Max-Planck-Institut
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by

Peter Gilkey, Klaus Kirsten, and Dmitri Vassilevich

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SUPERTRACE DIVERGENCE TERMS FOR THE WITTEN LAPLACIAN

P. GILKEY, K. KIRSTEN, AND D. VASSILEVICH

Abstract. We use invariance theory to compute the divergence term $a^{d+\delta}_{m+2,m}$ in the super trace for the twisted de Rham complex for a closed Riemannian manifold.

1. Introduction

Let (M,g) be a compact m dimensional Riemannian manifold without boundary. The fundamental solution of the heat equation e^{-tD} for an operator of Laplace type D on M is an infinitely smoothing operator. Let $f\in C^\infty(M)$ be an auxiliary smooth 'smearing' function. Work of Seeley [15] shows the smeared heat trace has a complete asymptotic expansion as $t\downarrow 0$ of the form:

$$\mathrm{Tr}_{L^2}(fe^{-tD})\sim\textstyle\sum_{n\geq 0}a_{n,m}(f,D)t^{(n-m)/2}.$$

The *heat trace invariants* $a_{n,m}$ vanish if n is odd; if n is even, there are local invariants $a_{n,m}(x,D)$ so that

$$a_{n,m}(f,D)=\textstyle\int_M f(x)a_{n,m}(x,D)\,\mathrm{dvol}_g(x).$$

The function f localizes the problem and permits us to recover divergence terms which would otherwise not be detected.

Let ϕ be an auxiliary smooth function called the dilaton. We twist the exterior derivative d and the coderivative δ_g to define

$$d_\phi:=e^{-\phi}de^{\phi}\quad\text{and}\quad\delta_{\phi,g}:=e^{\phi}\delta_g e^{-\phi}.$$

We denote the associated Laplacian by $\Delta^p_{\phi,g}$ on $C^\infty(\Lambda^p(M))$. It appears in supersymmetric quantum mechanics [2] and in the study of Morse theory [19]. It also is used to study quantum p form fields interacting with a background dilaton [11, 17].

Let $\chi(M):=\sum_p(-1)^p\dim H^p(M;\mathbb{R})$ be the Euler-Poincaré characteristic of M. Arguments of McKean and Singer [12] extend to the twisted setting to show

$$\textstyle\sum_p(-1)^p\mathrm{Tr}_{L^2}(e^{-t\Delta^p_{\phi,g}})=\chi(M).\tag{1.a}$$

We define the local supertrace asymptotics by setting:

$$a^{d+\delta}_{n,m}(\phi,g)(x):=\textstyle\sum_p(-1)^p a_{n,m}(x,\Delta^p_{\phi,g}).$$

We expand the left hand side of equation (1.a) and then equate powers of t to see:

$$\textstyle\int_M a^{d+\delta}_{n,m}(\phi,g)(x)\,\mathrm{dvol}_g(x)=\begin{cases}\chi(M) & \text{if}\quad n=m,\\ 0 & \text{if}\quad n\neq m.\end{cases}\tag{1.b}$$

Let R_{ijkl} be the components of the Riemann curvature tensor relative to a local orthonormal frame for the tangent bundle with the sign convention that $R_{1221}=+1$ on the unit sphere $S^2\subset\mathbb{R}^3$. We adopt the Einstein convention and sum over repeated indices. If $I=(i_1,...,i_m)$ and $J=(j_1,...,j_m)$ are m tuples of indices, let

$$\varepsilon^I_J:=g(e_{i_1}\wedge...\wedge e_{i_m},e_{j_1}\wedge...\wedge e_{j_m})$$

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be the totally anti-symmetric tensor and let

$$\mathcal{R}_{J,s}^{I,t} := R_{i_s i_{s+1} j_{s+1} j_s} \dots R_{i_{t-1} i_t j_t j_{t-1}};$$

we set $\mathcal{R}_{J,s}^{I,t} = 1$ for $t < s$.

Theorem 1.1. (1) *If $n < m$ or if n is odd, then $a_{n,m}^{d+\delta}(\phi, g)(x) = 0$.*

(2) *If $m = 2\bar{m}$, then $a_{m,m}^{d+\delta}(\phi, g)(x) = \frac{1}{8^{\bar{m}}\pi^{\bar{m}}\bar{m}!}\varepsilon_I^J \mathcal{R}_{J,1}^{I,m}$.*

(3) *If $m = 2\bar{m}+1$, then $a_{m+1,m}^{d+\delta}(\phi, g)(x) = \frac{1}{\sqrt{\pi}8^{\bar{m}}\pi^{\bar{m}}\bar{m}!}\varepsilon_J^I \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{I,m}$.*

Assertions (1) and (2) were proved in the untwisted case ($\phi = 0$) by Atiyah, Bott, and Patodi [4], by Gilkey [6], and by Patodi [13]. This provided a heat equation proof of the classical Chern-Gauss-Bonnet [5] theorem. Assertions (1) and (2) in the twisted setting were established in [11] and the divergence term $a_{m+1,m}^{d+\delta}$ was identified in [10]. Our previous paper [10] dealt with the odd dimensional case for manifolds with boundary. The present paper computes $a_{m+2,m}^{d+\delta}$ for closed even dimensional manifolds. This requires significantly different techniques. We note that some information concerning $a_{m+2,m}^{d+\delta}$ was derived earlier in [8] using an entirely different approach.

