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Heat kernel coefficients and divergencies of the Casimir energy for the dispersive sphere

by

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Heat Kernel Coefficients and Divergencies of the Casimir Energy for the Dispersive Sphere

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Abstract

The first heat kernel coefficients are calculated for a dispersive ball whose permittivity at high frequency differs from unity by inverse powers of the frequency. The corresponding divergent part of the vacuum energy of the electromagnetic field is given and ultraviolet divergencies are seen to be present. Also in a model where the number of atoms is fixed the pressure exhibits infinities. As a consequence, the ground-state energy for a dispersive dielectric ball cannot be interpreted easily.

The ground-state energy for a dielectric ball shows ultraviolet divergencies still lacking physical understanding. This is an unsatisfactory situation, not only for general reasons but also in view of the rapid experimental progress.

The canonical way to investigate the ultraviolet divergencies is to calculate the corresponding heat kernel coefficients. For the dielectric (nondispersive) ball this had been done in [1] and for the dielectric cylinder in [2], where it had been shown, for instance, that the coefficient a_2 is zero in dilute order and nonzero beyond. In the present note we calculate the relevant heat kernel coefficients for a dielectric ball with dispersion.

Dispersion means a frequency dependent permittivity, $\epsilon(\omega)$. This is motivated by the expectation that for high frequency the permittivity tends to unity so

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that the ultraviolet modes contribute less to the ground-state energy and the divergencies become weaker.

It is reasonable to assume the asymptotic behavior of $\epsilon(\omega)$ to be

$$\epsilon(\omega) = 1 - \frac{\epsilon_1}{\omega^2} + \frac{\epsilon_2}{\omega^4} + \dots \tag{1}$$

for $\omega \to \infty$. Higher terms do not contribute to the ultraviolet divergencies. Let us note that this asymptotic behavior is typical for solid state models, the Drude and plasma models for instance. In the latter, the parameter ϵ_1 in Eq. (1) has the meaning of the plasma frequency squared.

We remind the reader of some basic formulas. In zeta functional regularization the ground-state energy is given by [3, 4]

$$E_0(s) = \frac{\mu^2}{2} \sum_j \lambda_j^{1-2s}, \tag{2}$$

where λ_j are the corresponding (discrete) energy eigenvalues. It can be expressed in terms of the corresponding zeta function,

$$E_0(s) = \frac{\mu^2}{2} \zeta(s - \frac{1}{2}). \tag{3}$$

Here μ is an arbitrary parameter with the dimension of a mass. Dropping the so called Minkowski space contribution the zeta function can be represented as

$$\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} (2l+1) \int_0^{\infty} dk\ k^{-2s} \frac{\partial}{\partial k} \ln f_l(ik), \tag{4}$$

where $f_l(ik)$ is the Jost function of the corresponding scattering problem. A detailed explanation of these and related formulas can be found in [1, 7, 8].

The heat kernel coefficients can be obtained from the zeta function by means of

$$a_n = (4\pi)^{3/2} \operatorname*{Res}_{s = \frac{3}{2} - n} \Gamma(s)\zeta(s) \tag{5}$$

and the divergent part of the ground-state energy in zeta functional regularization is given by (for a massless field)

$$E_0^{\text{div}}(s) = \frac{-a_2}{32\pi^2} \left(\frac{1}{s} + 2\ln\mu - 2 \right). \tag{6}$$

Here we drop contributions from $a_{1/2}$ and $a_{3/2}$, as these coefficients will turn out to vanish for the problem considered.

It is known that in the zeta function regularization there is a smaller number of singular contributions to the vacuum energy than in other regularization schemes. For example, in the regularization

$$E_0(\delta) = \frac{1}{2} \sum_j \lambda_j\ e^{-\delta \lambda_j} \tag{7}$$

with an exponentially damping function the divergent part of the ground-state energy is

$$E_0^{\rm div}(\delta) = \frac{1}{16\pi^2}\left(\frac{2}{\delta^2}a_1 + \ln\delta\ a_2\right), \tag{8}$$

again with the Minkowski space contribution already subtracted and dropping the terms with $a_{\frac{1}{2}}$ and $a_{\frac{3}{2}}$.

