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by

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1 Introduction

A famous theorem of H. Lebesgue states that a Lipschitz function $f : [0,1] \to \mathbb{R}$ is differentiable at almost every point. Approximation by linear functions at the scale r is measured by

$$\beta_f(x,r) = \inf_{a,b\in\mathbb{R}} \sup\{\frac{1}{r}|f(y) - (ay+b)| : y \in (x-r, x+r) \cap \mathrm{dom}(f)\}. \quad (1)$$

Thus Lebesgue's theorem implies that

$$\lim_{r\to 0} \beta_f(x,r) = 0,$$

for almost every $x \in [0,1]$. This conclusion was profoundly strengthened by C. Bishop and P. Jones who proved in [J, B-J] that

$$\sum_{k=1}^{\infty} \beta_f^2(x, 2^{-k}) < \infty,$$

for almost every $x \in [0,1]$. They also showed that this result is optimal within the class of estimates that hold almost everywhere. This however does not rule out the possibility that a better estimate holds on a small subset of $[0,1]$. In particular, the question remains open whether for an arbitrary Lipschitz function $f : [0,1] \to \mathbb{R}$ the estimate

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) < \infty \quad (2)$$

holds in *at least one point* $x \in [0,1]$.

This problem is linked to the ongoing efforts to provide geometric understanding for J. Bourgain's results ([B-1], [B-2]), that there exist points $x \in [0,1]$, at which bounded harmonic functions have finite radial variation. That is, when u is bounded and harmonic in the unit disk then there necessarily exists $x \in [0,1]$ such that

$$\int_0^1 |\nabla u(re^{2\pi i x})| dr < \infty.$$

The link is P. Jones' estimate [J1] that

$$\int_0^1 |\nabla u(re^{2\pi i x})| dr \le C \sum_{k=1}^{\infty} \beta_f(x, 2^{-k}),$$

where f is the Lipschitz function obtained by integrating the boundary values of u, in other words

$$f(x) = \int_0^x u(e^{2\pi i y}) dy.$$

In this paper we exhibit a Lipschitz function $f : [0, 1] \to \mathbb{R}$ for which $\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty$ at *every* $x \in [0, 1]$. The feedback to the result on radial variation is the clarification that Bourgain's proof does not find points where the Lipschitz functions f is particularily flat, but rather it exhibits points around which f is remarkably symmetric. Indeed, by our example points where f is flat might not exist; by Bourgain's results (see [B-1],[B-2]) points of symmetry do exist, and they even form a set of Hausdorff dimension one.

Beside this connection with the radial variations of harmonic function, our estimate (2) is also closely related to the original result about differentiability. In fact, it is the only sufficient condition in terms of decay of the $\beta_f(x, \cdot)$'s as we notice the following

Remark

i) Given an arbitrary function f and a point x from the interior of its domain. Then the condition (2) implies that f is at x differentiable. This is a consequence of the estimate

$$\left| \frac{f(y) - f(z)}{y - z} - \frac{f(y') - f(z')}{y' - z'} \right| \leq 16\, \beta_f(x, R)$$

if $0 < \frac{R}{4} < (y - z), (y' - z')$ and $y, y', z, z' \in [x - R, x + R]$, which is in turn implied by

$$\left| \frac{f(y) - f(z)}{y - z} - a_R \right| \leq 8\, \beta_f(x, R)$$

where a_R is an optimal slope in the definition (1) of $\beta_f(x, R)$.

ii) If $\beta_{k+1} \in (0, 2\,\beta_k)$, $|s_k - s_{k+1}| \leq \beta_k$ and $\sup_k |s_k| < \infty$ then the Lipschitz function f which satisfies $f(x) = s_k x$ for $|x| = 2^{-k}$ and is affine on any of the intervals $\pm[2^{-k-1}, 2^{-k}]$ also fulfills $\beta_f(0, 2^{-k}) \leq \beta_k$ for all k.

2 The construction of a rough Lipschitz function

The result of this paper is the following

Theorem 1 *There exists a Lipschitz function f on $[0, 1]$ such that*

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty,$$

at every $x \in [0, 1]$.

