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Charge Superselection Sectors for  
Scalar QED on the Lattice

by

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## Abstract

The lattice model of scalar quantum electrodynamics (Maxwell field coupled to a complex scalar field) in the Hamiltonian framework is discussed. It is shown that the algebra of observables  $\mathcal{O}(\Lambda)$  of this model is a  $C^*$ -algebra, generated by a set of gauge-invariant elements satisfying the Gauss law and some additional relations. Next, the faithful, irreducible and non-degenerate representations of  $\mathcal{O}(\Lambda)$  are found. They are labeled by the value of the total electric charge, leading to a decomposition of the physical Hilbert space into charge superselection sectors. In the Appendices we give a unified description of spinorial and scalar quantum electrodynamics and, as a byproduct, we present an interesting example of weakly commuting operators, which do not commute strongly.

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# 1 Introduction

The ideas of axiomatic and algebraic quantum field theory in the sense of Wightman, Haag and Kastler [1] have played an important role in clarifying basic nonperturbative structures of quantum physics. In particular, there has been developed a general scheme for superselection rules [2], which, however, does not apply to theories with massless particles. Thus, an extension of these ideas to realistic gauge theories is still a big challenge. Some partial results in this direction already exist, see a series of papers by Strocchi and Wightman ([3], [4] and [5]). In particular, in [3] Quantum Electrodynamics was considered. It was shown that if one insists in locality and Lorentz covariance, one is rather naturally led to a theory with indefinite metric. Within this scheme, the charge superselection rule for QED was proven, but a decomposition of the physical Hilbert space into a direct sum of subspaces carrying definite total charge was not obtained. For a deep discussion of charged states in QED we refer to [6] and further references therein. Studying simple toy models, e.g. a  $Z_2$ -gauge theory with  $Z_2$ -matter fields [7], one can realize the full programme, which one would like to implement for realistic theories. In [7], the authors were able to determine the ground state and charged states explicitly. Using methods of Euclidean quantum field theory, it was possible to show that – for some regions in the space of coupling constants – the thermodynamic limit for charged states can be controlled.

In this paper we also discuss a simplified model, we put scalar quantum electrodynamics on a finite lattice. In the context of lattice approximation complicated operator theoretic problems arising in (continuum) quantum field theory become simpler, whereas problems typical for gauge theories remain and can be, therefore, discussed separately. We consider scalar QED in the hamiltonian approach on a finite cubic (3-dimensional) lattice and work in the non-compact formulation, where the gauge potential remains Lie-algebra-valued on the lattice level. Our starting point are the commutation relations for canonically conjugate pairs of gauge-dependent lattice fields. Then, in a first step we construct the algebra of observables as the algebra of gauge invariant operators fulfilling the Gauss law and show that the charge superselection rule holds. Algebraically, the observable algebra is generated by electric and magnetic flux operators, together with gauge invariant operators bilinear in the matter fields. These gauge invariant generators fulfil a number of algebraic identities, which by using the technical tool of a lattice tree can be reduced essentially. It turns out that the observable algebra is

the tensor product of an associative algebra generated by canonical commutation relations (electromagnetic part) and an associative algebra generated by a certain Lie algebra (matter field part). Using Woronowicz's theory of  $C^*$ -algebras generated by unbounded elements (see [17]), we can endow the observable algebra with a  $C^*$ -structure.

Within this setting, we are able to classify all faithful, irreducible and non-degenerate representations of the observable algebra and to obtain the physical Hilbert space as a direct sum of representation spaces labeled by the total charge. We stress that the restriction to non-degenerate representations of  $C^*$ -algebras (or, equivalently, to *integrable* representations of the underlying Lie algebras) is of fundamental importance, as is shown in Appendix A.

We have obtained a similar result for the case of spinorial QED earlier, see [10] and [11]. However, from the mathematical point of view, the problems occurring in this paper are much more complicated due to the fact that the matter field part of the observable algebra is generated by a non-compact Lie algebra. Consequently, the observable algebra is infinite-dimensional and its representations are more difficult to control. For some earlier work on scalar lattice QED we refer to [9] and for basic notions concerning lattice gauge theories we refer to [19] and references therein. Recently, we have started to investigate lattice QCD in the above spirit, see [12].

