near ∂M. Let $\mathcal{S}$ be an auxiliary endomorphism of Π_+. We impose Robin boundary conditions on $V_+ := \mathrm{Range}(\Pi_+)$ and Dirichlet boundary conditions on $V_- := \mathrm{Range}(\Pi_-)$ to define the mixed boundary operator:

$$\mathcal{B} := \{\Pi_+(\nabla_m + \mathcal{S}) \oplus \Pi_-\}|_{\partial M}\,.$$

We refer to [6] for the proof of the following result which expresses the heat trace asymptotics in terms of this formalism for $n \le 3$:

Lemma 2.2.

(1) $a_0(f,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \mathrm{Tr}(f\,\mathrm{Id})dx$.
(2) $a_1(f,D,\mathcal{B}) = (4\pi)^{-(m-1)/2}\frac{1}{4}\int_{\partial M}\mathrm{Tr}(f\chi)dy$.
(3) $a_2(f,D,\mathcal{B}) = (4\pi)^{-m/2}\frac{1}{6}\int_M \mathrm{Tr}\{f(6E + R_{ijji}\,\mathrm{Id})\}dx$
$+(4\pi)^{-m/2}\frac{1}{6}\int_{\partial M}\mathrm{Tr}\{f(2L_{aa}\mathrm{Id} + 12\mathcal{S}) + 3f_{;m}\chi\}dy$.

(4) $a_3(f,D,\mathcal{B})=(4\pi)^{-(m-1)/2}\frac{1}{384}\int_{\partial M}\operatorname{Tr}\{f(96\chi E+16\chi R_{ijji}+8\chi R_{amam}$
$+[13\Pi_+-7\Pi_-]L_{aa}L_{bb}+[2\Pi_++10\Pi_-]L_{ab}L_{ab}+96\mathcal{S}L_{aa}+192\mathcal{S}^2$
$-12\chi_{:a}\chi_{:a})+f_{;m}([6\Pi_++30\Pi_-]L_{aa}+96\mathcal{S})+24\chi f_{;mm}\}dy.$

Explicit formulas are available [6, 7, 15, 20] for $n=4,5$ in terms of this formalism as well. What is crucial to our analysis, however, is the general form of these expressions. They are the trace of certain non-commutative polynomials in the covariant derivatives of the variables $\{R,E,\Omega,\mathcal{S},L\}$ with indices contracted in pairs.

The following is a technical result we shall need in Section 7.

Lemma 2.3. *Let D be an operator of Laplace type on a compact Riemannian manifold M with smooth boundary ∂M. Let $\mathcal{B}=\mathcal{B}_{\chi,\mathcal{S}}$ define mixed boundary conditions.*

(1) *We may expand $a_{n,m,k}(y,D,\mathcal{B})=\mathfrak{c}_{n,m,k}\operatorname{Tr}\{\mathcal{S}^{n-k-1}\}$ + other terms, where the coefficients $\mathfrak{c}_{n,m,k}$ are universal constants which are independent of the particular admissible Weyl spanning set chosen for the space of invariants.*
(2) *We have $\mathfrak{c}_{m+1,m,1}=\frac{\sqrt{\pi}}{\operatorname{vol}(S^m)m!}$.*

Proof. To proof assertion (1) we express the invariants $a_{n,m,k}$ in terms of a Weyl spanning set – we take traces of monomials involving the variables $\{E,\Omega,R,L,\mathcal{S},\chi\}$ and their covariant derivatives and then contract indices in pairs; this is discussed in more detail subsequently in Section 4. Formulas of this type are illustrated in Lemma 2.2. However, such formulas are not unique - we can commute covariant derivatives at the cost of introducing additional curvature terms and there are additional relations, as we shall discuss presently in Theorem 4.4. However, this indeterminacy does not affect the variables $\operatorname{Tr}\{\mathcal{S}^{n-k-1}\nabla_m^k f\}$ since there are no tangential indices present to be contracted nor is the curvature present; thus the coefficients of these terms are uniquely determined regardless of the particular admissible Weyl spanning set chosen. Lemma 2.3 (1) now follows.

Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M. We take Robin boundary conditions $\mathcal{B}$ for the scalar Laplacian D. Let $\mathcal{O}$ be a small neighborhood of a point $P_0\in\partial M$. We suppose that the metric is flat on $\mathcal{O}$, that $\partial M\cap\mathcal{O}$ is totally geodesic, that $\mathcal{S}$ is constant on $\mathcal{O}$, and that f is compactly supported in $\mathcal{O}$. The discussion in [4] (see display (16), Section II.B) then yields:

$$\operatorname{Tr}_{L^2}(fe^{-tD_{\mathcal{B}}})\sim\sum_{l,k}(4\pi)^{-(m-1)/2}\int_{\partial M\cap\mathcal{O}}\frac{\nabla_m^k f\operatorname{Tr}(\mathcal{S}^l)t^{(l+k-m+1)/2}}{2^{k+1}\Gamma(\frac{l+k}{2}+1)}. \tag{2.a}$$

Taking $k=1$ and $l=m-1$ in equation (2.a) then yields

$$\mathfrak{c}_{m+1,m,1}=(4\pi)^{-(m-1)/2}\frac{1}{4\Gamma(\frac{m}{2}+1)}. \tag{2.b}$$

We complete the proof by expressing $\mathfrak{c}_{m+1,m,1}$ in terms of $\operatorname{vol}(S^m)$; we use $\operatorname{vol}(S^m)$ rather than the Γ function to ensure compatibility with the formulas derived previously in Theorem 1.3. We have, see, for example, [12]:

$$\operatorname{vol}(S^{2j-1})=\frac{2\pi^j}{(j-1)!}\quad\text{and}\quad\operatorname{vol}(S^{2j})=\frac{j!\pi^j2^{2j+1}}{(2j)!}.$$

We prove assertion (2) if $m=2j-1$ by computing:

$$\begin{aligned}\frac{\sqrt{\pi}}{\operatorname{vol}(S^{2j-1})(2j-1)!}&=\frac{\sqrt{\pi}(j-1)!}{2\pi^j(2j-1)!}=\frac{\sqrt{\pi}}{2\pi^j2^{j-1}(2j-1)(2j-3)\dots1}\\&=\frac{\sqrt{\pi}}{2^{2j}\pi^j(j-\frac12)(j-\frac32)\dots(\frac12)}=\frac{1}{2^{2j}\pi^{j-1}\Gamma(j+\frac12)}=\frac{1}{4(4\pi)^{(m-1)/2}\Gamma(\frac{m}{2}+1)}.\end{aligned}$$

We complete the proof by computing for $m=2j$ that:

$$\frac{\sqrt{\pi}}{\operatorname{vol}(S^{2j})(2j)!}=\frac{\sqrt{\pi}}{j!\pi^j2^{2j+1}}=\frac{\sqrt{\pi}}{\pi^j2^{2j+1}\Gamma(j+1)}=\frac{1}{4(4\pi)^{(m-1)/2}\Gamma(\frac{m}{2}+1)}.$$ □

This Lemma is crucial to our investigations. Consequently, we will give a second proof in Appendix A which is entirely self-contained.

To apply Lemma 2.2 to the setting at hand, we must identify the structures which are involved for the twisted Laplacian. Let $\mathfrak{e}_i:\omega\to e_i\wedge\omega$ be exterior multiplication by the covector e_i and let $\mathfrak{i}_i$ be the dual operator, interior multiplication by e_i. Let $\gamma_i = \mathfrak{e}_i - \mathfrak{i}_i$ give the Clifford module structure on the exterior algebra. Extend the Levi-Civita connection to act on tensors of all types and let Ω_{ij} be the associated curvature operator.

Lemma 2.4. (1) $\Delta_{\phi,g} = \Delta_g + \phi_{;i}\phi_{;i} \cdot \mathrm{Id} + \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i)$.