The Jost function for the problem at hand is known. For convenience, during the calculation we put the radius R of the dielectric ball equal to one, $R = 1$. The dependence on R is recovered by dimensional arguments. With the notation $\nu = l + 1/2$, the Jost function consists of contributions from the TE and the TM modes,

$$f_l(ik) = \Delta_\nu^{TE}(ik)\ \Delta_\nu^{TM}(ik) \tag{9}$$

$(l = 1, 2, \ldots)$ with

$$\Delta_\nu^{TE}(ik) = \epsilon^{-\frac{\nu}{2}}\left(\sqrt{\epsilon}s'e - se'\right), \tag{10}$$

$$\Delta_\nu^{TM}(ik) = \epsilon^{-\frac{\nu}{2}}\left(\frac{1}{\sqrt{\epsilon}}s'e - se'\right). \tag{11}$$

Here the abbreviations

$$s \equiv s_l(q) = \sqrt{\frac{\pi q}{2}}I_\nu(q), \tag{12}$$

$$e \equiv e_l(k) = \sqrt{\frac{2k}{\pi}}K_\nu(k), \tag{13}$$

are used where $I_\nu(q)$ and $K_\nu(k)$ are the modified Bessel functions. The prime denotes the differentiation with repect to the argument of these functions. The arguments of the Bessel functions are related by

$$q = \sqrt{\epsilon}k. \tag{14}$$

The factors $\epsilon^{-\frac{\nu}{2}}$ in (10) and (11) follow from the normalization condition of the regular solution of the scattering problem, for details see the example of a square well potential in [7]. This is of importance since we consider ϵ depending on k. We mention that these representations hold in the presence of arbitrary frequency dispersion, as has been noted in [5] (see also [6]).

In order to get the residues according to Eq. (5) it is sufficient to approximate the Jost function by its uniform asymptotic expansion for large k and ν keeping $z \equiv \frac{k}{\nu}$ fixed. Using the well known expansions [10], we obtain

$$\begin{aligned} \ln f_l(ik) &= \nu\left(2\left(\eta(\sqrt{\epsilon}z) - \eta(z)\right) - \ln\epsilon\right) \\ &\qquad + \ln\left(\epsilon^{\frac{1}{4}}\tilde{s}'\tilde{e} - \epsilon^{-\frac{1}{4}}\tilde{s}\tilde{e}'\right) + \ln\left(\epsilon^{-\frac{1}{4}}\tilde{s}'\tilde{e} - \epsilon^{\frac{1}{4}}\tilde{s}\tilde{e}'\right) \\ &\equiv D_0 + D_{TE} + D_{TM}, \end{aligned} \tag{15}$$

where the tilde denotes the Bessel functions without the exponential factors. Now we use the expansion

$$\epsilon(ik) = 1 + \frac{\epsilon_1}{k^2} + \frac{\epsilon_2}{k^4} + \dots \tag{16}$$

as well as the known expression for η, for example $\eta'(z) = \sqrt{1+z^2}/z$, and obtain

$$D_0 = \frac{1}{\nu}\epsilon_1 \frac{\sqrt{1+z^2}-1}{z^2} + \frac{1}{\nu^3}\left(\epsilon_2 \frac{\sqrt{1+z^2}-1}{z^4} - \frac{\epsilon_1^2}{4}\,\frac{(\sqrt{1+z^2}-1)^2}{z^4\sqrt{1+z^2}}\right) + \dots \tag{17}$$