Our paper is organized as follows: In Sections 2 we discuss the standard second quantization procedure on the lattice. We define the field algebra, discuss the Gauss law and introduce the notion of boundary data. In Section 3 we construct observables in terms of field operators and give an abstract definition of the observable algebra in terms of generators and relations. Finally, we endow it with a  $C^*$ -structure. In Section 4 we find all faithful, irreducible and non-degenerate representations of the observable algebra and prove that they are labeled by the eigenvalues of the total charge operator. This yields the superselection structure. Finally, we discuss some perspectives of our approach. In Appendix A we give an example of a non-integrable representation carrying non-integer charge and in Appendix B we present a unified description for both bosonic and fermionic matter.

## 2 Scalar QED on the Lattice

Continuum scalar quantum electrodynamics (QED) is the theory of a complex-valued scalar field  $\phi$  interacting with the electromagnetic field  $A_\mu$ . The classical Lagrangian of this model is defined as follows

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - V(|\phi|^2), \quad (2.1)$$

where  $D^\mu\phi = \partial_\mu\phi + igA_\mu\phi$ ,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $g = e/\hbar$ . Local gauge transformations are given by

$$\tilde{\phi}(x) = e^{-ig\lambda(x)}\phi, \quad \tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x). \quad (2.2)$$

For a given Cauchy hyperplane  $\Sigma = \{t = \text{const}\}$  in Minkowski space, the above Lagrangian gives rise to an infinite-dimensional Hamiltonian system in variables  $(A_k, E^k, \phi, \pi)$  with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(E_k E^k + B_k B^k) + \frac{1}{2}|\pi|^2 + \frac{1}{2}|\vec{D}\phi|^2 + V(|\phi|^2), \quad (2.3)$$

where  $B = \text{curl } A$  is the magnetic field,  $E$  is the electric field (the momentum canonically conjugate to  $A$ ) and  $\pi$  denotes the momentum canonically conjugate to  $\phi$ .

Let us take a finite, regular, cubic lattice  $\Lambda$  contained in  $\Sigma$ , with lattice spacing  $a$ , and let us denote the set of  $n$ -dimensional lattice elements by  $\Lambda^n, n = 0, 1, 2, 3$ . Such elements are (in increasing order of  $n$ ) called sites, links, plaquettes and cubes. We approximate every continuous configuration  $(A_k, E^k, \phi, \pi)$  in the following way:

$$\Lambda^0 \ni x \longrightarrow \phi_x := \phi(x) \in \mathbb{C}, \quad (2.4)$$

$$\Lambda^0 \ni x \longrightarrow \pi_x := \pi(x) \in \mathbb{C}, \quad (2.5)$$

$$\Lambda^1 \ni (x, x + \hat{k}) \longrightarrow A_{x, x + \hat{k}} := \int_{(x, x + \hat{k})} A_k dl \in \mathbb{R}, \quad (2.6)$$

$$\Lambda^1 \ni (x, x + \hat{k}) \longrightarrow E_{x, x + \hat{k}} := \int_{\sigma(x, x + \hat{k})} E^k d\sigma_k \in \mathbb{R}. \quad (2.7)$$

Here  $\sigma(x, x + \hat{k})$  denotes a plaquette of the dual lattice, dual to the link  $(x, x + \hat{k}) \in \Lambda^1$ . A local gauge transformation of a lattice configuration is

given by:

$$\tilde{\phi}_x = \exp(-ig\lambda_x) \phi_x , \quad (2.8)$$

$$\tilde{\pi}_x = \exp(-ig\lambda_x) \pi_x , \quad (2.9)$$

$$\tilde{A}_{x,x+\hat{k}} = A_{x,x+\hat{k}} + \lambda_{x+\hat{k}} - \lambda_x , \quad (2.10)$$

where  $\Lambda^0 \ni x \longrightarrow \lambda_x \in \mathbb{R}$ . The electric field  $E$  is gauge invariant. Note that we have chosen the non-compact lattice approximation, where the potential and the field strength remain Lie-algebra-valued on the lattice level.