(2) *The Levi-Civita connection is the connection associated to $\Delta_{\phi,g}$.*

(3) $E_{\phi,g} := -\frac{1}{2}\gamma_i\gamma_j\Omega_{ij} - \phi_{;i}\phi_{;i} - \phi_{;ji}(\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_j\mathfrak{e}_i)$ *is the endomorphism for $\Delta_{\phi,g}$.*

(4) *Absolute boundary conditions are defined by taking Π_+ to be orthogonal projection on the tangential differential forms $\Lambda(\partial M)$ and taking $\mathcal{S} = -L_{ab}\mathfrak{e}_b\mathfrak{i}_a$.*

(5) *We have $\chi_{:a} = 2L_{ab}(\mathfrak{e}_b\mathfrak{i}_m + \mathfrak{e}_m\mathfrak{i}_b)$*

Proof. We extend the formulas $d = \mathfrak{e}_i\nabla_i$ and $\delta_g = -\mathfrak{i}_j\nabla_j$ to the twisted setting:

$$d_\phi = \mathfrak{e}_i\nabla_i + \mathfrak{e}_i\phi_{;i} \quad \text{and} \quad \delta_{\phi,g} = -\mathfrak{i}_i\nabla_i + \mathfrak{i}_i\phi_{;i}\,.$$

We use the commutation rules $\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i = \delta_{ij}$, the fact that $\nabla\mathfrak{e} = 0$, and the fact that $\nabla\mathfrak{i} = 0$ to prove assertion (1) by computing:

$$\begin{aligned}
\Delta_{\phi,g} &= \Delta_g + \mathfrak{e}_i\nabla_i\mathfrak{i}_j\phi_{;j} + \mathfrak{i}_j\phi_{;j}\mathfrak{e}_i\nabla_i - \mathfrak{i}_i\nabla_i\mathfrak{e}_j\phi_{;j} \\
&\quad -\mathfrak{e}_j\phi_{;j}\mathfrak{i}_i\nabla_i + (\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i)\phi_{;i}\phi_{;j} \\
&= \Delta_g + (\mathfrak{e}_i\mathfrak{i}_j + \mathfrak{i}_j\mathfrak{e}_i - \mathfrak{i}_i\mathfrak{e}_j - \mathfrak{e}_j\mathfrak{i}_i)\phi_{;j}\nabla_i + (\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_i\mathfrak{e}_j)\phi_{;ji} + \phi_{;i}\phi_{;i} \\
&= \Delta_g + (\mathfrak{e}_i\mathfrak{i}_j - \mathfrak{i}_i\mathfrak{e}_j)\phi_{;ji} + \phi_{;i}\phi_{;i}.
\end{aligned}$$

This shows that the associated connection does not depend on ϕ and hence is the Levi-Civita connection [12]. Since the standard Weitzenböck formulas yield $E(\Delta_g) = -\frac{1}{2}\gamma_i\gamma_j\Omega_{ij}$, assertion (3) follows.

We refer to [6] for the proof of assertion (4). Let $\omega_+ := e^{a_1} \wedge ... \wedge e^{a_\ell}$ and $\omega_- := e^m \wedge \omega_+$. We then have $\chi\omega_\pm = \pm\omega_\pm$. We use assertion (4) to prove assertion (5) by computing:

$$\begin{aligned}
(\nabla_{e_a}\chi - \chi\nabla_{e_a})\omega_+ &= (\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b + \Gamma_{abm}\mathfrak{e}_m\mathfrak{i}_b)\omega_+ - (\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b - \Gamma_{abm}\mathfrak{e}_m\mathfrak{i}_b)\omega_+ \\
&= 2L_{ab}\mathfrak{e}_m\mathfrak{i}_b\omega_+, \\
(\nabla_{e_a}\chi - \chi\nabla_{e_a})\omega_- &= -(\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b + \Gamma_{amb}\mathfrak{e}_b\mathfrak{i}_m)\omega_- + (\Gamma_{abc}\mathfrak{e}_c\mathfrak{i}_b - \Gamma_{amb}\mathfrak{e}_b\mathfrak{i}_m)\omega_- \\
&= 2L_{ab}\mathfrak{e}_b\mathfrak{i}_m\omega_-.
\end{aligned}$$

□

3. Properties of the super trace invariants

We begin our discussion with:

Lemma 3.1. (1) *For any m, $a^{d+\delta}_{0,m}(\phi, g) = 0$.*

(2) *On the circle, $a^{d+\delta}_{2,1} = \frac{1}{\sqrt{\pi}}\phi_{;11}$.*

(3) *We have $a^{d+\delta}_{n,m}(\phi, g)(x) = (-1)^m a^{d+\delta}_{n,m}(-\phi, g)(x)$.*

(4) *We have $\int_{\partial M} a^{d+\delta}_{m+1,m,0}(0, g)dy = 0$.*

Proof. We use Lemma 2.2 to prove assertion (1) by computing:

$$a^{d+\delta}_{0,m} = (4\pi)^{-m/2}\textstyle\sum_p(-1)^p \dim(\Lambda^p(M)) = 0.$$

Let $\theta \in [0, 2\pi]$ be the usual periodic parameter on the circle S^1. We set $\phi_\theta := \partial_\theta\phi$, $\phi_{\theta\theta} := \partial_\theta\phi_\theta$, etc. We use the canonical 1 form $d\theta$ to identify $C^\infty(\Lambda^1(S^1))$ with the trivial bundle and derive assertion (2) from Lemma 2.2 by computing:

$$\begin{aligned}
&d_\phi = \partial_\theta + \phi_\theta, && \delta_\phi = -\partial_\theta + \phi_\theta, \\
&\Delta^0_\phi = -(\partial^2_\theta + \phi_{\theta\theta} - \phi^2_\theta), && \Delta^1_\phi = -(\partial^2_\theta - \phi_{\theta\theta} - \phi^2_\theta), \\
&E^0_\phi = \phi_{\theta\theta} - \phi^2_\theta, && E^1_\phi = -\phi_{\theta\theta} - \phi^2_\theta.
\end{aligned}$$

Since the interior invariants $a^{d+\delta}_{n,m}(\phi,g)$ are local, we may suppose without loss of generality that M is a closed orientable manifold in the proof of assertion (3). Let $\tilde\star_g$ be the normalized Hodge operator defined by the metric. Then, the normalizations having taken into account the sign conventions, the usual intertwining relations extend to the twisted context to show

$$\tilde\star_g^2=\mathrm{id},\quad \tilde\star_g d_\phi\,\tilde\star_g=\delta_{\phi,g},\quad\text{and}\quad \tilde\star_g\,\delta_{\phi,g}\,\tilde\star_g=d_\phi\,.$$

Assertion (3) now follows from the intertwining relationship:

$$\tilde\star_g\,\Delta^{\ p}_{\phi,g}\,\tilde\star_g=\Delta^{\ m-p}_{-\phi,g}\,.$$

We note that $\tilde\star_g$ intertwines absolute and relative boundary conditions; thus we can not conclude a similar equivariance property for the boundary invariants.

We use Theorem 1.2 (1) if m is even and assertion (3) if m is odd to see $a^{d+\delta}_{m+1,m}(0,g)=0$. Assertion (4) now follows from Lemma 1.1. □

There are useful product formulas.

Lemma 3.2. *Let $(M,\phi,g):=(M_1\times M_2,\phi_1+\phi_2,g_1+g_2)$ where $\partial M_1=\emptyset$. Then*

1. $a^{d+\delta}_{n,m}(\phi,g)=\sum_{n_1+n_2=n}a^{d+\delta}_{n_1,m_1}(\phi_1,g_1)\cdot a^{d+\delta}_{n_2,m_2}(\phi_2,g_2)$,
2. $a^{d+\delta}_{n,m,k}(\phi,g)=\sum_{n_1+n_2=n}a^{d+\delta}_{n_1,m_1}(\phi_1,g_1)\cdot a^{d+\delta}_{n_2,m_2,k}(\phi_2,g_2)$.
3. *If m is even,* $a^{d+\delta}_{m+1,m}(\phi,g)=a^{d+\delta}_{m_1,m_1}(\phi_1,g_1)a^{d+\delta}_{m_2+1,m_2}(\phi_2,g_2)$ *and* $a^{d+\delta}_{m+1,m,k}(\phi,g)=a^{d+\delta}_{m_1,m_1}(\phi_1,g_1)a^{d+\delta}_{m_2+1,m_2,k}(\phi_2,g_2)$.
4. *If m is odd, then* $a^{d+\delta}_{m+1,m}(\phi,g)=a^{d+\delta}_{m_1+1,m_1}(\phi_1,g_1)a^{d+\delta}_{m_2,m_2}(\phi_2,g_2)$ *and* $a^{d+\delta}_{m+1,m,k}(\phi,g)=a^{d+\delta}_{m_1+1,m_1}(\phi_1,g_1)a^{d+\delta}_{m_2,m_2,k}(\phi_2,g_2)$.

