# Max-Planck-Institut
## für Mathematik
## in den Naturwissenschaften
## Leipzig


**Partial regularity for the Landau-Lifshitz equation in small dimensions**

(revised version: April 2002)

by


*Roger Moser*



Preprint no.: 26 2002

# Partial regularity for the Landau-Lifshitz equation in small dimensions

Roger Moser
MPI for Mathematics in the Sciences
Inselstr. 22–26, D-04103 Leipzig, Germany

March 13, 2002

**Abstract**

We show that in $n \leq 4$ space dimensions, weak solutions of the Landau-Lifshitz equation of the ferromagnetic spin chain are smooth in an open set with a complement of vanishing $n$-dimensional Hausdorff measure with respect to the parabolic metric.

## 1 Introduction

For $n = 2$, 3, or 4, let $\Omega \subset \mathbb{R}^n$ be an open set and $T > 0$. We consider solutions $u = (u^1, u^2, u^3) : \Omega \times (0, T) \to \mathbb{S}^2$ of the Landau-Lifshitz equation

$$\partial_t u = -\alpha\, u \times (u \times \Delta u) - \beta\, u \times \Delta u, \tag{1}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are given constants. Here $\mathbb{S}^2 \subset \mathbb{R}^3$ is the standard 2-sphere, and $\times$ denotes the vector product in $\mathbb{R}^3$. By rescaling the time axis, we can normalize the equation so that

$$\alpha^2 + \beta^2 = 1. \tag{2}$$

We will henceforth assume that (2) holds. It is then easy to see that for classical solutions, (1) is equivalent to

$$\partial_t u = \alpha\, \Delta u + \alpha\, |\nabla u|^2 u - \beta\, u \times \Delta u, \tag{3}$$

and also to

$$\alpha\, \partial_t u + \beta\, u \times \partial_t u = \Delta u + |\nabla u|^2 u. \tag{4}$$

We are in particular interested in weak solutions of the Landau-Lifshitz equation in the version (4). We define

$$H^1(\Omega', \mathbb{S}^2) = \left\{ u \in H^1(\Omega', \mathbb{R}^3) \colon\ |u| = 1 \text{ almost everywhere} \right\}$$

for any open subset $\Omega'$ of $\mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}$, and call a map $u \in H^1(\Omega \times (0, T), \mathbb{S}^2)$ a weak solution of the Landau-Lifshitz equation, if

$$\int_0^T \int_\Omega (\langle \alpha\, \partial_t u + \beta\, u \times \partial_t u - |\nabla u|^2 u, \phi \rangle + \langle \partial_\gamma u, \partial_\gamma \phi \rangle)\, dx\, dt = 0$$

for all $\phi \in C_0^\infty(\Omega \times (0,T), \mathbb{R}^3)$, where $\partial_\gamma = \frac{\partial}{\partial x^\gamma}$, and where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^3$. Here and throughout the paper we sum over repeated Greek indices from 1 to $n$. For compact manifolds (instead of $\Omega$) as domains, Guo–Hong [18] proved the existence of global weak solutions to the Cauchy problem for (4).

In the special case $\alpha = 1$ (and $\beta = 0$), the Landau-Lifshitz equation reads

$$\partial_t u = \Delta u + |\nabla u|^2 u, \tag{5}$$

which is the heat flow for harmonic maps, i. e. the negative $L^2$-gradient flow of the energy functional

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$$

for $u \in H^1(\Omega, \mathbb{S}^2)$. In general, (4) differs from (5) by a rotation of the vector $\partial_t u$ by the fixed angle $\arcsin \beta$ in the tangent space of $\mathbb{S}^2$ at $u$. Since $\alpha > 0$, the equation retains its parabolicity with this transformation.

For the heat flow for harmonic maps, there is the following partial regularity result, due to Feldman [12] (and proved in a different version independently by Chen–Li–Lin [5]). If a map $u \in H^1(\Omega \times (0,T), \mathbb{S}^2)$ satisfies (5) and a certain stability condition, then there exists an open set $\mathcal{R} \subset \Omega \times (0,T)$, such that $u \in C^\infty(\mathcal{R}, \mathbb{S}^2)$, and the $n$-dimensional Hausdorff measure of $(\Omega \times (0,T)) \backslash \mathcal{R}$ with respect to the parabolic metric $d((x,s),(y,t)) = \max\{|x-y|, \sqrt{|s-t|}\}$ (subsequently called the $n$-dimensional parabolic Hausdorff measure) vanishes. Even better results hold for the case $n = 2$. Namely, under certain conditions, weak solutions of (5) are smooth except at finitely many points. This follows from a uniqueness result of Freire [13, 14] for the Cauchy problem and the construction of such solutions by Struwe [22]. These results for dimension 2 have been extended to the Landau-Lifshitz equation by Chen–Guo [4], Chen–Ding–Guo [3], and Ding–Guo [8, 9] (with some inaccuracy in the arguments however; see Section 1.4 in [19]).