We deduce Theorem 1 from the following seemingly weaker statement. In fact we show that most (in the topological sense) Lipschitz functions satisfy the conclusion of Theorem 1.

Theorem 2 *For any K there exists a Lipschitz function h on $[0,1]$ with Lipschitz constant $\leq 1/K$ such that*

$$\sum_{k=1}^{\infty} \beta_h(x, 2^{-k}) \geq K,$$

at every $x \in [0,1]$.

We will first present the reduction of Theorem 1 to Theorem 2. Here we employ a general scheme based on Baire's category theorem. We work with the complete metric space formed by imposing the constraint $Lip(f) \leq 1$ on the unit ball of L^{∞}.

Proof of Theorem 1: Assuming that Theorem 2 has been proven we obtain from it Theorem 1. Obviously, that the metric space

$$X = (\{f \in L^{\infty}[0,1] : Lip(f) \leq 1\}, ||\cdot||_{\infty})$$

is complete. Moreover, notice that on the product $[0,1] \times [0,1] \times (X, ||\cdot||_{\infty})$, the mapping

$$(x, r, f) \to \beta_f(x, r),$$

is continuous due to the compactness of the slopes for which the β_f's are realized. We show first that for each $K \in \mathbb{N}$ the set

$$B_K = \{f \in X : \exists x \in [0,1], \sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) \leq K\}$$

is a closed subset of X. To this end we let f_n be a convergent sequence in B_K, such that $f_n \to f$ in the $||\cdot||_{\infty}-$ metric. We prove that then $f \in B_K$. Let $x_n \in [0,1]$, such that

$$\sum_{k=1}^{\infty} \beta_{f_n}(x_n, 2^{-k}) \leq K.$$

Then, let $x \in [0,1]$ be a cluster point of the sequence x_n. Without loss of generality we may assume that $x_n \to x$. To show that $f \in B_K$, we will verify that,

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) \leq K.$$

Select any $N \in \mathbb{N}$ and $\varepsilon > 0$. Recall now that $\beta_f(x,r)$ depends continuously on x and f. Hence we find $n = n(N)$, such that

$$|\beta_f(x, 2^{-k}) - \beta_{f_n}(x_n, 2^{-k})| \leq \frac{\varepsilon}{N},$$

for any $k \leq N$. Thus we estimate,

$$\begin{aligned}\sum_{k=1}^{N} \beta_f(x, 2^{-k}) &\leq \sum_{k=1}^{N} \beta_{f_n}(x_n, 2^{-k}) + \sum_{k=1}^{N} |\beta_f(x, 2^{-k}) - \beta_{f_n}(x_n, 2^{-k})| \\ &\leq K + N\frac{\varepsilon}{N}.\end{aligned}$$

As $N \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary we showed that $f \in B_K$.

Next we prove that none of the sets B_K, $K \in \mathbb{N}$, has an interior point in X. Indeed, let $K < \infty$, let $f \in B_K$ and let $\varepsilon > 0$. Next we choose $f_1 \in C^\infty$ such the $Lip(f_1)$ is *strictly (!)* less than 1, and

$$||f - f_1||_{L^\infty[0,1]} \leq \varepsilon/4.$$

Clearly, as $f_1 \in C^\infty$ we find a constant C_1 such that

$$\sup_{x\in[0,1]} \sum_{k=1}^{\infty} \beta_{f_1}(x, 2^{-k}) < C_1.$$

By Theorem 2, there exists f_2, so that $Lip(f_2) \leq (1 - Lip(f_1))/2$, and $||f_2||_\infty \leq \varepsilon/4$, and such that, for any $x \in [0, 1]$,

$$\sum_{k=1}^{\infty} \beta_{f_2}(x, 2^{-k}) \geq K + C_1 + 1.$$

Now define

$$g = f_1 + f_2.$$

Then we have that $Lip(g) \leq 1$, and $||f - g||_\infty \leq 3\varepsilon/4$, and for any $x \in [0, 1]$,

