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phase transitions

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# AN EXAMPLE IN THE GRADIENT THEORY OF PHASE TRANSITIONS

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ABSTRACT. We prove by giving an example that when  $n \geq 3$  the asymptotic behavior of functionals  $\int_{\Omega} \varepsilon |\nabla^2 u|^2 + (1 - |\nabla u|^2)^2 / \varepsilon$  is quite different with respect to the planar case. In particular we show that the one-dimensional ansatz due to Aviles and Giga in the planar case (see [2]) is no longer true in higher dimensions.

**Keywords:** Phase transitions,  $\Gamma$ -convergence, asymptotic analysis, singular perturbation, Ginzburg–Landau.

## 1. INTRODUCTION

This paper is devoted to the study of the asymptotic behavior of functionals

$$(1) \quad F_{\varepsilon}^{\Omega}(u) := \int_{\Omega} \left( \varepsilon |\nabla^2 u|^2 + \frac{(1 - |\nabla u|^2)^2}{\varepsilon} \right) \quad \Omega \subset \mathbf{R}^n$$

as  $\varepsilon \downarrow 0$ , where  $u$  maps  $\Omega$  into  $\mathbf{R}$ . This problem was raised by Aviles and Giga in [2] in connection with the mathematical theory of liquid crystals and more recently by Gioia and Ortiz in [8] for modeling the behavior of thin film blisters. Recently many authors have studied the planar case giving strong evidences that, as conjectured by Aviles and Giga in [2], the sequence  $(F_{\varepsilon})$   $\Gamma$ -converge (in the strong topology of  $W^{1,3}$ : see [1] for a discussion of such a choice and a rigorous setting) to the functional

$$F_{\infty}^{\Omega}(u) := \begin{cases} \frac{1}{3} \int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 d\mathcal{H}^{n-1} & \text{if } |\nabla u| = 1, u \in W^{1,\infty} \\ +\infty & \text{otherwise .} \end{cases}$$

Here  $J_{\nabla u}$  denotes the set of points where  $\nabla u$  has a jump and  $|\nabla u^+ - \nabla u^-|$  is the amount of this jump. Of course the first line of the previous definition makes sense only for particular choices of  $u$ , such as piecewise  $C^1$ . For a rigorous setting the reader should think about a suitable function space  $S$  which contains piecewise  $C^1$  functions and on which we can give a precise meaning to the above integral (for example a natural choice would be  $\{u | \nabla u \in BV\}$ ; however this space turns out

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not to be the natural one: we refer again to [1] for a discussion of this topic).

Partial results in proving Aviles and Giga's conjecture (i.e. compactness of minimizers of  $F_\varepsilon^\Omega$ , estimates from below on  $F_\varepsilon^\Omega(u_\varepsilon)$  and a suitable weak formulation for the problem of minimizing  $F$  subject to some boundary conditions) can be found in [1], [3], [5], [6], [7].

In their first work Aviles and Giga based their conjecture on the following ansatz (which they made in the case  $n = 2$ ):

**Conjecture 1.1.** *Let us choose a map  $w : \Omega \rightarrow \mathbf{R}$  (with  $\Omega \subset \mathbf{R}^n$  bounded open set containing 0) such that:*

- (a)  *$w$  is Lipschitz and satisfies the eikonal equation  $|\nabla w| = 1$ ;*
- (b)  *$\nabla w$  is constant in  $\{x_1 < 0\}$  and in  $\{x_1 > 0\}$ .*

*Let us define  $E := \inf\{\liminf_\varepsilon F_\varepsilon^\Omega(u_\varepsilon) : \|u_\varepsilon - w\|_{W^{1,3}} \rightarrow 0\}$ . Then there exists a family of functions  $w_\varepsilon$  such that*

- (i) *the component of  $\nabla w_\varepsilon$  perpendicular to  $(1, 0, \dots, 0)$  is constant;*
- (ii)  *$w_\varepsilon \rightarrow w$  in  $W^{1,3}$ ;*
- (iii)  *$\lim F_\varepsilon^\Omega(w_\varepsilon) = E$ .*

This ansatz has been proved by Jin and Kohn in [7] for  $n = 2$ . It reduces the problem of finding  $E$  to a one dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the result  $E = F_\infty^\Omega(w)$ , which means that at  $w$  the  $\Gamma$ -limit of  $F_\varepsilon^\Omega$  exists and coincides with  $F_\infty^\Omega(w)$ . With a standard cut and paste argument (see [4]) it can be proved that the same happens for every  $w$  which is piecewise affine. In the next section we will prove the following theorem