The main new result of this paper is the following:

Theorem 1.2. *Let M be a closed Riemannian manifold of dimension $m = 2\bar{m}$.*

$$\begin{aligned} a_{m+2,m}^{d+\delta} &= \tfrac{1}{\pi^{\bar{m}}8^{\bar{m}}\bar{m}!}\varepsilon_J^I \{4\bar{m}(\phi_{;i_1 j_1}\phi_{;i_2}\mathcal{R}_{J,3}^{I,m})_{;j_2} + \tfrac{1}{12}(\mathcal{R}_{J,1}^{I,m})_{;kk} \\ &+ \tfrac{\bar{m}}{6}(R_{i_1 i_2 k j_1;k}\mathcal{R}_{J,3}^{I,m})_{;j_2}\}. \end{aligned}$$

In Section 2, we recall combinatorial formulas for the invariants $a_{n,m}(x, D)$ for $n = 0, 2, 4, 6$ [9]; formulas for a_8, and for a_{10} are available [1, 3, 16]. These formulas become very complicated as n increases and it seems hopeless to try to establish Theorem 1.2 via direct computation even for $m = 6$ and $m = 8$. There are also closed formulas available due to Polterovich [14]. However, it does not seem possible to make a direct use of these formulas to derive Theorem 1.2. Instead, we proceed indirectly. In Section 3, we establish some functorial properties of these invariants and use invariance theory to prove the following result:

Lemma 1.3. *If m is even, then there exist universal constants so that*

$$\begin{aligned} a_{m+2,m}^{d+\delta}(\phi, g) &= c_{m+2,m}^1(\varepsilon_J^I \phi_{;i_1 j_1}\phi_{;i_2}\mathcal{R}_{J,3}^{I,m})_{;j_2} + c_{m+2,m}^2(\varepsilon_J^I \mathcal{R}_{J,1}^{I,m})_{;kk} \\ &\quad + c_{m+2,m}^3(\varepsilon_J^I R_{i_1 i_2 k j_1;k}\mathcal{R}_{J,3}^{I,m})_{;j_2}. \end{aligned}$$

We shall complete the proof of Theorem 1.2 in Section 4 by evaluating these normalizing constants. The new features of this investigation are that both H. Weyl's first and second main theorems of invariance theory [18] play a crucial role as does the analysis of the formal cohomology groups of spaces of p form valued invariants [7]. Thus we expect that the techniques presented in this paper will be useful in other similar investigations of this type.

2. Local Formulae for the heat trace invariants

If D is an operator of Laplace type, then there is a canonical connection ∇ on the underlying vector bundle that, together with the Levi-Civita connection, we use to covariantly differentiate tensors of all types - we denote multiple covariant differentiation by ';'. There is also a canonical endomorphism E so that

$$D = -(\mathrm{Tr}(\nabla^2) + E) \quad \text{i.e.} \quad Du = -(u_{;ii} + Eu).$$

Let Ω_{ij} be the curvature of the connection ∇. Let $\rho_{ij} = R_{ikkj}$ be the Ricci tensor, and let $\tau = \rho_{jj}$.

The heat trace invariants $a_{n,m}$ for an operator of Laplace type can be expressed in this formalism [9]:

Theorem 2.1.

(1) $a_{0,m}=(4\pi)^{-m/2}\operatorname{Tr}\{\mathrm{Id}\}$.

(2) $a_{2,m}=(4\pi)^{-m/2}\frac16\operatorname{Tr}\{6E+\tau\,\mathrm{Id}\}$.

(3) $a_{4,m}=(4\pi)^{-m/2}\frac1{360}\operatorname{Tr}\{60E_{;kk}+60\tau E+180E^2+(12\tau_{;kk}+5\tau^2$
$-2|\rho|^2+2|R|^2)\,\mathrm{Id}+30\Omega_{ij}\Omega_{ij}\}$.

(4) $a_{6,m}=(4\pi)^{-m/2}\operatorname{Tr}\{(\frac{18}{7!}\tau_{;iijj}+\frac{17}{7!}\tau_{;k}\tau_{;k}-\frac{2}{7!}\rho_{ij;k}\rho_{ij;k}-\frac{4}{7!}\rho_{jk;n}\rho_{jn;k}$
$+\frac{9}{7!}R_{ijkl;n}R_{ijkl;n}+\frac{28}{7!}\tau\tau_{;nn}-\frac{8}{7!}\rho_{jk}\rho_{jk;nn}+\frac{24}{7!}\rho_{jk}\rho_{jn;kn}$
$+\frac{12}{7!}R_{ijkl}R_{ijkl;nn}+\frac{35}{9\cdot7!}\tau^3-\frac{14}{3\cdot7!}\tau\rho^2+\frac{14}{3\cdot7!}\tau R^2-\frac{208}{9\cdot7!}\rho_{jk}\rho_{jn}\rho_{kn}$
$-\frac{64}{3\cdot7!}\rho_{ij}\rho_{kl}R_{ikjl}-\frac{16}{3\cdot7!}\rho_{jk}R_{jnli}R_{knli}-\frac{44}{9\cdot7!}R_{ijkn}R_{ijlp}R_{knlp}$
$-\frac{80}{9\cdot7!}R_{ijkn}R_{ilkp}R_{jlnp})\,\mathrm{Id}+\frac1{45}\Omega_{ij;k}\Omega_{ij;k}+\frac1{180}\Omega_{ij;j}\Omega_{ik;k}+\frac1{60}\Omega_{ij;kk}\Omega_{ij}$
$+\frac1{60}\Omega_{ij}\Omega_{ij;kk}-\frac1{30}\Omega_{ij}\Omega_{jk}\Omega_{ki}-\frac1{60}R_{ijkn}\Omega_{ij}\Omega_{kn}-\frac1{90}\rho_{jk}\Omega_{jn}\Omega_{kn}$
$+\frac1{72}\tau\Omega_{kn}\Omega_{kn}+\frac1{60}E_{;iijj}+\frac16 EE_{;ii}+\frac1{12}E_{;i}E_{;i}+\frac16E^3$
$+\frac1{12}E\Omega_{ij}\Omega_{ij}+\frac1{36}\tau E_{;kk}+\frac1{90}\rho_{jk}E_{;jk}+\frac1{30}\tau_{;k}E_{;k}+\frac1{12}EE\tau$
$+\frac1{30}E\tau_{;kk}+\frac1{72}E\tau^2-\frac1{180}E|\rho|^2+\frac1{180}E|R|^2\}$.