$$\sum_{k=1}^{\infty} \beta_g(x, 2^{-k}) \geq K + 1.$$

Summing up, we showed that $g \in X$ is $\varepsilon-$close to $f \in B_K$ and $g \notin B_K$. Thus $B_K \subseteq X$ does not contain an interior point. The union of the sets B_K, is then a first category set in the complete metric space X. By Baire's theorem we obtain that,

$$X \setminus \bigcup_{K\in\mathbb{N}} B_K \neq \emptyset.$$

This proves Theorem 1 since each $f \in X \setminus \bigcup_{K\in\mathbb{N}} B_K$, is a Lipschitz function for which,

$$\sum_{k=1}^{\infty} \beta_f(x, 2^{-k}) = \infty$$

for any $x \in [0, 1]$.

■

Comment: The use of Baire's Theorem shortened this paper considerably. Indeed, without Baire we would be forced construct a Lipschitz Function h in Theorem 2, which satisfies also

$$\sup_{x\in[0,1]} \sum_{k=1}^{\infty} \beta_h(x, 2^{-k}) < \infty.$$

The modifications necessary to obtain this are quite unpleasant, technically, and more importantly they are obscuring the nature of the problem at hand.

Before entering the proof of Theorem 2 we would like to point out that in 1980 M. Talagrand constructed a collection $\mathcal{E}$ of pairwise disjoint intervals in $[0, 1]$ covering a set E of measure $1/100$, and for almost every $x \in [0, 1] \setminus E$ there exists a sequence of intervals $I_n \in \mathcal{E}$ such that,

$$\sum_n \frac{|I_n|}{|I_n| + \operatorname{dist}(x, I_n)} \geq K \gg 1.$$

The proof below is based in Talagrand's method of construction, as presented in [J-M-T]. Here we have to review it carefully, since we need to iterate it, to handle an exceptional zero set and to describe the path from Talagrand's collection to the Lipschitz function required in Theorem 2.

Proof of Theorem 2: In this proof and the rest of the paper we abbreviate $\bigcup \mathcal{A}$ by $\mathcal{A}^*$.

Step 1. We fix a large constant $K \in \mathbb{N}$, and a sequence $\varepsilon_p > 0$ of small positive constants such that

$$\sum_{p=1}^{\infty} \varepsilon_p \leq \frac{1}{100}$$

and

$$\sum_{p=1}^{\infty} \varepsilon_p |\log \varepsilon_p| = \infty.$$

We start defining a collection $\mathcal{D}$ of disjoint closed subintervals of $[0, 1]$. Let $\mathcal{D}_1$ be a collection consisting of equidistant closed intervals in $[0, 1]$ of equal length l_1 such that $0, 1 \in D_1^*$, $|\mathcal{D}_1^*| = \varepsilon_1$ and define the function

$$b_1(x) = \frac{l_1}{\operatorname{dist}(x, \mathcal{D}_1^*) + l_1}.$$

After p steps of the construction we have arrived at collections

$$\mathcal{D}_1, \cdots, \mathcal{D}_p$$

where each $\mathcal{D}_i$ is a family of pairwise disjoint closed intervals of equal length l_i and also the covered closed sets $\mathcal{D}_i^*$ are pairwise disjoint subsets of $[0, 1]$. Together with these families we also have the functions

$$b_i(x) = \frac{l_i}{\operatorname{dist}(x, \mathcal{D}_i^*) + l_i}, \text{ where } b_i \equiv 0 \text{ if } \mathcal{D}_i = \emptyset.$$

Moreover, we are given a sequence of open sets

$$H_1, \cdots, H_{p-1}$$

with $H_i \subseteq H_{i-1} \subseteq [0,1]$ such that

$$(\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_j^*) \cap H_j = \emptyset \text{ for } j = 1, \ldots, p-1.$$

Now we define the new set

$$H_p = \{x \in [0,1] : \sum_{i=1}^{p} b_i(x) < K + 1 \text{ and } x \notin (\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_p^*)\} \subset (0,1).$$