The question that we study in this note is whether partial regularity can also be obtained for weak solutions of (4) in higher dimensions. The answer is yes if $n \leq 4$. The reason why we have to restrict ourselves to small dimensions is the following. For the equation (5), a main tool for proving regularity is a monotonicity formula which was discovered by Struwe [23], and certain estimates derived from it. For the Landau-Lifshitz equation, no such formula is available. If $n$ is however at most 4, we can nevertheless derive a monotonicity inequality that serves our purpose. Our main result then is that under a certain stability condition, any weak solution of (4) is smooth in an open set that has a complement of vanishing $n$-dimensional parabolic Hausdorff measure.

# 2 The stability condition

Even for solutions of the heat flow for harmonic maps, no partial regularity result holds without any additional conditions. Indeed there is an example, due to Rivière [21], of a weak solution of the elliptic problem (giving rise to a time-independent weak solution of (4) for any parameters $\alpha, \beta$), which is nowhere continuous.

In [12], Feldman proposed a stability condition for weak solutions of (5), which is a parabolic version of the usual stationarity condition for the elliptic case, and which allows to prove a local energy inequality and the monotonicity formula of Struwe [23] for such solutions. We impose a similar condition on weak solutions of (4).

But first, we introduce a convenient abbreviation.

*Notation.* For $p \in \mathbb{S}^2$, let $R_p : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ denote the rotation $R_p v = \alpha\, v + \beta\, p \times v$.

**Definition 2.1** *Let $u \in H^1(\Omega \times (0,T), \mathbb{S}^2)$ be a weak solution of (4). Consider for $\xi \in C_0^\infty(\Omega \times (0,T), \mathbb{R}^n)$ and $\tau \in C_0^\infty(\Omega \times (0,T), [0,\infty))$ the variation*

$$\tilde{u}_\sigma(x,t) = u(x + \sigma\xi(x,t), t + \sigma\tau(x,t)),$$

*which consists of maps in $H^1(\Omega\times(0,T), \mathbb{S}^2)$ for small $|\sigma|$. We say that u satisfies the stability condition, if for all such $\xi$ and $\tau$, the inequality*

$$\int_0^T \int_\Omega \left\langle R_u \partial_t u, (\tfrac{\partial}{\partial\sigma}\tilde{u}_\sigma)|_{\sigma=0} \right\rangle dx\,dt + \left( \partial_\sigma^+ \int_0^T E(\tilde{u}_\sigma(\cdot,t))\,dt \right)\bigg|_{\sigma=0} \leq 0$$

*holds, where $\partial_\sigma^+$ denotes the right hand derivative with respect to $\sigma$.*

*Remark.* A simple integration by parts shows that smooth solutions of the Landau-Lifshitz equation satisfy the stability condition.

**Lemma 2.1** *Let $u \in H^1(\Omega\times(0,T), \mathbb{S}^2)$ be a weak solution of (4) which satisfies the stability condition. Let $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$ and $\tau \in C_0^\infty(\Omega \times (0,T), [0,\infty))$. Then*

$$\int_{\Omega\times\{t\}} \left( \phi \cdot \langle R_u\partial_t u, \nabla u\rangle - \frac{1}{2} \operatorname{div}\phi\, |\nabla u|^2 + \partial_\gamma \phi^\delta \, \langle \partial_\gamma u, \partial_\delta u\rangle \right) dx = 0 \qquad (6)$$

*for almost every $t \in (0,T)$, and*

$$\begin{aligned} \int_{\Omega\times\{t_2\}} \tau|\nabla u|^2\,dx - \int_{\Omega\times\{t_1\}} \tau|\nabla u|^2\,dx \qquad\qquad (7) \\ \leq \int_{t_1}^{t_2} \int_\Omega (\partial_t\tau|\nabla u|^2 - \nabla\tau\cdot\langle\nabla u, \partial_t u\rangle - \alpha\,\tau|\partial_t u|^2)\,dx\,dt \end{aligned}$$