We refer to [10] for the proof of the following results:

Lemma 2.2. *On the circle,* $a_{2,1}^{d+\delta}=\frac{1}{\sqrt{\pi}}\phi_{;11}$.

Lemma 2.3. *Let* $M=(M_1\times M_2,\phi_1+\phi_2,g_1+g_2)$ *decouple as a product. Then* $a_{n,m}^{d+\delta}(\phi,g)(x_1,x_2)=\sum_{n_1+n_2=n,n_1\ge m_1,n_2\ge m_2}a_{n_1,m_1}^{d+\delta}(\phi_1,g_1)(x_1)\cdot a_{n_2,m_2}^{d+\delta}(\phi_2,g_2)(x_2)$.

3. Spaces of Invariants

Let $\mathcal{Q}_m$ be the space of $O(m)$ invariant polynomials in the components of the tensors $\{R,\nabla R,\nabla^2R,...,\phi,\nabla\phi,\nabla^2\phi,...\}$. Define a grading on $\mathcal{Q}_m$ by setting:

$$\mathrm{weight}(R_{ijkl;\beta})=|\beta|+2\ \text{and}\ \mathrm{weight}(\phi_{;\beta})=|\beta|.$$

An element $Q\in\mathcal{Q}_m$ is homogeneous of weight n if and only if

$$Q(\phi,c^{-2}g)=c^nQ(\phi,g).$$

Let $\mathcal{Q}_{n,m}\subset\mathcal{Q}_m$ be the set of all $O(m)$ invariant polynomials which are homogeneous of weight n; we then have a direct sum decomposition:

$$\mathcal{Q}_m=\oplus_n\mathcal{Q}_{n,m}.$$

We may use the $\mathbb{Z}_2$ action $\phi\to-\phi$ to decompose $\mathcal{Q}_{n,m}=\mathcal{Q}_{n,m}^+\oplus\mathcal{Q}_{n,m}^-$ where

$$\mathcal{Q}_{n,m}^\pm:=\{Q\in\mathcal{Q}_{n,m}:Q(-\phi,g)=\pm Q(\phi,g)\}.$$

The following natural restriction map $r:\mathcal{Q}_{n,m}\to\mathcal{Q}_{n,m-1}$ will play a crucial role. If (N,g_N,ϕ_N) are structures in dimension $m-1$, then we can define corresponding structures in dimension m by setting

$$(M,\phi_M,g_M):=(N\times S^1,\phi_N,g_N+d\theta^2)\,.$$

If $x\in N$ is the point of evaluation, we take the corresponding point $(x,1)\in M$ for evaluation; which point on the circle chosen is, of course, irrelevant as S^1 has a rotational symmetry. The restriction map is characterized dually by the formula:

$$r(Q)(\phi_N,g_N)(x)=Q(\phi_N,g_N+d\theta^2)(x,1)\,.$$

We can also describe the restriction map r in classical terms. H. Weyl's [18] first theorem of invariance theory implies orthogonal invariants are built by contracting indices in pairs, where the indices range from 1 through m. If P is given in terms of such a Weyl spanning set, then $r(P)$ is given in terms of the same Weyl spanning set by restricting the range of summation to be from 1 through $m-1$. Thus necessarily r is surjective. We refer to [10] for the proof of:

Lemma 3.1. *If m is even, then* $a_{m+2,m}^{d+\delta}\in\mathcal{Q}_{n,m}^+\cap\ker(r)$.

We use H. Weyl's second theorem to see $\mathcal{Q}^+_{m+2,m}\cap\ker(r)$ is generated by invariants where we contract $2m$ indices using the ε tensor and contract the remaining indices in pairs, we refer to the discussion in [10] for details. A direct calculation shows, after some additional work to eliminate dependencies, that:

$$\begin{aligned}\mathcal{Q}^+_{m+2,m}\cap\ker(r)=\operatorname{Span}\{&\varepsilon^I_J\phi_{;i_1j_1}\phi_{;i_2j_2}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_J\phi_{;k}\phi_{;k}\mathcal{R}^{I,m}_{J,1},\ \varepsilon^I_JR_{k\ell\ell k}\mathcal{R}^{I,m}_{J,1},\\ &\varepsilon^I_J\phi_{;i_1}\phi_{;j_1}R_{ki_2j_2k}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{i_1i_2j_2j_1;kk}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{i_1i_2j_2j_1;k}R_{i_3i_4j_4j_3;k}\mathcal{R}^{I,m}_{J,5},\\ &\varepsilon^I_JR_{ki_1j_1k}R_{\ell i_2j_2\ell}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{k\ell j_2j_1}R_{i_1i_2k\ell}\mathcal{R}^{I,m}_{J,3},\ \varepsilon^I_JR_{ki_1j_1\ell}R_{ki_2j_2\ell}\mathcal{R}^{I,m}_{J,3},\\ &\varepsilon^I_JR_{ki_1j_2j_1}R_{kj_3i_3i_2}R_{\ell i_4j_5j_4}R_{\ell j_6i_6i_5}\mathcal{R}^{I,m}_{J,7}\}.\end{aligned}$$

Although this is some gain in simplifying the question, the list of invariants is still quite long. We shall use equation (1.b) to further reduce the number of invariants to be considered and complete the proof of Lemma 1.3.