**Theorem 1.2.** *Let  $u$  be the function  $u(x_1, x_2, x_3) = |x_3|$  and  $C$  the cylinder  $\{|x_1|^2 + |x_2|^2 < 1\}$ . Then there exists  $(u_k)$  such that*

- (a) *every  $u_k$  is piecewise affine (being the union of a finite number of affine pieces) and satisfies the eikonal equation;*
- (b)  *$\lim_k F_\infty^C(u_k) < F_\infty^C(u)$ ;*
- (c)  *$u_k \rightarrow u$  strongly in  $W^{1,p}$  for every  $p < \infty$ .*

The proof can be easily generalized to every  $n \geq 3$ . As an easy corollary we get that the one-dimensional ansatz fails for  $n \geq 3$ . Moreover this failure means that  $F$  cannot be the  $\Gamma$ -limit of  $F_\varepsilon^\Omega$  for  $n \geq 3$ .

**Corollary 1.3.** *The one-dimensional ansatz is not true for  $n \geq 3$ .*

*Proof.* As already observed, being every  $u_k$  piecewise affine, there is a family of functions  $u_{k,\varepsilon}$  such that  $u_{k,\varepsilon}$  converge to  $u_k$  in  $W^{1,p}$  (for every  $p < \infty$ ) and  $\lim_\varepsilon F_\varepsilon^C(u_{k,\varepsilon}) = F_\infty^C(u_k)$ . A standard diagonal argument

gives a sequence  $(u_{k,\varepsilon(k)})$  strongly converging to  $u$  in  $W^{1,p}$  such that  $\lim_k F_{\varepsilon(k)}^C(u_{k,\varepsilon(k)}) < F_\infty^C(u)$ .  $\square$

## 2. THE EXAMPLE

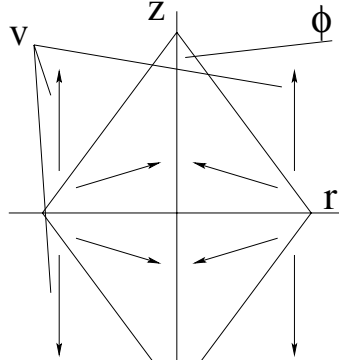
In this section we prove Theorem 1.2. First of all we recall the following fact.

(Curl) If  $v : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a piecewise constant vector field, then  $v$  is a gradient if and only if for every hyperplane of discontinuity  $\pi$  the right trace and the left trace of  $v$  have same component parallel to  $\pi$ .

The building block of the construction of Theorem 1.2 is the following vector field, depending on a parameter  $\phi \in (0, \pi/2)$ . First of all we fix in  $\mathbf{R}^3$  a system of cylindrical coordinates  $(r, \theta, z)$  and then we call  $A$  the cone given by  $\{z > 0, r < 1, (1-r) > z \tan \phi\}$  and  $A'$  the reflection of  $A$  with respect to the plane  $\{z = 0\}$ . Hence we put

$$\begin{aligned} v(r, \theta, z) &= (0, 0, 1) && \text{if } z > 0 \text{ and } (r, \theta, z) \notin A \\ v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, \cos(2\phi)) && \text{if } z > 0 \text{ and } z \in A \\ v(r, \theta, z) &= (0, 0, -1) && \text{if } z < 0 \text{ and } (r, \theta, z) \notin A' \\ v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, -\cos(2\phi)) && \text{if } z < 0 \text{ and } z \in A'. \end{aligned}$$

It is easy to see that  $v$  maps every plane  $\{\theta = \alpha\} \cup \{\theta = \alpha + \pi\}$  into itself. Moreover the restrictions of  $v$  to these planes all look like as in the following picture



planar section  
of the field  $v$

**Lemma 2.1.** *The vector field  $v$  is the gradient of a function  $w$ . Moreover there is a sequence of piecewise affine functions  $w_k$  such that*

- (a)  $w_k \rightarrow w$  strongly in  $W_{loc}^{1,p}$  for every  $p$ ;

(b)  $F_\infty^\Omega(w_k) \rightarrow F_\infty^\Omega(w)$  for every open set  $\Omega \subset\subset \mathbf{R}^3$ .