Proof. We may decompose

$$\Lambda(M)=\Lambda(M_1)\otimes\Lambda(M_2),\ d_\phi=d_1+d_2,\ \text{and}\ \delta_{\phi,g}=\delta_1+\delta_2$$

where, on $C^\infty(\Lambda^p(M_1)\otimes\Lambda^q(M_2))$, we have

$$\begin{aligned}d_1&:=d_{\phi_1}\otimes\mathrm{Id},\quad & d_2&:=(-1)^p\,\mathrm{Id}\otimes d_{\phi_2},\\ \delta_1&:=\delta_{\phi_1,g_1}\otimes\mathrm{Id},\quad & \delta_2&:=(-1)^p\,\mathrm{Id}\otimes\delta_{\phi_2,g_2}.\end{aligned}$$

Consequently these operators satisfy the commutation relations:

$$d_1d_2+d_2d_1=0,\ d_1\delta_2+\delta_2d_1=0,\ \delta_1d_2+d_2\delta_1=0,\ \delta_1\delta_2+\delta_2\delta_1=0\,.$$

Thus the associated Laplacian can be expressed in the form:

$$\Delta_{\phi,g}=\Delta_{\phi_1,g_1}\otimes\mathrm{Id}+\mathrm{Id}\otimes\Delta_{\phi_2,g_2}\,.$$

Assertions (1) and (2) now follow from Lemma 2.1 (1).

We set $n=m+1$. Assertions (3) and (4) now follow from assertions (1) and (2) and from the vanishing results of Theorem 1.3. □

Let $\mathcal{Q}^M_m$ be the space of all $O(m)$ invariant polynomials in the components of $(R,\nabla R,\nabla\phi,\nabla^2R,\nabla^2\phi,...)$ and let $\mathcal{Q}^{\partial M}_m$ be the space of all $O(m-1)$ polynomials in the components of $(R,L,\nabla R,\nabla L,\nabla\phi,\nabla^2R,\nabla^2L,\nabla^2\phi,...)$ where we only covariantly differentiate L tangentially. In light of Lemma 2.4 (1), we only permit monomials which either do not involve ϕ or which involve at least two covariant derivatives of ϕ.

These spaces have a natural filtration. If A is a monomial of degree (k_R,k_L,k_ϕ) in (R,L,ϕ) and if k_∇ explicit covariant derivatives appear, then the *weight* of A is $2k_R+k_L+k_\nabla$. An invariant polynomial Q is homogeneous of weight n if and only if it satisfies the scaling property:

$$Q(\phi,c^{-2}g)=c^nQ(\phi,g)\,.$$

We use the filtration provided by the weight to decompose

$$\mathcal{Q}_m^M = \oplus_n \mathcal{Q}_{n,m}^M \quad \text{and} \quad \mathcal{Q}_m^{\partial M} = \oplus_n \mathcal{Q}_{n,m}^{\partial M}\,.$$

Let $\mathcal{P}_{n,m}^M \subset \mathcal{Q}_{n,m}^M$ and $\mathcal{P}_{n,m}^{\partial M} \subset \mathcal{Q}_{n,m}^{\partial M}$ be the subspaces of those invariants which do not involve the auxiliary function ϕ. For example,

$$\phi_{;ii} \in \mathcal{Q}_{2,m}^M, \quad \phi_{;mm} \in \mathcal{Q}_{2,m}^{\partial M}, \quad \text{and} \quad L_{aa}R_{ijji} \in \mathcal{P}_{3,m}^{\partial M}\,.$$

There is a useful restriction property that expresses the fact that the Euler class is an unstable characteristic class. If (N, ϕ_N, g_N) are structures in dimension $m-1$, then we can define corresponding structures in dimension m by setting

$$(M, \phi_M, g_M) := (N \times S^1, \phi_N, g_N + d\theta^2)\,.$$

If $y \in \partial N$ is the point of evaluation, we take the corresponding point $(y,1) \in \partial M$ for evaluation. (It does not matter which point is chosen on the circle owing to the rotational symmetry.) The restriction maps

$$r : \mathcal{Q}_{n,m}^M \to \mathcal{Q}_{n,m-1}^N \quad \text{and} \quad r : \mathcal{Q}_{n,m}^{\partial M} \to \mathcal{Q}_{n,m-1}^{\partial N}$$

are characterized dually by the formula:

$$r(Q)(\phi_N, g_N)(x) = Q(\phi_N, g_N + d\theta^2)(x,1)\,. \tag{3.a}$$

We can also describe the restriction map r in classical terms. In Section 4, we will use H. Weyl's results [22] on the invariants of the orthogonal group to show that all orthogonal invariants are built by contracting indices in pairs. If Q is given in terms of a Weyl spanning set, then $r(Q)$ is given in terms of the same Weyl spanning set by restricting the range of summation. Thus r is surjective.

Lemma 3.3. *We have* $a_{n,m}^{d+\delta} \in \ker(r) \cap \mathcal{Q}_{n,m}^M$ *and* $a_{n,m,k}^{d+\delta} \in \ker(r) \cap \mathcal{Q}_{n-k-1,m}^{\partial M}$.

Proof. The Greiner-Seeley calculus [14, 19] can be used to show that

$$a_{n,m}^{d+\delta} \in \mathcal{Q}_{n,m}^{\partial M} \quad \text{and} \quad a_{n,m,k}^{d+\delta} \in \mathcal{Q}_{n-k-1,m}^{\partial M}.$$

By Lemma 3.1 (1), $a_{0,1}^{d+\delta} = 0$. The metric is flat on the circle. Thus if $\phi = 0$, $a_{n,1}^{d+\delta} = 0$ for $n \geq 1$. Consequently, by Lemma 3.2 (1,2),

$$a_{n,m}^{d+\delta}(\phi, g)(x) = 0 \text{ and } a_{n,m,k}^{d+\delta}(\phi, g)(y) = 0 \quad \text{on} \quad (S^1 \times N, \phi_N, d\theta^2 + ds_N^2)\,.$$

Therefore equation (3.a) implies $r(a_{n,m}^{d+\delta}) = 0$ and $r(a_{n,m,k}^{d+\delta}) = 0$. □

4. Invariance theory

Let V be an m dimensional real vector space which is equipped with a positive definite inner product $g(\cdot,\cdot)$. Let $O(V)$ be the associated orthogonal group. One says that a polynomial map $f : \times^k V \to \mathbb{R}$ is an *orthogonal invariant* if

$$f(\xi v^1, ..., \xi v^k) = f(v^1, ..., v^k) \quad \forall \xi \in O(V) \quad \text{and} \quad \forall (v^1, ..., v^k) \in \times^k V\,.$$

Weyl's first theorem of invariants [22] (Theorem 2.9.A) is the following:

Theorem 4.1. *Every orthogonal invariant depending on k vectors $(v_1, ..., v_k)$ in $\times^k V$ is expressible in terms of the k^2 scalar invariants $g(v_i, v_j)$.*

Let $\mathcal{I}_{k,m}$ be the set of all multilinear invariant maps from $\times^k V$ to $\mathbb{R}$, i.e.

$$\mathcal{I}_{k,m} := \operatorname{Hom}_{\mathbb{R}}^{O(V)}(\otimes^k V, \mathbb{R});$$

only the dimension m of V is really relevant so we suppress V from the notation. Given our interest is in $O(V)$ and not $SO(V)$ invariance, we have $\mathcal{I}_{k,m} = \{0\}$ if k is odd and we suppose $k = 2\bar{k}$ is even henceforth. Let Σ_k be the group of all permutations of the set $\{1, ..., k\}$. We define a multi-linear invariant map $p_{k,\sigma}$ for any permutation $\sigma \in \Sigma_k$ by setting:

$$p_{k,\sigma}(v_1, ..., v_k) := g(v_{\sigma(1)}, v_{\sigma(2)}) \cdots g(v_{\sigma(k-1)}, v_{\sigma(k)})\,.$$

Theorem 4.2. $\mathcal{I}_{k,m} = \mathrm{span}_{\sigma\in\Sigma_k}\{p_{k,\sigma}\}$.