Let $\mathcal{Q}^{p,+}_{n,m}$ be the space of p form valued invariants in the curvature tensor, the covariant derivatives of the curvature tensor, and the covariant derivatives of ϕ which are even in ϕ. The exterior co-derivative δ_g induces a natural map

$$\delta_g:\mathcal{Q}^{p,+}_{n,m}\to\mathcal{Q}^{p-1,+}_{n+1,m}.$$

Let $i:N\to N\times\{1\}\subset N\times S^1$. The analysis of [4] shows that p form valued invariants are constructed by alternating p indices and by contracting the remaining indices in pairs. The restriction map

$$r:\mathcal{Q}^p_{n,m}\to\mathcal{Q}^p_{n,m-1}$$

is defined by restricting the range of summation of the indices involved; it is characterized by the identity:

$$r(Q)(\phi_N,g_N)=i^*Q(\phi_N,g_N+d\theta^2).$$

One verifies that r is surjective and that

$$r\circ\delta_{g_M}=\delta_{g_N}\circ r\,.$$

The analysis of the formal cohomology groups of the spaces of invariants of Riemannian manifolds, which was given in [7], then extends immediately to this more general setting to yield:

Lemma 3.2. (1) *If $Q\in\mathcal{Q}^+_{n,m}$, if $\int_M Q(\phi,g)=0$ for all (ϕ,g), and if $n\neq m$, then there exists $Q^1\in\mathcal{Q}^{1,+}_{n-1,m}$ so that $\delta_gQ^1=Q$.*

(2) *If $Q^1\in\mathcal{Q}^{1,+}_{n,m}$, if $\delta_gQ^1=0$, and if $n\neq m-1$, then there exists $Q^2\in\mathcal{Q}^{2,+}_{n-1,m}$ so that $Q^1=\delta_gQ^2$.*

The first assertion shows that any scalar invariant which always integrates to zero is canonically a divergence and that any 1 form valued invariant which is co-closed is canonically co-exact. The restriction on the weight is a technical one which plays no role as we shall take $n=m+2$ in assertion (1) and $n=m+1$ in assertion (2).

We use this result to show

Lemma 3.3. *If m is even, then there exists a 1 form valued invariant $Q^1_{m+1,m}$ in $\mathcal{Q}^{1,+}_{m+1,m}\cap\ker(r)$ so that $\delta_gQ^1_{m+1,m}=a^{d+\delta}_{m+2,m}$.*

Proof. By equation (1.b) and Lemma 3.2 (1), there exists $\bar{Q}^1_{m+1,m}\in\mathcal{Q}^{1,+}_{m+1,m}$ so

$$\delta_g\bar{Q}^1_{m+1,m}(\phi,g)=a^{d+\delta}_{m+2,m}(\phi,g).\tag{3.a}$$

Unfortunately, $r(\bar{Q}^1_{m+1,m})$ need not be zero and we must correct for this. Since $r(a^{d+\delta}_{m+2,m})=0$,

$$0=r(\delta_g\bar{Q}^1_{m+1,m})=\delta_gr(\bar{Q}^1_{m+1,m}).$$

Thus by Lemma 3.2 (2), there exists $\bar Q^2_{m,m-1}\in\mathcal{Q}^2_{m,m-1}$ so that

$$r(\bar Q^1_{m+1,m})=\delta_g(\bar Q^2_{m,m-1}).$$

We observed above that r is surjective. Thus we can find $Q^2_{m,m}\in\mathcal{Q}^{2,+}_{m,m}$ so that:

$$r(Q^2_{m,m})=\bar Q^2_{m,m-1}.$$

We complete the proof by setting $Q^1_{m+1,m}:=\bar Q^1_{m+1,m}-\delta_g(Q^2_{m,m})$ and computing:

$$\begin{aligned}
&\delta_g(Q^1_{m+1,m})=\delta_g(\bar Q^1_{m+1,m})-\delta_g^2(Q^2_{m,m})=\delta_g(\bar Q^1_{m+1,m})=a^{d+\delta}_{m+2,m},\\
&r(Q^1_{m+1,m})=r(\bar Q^1_{m+1,m})-r(\delta_g(Q^2_{m,m}))=r(\bar Q^1_{m+1,m})-\delta_g(r(Q^2_{m,m}))\\
&\quad=r(\bar Q^1_{m+1,m})-\delta_g(\bar Q^2_{m,m-1})=0.
\end{aligned}$$

□

For m even, we define elements of $\mathcal{Q}^{1,+}_{m+1,m}\cap\ker(r)$ by setting:

$$\begin{array}{ll}
\Xi^{1,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3}e^{j_2} & (\ell\text{ even}),\\
\Xi^{2,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3}e^{k} & (\ell\text{ even}),\\
\Xi^{3,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3}e^{j_2} & (\ell\text{ even}),\\
\Xi^{4,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^{k}, & (\ell\text{ odd}),\\
\Xi^{5,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kkj_2}\mathcal{R}^{I,m}_{J,3}e^{j_1} & (\ell\text{ odd}).
\end{array}$$

Lemma 3.4. *If $m=2\bar m$ is even, then $\{\mathcal{Q}^{1,+}_{m+1,m}\cap\ker(r)\}=\operatorname{Span}\{\Xi^{i,\ell}_{m+1,m}\}_{i,\ell}$.*

Proof. We use H. Weyl's theorem on the invariants of the orthogonal group. Let

$$A=\phi^\ell\phi_{;\alpha_1}...\phi_{;\alpha_u}R_{i_1j_1k_1l_1;\beta_1}...R_{i_vj_vk_vl_v;\beta_v}e^h$$

be a typical 1 form valued monomial where $|\alpha_\nu|\ge1$ and $\ell+u$ is even. Note that:

$$n=\textstyle\sum_\mu|\alpha_\mu|+\sum_\nu(|\beta_\nu|+2).$$

We must contract $2m$ indices using the ε tensor and contract the remaining indices in pairs; we refer to [10] where this was discussed in some detail for scalar invariants – the extension to 1 form valued invariants is similar. We may estimate:

$$\begin{array}{lrcl}
\text{(3.b)} & 2m & \le & \text{number of indices in }A\\
& & = & \sum_\mu|\alpha_\mu|+\sum_\nu(|\beta_\nu|+4)+1=n+2v+1\\
\text{(3.c)} & & = & 2n+1-\sum_\mu|\alpha_\mu|-\sum_\nu|\beta_\nu|\le 2n+1
\end{array}$$

We set $n=m+1$. Since $2m$ and $m+1+2v+1$ are both even, the inequality in equation (3.c) must be strict and represents an increase either of 1 or of 3.