*Proof.* We consider the restriction of  $v$  to the plane  $P := \{\theta = 0\} \cup \{\theta = \pi\}$ . As already noticed  $v$  maps this plane into itself. Moreover its restriction to it satisfies condition (Curl), hence on  $P$   $v$  is the gradient of a scalar function  $w$ . Moreover we can find such a  $w$  so that it is identically zero on the line  $\{z = 0\} \cap P$ . Hence  $w$  is symmetric with respect to the  $z$  axis and so we can extend  $w$  to the whole three-dimensional space so to build a cylindrically symmetric function. It is easy to check that the gradient of such a function is equal to  $v$ .

We call this function  $w$  as well and we will prove that it satisfies conditions (a) and (b) written above.

(a) Our goal is approximating  $v$  with piecewise constant gradient fields. First of all we do it in the upper half-space  $\{z > 0\}$ . For every  $n$  we take a regular  $n$ -agon  $B_n$  which is inscribed to the circle of radius 1 and lies on the plane  $\{z = 0\}$ . The vertices of this  $n$ -agon are given by  $V_i := (1, 2i\pi/n, 0)$ .

Hence we construct the pyramid  $A^n$  with vertex  $V := (0, 0, \cot \phi)$  and base  $B_n$ . In the pyramid we identify  $n$  different regions  $A_1^n, \dots, A_n^n$ , where every  $A_i^n$  is given by the tetrahedron with vertices  $(0, 0, 0)$ ,  $V$ ,  $V_i$ ,  $V_{i+1}$ . After this we put  $v_n$  equal to  $(0, 0, 1)$  outside  $A^n$  and in every  $A_i^n$  we put

$$v_n(r, \theta, z) \equiv (\sin 2\phi, \pi + (2i + 1)\pi/n, \cos 2\phi).$$

It is easy to see that  $v_n$  satisfies condition (Curl), hence it is the gradient of some function  $w_n$ . Moreover we can choose  $w_n$  in such a way that it is identically 0 on  $\{z = 0\}$ . Then we extend  $w_n$  to the lower half space  $\{z < 0\}$  just by imposing  $w_n(r, \theta, -z) = w_n(r, \theta, z)$ . It is not difficult to see that  $\nabla w_n$  converges strongly to  $\nabla w$  in  $L_{loc}^p$  for every  $p$ .

(b) Now we check that the previous construction satisfies also the second condition of the Lemma. We fix an open set  $\Omega \subset\subset \mathbf{R}^3$  and we observe that both  $w_k$  and  $w$  satisfy the eikonal equation in  $\Omega$ . Moreover we call  $L_i^n$  the triangle with vertices  $V$ ,  $V_i$ ,  $V_{i+1}$  and  $L^n$  the union of  $L_i^n$  (so  $L^n$  is the ‘‘lateral surface’’ of the pyramid  $A^n$ ). Finally we denote by  $L$  the lateral surface of the cone  $A$ , i.e. the set  $\{(1 - r) = z \tan \phi\}$ .

- (i) The amount of jump of  $v_n$  (i.e.  $|v_n^+ - v_n^-|$ ) on  $L^n$  is constant and equal to the value of  $|v^+ - v^-|$  on  $L$ . Moreover the area of  $L^n$  is converging to the area of  $L$ . The same happens on the symmetric sets in the lower half-space  $\{z < 0\}$ .
- (ii) Let us call  $B$  the base of the cone. The right and left traces of  $v_n$  coincides with those of  $v$  on  $B_n \cup (\{z = 0\} \setminus B)$ . Moreover the area of  $B \setminus B_n$  is converging to zero.

- (iii) The vector fields  $v_n$  are discontinuous also on the triangles  $T_i^n$  joining  $V$ ,  $(0, 0, 0)$  and  $V_i$  (and on the symmetric triangles lying on  $\{z < 0\}$ ). The amount of jump of  $v_n$  on each of these triangles is given by

$$|v_n^+ - v_n^-| = 2 \sin(\pi/n).$$

Moreover the area of everyone is given by  $(\cot \phi)/2$ . Hence

$$\int_{\cup_i T_i^n} |v_n^+ - v_n^-|^3 d\mathcal{H}^2 = 4n \cot \phi \sin^3 \pi/n.$$

The right hand side goes to zero as  $n \rightarrow \infty$  and this completes the proof. □

**Proof of Theorem 1.2** First of all we pass from the cartesian coordinates of the statement to the cylindrical coordinates  $(r, \theta, z)$  given by  $x_3 = z$ ,  $x_1 = r \cos \theta$ ,  $x_2 = r \sin \theta$  (and sometimes we will denote the elements of  $\mathbf{R}^3$  with  $(y, z)$ , where  $y \in \mathbf{R}^2$  and  $z \in \mathbf{R}$ ).