*for almost all $t_1, t_2$ with $0 \leq t_1 \leq t_2 \leq T$.*

*Proof.* Inequality (7) is proved exactly like Proposition 8 in [12]. Like Proposition 7 in [12], we prove the equality

$$\int_{\Omega\times(t_1,t_2)} \left( \xi \cdot \langle R_u\partial_t u, \nabla u\rangle - \frac{1}{2}\operatorname{div}\xi\,|\nabla u|^2 + \partial_\gamma\xi^\delta\,\langle\partial_\gamma u, \partial_\delta u\rangle \right) dz = 0$$

for all $\xi \in C_0^\infty(\Omega \times (0,T), \mathbb{R}^n)$ and almost all $t_1, t_2$ with $0 \leq t_1 \leq t_2 \leq T$. From this, (6) follows immediately. □

From (6) we can now deduce a monotonicity inequality.

**Lemma 2.2** *For $n = 3$ or 4, let $u \in H^1(\Omega \times (0,T), \mathbb{S}^2)$ be a weak solution of (4) which satisfies the stability condition. Suppose $B_s(x_0) \subset B_r(x_0) \subset \Omega$. Set*

$$\Phi(\rho, t) = \rho^{2-n} \int_{B_\rho(x_0)\times\{t\}} \left( |\nabla u|^2 - \frac{2}{n-2} \langle (x - x_0) \cdot \nabla u, R_u \partial_t u \rangle \right) dx$$

*for $s \le \rho \le r$, and for all $t \in (0,T)$ such that this is well-defined. Then*

$$\begin{aligned} &\Phi(r,t) - \Phi(s,t) \\ &\quad = 2 \int_{B_r(x_0)\setminus B_s(x_0)} \left( \frac{|(x-x_0)\cdot \nabla u|^2}{|x-x_0|^n} - \frac{\langle (x-x_0)\cdot\nabla u, R_u \partial_t u\rangle}{(n-2)|x-x_0|^{n-2}} \right) dx. \end{aligned} \tag{8}$$

*for almost every $t \in (0,T)$. In particular, we have*

$$\begin{aligned} s^{2-n} \int_{B_s(x_0)\times\{t\}} |\nabla u|^2 \, dx \le 4r^{2-n} &\int_{B_r(x_0)\times\{t\}} |\nabla u|^2 \, dx \\ &+ 8r^{4-n} \int_{B_r(x_0)\times\{t\}} |\partial_t u|^2 \, dx \end{aligned} \tag{9}$$

*for almost every $t \in (0,T)$.*

*Proof.* The estimate (9) follows from (8) by Young's inequality. To prove (8), we use the usual arguments.

The following can be done for almost every fixed $t \in (0,T)$. Set $v(x) = u(x,t)$ and $w(x) = R_{u(x,t)} \partial_t u(x,t)$. We assume for simplicity that $x_0 = 0$. Inserting test functions of the form $\phi(x) = \eta_k(|x|)x$ into (6) for smooth functions $\eta_k : [0,\infty) \to [0,\infty)$ which converge to the characteristic function of $[0,\rho)$ (where $s \le \rho \le r$), we prove that

$$\int_{B_\rho(0)} \left( (n-2)|\nabla v|^2 - 2\langle x\cdot\nabla v, w\rangle \right) dx = \int_{\partial B_\rho(0)} \left( \rho |\nabla v|^2 - \frac{2}{\rho} |x \cdot \nabla v|^2 \right) do.$$

Hence

$$\begin{aligned} &\frac{d}{d\rho} \left( \rho^{2-n} \int_{B_\rho(0)} \left( |\nabla v|^2 - \frac{2}{n-2} \langle x \cdot \nabla v, w \rangle \right) dx \right) \\ &\qquad = 2 \int_{\partial B_\rho(0)} \left( \rho^{-n} |x\cdot\nabla v|^2 - \frac{\rho^{2-n}}{n-2} \langle x\cdot\nabla v, w\rangle \right) do. \end{aligned}$$

Integrating this over the interval $(s,r)$ yields (8), and the proof is complete. □

*Remarks.*

(i) Whereas everything else in this paper works regardless of the dimension of the domain, this is where we use the restriction $n \le 4$.