Suppose first that equation (3.b) is an equality. Then all the $2m$ indices present in A are contracted using the ε tensor. We can commute covariant derivatives at the cost of introducing additional curvature terms. Thus since all indices are to be contracted using the ε tensor, we may assume $|\alpha_\mu|\le2$ for all μ. Furthermore, by the first and second Bianchi identity, at most 2 indices can be alternated in $R_{ijkl;\beta}$. Thus $|\beta_\nu|=0$ for all ν so $\sum_\mu|\alpha_\mu|=3$. This leads to the invariants $\Xi^{1,\ell}_{m+1,m}$.

Suppose next that equation (3.b) is not an equality. Then there are $2m+2$ indices and one explicit covariant derivative present in A; $2m$ indices are contracted using the ε tensor and two indices are contracted as a pair. This yields the invariants $\Xi^{i,\ell}_{m+1,m}$ for $i=2,3,4,5$ and the additional invariants:

$$\begin{array}{ll}
\Theta^{1,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1} & (\ell\text{ odd}),\\
\Theta^{2,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2j_1}\mathcal{R}^{I,m}_{J,3}e^{k} & (\ell\text{ odd}),\\
\Theta^{3,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_3j_2}R_{i_3i_4j_4k}\mathcal{R}^{I,m}_{J,5}e^{j_1} & (\ell\text{ odd}),\\
\Theta^{4,\ell}_{m+1,m}:=\varepsilon^I_J\phi^\ell R_{i_1i_2kj_2}R_{i_3i_4j_4j_3;k}\mathcal{R}^{I,m}_{J,5}e^{j_1} & (\ell\text{ even}).
\end{array}$$

To complete the proof, we must show the invariants $\Theta^{i,\ell}_{m+1,m}$ play no role. Let U and V be collections of $m+1$ indices. Since $\varepsilon^U_V=0$, we have

$$0=\varepsilon^U_V\phi^\ell\phi_{;u_1}\mathcal{R}^{U,m+1}_{V,2}e^{v_1}.$$

We set $u_1=k$ and then set $v_1=k$, $v_2=k$, ..., and $v_{m+1}=k$ in turn to see:

$$\begin{aligned}0&=\varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^k-m\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\\&=\Xi^{4,\ell}_{m+1,m}-m\Theta^{1,\ell}_{m+1,m}.\end{aligned}$$

We set $u_2=k$ and expand in v to see:

$$\begin{aligned}0&=\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2j_1}\mathcal{R}^{I,m}_{J,3}e^k-2\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\\&-(m-2)\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_3j_2}R_{i_3i_4j_4k}\mathcal{R}^{I,m}_{J,5}e^{j_1}\\&=\Theta^{2,\ell}_{m+1,m}+2\Xi^{5,\ell}_{m+1,m}-(m-2)\Theta^{3,\ell}_{m+1,m}.\end{aligned}$$

Next, we set $v_1=k$ and expand in u to see:

$$\begin{aligned}0&=\varepsilon^I_J\phi^\ell\phi_{;k}\mathcal{R}^{I,m}_{J,1}e^k+m\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2j_1}\mathcal{R}^{I,m}_{J,3}e^k\\&=\Xi^{4,\ell}_{m+1,m}+m\Theta^{2,\ell}_{m+1,m}.\end{aligned}$$

Finally, we set $v_2=k$ and expand in u to see:

$$\begin{aligned}0&=\varepsilon^I_J\phi^\ell\phi_{;k}R_{i_1i_2j_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}+2\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2kj_2k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\\&+(m-2)\varepsilon^I_J\phi^\ell\phi_{;i_1}R_{i_2i_3j_2k}R_{i_4kj_4j_3}\mathcal{R}^{I,m}_{J,5}e^{j_1}\\&=\Theta^{1,\ell}_{m+1,m}-2\Xi^{5,\ell}_{m+1,m}+(m-2)\Theta^{3,\ell}_{m+1,m}.\end{aligned}$$

We can show that

$$\{\Theta^{1,\ell}_{m+1,m},\Theta^{2,\ell}_{m+1,m},\Theta^{3,\ell}_{m+1,m}\}\subset\operatorname{Span}\{\Xi^{i,\ell}_{m+1,m}\}_{i,\ell}$$

by computing:

$$\begin{aligned}&\Theta^{1,\ell}_{m+1,m}=\tfrac1m\Xi^{4,\ell}_{m+1,m}=2\Xi^{5,\ell}_{m+1,m}-(m-2)\Theta^{3,\ell}_{m+1,m},\\&\Theta^{2,\ell}_{m+1,m}=-\tfrac1m\Xi^{4,\ell}_{m+1,m}=-2\Xi^{5,\ell}_{m+1,m}+(m-2)\Theta^{3,\ell}_{m+1,m}.\end{aligned}$$