For D_{TE} and D_{TM} we obtain

$$D_{TE} = \ln\left\{\frac{1}{2}\left[\left(\frac{1+\epsilon z^2}{1+z^2}\right)^{\frac{1}{4}}(1+C)\,(1+B) + \left(\frac{1+\epsilon z^2}{1+z^2}\right)^{-\frac{1}{4}}(1+A)\,(1+D)\right]\right\} \tag{18}$$

and

$$\begin{aligned} D_{TM} &= \ln\left\{\frac{1}{2}\left[\epsilon^{-\frac{1}{2}}\left(\frac{1+\epsilon z^2}{1+z^2}\right)^{\frac{1}{4}}(1+C)\,(1+B) + \epsilon^{\frac{1}{2}}\left(\frac{1+\epsilon z^2}{1+z^2}\right)^{-\frac{1}{4}}(1+A)\,(1+D)\right]\right. \\ &\quad \left. + \left(\epsilon^{-\frac{1}{2}} - \epsilon^{\frac{1}{2}}\right)\frac{(1+A)(1+B)}{4\nu((1+z^2)(1+\epsilon z^2))^{\frac{1}{4}}}\right\}, \end{aligned} \tag{19}$$

where we used the same abbreviations for the Debye polynomials as in [1]. We need them in the first nontrivial order only, $A = u_1(t_q)/\nu$, $B = -u_1(t)/\nu$, $C = v_1(t_q)/\nu$ and $D = -v_1(t)/\nu$ with $t = 1/\sqrt{1+z^2}$ and $t_q = 1/\sqrt{1+\epsilon z^2}$. Inserting now the expansion (16) of ϵ we obtain finally

$$D_{TE} = -\frac{1}{\nu^3}\frac{\epsilon_1}{16}\frac{z^2}{(1+z^2)^{5/2}} + \dots \tag{20}$$

and

$$D_{TM} = -\frac{1}{\nu^3}\frac{\epsilon_1}{16}\frac{z^4+4z^2+4}{z^2(1+z^2)^{5/2}} + \dots\ . \tag{21}$$

We have to insert these expansions, Eqs. (17), (20), (21), into the Jost function, Eq. (9), and the latter, then, into the zeta function, Eq. (4). Performing there the variable substitution $k = \nu z$, the sum over ν can be carried out which gives Hurwitz zeta functions,

$$\zeta_H(s;\frac{3}{2}) = \sum_{l=1}^{\infty}\left(l+\frac{1}{2}\right)^{-s}. \tag{22}$$

In summary, we obtain

$$\zeta(s) = 2\frac{\sin\pi s}{\pi}\left\{\zeta_H(2s;\frac{3}{2})\int_0^\infty dz\ z^{-2s}\frac{\partial}{\partial z}\epsilon_1\frac{\sqrt{1+z^2}-1}{z^2}\right. \tag{23}$$

$$+\zeta_H(2s+2;\frac{3}{2})\int_0^\infty dz\ z^{-2s}\frac{\partial}{\partial z}\left[\epsilon_2\frac{\sqrt{1+z^2}-1}{z^4}-\frac{\epsilon_1^2}{4}\frac{(\sqrt{1+z^2}-1)^2}{z^4\sqrt{1+z^2}}\right.$$
$$\left.\left.-\frac{\epsilon_1}{8}\frac{z^4+2z^2+2}{z^2(1+z^2)^{5/2}}\right]\right\}\ .$$

Now it is easy to extract the heat kernel coefficients using Eq. (5). The rightmost pole is at $s = 1/2$ resulting from the first Hurwitz zeta function. It yields

$$a_1 = -\frac{8\pi}{3}\epsilon_1. \tag{24}$$

The next pole is at $s = -\frac{1}{2}$. It results from the integral in the first line of Eq. (23),

$$\int_o^\infty dz\ z^{-2s}\frac{\partial}{\partial z}\frac{\sqrt{1+z^2}-1}{z^2} = \frac{-1}{2\sqrt{\pi}}s\Gamma(-1-s)\Gamma(s+\frac{1}{2}).$$

Further contributions result from the pole of the second Hurwitz zeta function. Taking all contributions together we obtain

$$a_2 = \frac{4\pi}{3}\epsilon_1^2 + \frac{16\pi}{3}\epsilon_2, \tag{25}$$

where a term linear in ϵ_1 cancelled out between TE and TM contributions. To a_1, (24), and a_2, (25), the TE and TM modes give equal contributions.