We define second quantization of the lattice theory by postulating the following canonical commutation relations for the lattice quantum fields  $(\hat{A}, \hat{E}, \hat{\phi}, \hat{\pi})$  corresponding to the classical lattice fields given by (2.4) – (2.7):

$$\left[ \hat{\phi}_x^*, \hat{\pi}_y \right] = 2i\hbar \delta_{xy} \mathbb{1} = \left[ \hat{\phi}_x, \hat{\pi}_y^* \right] , \quad (2.11)$$

$$\left[ \hat{A}_{x,x+\hat{k}}, \hat{E}_{y,y+\hat{l}} \right] = i\hbar \delta_{(x,x+\hat{k}), (y,y+\hat{l})} \mathbb{1} . \quad (2.12)$$

The remaining commutators have to vanish. Here,  $\delta_{(x,x+\hat{k}), (y,y+\hat{l})} = 0$  if  $(x, x+\hat{k})$  and  $(y, y+\hat{l})$  are different links,  $\delta_{(x,x+\hat{k}), (y,y+\hat{l})} = 1$  if they coincide and have the same orientation and  $\delta_{(x,x+\hat{k}), (y,y+\hat{l})} = -1$  if they coincide and have opposite orientations.

All irreducible representations in the strong (Weyl) sense of the above algebra are equivalent to the Schrödinger representation, see [8], of wave functions  $\Psi \in \mathcal{H}_0 \equiv L^2(A, \phi)$ . We denote the field algebra of bounded operators on  $\mathcal{H}_0$ , generated by  $(\hat{A}, \hat{E}, \hat{\phi}, \hat{\pi})$  and fulfilling (2.11) and (2.12), by  $\mathcal{F}(\Lambda)$ . It is endowed with a natural  $*$ -operator, such that  $\hat{A}$  and  $\hat{E}$  are self-adjoint. Obviously,  $\mathcal{F}(\Lambda)$  contains a lot of unphysical (gauge-dependent) elements. Moreover, the above electric field  $\hat{E}$  does not automatically satisfy the Gauss law. In what follows we will present an explicit construction of the algebra  $\mathcal{O}(\Lambda)$  of observables (gauge invariant operators satisfying the Gauss law), together with a complete classification of its irreducible representations.

The group of local gauge transformations acts on  $\mathcal{F}(\Lambda)$  by automorphisms, whose generators are given by

$$\Lambda^0 \ni x \longrightarrow \hat{\mathcal{G}}_x := -\frac{i}{\hbar} \left( \sum_{\hat{k}} \hat{E}_{x,x+\hat{k}} - \hat{q}_x \right) \in \text{End}(\mathcal{F}(\Lambda)) , \quad (2.13)$$

with

$$\hat{q}_x = \frac{e}{\hbar} \left( \text{Im}(\hat{\phi}_x^* \hat{\pi}_x) - \hbar \mathbb{1} \right) \quad (2.14)$$



denoting the operator of electric charge at  $x$ . Thus, the corresponding (local) Gauss law constraint, which has to be imposed on observables, has the following form

$$\sum_{\hat{k}} \hat{E}_{x, x+\hat{k}} = \hat{q}_x . \quad (2.15)$$

We define the operator  $\hat{Q}$  of total electric charge putting

$$\hat{Q} := \sum_{x \in \Lambda^0} \hat{q}_x . \quad (2.16)$$

Summing up the local Gauss laws over all lattice sites, we see that nontrivial values of the total charge  $\hat{Q}$  can only arise from nontrivial boundary data, which we are now going to introduce. For this purpose we consider also external links of  $\Lambda$ , connecting lattice sites belonging to the boundary  $\partial\Lambda$  with “the rest of the world”. We denote these external links by  $(x, \infty)$  and consider their electric fluxes  $E_{x, \infty}$ . Then we obtain from the Gauss law

$$\hat{Q} = \sum_{x \in \partial\Lambda^0} \hat{E}_{x, \infty} , \quad (2.17)$$

where we denote  $\partial\Lambda^n := \partial\Lambda \cap \Lambda^n$ . In this paper we assume that the fluxes  $E_{x, \infty}$  are constant in time.