Finally, we put $u_1=k$ in the identity

$$0=\varepsilon^U_V\phi^\ell R_{u_2u_3v_3v_2;u_1}\mathcal{R}^{U,m+1}_{V,4}e^{v_1}$$

and expand in v to show

$$\begin{aligned}0&=\varepsilon^I_J\phi^\ell R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3}e^k-2\varepsilon^I_J\phi^\ell R_{i_1i_2j_2k;k}\mathcal{R}^{I,m}_{J,3}e^{j_1}\\&-(m-2)\varepsilon^I_J\phi^\ell R_{i_1i_2j_3j_2;k}R_{i_3i_4j_4k}\mathcal{R}^{I,m}_{J,5}e^{j_1}\\&=\Xi^{2,\ell}_{m+1,m}-2\Xi^{3,\ell}_{m+1,m}+(m-2)\Theta^{4,\ell}_{m+1,m}.\end{aligned}$$

This establishes the lemma. □

We now prove Lemma 1.3. Let $m=2\bar m$ be even. We apply Lemma 3.3 and Lemma 3.4 to see there exist universal constants so

$$a^{d+\delta}_{m+2,m}(\phi,g)=\textstyle\sum_{i,\ell}c^{i,\ell}_{m+1,m}\delta_g\{\Xi^{i,\ell}_{m+1,m}\},$$

where ℓ is chosen so ϕ appears an even number of times in each expression. Terms which are linear in the 2 jets of ϕ and which are of total weight 2 in ϕ can arise only from $i=4$ and $i=5$. Consequently we have

$$a^{d+\delta}_{m+2,m}(\phi,g)=-\textstyle\sum_\ell\phi^\ell\varepsilon^I_J\{c^{4,\ell}_{m+1,m}\phi_{;kk}\mathcal{R}^{I,m}_{J,1}+c^{5,\ell}_{m+1,m}\phi_{;i_1j_1}R_{i_2kkj_2}\mathcal{R}^{I,m}_{J,3}\}+\dots$$

where ℓ is odd. Replacing ϕ by $\phi+c$ does not change d_ϕ and $\delta_{g,\phi}$. Thus ϕ^μ does not appear in the formula for $a^{d+\delta}_{m+2,m}$ for $\mu>0$. Consequently,

$$0=c^{4,\ell}_{m+1,m}\varepsilon^I_J\phi_{;kk}\mathcal{R}^{I,m}_{J,1}+c^{5,\ell}_{m+1,m}\varepsilon^I_J\phi_{;i_1j_1}R_{i_2kkj_2}\mathcal{R}^{I,m}_{J,3}.\tag{3.d}$$

We consider the expressions:

$$A_1 := \phi_{;11}R_{1221}R_{3443}...R_{m-1,mm,m-1}, \quad \text{and}$$
$$A_2 := \phi_{;12}R_{1332}R_{3443}...R_{m-1,mm,m-1}.$$

We may then expand

$$\begin{aligned}\phi_{;kk}\mathcal{R}^{I,m}_{J,1} &= 4^{\bar m}\bar m!A_1 + 0A_2 + ...,\\ \phi_{;i_1j_1}R_{i_2kkj_2}\mathcal{R}^{I,m}_{J,3} &= 4^{\bar m-1}(\bar m-1)!A_1 - 4\cdot 4^{\bar m-1}(\bar m-1)!A_2 + ...\end{aligned}$$

Consequently equation (3.d) implies $c^{4,\ell}_{m+1,m} = 0$ and $c^{5,\ell}_{m+1,m} = 0$.

We argue similarly to show that if $\ell > 0$, then

$$\begin{aligned}0 &= c^{1,\ell}_{m+1,m}\varepsilon^I_J\phi_{;j_2}\phi_{;i_1j_1}\phi_{;i_2}\mathcal{R}^{I,m}_{J,3}, \quad \text{and}\\ 0 &= c^{2,\ell}_{m+1,m}\varepsilon^I_J\phi_{;k}R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3} + c^{3,\ell}_{m+1,m}\varepsilon^I_J\phi_{;j_2}R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3}.\end{aligned}$$

This shows $c^{1,\ell}_{m+1,m} = 0$ for $\ell > 0$. We consider the expressions:

$$B_1 := \phi_{;1}R_{1221;1}R_{3443}...R_{m-1,mm,m-1}, \quad \text{and}$$
$$B_2 := \phi_{;3}R_{1221;3}R_{3443}...R_{m-1,mm,m-1}$$

and expand

$$\begin{aligned}\varepsilon^I_J\phi_{;k}R_{i_1i_2j_2j_1;k}\mathcal{R}^{I,m}_{J,3} &= 4^{\bar m}(\bar m-1)!B_1 + 4^{\bar m}(\bar m-1)!B_2 + \dots,\\ \varepsilon^I_J\phi_{;j_2}R_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3} &= 2\cdot 4^{\bar m-1}(\bar m-1)!B_1 + 0B_2 + \dots\end{aligned}$$

to see $c^{2,\ell}_{m+1,m} = 0$ and $c^{3,\ell}_{m+1,m} = 0$ for $\ell > 0$; Lemma 1.3 now follows. □

4. Determining the normalizing constants

We complete the proof of Theorem 1.2 by evaluating the normalizing constants of Lemma 1.3:

Lemma 4.1. *Let $m = 2\bar m$. Then*

(1) $c^1_{m+2,m} = \frac{4\bar m}{\pi^{\bar m}8^{\bar m}\bar m!}$.
(2) $c^2_{m+2,m} = \frac{1}{12}\frac{1}{\pi^{\bar m}8^{\bar m}\bar m!}$.
(3) $c^3_{m+2,m} = \frac{\bar m}{6}\frac{1}{\pi^{\bar m}8^{\bar m}\bar m!}$.