(ii) The same computations give a monotonicity inequality similar to (9) for any solution $u \in H^1(\Omega, \mathbb{S}^2)$ of the equation

$$\Delta u + |\nabla u|^2 u = w$$

with $w \in L^2(\Omega, \mathbb{R}^3)$, or the corresponding equation for different target manifolds, provided that a condition similar to (6) is satisfied. This might be of independent interest, in particular in view of certain inequalities in [20].

## 3 Energy decay

Many of the arguments which follow are well-known and have been used before to prove partial regularity for harmonic maps or for the heat flow for harmonic maps (cf. [1, 5, 11, 12, 20]). We only have to adapt them to the present situation.

The following inequality was proved in this form by Feldman [12] (cf. also [2, 6]).

**Lemma 3.1** *Let $f, h \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, such that $\operatorname{div} g \in L^2(\mathbb{R}^n)$ in the distribution sense, and*

$$\sup_{x_0 \in \mathbb{R}^n, r>0} \left( r^{2-n} \int_{B_r(x_0)} |\nabla h|^2 \, dx \right) = A^2 < \infty.$$

*Then*

$$\left| \int_{\mathbb{R}^n} f\, g \cdot \nabla h \, dx \right| \leq CA(\|\nabla f\|_{L^2} \|g\|_{L^2} + \|f\|_{L^2} \|\operatorname{div} g\|_{L^2})$$

*for a universal constant $C$.*

Using this, we can estimate the energy of solutions of (4) which satisfy the stability condition as follows.

**Lemma 3.2** *For any $\delta > 0$ there exists a number $\epsilon_0 > 0$, such that for any weak solution $u \in H^1(P_r(z_0), \mathbb{S}^2)$ of (4) satisfying the stability condition, the inequality*

$$r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \, dz \leq \epsilon^2 \leq \epsilon_0^2 \tag{10}$$

*implies*

$$r^{-n} \int_{P_{r/8}(z_0)} |\nabla u|^2 \, dz \leq \delta \epsilon^2 + C_1 r^{-n-2} \int_{P_r(z_0)} |u - (u)_{P_r(z_0)}|^2 \, dz$$

*for a constant $C_1 = C_1(\alpha, \delta)$, where*

$$(u)_{P_r(z_0)} = \frac{1}{|P_r(z_0)|} \int_{P_r(z_0)} u \, dz.$$

*Proof.* Note first that all the quantities appearing in the lemma are invariant under the transformation $(x,t) \mapsto (rx + x_0, r^2 t + t_0)$. We may thus assume that $P_r(z_0) = P_1(0)$.

Given a number $\lambda \in (0,1)$, we infer from (7) that there exists a set $\Lambda \subset (-\frac{1}{2}, \frac{1}{2})$ of measure $|\Lambda| \leq \lambda$, such that

$$\int_{B_{1/2}(0) \times \{t\}} |\partial_t u|^2 \, dx \leq \frac{C\epsilon^2}{\lambda}$$

for all $t \in (-\frac{1}{2}, \frac{1}{2}) \backslash \Lambda$. Here and in the sequel, we denote by $C$ indiscriminately any constant which depends only on the parameter $\alpha$. Moreover, for almost every $t \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$\int_{B_{1/2}(0) \times \{t\}} |\nabla u|^2 \, dx \leq C\epsilon^2$$

by the same inequality, and for almost all $t \notin \Lambda$, we even obtain the estimate

$$\sup_{B_r(x_0) \subset B_{1/4}(0)} \left( r^{2-n} \int_{B_r(x_0) \times \{t\}} |\nabla u|^2 \, dx \right) \leq \frac{C\epsilon^2}{\lambda}$$

from (9). Pick a $t$ with these properties.