Proof. We use Theorem 4.1 to express $p \in \mathcal{I}_{k,m}$ in terms of monomials involving the inner products $g(v_i, v_j)$. Since p is multi-linear,

$$p(cv_1, v_2, ..., v_k) = cp(v_1, v_2, ..., v_k)\,.$$

This implies that we need only consider monomials where the variable v_1 appears exactly once because otherwise we contradict multi-linearity. A similar observation holds for the remaining indices and these are exactly the expressions $p_{k,\sigma}$ defined above. □

In view of Theorem 4.2, one says 'invariant multilinear maps are given by contractions of indices' as, relative to an orthonormal basis, the inner products involved correspond to contraction of indices in pairs. Let $\{e_i\}$ be an orthonormal basis for the vector space V and let $\omega = \omega_{i_1i_2...i_k}e_{i_1}\otimes...\otimes e_{i_k} \in \otimes^kV$. We have, for example:

$$\mathcal{I}_{2,m} := \mathrm{Span}\{\omega_{ii}\} \quad \text{and} \quad \mathcal{I}_{4,m} := \mathrm{Span}\{\omega_{iijj}, \omega_{ijij}, \omega_{ijji}\}.$$

Atiyah, Bott, and Patodi [1] applied this formalism to study the spaces $\mathcal{P}_{n,m}$. In geodesic coordinate systems, all jets of the metric can be computed in terms of the covariant derivatives of the curvature tensor and vice versa. Thus, for example, if $n = 4$, an invariant $P \in \mathcal{P}_{4,m}$ can be regarded as a map from a certain subspace

$$W \subset \{\otimes^6T(M)\} \oplus \{\otimes^8T(M)\}$$

to $\mathbb{R}$ which is invariant under the action of the orthogonal group; here W is generated by the algebraic covariant derivatives $\nabla^2R \subset \otimes^6T(M)$ and by the algebraic curvature tensors $R\otimes R \subset \otimes^8T(M)$. As the subspace W is orthogonally invariant, extending P to be zero on $W^\perp$ defines an orthogonally invariant map to which Theorem 4.2 applies. Thus, for example, after taking into account the appropriate curvature symmetries, one has:

$$\mathcal{P}_{2,m} = \mathrm{Span}\{\tau := R_{ijji}\},$$
$$\mathcal{P}_{4,m} = \mathrm{Span}\{\tau^2,\ |\rho^2| := R_{ijjk}R_{illk},\ |R|^2 := R_{ijkl}R_{ijkl},\ \Delta\tau := -R_{ijji;kk}\}.$$

They extended this analysis to form valued invariants with coefficients in an auxiliary vector bundle to give a heat equation proof of the index theorem for the classical elliptic complexes based on H. Weyl's first main theorem of invariants.

What is relevant to our analysis, however, is Weyl's second main theorem [22] (Theorem 2.17.A) as this will permit us to study the restriction map r:

Theorem 4.3. *Every relation among scalar products is an algebraic consequence of the relations*

$$0 = \det\begin{pmatrix} g(v_1, w_1) & g(v_2, w_1) & ... & g(v_{m+1}, w_1) \\ g(v_1, w_2) & g(v_2, w_2) & ... & g(v_{m+1}, w_2) \\ ... & ... & ... & ... \\ g(v_1, w_{m+1}) & g(v_2, w_{m+1}) & ... & g(v_{m+1}, w_{m+1}) \end{pmatrix}.$$

We remark that this relation can also be expressed in the form:

$$0 = g(v_1 \wedge ... \wedge v_{m+1}, w_1 \wedge ... \wedge w_{m+1}). \tag{4.a}$$

Let W be a vector space of dimension $m-1$. Choose an inner product preserving inclusion $i : W \subset V$ and a corresponding embedding $O(W) \subset O(V)$. We define:

$$\mathfrak{R} : \mathcal{I}_{k,m} \to \mathcal{I}_{k,m-1}$$

which is characterized dually by the property:

$$\mathfrak{R}(p)(w_1, ..., w_k) = p(i(w_1), ..., i(w_k))\,.$$

If p is given by contractions of indices which range from 1 to m, then $\mathfrak{R}(p)$ is given by restricting the range of summation to range from 1 to $m-1$. (This directly relates to the restriction map r in the geometric context.) Consequently, the map $\mathfrak{R}$ is surjective. If $k \geq 2m$ and if $\sigma \in \Sigma_k$, define:

$$\begin{aligned} r_{k,m,\sigma}(v_1, ..., v_k): &= g(v_{\sigma(1)} \wedge ... \wedge v_{\sigma(m)}, v_{\sigma(m+1)} \wedge ... \wedge v_{\sigma(2m)}) \\ &\times g(v_{\sigma(2m+1)}, v_{\sigma(2m+2)}) \cdots g(v_{\sigma(k-1)}, v_{\sigma(k)}). \end{aligned}$$

Theorem 4.4. *Let $m \geq 2$.*

(1) $\mathfrak{R}: \mathcal{I}_{k,m} \to \mathcal{I}_{k,m-1}$ *is surjective.*
(2) $\mathfrak{R}: \mathcal{I}_{k,m} \to \mathcal{I}_{k,m-1}$ *is injective if $k < 2m$.*
(3) *If $k \geq 2m$, then* $\ker(\mathfrak{R}) \cap \mathcal{I}_{k,m} = \text{span}_{\sigma\in\Sigma_k}\{r_{k,m,\sigma}\}$.

Proof. We have already verified assertion (1). To prove assertion (2), we use Theorem 4.1 to express $p \in \mathcal{I}_{k,m}$ in terms of inner products. We use Theorem 4.3, after making an appropriate dimension shift, to see that $\mathfrak{R}(p)$ vanishes if and only if it can be written as sums of terms each of which is divisible by an appropriate determinant J of size $m \times m$. The desired result now follows from equation (4.a) and from the same arguments used to prove Theorem 4.2. □

5. The proof of Lemma 1.6

We may use the $\mathbb{Z}_2$ action $\phi \to -\phi$ to decompose $\mathcal{Q}_{n,m}^M = \mathcal{Q}_{n,m}^{M,+} \oplus \mathcal{Q}_{n,m}^{M,-}$ where

$$\mathcal{Q}_{n,m}^{M,\pm} := \{Q \in \mathcal{Q}_{n,m} : Q(\phi, g) = \pm Q(-\phi, g)\}.$$

We use Lemma 3.1 (3) and Lemma 3.3 to see that:

$$a_{n,m}^{d+\delta} \in \mathcal{Q}_{n,m}^{M,-} \cap \ker(r) \text{ if } m \text{ is odd}, \quad \text{and} \quad a_{n,m,k}^{d+\delta} \in \mathcal{Q}_{n-k-1,m}^{\partial M} \cap \ker(r).$$

Thus Lemma 1.6 will follow from:

Lemma 5.1. (1) $\mathcal{Q}_{m+1,m}^{M,-} \cap \ker(r) = \text{Span}\{\varepsilon_J^I \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{I,m}\}$ *for m odd.*

(2) *We have* $\mathcal{Q}_{m-1,m}^{\partial M} \cap \ker r = \text{Span}\{\varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,m-1}\}_k$.
(3) *We have* $\mathcal{Q}_{m,m}^{\partial M} \cap \ker(r) = \text{Span}\{\mathcal{F}_{m,m}^{i,k}\}_{i,k}$.

Proof. The same argument based on work of [1] which extends Theorem 4.2 from the algebraic to the geometric context can be used to extend Theorem 4.4 from the algebraic to the geometric context. One can also include ϕ and L. Consequently, elements of $\mathcal{Q}_{n,m}^M \cap \ker(r)$ or of $\mathcal{Q}_{n,m}^{\partial M} \cap \ker(r)$ are formed by contracting some indices using the ε tensor and by contracting the remaining indices in pairs.