If $|H_p| \leq 4\varepsilon_{p+1}$, then we put $\mathcal{D}_{p+1} = \emptyset$ and are ready to iterate the procedure. Else, we pick a collection $\mathcal{H}_p$ of disjoint open intervals of equal length $L_{p+1} \in (0, l_p/50)$ such that $|\mathcal{H}_p^*| > |H_p|/2$ and that $x \in H_p$ provided $\operatorname{dist}(x, \mathcal{H}_p^*) < L_{p+1}$. For each $J \in \mathcal{H}_p$ we define $\mathcal{D}_{p+1}(J)$ to consist of the single closed intervals of length $l_{p+1} = L_{p+1}\varepsilon_{p+1}/|\mathcal{H}_p^*| < L_{p+1}/2$ concentric with J. We put

$$\mathcal{D}_{p+1} = \bigcup_{J \in \mathcal{H}_p} \mathcal{D}_{p+1}(J),$$

and we define as before

$$b_{p+1}(x) = \frac{l_{p+1}}{\operatorname{dist}(x, \mathcal{D}_{p+1}^*) + l_{p+1}}.$$

For b_{p+1} we have in this case the following crucial estimate:

$$\int_{H_p} b_{p+1}(x) \geq 2\varepsilon_{p+1} |\log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}}|. \tag{3}$$

Indeed for each $J \in \mathcal{H}_p$ we obtain, by integrating $\frac{1}{x}$,

$$\int_J b_{p+1}(x) \geq |\mathcal{D}_{p+1}(J)^*| |\log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}}|.$$

Then we use that

$$\sum_{J \in \mathcal{H}_p} |\mathcal{D}_{p+1}(J)^*| = \varepsilon_{p+1},$$

to arrive at (3). Again we are ready to iterate our construction and keep on doing so.

In this way we obtain the full family $\mathcal{D} = \bigcup_{p=1}^{\infty} \mathcal{D}_p$ and the exceptional set $H = \bigcap_{p=1}^{\infty} H_p$. Notice that $|\mathcal{D}^*| \leq 1/100$ and that we can suppose

$$|I| \leq 100^{-k}|I'| \text{ if } I \in \mathcal{D}_p \text{ and } I' \in \mathcal{D}_{p+k} \text{ with } k \geq 0. \tag{4}$$

We now show that $|H| = \lim_{p\to\infty} |H_p| = 0$. Indeed, otherwise we see that (3) will hold for all p sufficiently large. We would then obtain a contradiction from the following chain of estimates which uses the fact that the sequence $\{H_p\}$ is decreasing

$$\begin{aligned} \sum_{p=p_0}^{\infty} 2\varepsilon_{p+1} |\log \frac{|\mathcal{H}_p^*|}{\varepsilon_{p+1}}| &\leq \sum_{p=1}^{\infty} \int_{H_p} b_{p+1}(x) \\ &= \sum_{p=1}^{\infty} \int_{H_p \setminus H_{p+1}} \sum_{j=2}^{p+1} b_j(x) \\ &\leq K + 2. \end{aligned}$$

Step 2 Here we want to understand how the functions b_p defined before can be used to get a lower bound on β_f. We have the following simple statement.

Lemma 1 *Let $x, a, a + 2b \in [0,1]$ and $b > 0$. Suppose the measureable set $M \subset [a, a + 2b]$ satisfies $|M \Delta [a+b, a+2b]| < b/49$ and that the lipschitz function $f : [0,1] \to \mathbb{R}$ fulfils $|f' - \chi_M| \leq 1/8$ a.e. on $[a, a + 2b]$. Then*

$$\beta_f(x, 2^{-k}) \geq \left(\frac{2}{49}\right) \frac{2b}{2b + \operatorname{dist}(x, [a, a + 2b])},$$

if $\max(|x - a|, |x - a - 2b|) \in (2^{-k-1}, 2^{-k}]$.

For the proof it is sufficient to notice that on one of the subintervals $[a, a + b]$, $[a + b, a + 2b]$ the gradient of f differs, up to a subset of measure $b/49$, with a fixed sign and in modulus at least $3/8$ from the best approximating slope a (as occuring in (1)), which moreover has to be in $[-1/8, 9/8]$. This shows that f minus its best affine approximation oscillates at least $(3/8)(48b/49) - 2(b/49) = 16b/49$ and so the difference can not everywhere be smaller than $8b/49$. Since $2^{-k-1} < 2b + \operatorname{dist}(x, [a, a + 2b])$, the lemma follows.