Choose $\zeta \in C_0^\infty(B_{1/4}(0))$ with $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $B_{1/8}(0)$, and $|\nabla \zeta| \leq 16$. Note that

$$\begin{aligned} \int_{B_1(0) \times \{t\}} \zeta |\nabla u|^2 \, dx &= -\int_{B_1(0) \times \{t\}} \zeta \left\langle u - (u)_{P_1(0)}, R_u \partial_t u \right\rangle \, dx \\ &\quad + \int_{B_1(0) \times \{t\}} \zeta \left\langle u - (u)_{P_1(0)}, |\nabla u|^2 u \right\rangle \, dx \qquad (11) \\ &\quad - \int_{B_1(0) \times \{t\}} \nabla \zeta \cdot \left\langle u - (u)_{P_1(0)}, \nabla u \right\rangle \, dx. \end{aligned}$$

Since $u$ takes values in $\mathbb{S}^2$ almost everywhere, we have

$$|\nabla u|^2 u^i = \sum_{j=1}^{3} \nabla u^j \cdot (u^i \nabla u^j - u^j \nabla u^i).$$

Furthermore,

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = u^i w^j - u^j w^i,$$

where $w = R_u \partial_t u$, and hence

$$\begin{aligned} \| \operatorname{div}(\zeta(u^i \nabla u^j - u^j \nabla u^i))\|_{L^2} &\leq 32 \|\nabla u(\cdot, t)\|_{L^2(B_{1/4}(0))} + 2 \|\partial_t u(\cdot, t)\|_{L^2(B_{1/4}(0))} \\ &\leq \frac{C\epsilon}{\sqrt{\lambda}}. \end{aligned}$$

Extending the functions $u - (u)_{P_1(0)}$ and $\nabla u$ appropriately to $\mathbb{R}^n$ and applying Lemma 3.1, we find that

$$\int_{B_1(0) \times \{t\}} \zeta \left\langle u - (u)_{P_1(0)}, |\nabla u|^2 u \right\rangle \, dx \leq \frac{C\epsilon^3}{\lambda} + C\epsilon^2 \int_{B_1(0) \times \{t\}} |u - (u)_{P_1(0)}| \, dx.$$

Using Hölder's and Young's inequality to estimate the other terms on the right hand side of (11) and the second term on the right hand side above, we obtain

$$\int_{B_{1/8}(0) \times \{t\}} |\nabla u|^2 \, dx \leq \left( \frac{C\epsilon}{\lambda} + \frac{\delta}{2} \right) \epsilon^2 + \frac{C}{\delta \lambda} \int_{B_1(0) \times \{t\}} |u - (u)_{P_1(0)}|^2 \, dx.$$

Hence

$$\int_{P_{1/8}(0)} |\nabla u|^2 \, dx \leq \left( \frac{C\epsilon}{\lambda} + C\lambda + \frac{\delta}{2} \right) \epsilon^2 + \frac{C}{\delta \lambda} \int_{P_1(0)} |u - (u)_{P_1(0)}|^2 \, dx.$$

With the right choice of $\lambda$ and $\epsilon_0$, this implies the claim. □

**Lemma 3.3** *There exists a constant $c > 0$, such that for any $\theta \in (0, \frac{1}{2}]$, there is a number $\epsilon_0 > 0$ with the following property. For any weak solution $u \in H^1(P_r(z_0), \mathbb{S}^2)$ of (4), satisfying (7) and the small energy condition (10), we have*

$$(\theta r)^{-n-2} \int_{P_{\theta r}(z_0)} |u - (u)_{P_{\theta r}(z_0)}|^2 \, dz \leq c \theta^2 \epsilon^2.$$

*Proof.* We may assume again that $P_r(z_0) = P_1(0)$. Suppose the claim were false. Then for any fixed $c > 0$ we could find a number $\theta \in (0, \frac{1}{2}]$ and weak solutions $u_k \in H^1(P_1(0), \mathbb{S}^2)$ of (4), satisfying (7), such that

$$\int_{P_1(0)} |\nabla u_k|^2 \, dz =: \epsilon_k^2 \to 0 \quad \text{as } k \to \infty, \tag{12}$$

but

$$\int_{P_\theta(0)} |u_k - (u_k)_{P_\theta(0)}|^2 \, dz > c\theta^{n+4}\epsilon_k^2. \tag{13}$$

Set $v_k = \frac{1}{\epsilon_k}(u_k - (u_k)_{P_\theta(0)})$. This sequence is bounded in $H^1(P_{1/2}(0), \mathbb{R}^3)$ by (12) and (7), thus we may assume that it converges weakly in $H^1(P_{1/2}(0), \mathbb{R}^3)$ and strongly in $L^2(P_{1/2}(0), \mathbb{R}^3)$ to a map $v \in H^1(P_{1/2}(0), \mathbb{R}^3)$. Obviously,

$$\int_{P_\theta(0)} v \, dz = 0 \quad \text{and} \quad \int_{P_{1/2}(0)} |\nabla v|^2 \, dz \le 1.$$