For purposes of the construction of the complete field theory *via* a limit of lattice approximations, we may treat  $\Lambda$  as a piece of a bigger lattice  $\tilde{\Lambda}$ . Then the boundary flux operators  $\hat{E}_{x, \infty}$  belong to  $\mathcal{F}(\tilde{\Lambda})$  and – due to locality of the theory – must commute with all elements of the field algebra  $\mathcal{F}(\Lambda)$ . Physically, these external fluxes measure the “violation of the local Gauss law” on the boundary  $\partial\Lambda$ ,

$$\hat{E}_{x, \infty} := \hat{q}_x - \sum_{\hat{k}} \hat{E}_{x, x+\hat{k}} . \quad (2.18)$$

This is due to the fact, that the “world outside of  $\Lambda$ ” has been discarded on this level of approximation. According to the above discussion, we assume that the above elements belong to the center of the algebra  $\mathcal{F}(\Lambda)$ . Mathematically, admitting non-vanishing elements of this type is equivalent to admitting gauge dependence of quantum states under the action of boundary gauges  $\partial\Lambda^0 \ni x \rightarrow \xi(x) \in U(1)$ .

As will be shown, the charge operator  $\hat{Q}$  defines a superselection rule, giving  $\hat{Q} = Q\hat{\mathbb{1}}$  on every superselection sector. Consequently, the only consistent choice for the external fluxes is  $\hat{E}_{x,\infty} = E_{x,\infty}\hat{\mathbb{1}}$  on every superselection sector, where  $E_{x,\infty}$  are  $c$ -numbers fulfilling

$$Q = \sum_{x \in \partial\Lambda^0} E_{x,\infty} . \quad (2.19)$$

Therefore, we treat external fluxes as prescribed, classical boundary conditions and show that representations characterized by the same value  $Q$ , but corresponding to different external flux distributions fulfilling (2.17) are equivalent. For a more detailed discussion of this point we refer to [12].

When considering  $\Lambda$  as a piece of a bigger lattice  $\tilde{\Lambda}$ , we must also prescribe the interaction of the magnetic degrees of freedom on  $\Lambda$  with the rest of the world: in continuum theory, an additional condition for  $B^\parallel$  or  $B^\perp$  on the boundary is necessary. In lattice theory, these quantities live on external plaquettes of lattice cubes, adjacent to  $\partial\Lambda$ . In what follows, we simply assume the boundary condition  $B^\parallel = 0$  over the whole boundary  $\partial\Lambda$ . Due to the Maxwell equation

$$\dot{E}^\perp = \text{curl}_2 B^\parallel ,$$

this condition is compatible with the fact that  $\hat{E}_{x,\infty}$  are time-independent. We stress that other boundary conditions could be considered as well.

### 3 The Observable Algebra

The observable algebra  $\mathcal{O}(\Lambda)$  will be defined by imposing the local Gauss law and gauge invariance. To implement gauge invariance we have to take those elements of  $\mathcal{F}(\Lambda)$ , which commute with all generators  $\hat{\mathcal{G}}_x$ . In Subsection 3.1 we give a complete list of gauge invariant generators, built from elements of the field algebra. These generators are not independent, they have to fulfil a number of relations. Moreover, as an additional relation we impose the Gauss law. We define  $\mathcal{O}(\Lambda)$  as a  $C^*$ -algebra generated by unbounded elements, fulfilling these relations.