We first consider the interior invariants. We consider a typical expression in the covariant derivatives of ϕ and R:

$$A = \phi_{;\alpha_1} ... \phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1;\beta_1} ... R_{i_v j_v k_v \ell_v;\beta_v}$$

where α_μ and β_ν denote appropriate collections of indices. The weight of A is then:

$$n = \textstyle\sum_\mu |\alpha_\mu| + \sum_\nu (2 + |\beta_\nu|).$$

We form a spanning set for $\ker(r) \cap \mathcal{Q}_{n,m}^M$ by contracting $2m$ indices using the ε tensor and contracting the remaining indices in pairs. Thus, at least $2m$ indices must appear in the monomial A. We count indices to see:

$$\begin{aligned} &(5.a) \quad 2m &\leq \; &\text{number of indices in A} = \textstyle\sum_\mu |\alpha_\mu| + \sum_\nu (|\beta_\nu| + 4) = n + 2v \\ &(5.b) &= \; &2n - \textstyle\sum_\mu |\alpha_\mu| - \sum_\nu |\beta_\nu| \leq 2n. \end{aligned}$$

This is not possible, of course, if $n < m$, and this observation can be used to give a slightly different proof of Theorem 1.2 (3) than was given in [13].

To prove assertion (1), we assume $m = 2\bar{m}+1$ and $n = m+1$. Since $2m$, $n+2v$, and $2n$ are all even, only one of the two inequalities given above can be strict. Since we are studying $\mathcal{Q}^{M,-}_{m+1,m}$, u must be *odd*. Thus $\sum_\mu |\alpha_\mu| > 0$ so the inequality in equation (5.b) is strict and equation (5.a) must be an equality. Thus exactly $2m$ indices appear in A and all are contracted using the ε tensor. The first and second Bianchi identity now imply $R_{****;\beta} = 0$ if we alternate 3 indices. Thus at most two i indices and at most two j indices can appear in each $R_{****;\beta}$ variable. This shows that all the $|\beta_\nu| = 0$. Furthermore, the two possibilities are $R_{i_1 i_2 j_2 j_1}$ or $R_{i_1 j_1 i_2 j_2}$; the first Bianchi identity can then be used to express the second variable in terms of the first. (We will apply a similar argument subsequently.) Since $u \le |\alpha_1|+...+|\alpha_u| = 2$ and u is odd, we see $u = 1$ and $|\alpha_1| = 2$. Thus the only possibility is $\varepsilon^I_J \phi_{;i_1 j_1}\mathcal{R}^{I,m}_{J,2}$.

Next we study the boundary invariants. We consider a typical expression in the covariant derivatives of $\{\phi, R, L\}$ which is to be homogeneous of degree $m-1$:

$$A := \phi_{;\alpha_1}\cdots\phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1;\beta_1}\cdots R_{i_v j_v k_v \ell_v;\beta_v} L_{a_1 b_1:\gamma_1}\cdots L_{a_w b_w:\gamma_w},$$

where $|\alpha_\mu| \ge 1$. We must contract $2(m-1)$ tangential indices using the ε tensor, the remaining tangential indices must be contracted in pairs, and the normal index 'm' can stand alone and unchanged. We may therefore estimate:

$$\begin{array}{llll}
(5.c) & 2m-2 & \le & \text{number of tangential indices in A}\\
(5.d) & & \le & \text{total number of indices in A}\\
 & & = & \sum_\mu |\alpha_\mu| + \sum_\nu(|\beta_\nu|+4) + \sum_\sigma(|\gamma_\sigma|+2) = n+2v+w\\
(5.e) & & = & 2n - \sum_\mu |\alpha_\mu| - \sum_\mu |\beta_\nu| - \sum_\sigma |\gamma_\sigma| \le 2n.
\end{array}$$

Clearly this is not possible if $n < m-1$ and this observation can be used to give a slightly different proof of Theorem 1.2 (4) than that given in [13]. Furthermore, if $n = m-1$, then the inequalities of equations (5.c)-(5.e) must have been equalities. Thus in particular there are no covariant derivatives and the ϕ variables do not appear. All the indices are tangential and are contracted using the ε tensor. After using the first Bianchi identity, we see that this leads to the invariants $\varepsilon^A_B \mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}$ and proves assertion (2).

We set $n = m$ to prove the final assertion. We consider various cases. Suppose first that $2m-2$ tangential indices appear and that no normal index appears. Then the inequalities of displays (5.c) and (5.d) are equalities. Thus the inequality in display (5.e) represents an increase by 2 so there are exactly 2 explicit covariant derivatives. We have, see for example [6],

$$L_{c_2 c_3:c_1} - L_{c_1 c_3:c_2} = R_{c_1 c_2 c_3 m}. \tag{5.f}$$

Since we contract every tangential index using the tensor ε, we must alternate at least 2 of the indices in $L_{**:*}$. We use equation (5.f) to see that we may assume $|\gamma_\sigma| = 0$. Taking into account the Bianchi identities, we see $|\beta_\nu| = 0$. Consequently $\sum_\mu |\alpha_\mu| = 2$. This leads to the invariants $\mathcal{F}^{1,k}_{m,m}$ and $\mathcal{F}^{2,k}_{m,m}$.

Suppose $2m-2$ tangential indices appear and that the normal index m appears exactly once. Then the inequality given in (5.c) is an equality and the inequality given in equation (5.d) represents an increase by 1. Thus the inequality given in equation (5.e) also represents an increase by 1. Consequently exactly one explicit covariant derivative appears. If ϕ appears, the ϕ terms must have total weight at least 2. Since there is only one explicit covariant derivative, ϕ does not appear. We contract every tangential index using the ε tensor. By the second Bianchi identity and equation (5.f), the only possibility is $\mathcal{F}^{3,k}_{m,m}$.

Suppose $2m-2$ tangential indices appear. If the normal index m appears twice, the inequality of equation (5.d) represents an increase of 2 so equation (5.e) is an equality. Thus no explicit covariant derivatives appear; we have $\mathcal{F}^{4,k}_{m,m}$ and $\mathcal{F}^{5,k}_{m,m}$.

Suppose $2m$ tangential derivatives appear. Then equation (5.c) represents an increase of 2 and hence equation (5.d) and equation (5.e) are equalities. Thus the normal index m does not appear nor are there any explicit covariant derivatives. There is one pair of tangential indices which is contracted using the metric tensor, the remaining tangential indices are contracted using the tensor ε and we have $\mathcal{F}^{6,k}_{m,m}$, $\mathcal{F}^{7,k}_{m,m}$, $\mathcal{F}^{8,k}_{m,m}$, $\Xi^{1,k}_{m,m}$, and $\Xi^{2,k}_{m,m}$ where

$$\Xi^{1,k}_{m,m} := \varepsilon^A_B R_{a_1 c c b_1} \mathcal{R}^{A,2k-1}_{B,2} \mathcal{L}^{A,m-1}_{B,2k} \quad \text{and} \quad \Xi^{2,k}_{m,m} := \varepsilon^A_B L_{cc} \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1}.$$

We complete the proof by showing

$$\Xi^{1,k}_{m,m} \in \operatorname{Span}\{\mathcal{F}^{i,j}_{m,m}\}_{i,j} \quad \text{and} \tag{5.g}$$

$$\Xi^{2,k}_{m,m} \in \operatorname{Span}\{\mathcal{F}^{i,j}_{m,m}\}_{i,j}. \tag{5.h}$$

Let $U = (u_1, ..., u_m)$ and $V = (v_1, ..., v_m)$ be m-tuples of tangential indices. Since the indices u_i and v_i range from 1 to $m-1$, the tensor $\varepsilon^U_V = 0$ and we have:

$$0 = \varepsilon^V_U R_{u_1 u_2 v_2 v_1} ... R_{u_{2k-1} u_{2k} v_{2k} v_{2k-1}} L_{u_{2k+1} v_{2k+1}} .... L_{u_m v_m}. \tag{5.i}$$

We set $u_1 = c$ and expand equation (5.i) by replacing v_1, v_2, ..., and v_m in turn by the index c to see:

$$\begin{aligned} 0 &= 2\varepsilon^A_B R_{c a_1 b_1 c} \mathcal{R}^{A,2k-1}_{B,2} \mathcal{L}^{A,m-1}_{B,2k} \\ &+ (2k-2)\varepsilon^A_B R_{c a_1 b_2 b_1} R_{a_2 a_3 b_3 c} \mathcal{R}^{A,2k-1}_{B,4} \mathcal{L}^{A,m-1}_{B,2k} \\ &+ (m-2k)\varepsilon^A_B R_{c a_1 b_2 b_1} L_{a_2 c} \mathcal{R}^{A,2k}_{B,3} \mathcal{L}^{A,m-1}_{B,2k+1} \\ &= 2\Xi^{1,k}_{m,m} + (2k-2)\mathcal{F}^{8,k-1}_{m,m} - (m-2k)\mathcal{F}^{7,k}_{m,m}. \end{aligned}$$