Moreover we may assume that $u_k$ converges strongly in $L^2(P_{1/2}(0), \mathbb{R}^3)$ to some constant $p \in \mathbb{S}^2$ as $k \to \infty$. Then for any $\phi \in C_0^\infty(P_{1/2}(0), \mathbb{R}^3)$, we have

$$\begin{aligned}
&\int_{P_{1/2}(0)} (\langle \alpha \, \partial_t v + \beta \, p \times \partial_t v, \phi \rangle + \langle \partial_\gamma v, \partial_\gamma \phi \rangle) \, dz \\
&\qquad = \lim_{k\to\infty} \frac{1}{\epsilon_k} \int_{P_{1/2}(0)} (\langle \alpha \, \partial_t u_k + \beta \, u_k \times \partial_t u_k, \phi \rangle + \langle \partial_\gamma u_k, \partial_\gamma \phi \rangle) \, dz \\
&\qquad = \lim_{k\to\infty} \frac{1}{\epsilon_k} \int_{P_{1/2}(0)} |\nabla u_k|^2 \langle u_k, \phi \rangle \, dz = 0.
\end{aligned}$$

Thus $v$ satisfies

$$\alpha \, \partial_t v + \beta \, p \times \partial_t v - \Delta v = 0,$$

or, equivalently,

$$\partial_t v + \alpha \, p \times (p \times \Delta v) + \beta \, p \times \Delta v - \frac{1}{\alpha} \langle p, \Delta v \rangle \, p = 0.$$

This is a linear parabolic system, and standard estimates yield

$$\int_{P_\theta(0)} |v|^2 \, dz \le C\theta^{n+4}.$$

Choosing $c > C$, we obtain a contradiction to (13) by the strong $L^2$-convergence of $v_k$ to $v$. □

Combining Lemma 2.1, Lemma 3.2, and Lemma 3.3, we obtain immediately the following energy decay estimate.

**Proposition 3.1** *There exists a constant $c > 0$, such that for every $\theta \in (0, 1]$ there is a number $\epsilon_0 > 0$ with the following property. If $u \in H^1(P_r(z_0), \mathbb{S}^2)$ is a solution of (4) which satisfies the stability condition, then (10) implies*

$$(\theta r)^{-n} \int_{P_{\theta r}(z_0)} |\nabla u|^2 \, dz \le c\theta^2\epsilon^2.$$

## 4 Partial Regularity

Finally, we are able to prove the main results.

**Proposition 4.1** *There exist constants* $\epsilon_0 > 0$ *and* $c_{kl} < \infty$ *(*$k, l = 0, 1, 2, \ldots$*), such that any weak solution* $u \in H^1(P_r(z_0), \mathbb{S}^2)$ *of (4), which satisfies the stability condition and (10), is smooth in* $P_{r/2}(z_0)$ *with*

$$\|\partial_t^l \nabla^k u\|_{L^\infty(P_{r/2}(z_0))} \leq c_{kl} r^{-k-2l} \epsilon, \quad k, l = 0, 1, 2, \ldots \tag{14}$$

*Proof.* Proposition 3.1 implies that for any $\lambda \in (0, 1)$, if $\epsilon_0 > 0$ is sufficiently small, we have under the conditions above

$$\int_{P_s(z_1)} (|\nabla u|^2 + s^2 |\partial_t u|^2) \, dz \leq C_1 s^{n+2\lambda}$$

for any $z_1 \in P_{3r/4}(z_0)$ and $s \in (0, \frac{r}{4})$, where $C_1$ is a constant depending only on $\lambda$ and $\alpha$. By Lemma 4.1 in [5], $u$ is $\lambda$-Hölder continuous in $P_{3r/4}(z_0)$ with respect to the parabolic metric. In particular it is the solution of a parabolic systems with Hölder continuous leading coefficients. Lipschitz continuity for $u$ can now be proved like in [12] (Lemma 21), using the fundamental solutions for general parabolic systems, as constructed e. g. in Chapter 9 of [15], instead of the fundamental solution for the heat equation. A bootstrapping argument eventually gives higher regularity. The bounds in (14) follow from a scaling argument. We omit the details. □