### 3.1 Generators and defining relations

We start with giving a complete list of generators of  $\mathcal{O}(\Lambda)$  on a purely algebraic level. Obviously, the electric field

$$\Lambda^1 \ni (x, x + \hat{k}) \rightarrow \hat{E}_{x, x + \hat{k}} \in \mathbb{R},$$

as well as the magnetic flux through the rectangular *plaquette*  $(x; \hat{k}, \hat{l})$  at  $x$ , spanned by the vectors  $\hat{k}$  and  $\hat{l}$ ,

$$\Lambda^2 \ni (x; \hat{k}, \hat{l}) \rightarrow \hat{B}_{x; \hat{k}, \hat{l}} \in \mathbb{R},$$

defined by

$$\hat{B}_{x; \hat{k}, \hat{l}} = \hat{A}_{x, x + \hat{k}} + \hat{A}_{x + \hat{k}, x + \hat{k} + \hat{l}} + \hat{A}_{x + \hat{k} + \hat{l}, x + \hat{l}} + \hat{A}_{x + \hat{l}, x} \quad (3.1)$$

are gauge invariant. With every lattice path  $\gamma$ , starting at  $x$  and ending at  $y$ , we associate the following set of generators:

$$\hat{\mathcal{L}}_\gamma = \hat{\phi}_x^* \exp(ig \int_\gamma \hat{A}) \hat{\phi}_y, \quad (3.2)$$

$$\hat{\mathcal{M}}_\gamma = \hat{\phi}_x^* \exp(ig \int_\gamma \hat{A}) \hat{\pi}_y, \quad (3.3)$$

$$\hat{\mathcal{N}}_\gamma = \hat{\pi}_x^* \exp(ig \int_\gamma \hat{A}) \hat{\phi}_y, \quad (3.4)$$

$$\hat{\mathcal{R}}_\gamma = \hat{\pi}_x^* \exp(ig \int_\gamma \hat{A}) \hat{\pi}_y. \quad (3.5)$$

We will use the symbol  $\hat{\mathcal{P}}_\gamma$  as a place holder for any of  $\hat{\mathcal{L}}_\gamma$ ,  $\hat{\mathcal{M}}_\gamma$ ,  $\hat{\mathcal{N}}_\gamma$  or  $\hat{\mathcal{R}}_\gamma$ . It is clear that the set  $(\hat{B}, \hat{E}, \hat{\mathcal{P}}_\gamma)$  generates the observable algebra.

Gauge invariance of the theory, together with canonical commutation relations fulfilled by elements of field algebra generators  $(\hat{A}, \hat{E}, \hat{\phi}, \hat{\pi})$ , impose many relations between generators  $(\hat{B}, \hat{E}, \hat{\mathcal{P}}_\gamma)$  of the observable algebra. The proposition below contains a set of relations, which is minimal in the following sense: Taking them, together with the Gauss law, as defining relations of  $\mathcal{O}(\Lambda)$ , we show that its faithful irreducible representations are unique, for a given value of the total charge.

**Proposition 3.1.** *The following relations between generators of the observable algebra hold:*

1. For any lattice site  $x$  we have the Gauss law:

$$\sum_{\hat{k}} \hat{E}_{x,x+\hat{k}} = \frac{e}{\hbar} \left( \text{Im}(\hat{\mathcal{M}}_{xx}) \right) - e \hat{\mathbb{1}}, \quad (3.6)$$

where the index “ $xx$ ” on the right hand side stands for the trivial path at  $x$ .

2. For any lattice site  $x$  and three independent vectors  $(\hat{k}, \hat{l}, \hat{n})$ , spanning a lattice cube at  $x$ , we have the Bianchi identity:

$$\hat{B}_{x;\hat{k},\hat{l}} + \hat{B}_{x;\hat{n},\hat{k}} + \hat{B}_{x;\hat{l},\hat{n}} + \hat{B}_{x+\hat{n};\hat{l},\hat{k}} + \hat{B}_{x+\hat{l};\hat{k},\hat{n}} + \hat{B}_{x+\hat{k};\hat{n},\hat{l}} = 0. \quad (3.7)$$

3. For two different lattice paths  $\gamma_1$  and  $\gamma_2$ , both from  $x$  to  $y$ , we have

$$\hat{\mathcal{P}}_{\gamma_2} = \exp(ig\hat{B}_{\gamma_1^{-1}\gamma_2})\hat{\mathcal{P}}_{\gamma_1}, \quad (3.8)$$

where  $\gamma_1^{-1}\gamma_2$  is the loop composed of  $\gamma_1^{-1}$  and  $\gamma_2$  ( $\gamma_2$  is adjoint to the end of  $\gamma_1^{-1}$ ), and  $\hat{B}_\gamma = \int_\gamma \hat{A}$ .