Proof. We shall apply Theorem 1.1, Theorem 2.1, Lemma 2.2, and Lemma 2.3. We use the method of universal examples. Give $M := S^{m-2}\times S^1\times S^1$ the product metric. Let $\phi = \phi_1(\theta_1) + \phi_2(\theta_2)$. Then:

$$\begin{aligned}a^{d+\delta}_{m+2,m}(\phi,g) &= 2c^1_{m+2,m}2^{\bar m-1}(m-2)!\phi_{;m-1m-1}\phi_{;mm}\\ &= a^{d+\delta}_{m-2,m-2}(0,g_{S^{m-2}})\cdot a^{d+\delta}_{2,1}(\phi_1,d\theta_1^2)\cdot a^{d+\delta}_{2,1}(\phi_2,d\theta_2^2)\\ &= \tfrac{1}{8^{\bar m-1}\pi^{\bar m-1}(\bar m-1)!}\cdot 2^{\bar m-1}(m-2)!\tfrac{1}{\pi}\phi_{1;m-1,m-1}\phi_{2;mm}.\end{aligned}$$

We solve this equation for $c^1_{m+2,m}$ to establish assertion (1).

For the remainder of the proof of the Lemma, we set $\phi = 0$ to consider only metric invariants. We express

$$a^{d+\delta}_{m,m} = \mathcal{E}_{m,m}c_{m,m} \quad \text{for} \quad \mathcal{E}_{m,m} = \varepsilon^I_J\mathcal{R}^{I,m}_{J,1} \quad \text{and} \quad c_{m,m} = \tfrac{1}{8^{\bar m}\pi^{\bar m}\bar m!}.$$

If $m = 2$, then the invariants $(\varepsilon^I_J\mathcal{R}^{I,m}_{J,1})_{;kk}$ and $(\varepsilon^I_JR_{i_1i_2kj_1;k}\mathcal{R}^{I,m}_{J,3})_{;j_2}$ are not linearly independent. If (N,g_N) is a Riemann surface, then we may establish assertions (2) and (3) for $m = 2$ by computing:

$$\begin{aligned}a^{d+\delta}_{2,2}(g_N) &= \tfrac{1}{4\pi}\textstyle\sum_p(-1)^p\mathrm{Tr}(E^p)\\ a^{d+\delta}_{4,2}(g_N) &= \tfrac{1}{4\pi}\tfrac{1}{6}\{\textstyle\sum_p(-1)^p\mathrm{Tr}(E^p)\}_{;kk} + O(R^2)\\ &= \tfrac{1}{6}\{a^{d+\delta}_{2,2}\}_{;kk} + O(R^2) = \tfrac{1}{4\pi}\tfrac{1}{6}R_{ijji;kk} + O(R^2).\end{aligned}$$

Suppose now that $m=4$. Since $\sum_p(-1)^p\mathrm{Tr}(\dim(\Lambda^p))=0$, we compute:

$$\begin{aligned}
&0=a_{2,4}^{d+\delta}(0,g)=\tfrac{1}{8\pi^2}\textstyle\sum_p(-1)^p\mathrm{Tr}(E^p),\\
&a_{4,4}^{d+\delta}=\tfrac{1}{4^2\pi^2}\textstyle\sum_p(-1)^p\mathrm{Tr}\{\tfrac12E^pE^p+\tfrac1{12}\mathrm{Tr}(\Omega_{ij}^p\Omega_{ij}^p)\},\\
&a_{6,4}^{d+\delta}=\tfrac{1}{4^2\pi^2}\textstyle\sum_p(-1)^p\mathrm{Tr}\{\tfrac1{45}\Omega_{ij;k}^p\Omega_{ij;k}^p+\tfrac1{180}\Omega_{ij;j}^p\Omega_{ik;k}^p+\tfrac1{60}\Omega_{ij;kk}^p\Omega_{ij}^p\\
&\qquad+\tfrac1{60}\Omega_{ij}^p\Omega_{ij;kk}^p+\tfrac16E^pE_{;ii}^p+\tfrac1{12}E_{;i}^pE_{;i}^p\}+O(R^3).
\end{aligned}$$

We study the expressions $C_1:=R_{1221}R_{3443}$ and $C_2:=R_{1221;2}R_{3443;2}$ and suppress other terms. Only the term E^pE^p can give rise to the expression A_2 and only the term $E_{;i}^pE_{;i}^p$ can give rise to the expression A_3. We prove assertion (2) if $m=4$ by computing:

$$\begin{aligned}
a_{4,4}^{d+\delta}&=\tfrac{1}{2\cdot4^2\pi^2}\textstyle\sum_p(-1)^p\mathrm{Tr}(E^pE^p)+\ldots=\tfrac{32}{8^2\pi^22!}R_{1221}R_{3443}+\ldots\\
a_{6,4}^{d+\delta}&=\tfrac{1}{4^2\pi^2}\tfrac1{12}\textstyle\sum_p(-1)^p\mathrm{Tr}(E_{;i}^pE_{;i}^p)+\ldots=\tfrac{1}{4^2\pi^2}\tfrac1{24}\sum_p(-1)^p\mathrm{Tr}(E^pE^p)_{;ii}+\ldots\\
&=\tfrac1{12}(a_{4,4}^{d+\delta})_{;kk}+\ldots=\tfrac16\tfrac{32}{8^2\pi^22!}R_{1221;2}R_{3443;2}+\ldots\\
&=2c_{6,4}^2\varepsilon_J^IR_{i_1i_2j_2j_1;k}R_{i_3i_4j_4j_3;k}+\ldots=64c_{6,4}^2R_{1221;2}R_{3443;2}+\ldots,\quad\text{so}\\
c_{6,4}^2&=\tfrac1{12}\tfrac{1}{8^2\pi^22!}.
\end{aligned}$$