**Theorem 4.1** *Let* $u \in H^1(\Omega \times (0, T), \mathbb{S}^2)$ *be a weak solution of (4), satisfying the stability condition. There exists an open set* $\mathcal{R} \subset \Omega \times (0, T)$ *with a complement of vanishing* $n$*-dimensional parabolic Hausdorff measure, such that* $u \in C^\infty(\mathcal{R}, \mathbb{S}^2)$*.*

*Proof.* Consider the relatively closed set $\mathcal{S}$ of all points $z_0 \in \Omega \times (0, T)$ such that

$$\liminf_{r \searrow 0} \left( r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \, dz \right) \geq \epsilon_0^2,$$

where $\epsilon_0 > 0$ is the constant from Proposition 4.1. Then the $n$-dimensional parabolic Hausdorff measure of $\mathcal{S}$ vanishes. This is proved by a standard covering argument (cf. Lemma 11 in [16]).

If $z_0 \in \mathcal{R} = (\Omega \times (0, T)) \backslash \mathcal{S}$, then we can find a radius $r > 0$, such that the conditions of Proposition 4.1 are satisfied. Regularity in $\mathcal{R}$ thus follows immediately. □

*Acknowledgement.* This work was supported by a fellowship of the Swiss National Science Foundation.

## References

[1] F. Bethuel, *On the singular set of stationary harmonic maps*, Manuscripta Math. **78** (1993), 417–443.

[2] S. Chanillo, *Sobolev inequalities involving divergence free maps*, Comm. Partial Differential Equations **16** (1991), 1969–1994.

[3] Y. Chen, S. Ding, and B. Guo, *Partial regularity for two-dimensional Landau-Lifshitz equations*, Acta Math. Sinica (N.S.) **14** (1998), 423–432.

[4] Y. Chen and B. Guo, *Two-dimensional Landau-Lifshitz equation*, J. Partial Differential Equations **9** (1996), 313–322.

[5] Y. Chen, J. Li, and F.-H. Lin, *Partial regularity for weak heat flows into spheres*, Comm. Pure Appl. Math. **48** (1995), 429–448.

[6] R. Coifman, P. L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[7] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto, *Magnetic microstructures—a paradigm of multiscale problems*, ICIAM 99 (Edinburgh), Oxford Univ. Press, Oxford, 2000, pp. 175–190.

[8] S. Ding and B. Guo, *Initial-boundary value problem for the Landau-Lifshitz system. I. Existence and partial regularity*, Progr. Natur. Sci. (English Ed.) **8** (1998), 11–23.

[9] ———, *Initial-boundary value problem for the Landau-Lifshitz system. II. Uniqueness*, Progr. Natur. Sci. (English Ed.) **8** (1998), 147–151.

[10] W. E and C. J. García-Cervera, *Effective dynamics for ferromagnetic thin films*, J. Appl. Phys. **171** (2001), 370–374.

[11] L. C. Evans, *Partial regularity for stationary harmonic maps into spheres*, Arch. Rational Mech. Anal. **116** (1991), 101–113.

[12] M. Feldman, *Partial regularity for harmonic maps of evolution into spheres*, Comm. Partial Differential Equations **19** (1994), 761–790.

[13] A. Freire, *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment. Math. Helv. **70** (1995), 310–338, correction in Comment. Math. Helv. **71** (1996), 330–337.

[14] ———, *Uniqueness for the harmonic map flow in two dimensions*, Calc. Var. Partial Differential Equations **3** (1995), 95–105.

[15] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, 1964.

[16] M. Giaquinta and E. Giusti, *Partial regularity for the solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. (4) **97** (1973), 253–266.

[17] G. Gioia and R. D. James, *Micromagnetics of very thin films*, Proc. R. Soc. Lond. A **453** (1997), 213–223.

[18] B. Guo and M. C. Hong, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Calc. Var. Partial Differential Equations **1** (1993), 311–334.

[19] P. Harpes, *Ginzburg-Landau type approximatins for the Landau-Lifshitz flow and the harmonic map flow in two space dimensions*, Ph.D. thesis, ETH Zürich, 2001.

[20] R. Moser, *Regularity for the approximated harmonic map equation and application to the heat flow for harmonic maps*, to appear in Math. Z.

[21] T. Rivière, *Everywhere discontinuous harmonic maps into spheres*, Acta Math. **175** (1995), 197–226.

[22] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985), 558–581.

[23] ______, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. **28** (1988), 485–502.