4. The following commutation relations between the electromagnetic and the matter field generators hold:

$$\left[ \hat{\mathcal{P}}_\gamma, \hat{B}_{x;\hat{k},\hat{l}} \right] = 0, \quad (3.9)$$

$$\left[ \hat{\mathcal{P}}_\gamma, \hat{E}_{x,x+\hat{k}} \right] = e\hbar\delta_{\gamma,(x,x+\hat{k})}\hat{\mathcal{P}}_\gamma, \quad (3.10)$$

$$\left[ \hat{B}_{x;\hat{k},\hat{l}}, \hat{E}_{y,y+\hat{n}} \right] = i\hbar\delta_{\partial(x;\hat{k},\hat{l}), (y,y+\hat{n})}. \quad (3.11)$$

Here  $\delta_{\gamma,(x,x+\hat{k})} = 0$  if  $(x, x+\hat{k}) \notin \gamma$ ,  $\delta_{\gamma,(x,x+\hat{k})} = 1$  if  $(x, x+\hat{k}) \in \gamma$  and has the same orientation as  $\gamma$  and  $\delta_{\gamma,(x,x+\hat{k})} = -1$  otherwise. In the last formula,  $\partial(x;\hat{k},\hat{l})$  denotes the boundary of the oriented plaquette  $(x;\hat{k},\hat{l})$ .

5. The following commutation relations between the matter field generators hold: If  $\gamma$  is a path from  $x$  to  $y$ , and  $\gamma'$  is a path from  $x'$  to  $y'$ ,

then we have

$$[\hat{\mathcal{L}}_\gamma, \hat{\mathcal{L}}_{\gamma'}] = 0 \quad (3.12)$$

$$[\hat{\mathcal{L}}_\gamma, \hat{\mathcal{M}}_{\gamma'}] = 2i\hbar\delta_{xy'}\hat{\mathcal{L}}_{\gamma'\gamma} \quad (3.13)$$

$$[\hat{\mathcal{L}}_\gamma, \hat{\mathcal{N}}_{\gamma'}] = 2i\hbar\delta_{x'y}\hat{\mathcal{L}}_{\gamma\gamma'} \quad (3.14)$$

$$[\hat{\mathcal{L}}_\gamma, \hat{\mathcal{R}}_{\gamma'}] = 2i\hbar(\delta_{xy'}\hat{\mathcal{N}}_{\gamma'\gamma} + \delta_{x'y}\hat{\mathcal{M}}_{\gamma\gamma'}) \quad (3.15)$$

$$[\hat{\mathcal{M}}_\gamma, \hat{\mathcal{M}}_{\gamma'}] = 2i\hbar(\delta_{xy'}\hat{\mathcal{M}}_{\gamma'\gamma} - \delta_{x'y}\hat{\mathcal{M}}_{\gamma\gamma'}) \quad (3.16)$$

$$[\hat{\mathcal{M}}_\gamma, \hat{\mathcal{N}}_{\gamma'}] = 0 \quad (3.17)$$

$$[\hat{\mathcal{M}}_\gamma, \hat{\mathcal{R}}_{\gamma'}] = 2i\hbar\delta_{xy'}\hat{\mathcal{R}}_{\gamma'\gamma} \quad (3.18)$$

$$[\hat{\mathcal{N}}_\gamma, \hat{\mathcal{N}}_{\gamma'}] = 2i\hbar(\delta_{x'y}\hat{\mathcal{N}}_{\gamma\gamma'} - \delta_{xy'}\hat{\mathcal{N}}_{\gamma'\gamma}) \quad (3.19)$$

$$[\hat{\mathcal{N}}_\gamma, \hat{\mathcal{R}}_{\gamma'}] = 2i\hbar\delta_{x'y}\hat{\mathcal{R}}_{\gamma\gamma'} \quad (3.20)$$

$$[\hat{\mathcal{R}}_\gamma, \hat{\mathcal{R}}_{\gamma'}] = 0. \quad (3.21)$$

Thus, the invariant fields  $\hat{\mathcal{P}}_\gamma$  generate a Lie algebra.