Step 3 From Step 1 and Step 2 it is clear that we can hope to achieve a big sum of the β_f's only where $\sum_p b_p$ became large, i.e. at all $x \in [0, 1 \setminus (\mathcal{D}^* \cup H)$. Since the remaining set, in particular $\mathcal{D}^*$, is fairly large we have to iterate the construction from Step 1 in order to get a large $\sum_k \beta_f(\cdot, 2^{-k})$ also there.

For this purpose we associate with any closed interval $I \in [0, 1]$ the affine map ϕ_I such that $I = [\phi_I(0), \phi_I(1)]$ and put $\mathcal{D}^1 = \mathcal{D}$. Now let

$$\mathcal{D}^{i+1} = \{\phi_I(J) \ : \ I \in \mathcal{D}^i \text{ and } J \in \mathcal{D}\} \text{ for } i \geq 1.$$

We define

$$\mathcal{D}^\infty = \bigcup_{i=1}^{\infty} \mathcal{D}^i, H_I = \phi_I(H) \text{ and } H^\infty = \bigcup_{I \in \mathcal{D}^\infty} H_I.$$

As $\mathcal{D}^\infty$ is countable, H^∞ is again of measure zero. We also have $(\mathcal{D}^{i+1})^* \subset (\mathcal{D}^i)^*$ and more precisely, for $I \in \mathcal{D}^i$ we get $|(\mathcal{D}^{i+1})^* \cap I| \leq |I|/100$ and hence $|(\bigcup_{j>i} \mathcal{D}^j)^* \cap I| \leq |I|/99$.

We define M_0 to be the set of all x such that there are $i \geq 1$ and $[a, a+2b] \in \mathcal{D}^i$ with $x \in [a+b, a+2b] \setminus (\mathcal{D}^{i+1})^*$ and choose the lipschitz function

$$f_0(x) = \int_0^x \chi_{M_0}(t)\, dt = |M_0 \cap [0, x]|.$$

To ensure that (2) fails also in all $x \in H^\infty$ we use a slight modification of an idea of C.Goffman ([G]). Since $|H^\infty| = 0$, we find $\{G_k\}_{k=1}^\infty$ open sets such that $|G_k| < 2^{-k}$ and $H^\infty \subset G_{k+1} \subset G_k \subset [0, 1]$. We can even assume that $|G_{k+1} \cap I| < |I|/100$ for all connected components I of G_k. Indeed, once this is true for $G_1, \ldots, G_k$ we can modify G_{k+1} by replacing for each connected component I the set $G_{k+1} \cap I$ by $G_{k'} \cap I$, k' sufficiently large. In this way we get a new open set $\tilde{G}_{k+1} \supset H^\infty$ contained in G_{k+1} and sufficiently small in any component of G_k.

Let $\mathcal{I}_k$ be the system of all connected components of G_k, so $\{\mathcal{I}_k\}_{k=1}^\infty$ forms a sequence of nested families of open intervals. It is now easy to see that we inductively find numbers $w_I \in [-1/8, 1/8]$, $I \in \bigcup_k \mathcal{I}_k$, such that for $I = (a, b) \subset I' = (a', b')$, $I \in \mathcal{I}_{k+1}$, $I' \in \mathcal{I}_k$

$$\left| \frac{f_{k+1}(b) - f_{k+1}(a)}{b - a} - \frac{f_k(b') - f_k(a')}{b' - a'} \right| > \frac{1}{8} \tag{5}$$

where

$$f_l(x) = f_0(x) + \sum_{j=1}^{l} \int_0^x \sum_{I \in \mathcal{I}_j} w_I \chi_{I \setminus G_{j+1}}(t)\, dt.$$

Our final function f_∞ will then be given by $f_0(x) = \int_0^x g_\infty(t)\, dt$ with

$$g = \chi_{M_0} + \sum_{j=1}^{\infty} \sum_{I \in \mathcal{I}_j} w_I \chi_{I \setminus G_{j+1}}.$$