Formulas (24) and (25) are the main result of this paper. In order to draw more physical conclusions we restore the dependence on R. From $[\epsilon_1] = R^{-2}$, $[\epsilon_2] = R^{-4}$, $[a_1] = R$ and $[a_2] = R^{-1}$ we get

$$a_1 = -\frac{8\pi}{3}\epsilon_1 R^3, \qquad a_2 = \frac{4\pi}{3}\epsilon_1^2 R^3 + \frac{16\pi}{3}\epsilon_2 R^3.$$

Now the divergent part of the ground-state energy reads in zeta functional regularization

$$E_0^{\rm div}(s) = \frac{-1}{32\pi^2}\left(\frac{1}{s}+2\ln\mu-2\right)\left(\epsilon_1^2+4\epsilon_2\right)V, \tag{26}$$

where $V = \frac{4\pi}{3}R^3$ is the volume of the ball, and in the regularization Eq. (7), using the exponentially damping function, it is

$$E_0^{\rm div}(\delta) = \frac{1}{16\pi^2}\left(\frac{-2\epsilon_1}{\delta^2}+\left(\epsilon_1^2+4\epsilon_2\right)\ln\delta\right)V. \tag{27}$$

These results show that for a dispersive dielectric ball ultraviolet divergencies are present in a fashion similar to that for the nondispersive case.

We add some remarks.

1. In both regularizations the ground-state energy energy is divergent. Because of $a_2 \neq 0$, even after the removal of the diverging contributions an arbitrariness (e.g., $\ln \mu$ in Eq. (26)) remains in the finite part.

   The model, Eq. (1), chosen for the permittivity reflects the physical assumption that for high frequencies the dielectric ball becomes transparent. From the results, Eqs. (26) and (27), in particular from the contribution of ϵ_2, it follows that this is insufficient in order to get a satisfactory physical interpretation. Although the dependence of the divergencies on the dielectric properties is considerably simpler than in [1], we are left with the same conclusions as for the nondispersive case that for a dielectric body substantial changes in the physical context are necessary [1].

2. In the sense of renormalization one might try to absorb the divergent contributions into some classical part. For example, in the bag model, ultraviolet divergent contributions proportional to the volume like Eqs. (26) and (27) can be put into a redefinition of the bag constant. However, in the present case we do not have any classical energy which could be associated with the dielectric ball. In addition, we don't have any normalization condition which might help to fix the arbitrariness.

3. As compared with the nondispersive case, [1], the non vanishing of a_2 is a common feature. The only known exception is the vanishing of a_2 in the dilute approximation, i.e., to order $(\epsilon - 1)^2$ for $\epsilon \to 1$, which allowed the ground-state energy to have a physical meaning and ensures that the results of different calculations coincide. However, as it was shown in [9] and [1], this is a peculiarity of a ball with sharp boundaries. For a dielectric body with non sharp boundaries, i.e., with the permittivity $\epsilon(r)$ being a smooth function of the radius, a_2 is non zero even in the dilute approximation.

4. One might hope that the pressure (force per unit surface) is ultraviolet finite rather than the vacuum energy itself. For this end one has to divide by the surface $(4\pi R^2)$ of the ball and to take the derivative with respect to the radius. As the divergent part of the ground-state energy is proportional to the volume of the ball the pressure contains a divergent constant.

5. One may assume the dielectric ball to be an idealization of a number of polarizable atoms. Then a change in the radius leaving the number of atoms fixed requires a change in ϵ according to $(\epsilon - 1)V = const$ as discussed, for instance, in [11, 12, 13]. In this case, by means of Eq. (1), $\epsilon_{1,2}V = const$ follows. Due to the presence of ϵ_1^2 in (26) and (27), again, a divergent contribution is present.

6. An investigation similar to the present one had been recently carried out in [14], where the divergent part of the Casimir energy had been calculated

for the plasma model in zeta functional regularization. This is equivalent to calculate the contribution of ϵ_1 to the heat kernel coefficient a_2 and is in agreement with Eq. (25).

7. In the perturbative approach described in [11] divergencies proportional to the surface area were found. It would be very desirable to understand the origin of the different predictions in the different schemes used.

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