This establishes equation (5.g). We set $u_{2k+1} = c$ and compute, after taking into account the sign of the permutation involved, that:

$$\begin{aligned} 0 &= \varepsilon^A_B L_{cc} \mathcal{R}^{A,2k}_{B,1} \mathcal{L}^{A,m-1}_{B,2k+1} \\ &+ 2k\varepsilon^A_B L_{c b_1} R_{a_1 a_2 b_2 c} \mathcal{R}^{A,2k}_{B,3} \mathcal{L}^{A,m-1}_{B,2k+1} \\ &+ (m-2k-1) L_{c b_1} L_{a_1 c} \mathcal{R}^{A,2k+1}_{B,2} \mathcal{L}^{A,m-1}_{B,2k+2} \\ &= \Xi^{2,k}_{m,m} + 2k\mathcal{F}^{7,k}_{m,m} + (m-2k-1)\mathcal{F}^{6,k}_{m,m}. \end{aligned}$$

This establishes equation (5.g) and completes the proof. □

6. The proof of Theorem 1.4 (1,2)

By Lemma 1.6, there are universal constants so that:

$$a^{d+\delta}_{m+1,m}(\phi, g) = c_{m+1,m} \varepsilon^I_J \phi_{;i_1 j_1} \mathcal{R}^{I,m}_{J,2} \text{ for } m \text{ odd}$$
$$a^{d+\delta}_{m+1,m,0}(\phi, g) = \textstyle\sum_{i,k} c^{i,k}_{m,m} \mathcal{F}^{i,k}_{m,m}.$$

We complete the proof of Theorem 1.4 (1,2) by evaluating $c_{m+1,m}$ and $c^{i,k}_{m,m}$.

Lemma 6.1. (1) $c_{m+1,m} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{8^{\bar m} \pi^{\bar m} \bar m!}$.

(2) $c^{1,k}_{m,m} = \frac{1}{\sqrt{\pi}} \frac{1}{(8\pi)^k k! (m-2k-2)! \mathrm{vol}(S^{m-2k-2})}$.

(3) $c^{i,k}_{m,m} = 0$ *for* $i \geq 2$.

Proof. We use the method of universal examples. We adopt the following notational conventions. Let $ds^2_{S^\mu}$ and $ds^2_{D^\nu}$ be the standard metrics on the sphere S^μ and on the disk D^ν, respectively. Let r be the radial parameter on the ball; the boundary $S^{\nu-1} = \partial D^\nu$ is then given by $r = 1$. The case $\nu = 1$ is a bit exceptional and we shall always take the same structures at $r = 0$ as at $r = 1$ if $\nu = 1$. If $\mu = 0$, we take S^μ to be a single point and ignore the structures here. Let θ_i be the usual periodic parameters on the torus $T^\sigma := S^1 \times ... \times S^1$. Let $\phi = \phi(\theta) \in C^\infty(S^1)$.

Let $m=2\bar m+1$. Give $M:=S^1\times S^{2\bar m}$ the product structures. We use Theorem 1.3 (1) and Lemma 3.2 (4) to compute:

$$\begin{aligned} a^{d+\delta}_{m+1,m}(\phi,g) &= c_{m+1,m}\cdot 2^{\bar m}(m-1)!\cdot\phi_{;11}\\ &= a^{d+\delta}_{2,1}(\phi,d\theta^2)\cdot a^{d+\delta}_{m-1,m-1}(0,g_{S^{2\bar m}})\\ &= \tfrac{1}{\sqrt{\pi}}\cdot\tfrac{1}{8^{\bar m}\pi^{\bar m}\bar m!}\cdot 2^{\bar m}(m-1)!\cdot\phi_{;11}. \end{aligned}$$

We solve for $c_{m+1,m}$ to prove assertion (1).

Give $M=S^1\times S^{2k}\times D^{m-2k-1}$ the product structures. We compute similarly:

$$\begin{aligned} a^{d+\delta}_{m+1,m,0}(\phi,g) &= \{c^{1,k}_{m,m}\phi_{;11}+c^{2,k}_{m,m}\phi_{;1}\phi_{;1}\}2^k(2k)!(m-2k-2)!\\ &= a^{d+\delta}_{2,1}(\phi,d\theta^2)\cdot a^{d+\delta}_{m-1,m-1,0}(0,g_{S^{2k}\times D^{m-2k-1}})\\ &= \tfrac{1}{8^k\pi^k k!(m-2k-2)!\mathrm{vol}(S^{m-2k-2})}2^k(2k)!(m-2k-2)!\cdot\tfrac{1}{\sqrt{\pi}}\phi_{;11} \end{aligned}$$

We solve for $c^{1,k}_{m,m}$ to prove assertion (2) and to see $c^{2,k}_{m,m}=0$.

We set $\phi=0$ and consider purely metric invariants to study the remaining coefficients. We apply Lemma 3.1 (4):

$$\textstyle\int_{\partial M}a^{d+\delta}_{m+1,m,0}(0,g)dy=0.$$

Give $M_1:=S^1\times S^{2k}\times D^{m-2k-1}$ the metric:

$$g_1=e^{2f(r)}d\theta^2+ds^2_{S^{2k}}+ds^2_{D^{m-2k-1}}\text{ for }f(1)=0,\ f'(1)=\varepsilon_1,\ f''(1)=\varepsilon_2\,.$$

The relevant tensors are $L_{11}=\varepsilon_1$ and $R_{1mm1}=-\varepsilon_2+O(\varepsilon_1^2)$. The invariants $\mathcal{F}^{3,j}_{m,m}$ and $\mathcal{F}^{7,j}_{m,m}$ vanish on g_1 since they both involve a linkage of at least 2 tangential variables a_i with the normal variable through the curvature tensor and second fundamental form. The variables $\mathcal{F}^{5,j}_{m,m}$ and $\mathcal{F}^{8,j}_{m,m}$ vanish since they involve a linkage of an odd number, which is at least 3, of indices a_i through the curvature tensor. Finally, $\mathcal{F}^{4,j}_{m,m}$ and $\mathcal{F}^{6,j}_{m,m}$ vanish for $j\neq k$ since they involve the wrong partition of tangential variables into curvature and second fundamental form terms. Thus only $\mathcal{F}^{4,k}_{m,m}$ and $\mathcal{F}^{6,k}_{m,m}$ survive. As only R_{1mm1} involves ε_2,

$$a^{d+\delta}_{m+1,m,0}(0,g_1)=-2^k(2k)!(m-2k-2)!\varepsilon_2\cdot c^{4,k}_{m,m}+...\quad\text{so }c^{4,k}_{m,m}=0.$$

As $L_{11}=\varepsilon_1$, we have

$$a^{d+\delta}_{m+1,m,0}(0,g_1)=2^k(2k)!(m-2k-2)!\cdot\varepsilon_1^2\cdot c^{6,k}_{m,m}+\mathcal{O}(\varepsilon_1)\quad\text{so }c^{6,k}_{m,m}=0.$$

Give $M_2:=S^2\times S^{2k-2}\times D^{m-2k}$ the metric

$$g_2=e^{2f(r)}ds^2_{S^2}+ds^2_{S^{2k-2}}+ds^2_{D^{m-2k}}\text{ for }f(1)=0,\ f'(1)=\varepsilon_1,\ f''(1)=\varepsilon_2.$$

The only possibly non-zero invariants are $c^{3,k}_{m,m}\mathcal{F}^{3,k}_{m,m}$ and $c^{7,k}_{m,m}\mathcal{F}^{7,k}_{m,m}$ because $\mathcal{F}^{5,j}_{m,m}$ and $\mathcal{F}^{8,j}_{m,m}$ involve linking an odd number of tangential indices a_i through the curvature tensor. Furthermore, $\mathcal{F}^{3,j}_{m,m}$ and $\mathcal{F}^{7,j}_{m,m}$ vanish for $j\neq k$ as they involve the wrong partition of tangential variables into curvature and second fundamental form terms. Only $R_{1221;m}=-2\varepsilon_1\varepsilon_2+...$ involves ε_2. Thus

$$a_{m+1,m,0}(0,g_2)=-8\cdot 2^{k-1}\cdot(2k-2)!(m-2k-1)!c^{3,k}_{m,m}\varepsilon_1\varepsilon_2+...\text{ so }c^{3,k}_{m,m}=0.$$