Note that $\|g_\infty - \chi_{M_0}\|_\infty \leq 1/8$ because the family $\{I \setminus G_{j+1} \ : \ I \in \mathcal{I}_k, k \geq 1\}$ is disjointed and that for $(a, b) \in \mathcal{I}_k$

$$|f_k(b) - f_k(a) - (f_\infty(b) - f_\infty(a))| \leq \frac{1}{8}|(a, b) \cap G_{k+1}| \leq \frac{1}{800}|b - a|.$$

Together with (5) this gives

$$\left| \frac{f_\infty(b) - f_\infty(a)}{b - a} - \frac{f_\infty(b') - f_\infty(a')}{b' - a'} \right| > \frac{1}{9}$$

for $I = (a, b) \subset I' = (a', b')$, $I \in \mathcal{I}_{k+1}$, $I' \in \mathcal{I}_k$. This shows that f_∞ is not differentiable at any $x \in H^\infty$. Thus due to part (i) of the Remark in the Introduction we have $\sum_{k=1}^\infty \beta_{f_\infty}(x, 2^{-k}) = \infty$

for all $x \in H^\infty$. It is also clear from Lemma 1 that $\limsup_{k\to\infty} \beta_{f_\infty}(x, 2^{-k}) \geq 2/49$ if $x \in \bigcap_i (\mathcal{D}^i)^*$. Since the Lipschitz constant of f_∞ is not more than 2, Theorem 2 will be established if we show that $\sum_{k=1}^\infty \beta_{f_\infty}(x, 2^{-k}) > K/50$ for all $x \in [0,1] \setminus (H^\infty \cup \bigcap_i (\mathcal{D}^i)^*)$. Therefore, we can suppose $x \in I \setminus (\phi_I(\mathcal{D}^*) \cup \phi_I(H_p))$ for some $I \in \mathcal{D}^i$ and $p \geq 1$. This ensures of course that we can find $I_j \in \mathcal{D}_j$ (as defined in Step 1) such that

$$K + 1 \leq \sum_{j=1}^{p} \frac{|I_j|}{|I_j| + \operatorname{dist}(I_j, \phi_I^{-1}(x))} = \sum_{j=1}^{p} \frac{|\tilde{I}_j|}{|\tilde{I}_j| + \operatorname{dist}(\tilde{I}_j, x)}, \text{ where } \tilde{I}_j = \phi_I(I_j).$$

Denoting for $j \in \{1, \ldots, p\}$ by k_j the largest k such that $\tilde{I}_j \subseteq [x - 2^{-k}, x + 2^{-k}]$ the required estimate follows from Lemma 1 and the construction of M_0 if we show that for all $k \geq 1$

$$\sum_{\{j : k_j = k\}} \frac{|\tilde{I}_j|}{|\tilde{I}_j| + \operatorname{dist}(\tilde{I}_j, x))} \leq 2 \frac{|\tilde{I}_{j_0}|}{|\tilde{I}_{j_0}| + \operatorname{dist}(\tilde{I}_{j_0}, x)}, \text{ where } j_0 = \min\{j \,:\, k_j = k\}. \tag{6}$$

But if $k_j = k$ and $j > j_0$, then (4) implies $|\tilde{I}_j| \leq |\tilde{I}_{j_0}|/100 \leq 2^{-k-4}$ and hence $\tilde{I}_j \cap [x - 2^{-k-2}, x + 2^{-k-2}] = \emptyset$. Therefore, we obtain for such j that

$$\frac{|\tilde{I}_j|}{|\tilde{I}_j| + \operatorname{dist}(\tilde{I}_j, x)} \leq \frac{100^{j_0 - j} |\tilde{I}_{j_0}|}{2^{-k-2}} \leq 100^{j_0 - j} \frac{8 |\tilde{I}_{j_0}|}{|\tilde{I}_{j_0}| + \operatorname{dist}(\tilde{I}_{j_0}, x)},$$

which gives the desired inequality (6) even with a factor $1 + 8/99 < 2$. This finishes our proof.

■

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