6. The  $*$ -operator acts on the matter field generators as follows:

$$(\hat{\mathcal{L}}_\gamma)^* = \hat{\mathcal{L}}_{\gamma^{-1}}, (\hat{\mathcal{M}}_\gamma)^* = \hat{\mathcal{N}}_{\gamma^{-1}}, (\hat{\mathcal{N}}_\gamma)^* = \hat{\mathcal{M}}_{\gamma^{-1}}, (\hat{\mathcal{R}}_\gamma)^* = \hat{\mathcal{R}}_{\gamma^{-1}}. \quad (3.22)$$

7. For any path  $\gamma$  from  $x$  to  $y$  and any path  $\gamma'$  from  $y$  to  $z$ , the following identity holds:

$$\hat{\mathcal{L}}_\gamma \hat{\mathcal{L}}_{\gamma'} = \hat{\mathcal{L}}_{y\gamma} \hat{\mathcal{L}}_{\gamma\gamma'}, \quad (3.23)$$

**Proof:** by a number of lengthy, but simple calculations, which we leave to the reader.  $\square$

In particular, equation (3.6) follows from equation (2.14), with the right-hand-side expressed in terms of the generators  $\hat{\mathcal{M}}$ ,

$$\hat{q}_x = \frac{e}{\hbar} \text{Im}(\hat{\mathcal{M}}_{xx}) - e \hat{\mathbb{1}}. \quad (3.24)$$

Summing over all lattice points yields the expression for the global charge:

$$\hat{Q} = \frac{e}{\hbar} \sum_{x \in \Lambda^0} \text{Im}(\hat{\mathcal{M}}_{xx}) - Ne \hat{\mathbb{1}}, \quad (3.25)$$

with  $N$  being the number of lattice points.

Now we are able to give an abstract definition of the observable algebra, which does not refer any more to the field algebra we started with.

**Definition 3.2.** The observable algebra  $\mathcal{O}(\Lambda)$  is a  $C^*$ -algebra generated by abstract elements  $(\hat{B}, \hat{E}, \hat{\mathcal{P}}_\gamma)$ . The generators satisfy the following axioms:

1. The Gauss law (3.6).
2. The Bianchi identities (3.7).
3. Identities (3.8), relating generators  $\hat{\mathcal{P}}_\gamma$  for two different paths with common end points.
4. The commutation relations (3.9) – (3.11) between electromagnetic and matter field generators.
5. The Lie algebra commutation relations (3.12) – (3.21) among the matter field generators.
6. The generators  $\hat{E}$  and  $\hat{B}$  are self-adjoint,  $\hat{E}^* = \hat{E}$  and  $\hat{B}^* = \hat{B}$ , whereas the matter field generators fulfil (3.22).
7. There is a single pair  $(\gamma_0, \gamma'_0)$  of non-trivial paths with a common intermediate point  $y_0$ ,  $\gamma_0$  connecting  $x_0$  with  $y_0 \neq x_0$  and  $\gamma'_0$  connecting  $y_0$  with  $z_0 \neq y_0$ , such that the following identity holds:

$$\hat{\mathcal{L}}_{\gamma_0} \hat{\mathcal{L}}_{\gamma'_0} = \hat{\mathcal{L}}_{y_0 y_0} \hat{\mathcal{L}}_{\gamma_0 \gamma'_0} . \quad (3.26)$$

The notion “ $C^*$ -algebra generated by abstract elements” is meant in the sense of Woronowicz [17] and will be explained in Subsection 3.3.