Since $L_{11}=L_{22}=\varepsilon_1$,

$$a_{m+1,m,0}(0,g_2)=-4\cdot 2^{k-1}(2k-2)!(m-2k-1)!\varepsilon_1\cdot c^{7,k}_{m,m}+...\text{ so }c^{7,k}_{m,m}=0.$$

Give $M_3=S^3\times S^{2k-2}\times D^{m-2k-1}$ the standard metric. As $R_{a_1mb_2b_1}=0$, only $\mathcal{F}^{8,k}_{m,m}$ survives. Consequently,

$$a^{d+\delta}_{m+1,m,0}(0,g_3)=-c^{8,k}_{m,m}\cdot 24\cdot 2^{k-1}(2k-2)!(m-2k-2)!\quad\text{so }c^{8,k}_{m,m}=0.$$

Finally, give $M_4=T^3\times S^{2k-2}\times D^{m-2k-1}$ the metric

$$g_4=e^{2f(r,\theta_3)}(d\theta_1^2+d\theta_2^2)+d\theta_3^2+ds^2_{S^{2k-2}}+ds^2_{D^{m-2k-1}}$$

for $f(1,\theta_3)=0$, $\partial_r f(1,\theta_3)=\varepsilon F(\theta_3)$. The only remaining terms of interest are

$$\begin{aligned}&R_{13m1}=R_{23m2}=-\varepsilon F'(\theta_3)+O(\varepsilon^2),\\&L_{11}=L_{22}=\varepsilon F(\theta_3),\text{ and }R_{1331}=R_{2332}=O(\varepsilon^2)\,.\end{aligned}$$

As we must involve all three of the indices $\{1,2,3\}$ we see:

$$a^{d+\delta}_{m+1,m,0}(0,g_4)=c^{5,k}_{m,m}F'(\theta_3)^2 8\varepsilon^2 2^{k-1}(2k-2)!(m-2k-2)!+O(\varepsilon^3).$$

Since this has to integrate to zero for any function of θ_3, $c^{5,k}_{m,m}=0$. □

7. The proof of Theorem 1.4 (3)

By Lemma 1.6, ϕ plays no role in $a^{d+\delta}_{m+1,m,1}$ so we set $\phi=0$. We also suppose for the moment that $M\subset\mathbb{R}^m$ and set $R=E=\Omega=0$ in the formalism of Section 2. Motivated by Lemma 2.2, we consider the set $\mathcal{A}_{n,m}$ of all non-commutative endomorphism valued monomials in the tangential covariant derivatives of $\{\mathcal{S},\chi,L\}$. We contract indices in pairs, cyclically permute variables, and take the trace. For example, we use the identities $\mathcal{S}\chi=\mathcal{S}$ and $\chi^2=\mathrm{Id}$ to see:

$$\begin{aligned}\mathcal{A}_{2,m}\ =\ &\{\mathrm{Tr}(\mathcal{S}^2),\mathrm{Tr}(L_{aa}\mathcal{S}),\mathrm{Tr}(L_{aa}L_{bb}\mathrm{Id}),\mathrm{Tr}(L_{ab}L_{ab}\mathrm{Id}),\\&\mathrm{Tr}(L_{aa}L_{bb}\chi),\mathrm{Tr}(L_{ab}L_{ab}\chi),\mathrm{Tr}(\chi_{:a}\chi_{:a}),\mathrm{Tr}(\chi\chi_{:a}\chi_{:a}),\\&\mathrm{Tr}(\chi_{:aa}),\mathrm{Tr}(\chi_{:aa}\chi\}.\end{aligned}$$

Let $\mathcal{S}_p$ and χ_p be defined by Δ^p_g. By Lemma 2.4:

$$\begin{array}{llll}\chi_p=+1 & \text{on }\Lambda^p(\partial M), & \chi_p=-1 & \text{on }\Lambda^{p-1}(\partial M)\wedge e^m,\\ \mathcal{S}_p=-L_{ab}\mathfrak{e}_b\mathfrak{i}_a & \text{on }\Lambda^p(\partial M), & \mathcal{S}_p=0 & \text{on }\Lambda^{p-1}(\partial M)\wedge e^m.\end{array}\tag{7.a}$$

In Lemma 2.3 we showed that:

$$a_{n,m,k}(y,D,\mathcal{B})=\mathfrak{c}_{n,m,k}\mathrm{Tr}\{\mathcal{S}^{n-k-1}\}+\text{other terms}$$

Only these constants are crucial because as we now show the supertrace of the 'other terms' vanishes.

Lemma 7.1. *Let M^m be a compact submanifold of $\mathbb{R}^m$ with smooth boundary.*

1. *If $A\in\mathcal{A}_{m-1,m}$ and if $A\neq\mathcal{S}^{m-1}$, then $\sum_p(-1)^p\mathrm{Tr}(A_p)(g_M)=0$.*
2. *We have $a^{d+\delta}_{m,m,0}(g_M)=\mathfrak{c}_{m,m,0}\sum_p(-1)^p\mathrm{Tr}(\mathcal{S}^{m-1}_p)$.*
3. *We have $a^{d+\delta}_{m+1,m,1}(g_M)=\mathfrak{c}_{m+1,m,1}\sum_p(-1)^p\mathrm{Tr}(\mathcal{S}^{m-1}_p)$.*

Proof. Let $A\in\mathcal{A}_{m-1,m}$ and $A\neq\mathcal{S}^{m-1}$. Since $\sum_p(-1)^p\mathrm{Tr}(A_p)\in\mathcal{P}^{\partial M}_{m-1,m}\cap\ker(r)$ and since we are setting $R=0$,

$$\textstyle\sum_p(-1)^p\mathrm{Tr}(A_p)(g_M)=C\varepsilon^B_A\mathcal{L}^{A,m-1}_{B,1}(g_M)\,.$$

In particular, we may ignore all the ∇L terms since they contribute nothing.

By Lemma 2.4, $\chi_{:a}=2L_{ab}(\mathfrak{e}_b\mathfrak{i}_m+\mathfrak{e}_m\mathfrak{i}_b)$. As we are suppressing terms in ∇L, multiple covariant differentiation yields an expression of the form:

$$\chi_{:a_1\dots a_k}=L_{a_1b_1}L_{a_2b_2}\dots L_{a_kb_k}\mathcal{E}_{b_1\dots b_k}+\dots\tag{7.b}$$

where the precise form of $\mathcal{E}_{b_1\dots b_k}$ is not relevant. Generically, the principle curvatures $\kappa_1\dots\kappa_{m-1}$ are all distinct so:

$$\textstyle\sum_p(-1)^p\mathrm{Tr}(A^p)(g_M)=C(m-1)!\kappa_1\dots\kappa_{m-1}\,.$$

If a $\chi_{:a_1\dots}$ term appears, we must contract it with another index a_1; equation (7.b) contains no $L_{a_1a_1}$ term and consequently this contraction involves a different variable and consequently must produce a $\kappa^2_{a_1}$ term and eliminates the possibility that the monomial $\det(L)=\kappa_1\dots\kappa_{m-1}$ appears. This implies $C=0$. Thus $\nabla\chi$ does not appear; similarly $\nabla\mathcal{S}$ does not appear. Consequently $A=A(L,\mathcal{S})$.

If two terms $L_{a_1b_1}L_{a_2b_2}$ appear (it does not matter whether $b_1=b_2$ or $b_1\neq b_2$), then again each monomial of A must be divisible by $\kappa_{a_1}^2$. If the term L_{aa} appears, then we may factor $A=L_{aa}A_0$. Since

$$\textstyle\sum_p(-1)^p\mathrm{Tr}(A_{0,p})\in\mathcal{P}^{\partial M}_{m-2,m}\cap\ker(r)=\{0\}$$

this possibility is eliminated. Thus the only remaining possibility is $A=\mathcal{S}^{m-1}$. Assertion (1) follows; assertions (2) and (3) now follow from Lemma 2.3. □

The following combinatorial Lemma will be useful in our investigations.