We take  $w$  as in the previous lemma. First of all let us compute  $F_\infty^C(w)$  where  $C$  is the cylinder  $\{r < 1\}$ . As in the previous proof we call  $L$  the lateral surface of the cone, that is the set  $\{r - 1 = z \tan \phi\}$ . The value of  $|\nabla w^+ - \nabla w^-|$  on the surface  $L$  is given by  $2 \sin \phi$  and the area of  $L$  is given by  $\pi / \sin \phi$ : the same happens for the symmetric of  $L$  lying on the half-space  $\{z < 0\}$ . On the base of the cylinder we have  $|\nabla w^+ - \nabla w^-| = 2 |\cos 2\phi|$ . Hence

$$a(\phi) := F_\infty^C(u) - F_\infty^C(w) = \frac{\pi}{3} [8 - 8 \cos^3 2\phi - 16 \sin^2 \phi]$$

and it can be easily checked that for  $\phi$  close enough to zero,  $a(\phi)$  is positive.

Therefore let us fix an  $\alpha$  for which  $a(\alpha) > 0$  and let us agree that  $w$  is constructed as in the previous lemma by choosing  $\phi = \alpha$ . Given  $\rho > 0$  and  $x \in \mathbf{R}^2$  we define  $w_{x,\rho}$  in the cylinder  $C_{x,\rho} := \{(y, z) : |y - x| \leq \rho\} \subset \mathbf{R}^3$  as  $w_{x,\rho}(y, z) = \rho w((y - x)/\rho, z/\rho)$ . It is easy to see that

$$(2) \quad F_\infty^{C_{x,\rho}}(u) - F_\infty^{C_{x,\rho}}(w_{x,\rho}) = a(\alpha) \rho^2$$

Let us fix  $\varepsilon$  and take  $\rho$  such that  $\rho \cot \alpha < \varepsilon$ . Thanks to Besicovitch Covering Lemma we can cover  $\mathcal{H}^2$  almost all  $D := \{z = 0, r \leq 1\}$  with a disjoint countable family of closed discs  $D_i$  such that every  $D_i$  has radius  $r_i < \rho$ , center  $x_i$  and is contained in  $D$ . We construct  $u_\varepsilon$  by putting  $u_\varepsilon \equiv w_{x_i,\rho_i}$  in the cylinder  $C_{x_i,\rho_i}$ .

Since  $\nabla u_\varepsilon$  coincides with  $\nabla u$  in  $\{z \geq \varepsilon\}$  and satisfies the eikonal equation, it is easy to see that  $u_\varepsilon \rightarrow u$  locally in the strong topology of

$W^{1,p}$ . Moreover equation (2) implies that

$$F_\infty^C(u) - F_\infty^C(u_\varepsilon) = \sum_i a(\alpha) r_i^2 = a(\alpha).$$

At this point, using the previous Lemma we can approximate the function  $u_\varepsilon$  in the cylinders  $C_{x_i, \rho_i}$  with piecewise affine functions in such a way that their traces coincide with the trace of  $u_\varepsilon$  on the boundary of  $C_{x_i, \rho_i}$ . Using standard diagonal arguments for every  $\varepsilon$  we can find a sequence of piecewise affine functions  $u_\varepsilon^k$  which converge in  $W^{1,p}$  to  $u_\varepsilon$  and such that  $F_\infty^C(u_\varepsilon^k) \rightarrow F_\infty^C(u_\varepsilon)$ . Moreover, again using diagonal arguments, we can construct the sequence  $u_\varepsilon^k$  so that each one is a finite union of affine pieces.

Finally, one last diagonal argument, gives a sequence  $\tilde{u}_k$  such that

- (a)  $\tilde{u}_k$  is a finite union of affine pieces;
- (b)  $\lim_k F_\infty^C(\tilde{u}_k) < F_\infty^C(u)$ ;
- (c)  $\tilde{u}_k \rightarrow u$  strongly in  $W^{1,p}$  for every  $p < \infty$ .

□

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