If $m>4$, let $(M,g):=(N^4\times S^{m-4},g_N+g_0)$. Assertion (2) follows in general from:

$$\begin{aligned}
a_{m+2,m}^{d+\delta}(g)&=\bar m(\bar m-1)c_{m+2,m}^2\varepsilon_J^I(R_{i_1i_2j_2j_1;k}R_{i_3i_4j_4j_3;k})(g_N)\mathcal{E}_{m-4,m-4}(g_0)+\ldots\\
&=a_{6,4}^{d+\delta}(g_N)a_{m-4,m-4}^{d+\delta}(g_0)+\ldots\\
&=2c_{6,4}^2c_{m-4,m-4}\varepsilon_J^I(R_{i_1i_2j_2j_1;k}R_{i_3i_4j_4j_3;k})(g_N)\mathcal{E}_{m-4,m-4}(g_0)+\ldots\quad\text{so}\\
c_{m+2,m}^2&=\tfrac{2}{\bar m(\bar m-1)}\tfrac1{12}\tfrac{1}{\pi^28^22!}\cdot\tfrac{1}{\pi^{\bar m-2}8^{\bar m-2}(\bar m-2)!}=\tfrac1{12}\tfrac{1}{\pi^{\bar m}8^{\bar m}\bar m!}.
\end{aligned}$$

Let $(M,g)=(N^2\times S^{m-2},g_N+g_0)$. We derive a relation between the invariants $c_{m+2,m}^2$ and $c_{m+2,m}^3$ to complete the proof of assertion (3):

$$\begin{aligned}
&a_{m+2,m}^{d+\delta}(g)=(2\bar mc_{m+2,m}^2+c_{m+2,m}^3)R_{ijji;kk}(g_N)\cdot\mathcal{E}_{m-2,m-2}(g_0)\\
&\quad=a_{4,2}^{d+\delta}(g_N)a_{m-2,m-2}^{d+\delta}(g_0)=\tfrac1{4\pi}\tfrac16R_{ijji;kk}(g_N)\cdot c_{m-2,m-2}\mathcal{E}_{m-2,m-2}(g_0)\quad\text{so}\\
&2\bar mc_{m+2,m}^2+c_{m+2,m}^3=\tfrac16\tfrac1{4\pi}\tfrac{1}{\pi^{\bar m-1}8^{\bar m-1}(\bar m-1)!}=\tfrac{\bar m}{3}\tfrac{1}{\pi^{\bar m}8^{\bar m}\bar m!}.\qquad\square
\end{aligned}$$

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References

[1] P. Amsterdamski, A. Berkin, and D. O'Connor, b_8 *Hamidew coefficient for a scalar field*, Classical Quantum Grav., **6** (1989), 1981–1991.

[2] A. A. Andrianov, N. V. Borisov and M. V. Ioffe, *Factorization Method And Darboux Transformation For Multidimensional Hamiltonians,* Theor. Math. Phys. **61** (1984) 1078 [Teor. Mat. Fiz. **61** (1984) 183].

[3] I. G. Avramidi, *The covariant technique for the calculation of the heat kernel asymptotic expansion*, Physics Letters B., **238** (1990), 92–97.

[4] M. F. Atiyah, R. H. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330; Errata **28** (1975), 277–280.

[5] S. Chern, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, Ann. of Math. **45** (1944), 741–752.

[6] P. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math, **10** (1973), 344–382.

[7] —, Local invariants of an embedded Riemannian manifold, *Annals of Math.*, **102** (1971), 187–203.

[8] —, Curvature and the heat equation for the DeRham complex, in **Geometry and Analysis (Papers dedicated to the memory of V.K.Patodi)**, Indian Academy of Sciences (1979), 47–80.

[9] —, **Invariance Theory, the heat equation, and the Atiyah-Singer index theorem** 2^{nd} ed., CRC Press ISBN 0-8493-7874-4 (1994), 516pp.

[10] P. Gilkey, K. Kirsten, and D. Vassilevich, *Divergence terms in the supertrace heat asymptotics for the de Rham complex on a manifold with boundary*, math-ph/0211020.

[11] P. Gilkey, K. Kirsten, D. Vassilevich, and A. Zelnikov, *Duality symmetry of the p-form effective action and super trace of the twisted de Rham complex*, hep-th/0209125, to appear in Nucl. Phys. B.

[12] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geo., **1** (1967), 43–69.

[13] V. K. Patodi, Curvature and the fundamental solution of the heat operator, J. Indian Math. Soc. **34** (1970), 269–285.

[14] I. Polterovich, Heat invariants of Riemannian manifolds, *Isr. J. Math.* **119** (2000), 239-252.

[15] R. Seeley, *Complex powers of an elliptic operator*, in **Amer. Math. Soc. Proc. Symp. Pure Math**, **10** (1968), 288–307.

[16] A. E. van de Ven, *Index-free heat kernel coefficients*, Class. Quant. Grav., **15** (1998), 2311–2344, hep-th/9708152.

[17] D. Vassilevich and A. Zelnikov, *Discrete symmetries of functional determinants*, Nucl. Phys. B **594** (2000), 501–517.

[18] H. Weyl, **The Classical Groups**, Princeton Univ. Press, Princeton, 1946.

[19] E. Witten, *Supersymmetry and Morse Theory*, J. Diff. Geom. **17** (1982), 661–692.

PG: Mathematics Department, University of Oregon, Eugene Or 97403 USA
E-mail address: gilkey@darkwing.uoregon.edu

KK: Department of Mathematics, Baylor University, Waco, TX 76798 USA and Max-Planck-Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig Germany
E-mail address: klaus.kirsten@mis.mpg.de and Klaus_Kirsten@baylor.edu

DV: Max-Planck-Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig Germany
E-mail address: vassil@itp.uni-leipzig.de