Lemma 7.2. (1) *We have* $\sum_p(-1)^pp^\ell\frac{\ell!}{p!(\ell-p)!}=(-1)^\ell\ell!$

(2) *We have* $\sum_p(-1)^p\mathrm{Tr}(\mathcal{S}_p^{m-1})(g_{D^m})=(m-1)!$.

Proof. We expand:

$$(1-x)^\ell=\textstyle\sum_p(-1)^px^p\frac{\ell!}{p!(\ell-p)!}.$$

We establish assertion (1) by computing:

$$\textstyle\sum_p(-1)^pp^l\frac{\ell!}{p!(\ell-p)!}=\left(x\frac{d}{dx}\right)^l(1-x)^l\Big|_{x=1}=(-1)^ll!\,.$$

Give $M=D^m$ the standard metric; $L_{ab}=\delta_{ab}$. By Lemma 2.4, $\mathcal{S}_p=-p\cdot\mathrm{Id}$ on $\Lambda^p(\partial M)$. Assertion (2) now follows from assertion (1). □

We can now complete the proof of Theorem 1.4. We use Lemma 1.6, Lemma 2.3, and Lemma 7.2 to compute:

$$\begin{aligned}a^{d+\delta}_{m+1,m,1}(0,g_{D^m})&=\textstyle\sum_k c^k_{m+1,m,1}\varepsilon^A_B\mathcal{R}^{A,2k}_{B,1}\mathcal{L}^{A,m-1}_{B,2k+1}=c^0_{m+1,m,1}(m-1)!\\&=\mathfrak{c}_{m+1,m,1}\textstyle\sum_p(-1)^p\mathrm{Tr}(\mathcal{S}_p^{m-1})(g_{D^m})=\frac{\sqrt{\pi}}{\mathrm{vol}(S^m)m!}(m-1)!.\end{aligned}$$

We use Lemma 2.3 (2) to evaluate $\mathfrak{c}_{m+1,m,1}$. We solve this equation to see

$$c^0_{m+1,m,1}=\tfrac{\sqrt{\pi}}{\mathrm{vol}(S^m)m!}.$$

Let $M:=S^{2k}\times D^{m-2k}$. We compute:

$$\begin{aligned}&a^{d+\delta}_{m+1,m,1}(0,g_M)=c^k_{m+1,m,1}\cdot 2^k(2k)!(m-2k-1)!\\=\;&a^{d+\delta}_{2k,2k}(0,g_{S^{2k}})a^{d+\delta}_{m-2k+1,m-2k,1}(0,g_{D^{m-2k}})\\=\;&\tfrac{1}{8^k\pi^kk!}\tfrac{\sqrt{\pi}}{\mathrm{vol}(S^{m-2k})(m-2k)!}2^k(2k)!(m-2k-1)!.\end{aligned}$$

We solve this relation to determine $c^k_{m+1,m,1}$. □

Appendix

We conclude this paper by giving a second proof of Lemma 2.3 (2). Our purpose is twofold. First, this gives an independent check on our computations. Equally important, it illustrates the power of the methods we are employing. We recall our notational conventions. Let D be an operator of Laplace type on a compact Riemannian manifold M with smooth boundary ∂M. Let $B=B_{\chi,\mathcal{S}}$ define mixed boundary conditions. We expand

$$a_{n,m,k}=\mathfrak{c}_{n,m,k}\mathrm{Tr}(\mathcal{S}^{n-k-1})+\text{other terms}.$$

Lemma A.1.

(1) $\mathfrak{c}_{n,m,k}=(4\pi)^{-(m-1)/2}\mathfrak{c}_{n,1,k}$.

(2) *If* $n\geq 2$, *then* $\mathfrak{c}_{n,m,1}=\frac12\mathfrak{c}_{n,m,0}$.

(3) $\mathfrak{c}_{m,m,0}=\frac{1}{\mathrm{vol}(S^{m-1})(m-1)!}$, $\mathfrak{c}_{m,m,1}=\frac{1}{2\mathrm{vol}(S^{m-1})(m-1)!}$, $\mathfrak{c}_{m+1,m,1}=\frac{\sqrt{\pi}}{\mathrm{vol}(S^m)m!}$.

Proof. To prove assertion (1), we use product formulas. Let $M_1 = T^{m-1}$ be the torus and let D_1 be the scalar Laplacian. Since the structures are flat,

$$a_{n,m-1}(x_1, D_1) = \begin{cases} (4\pi)^{-(m-1)/2} & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n > 0. \end{cases}$$

Let $(M_2, D_2) = ([0,1], -\partial_r^2)$. Let $M = M_1 \times M_2$ and $D = D_1 + D_2$. Let $\mathcal{B} = \nabla_m + \mathcal{S}$ where $\mathcal{S}$ is constant and where e_m is the inward unit normal; $e_m = \partial_r$ when $r = 0$ and $e_m = -\partial_r$ when $r = 1$. Assertion (1) follows from the identity:

$$\begin{aligned} a_{n,m,k}(y, D, \mathcal{B}) &= \textstyle\sum_{n_1+n_2=n} a_{n_1,m-1}(x_1, D_1) \cdot a_{n_2,1,k}(y_2, D_2, \mathcal{B}) \\ &= (4\pi)^{-(m-1)/2} a_{n,1,k}(y_2, D_2, \mathcal{B}). \end{aligned}$$

In view of assertion (1), it suffices to take $m = 1$ in the proof of assertion (2) so $M = [0,1]$. We use a variational formula from [6]. Let $D(\varepsilon) := -e^{-2\varepsilon f}\partial_r^2$. We choose f so f vanishes on ∂M but $f_{;m} \neq 0$ on ∂M. We also assume that all the higher derivatives of f vanish on ∂M. To keep the boundary conditions invariant, we set $\mathcal{S}(\varepsilon) := \mathcal{S}_0 - \frac{1}{2}\varepsilon f_{;m} + O(\varepsilon^2)$. Since $E(\varepsilon) = O(\varepsilon^2)$,

$$\begin{aligned} a_{n,1}(1, D(\varepsilon)) &= \textstyle\int_{\partial M} \mathfrak{c}_{n,1,0}\{\mathcal{S}_0 - \frac{1}{2}\varepsilon f_{;m} dy + O(\varepsilon^2)\}^{n-1}, \\ a_{n,1}(f, D) &= \textstyle\int_{\partial M} \mathfrak{c}_{n,1,1}\mathcal{S}_0^{n-2} f_{;m} dy. \end{aligned}$$

Assertion (2) now follows from this equation and from Lemma 2.1 (2).

We use Theorem 1.3, Lemma 7.1 (2), and Lemma 7.2 (2) to compute:

$$\begin{aligned} &a_{m,m,0}^{d+\delta}(0, g_{D^m}) = \tfrac{1}{\mathrm{vol}(S^{m-1})(m-1)!}\varepsilon_B^A \mathcal{L}_{B,1}^{A,m-1}(0, g_{D^m}) = \tfrac{1}{\mathrm{vol}(S^{m-1})} \\ &\quad = \mathfrak{c}_{m,m,0} \textstyle\sum_p (-1)^p \mathrm{Tr}(\mathcal{S}_p^{m-1}) = \mathfrak{c}_{m,m,0}(m-1)!. \end{aligned}$$

We solve for $\mathfrak{c}_{m,m,0}$ to establish the first part of assertion (3). The second part now follows from assertion (2). The third part follows from assertion (1) and (2) by computing

$$\begin{aligned} \mathfrak{c}_{m+1,m,1} &= \frac{1}{2}\mathfrak{c}_{m+1,m,0} = \frac{1}{2}(4\pi)^{-(m-1)/2}\mathfrak{c}_{m+1,1,0} = \frac{1}{2}\sqrt{4\pi}(4\pi)^{-m/2}\mathfrak{c}_{m+1,1,0} \\ &= \sqrt{\pi}\mathfrak{c}_{m+1,m+1,0} = \frac{\sqrt{\pi}}{\mathrm{vol}(S^m)m!}. \end{aligned}$$

□

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