From the above axioms we obtain the following basic

**Lemma 3.3.**

1. All generators  $\hat{\mathcal{P}}_\gamma$  are normal:

$$[\hat{\mathcal{P}}_\gamma^*, \hat{\mathcal{P}}_\gamma] = 0 .$$

2. For any path  $\gamma$  from  $x$  to  $y$  and any path  $\gamma'$  from  $y$  to  $z$ , the following identities hold:

$$\hat{\mathcal{L}}_\gamma \hat{\mathcal{L}}_{\gamma'} = \hat{\mathcal{L}}_{yy} \hat{\mathcal{L}}_{\gamma\gamma'} , \quad (3.27)$$

$$\hat{\mathcal{R}}_\gamma \hat{\mathcal{R}}_{\gamma'} = \hat{\mathcal{R}}_{yy} \hat{\mathcal{R}}_{\gamma\gamma'} , \quad (3.28)$$

If, moreover, the end  $z$  of  $\gamma'$  differs from  $y$ , then the following identity holds:

$$\hat{\mathcal{L}}_\gamma \hat{\mathcal{M}}_{\gamma'} = \hat{\mathcal{L}}_{yy} \hat{\mathcal{M}}_{\gamma\gamma'} . \quad (3.29)$$

If, moreover, also the beginning  $x$  of  $\gamma$  differs from  $y$ , then the following identity holds:

$$\hat{\mathcal{N}}_\gamma \hat{\mathcal{M}}_{\gamma'} = \hat{\mathcal{L}}_{yy} \hat{\mathcal{R}}_{\gamma\gamma'} = \hat{\mathcal{R}}_{\gamma\gamma'} \hat{\mathcal{L}}_{yy} . \quad (3.30)$$

### 3. The element

$$\hat{\mathcal{Z}} := \frac{1}{\hbar} \sum_{x \in \Lambda^0} \text{Im}(\hat{\mathcal{M}}_{xx}) = \frac{1}{2i\hbar} \sum_{x \in \Lambda^0} (\hat{\mathcal{M}}_{xx} - \hat{\mathcal{N}}_{xx}) \quad (3.31)$$

commutes with all elements  $\hat{\mathcal{P}}$  and, therefore, belongs to the center of the observable algebra.

**Proof:** Points 1. and 3. are easily proved by direct inspection. To prove point 2. we have to consider two cases: the generic one, when all the three points  $x$ ,  $y$  and  $z$  are different and the non-generic one, when two of them coincide. We begin with the generic case. Suppose that  $x$  is different from  $x_0, y_0$  and  $z_0$ . Choose any path  $\alpha$  from  $x$  to  $x_0$ . Acting with  $\text{ad} \hat{\mathcal{M}}_\alpha$  to the right on both sides of (3.26), (i. e. taking the commutator with  $\frac{1}{2i\hbar} \hat{\mathcal{M}}_\alpha$ ) we obtain:

$$\frac{1}{2i\hbar} \left[ \left( \hat{\mathcal{L}}_{\gamma_0} \hat{\mathcal{L}}_{\gamma'_0} - \hat{\mathcal{L}}_{y_0 y_0} \hat{\mathcal{L}}_{\gamma_0 \gamma'_0} \right), \hat{\mathcal{M}}_\alpha \right] = \hat{\mathcal{L}}_{\alpha \gamma_0} \hat{\mathcal{L}}_{\gamma'_0} - \hat{\mathcal{L}}_{y_0 y_0} \hat{\mathcal{L}}_{\alpha \gamma_0 \gamma'_0} = 0 . \quad (3.32)$$

Using identity (3.8), we can replace the path  $\alpha \gamma_0$  by any other path from  $x$  to  $y_0$  and the path  $\gamma'_0$  by any other path from  $y_0$  to  $z_0$ . Acting successively with appropriate operators  $\text{ad} \hat{\mathcal{M}}$  or  $\text{ad} \hat{\mathcal{N}}$ , we may shift in the same way the endpoints  $y_0$  and  $z_0$  to any other generic positions  $y$  and  $z$ . We stress that the above procedure holds for both generic and non-generic (i. e.  $x_0 = z_0$ ) initial position. The cases  $x = y$  or  $y = z$  are trivial. Finally, the only nontrivial non-generic case of equation (3.27), namely  $x = z$ , may be obtained in a similar way from the generic case, by acting with  $\text{ad} \hat{\mathcal{M}}_\beta$ , where  $\beta$  is a path from  $z$  to  $x$ . This operation shifts point  $z$  to  $x$ . This ends the proof of formula (3.27).

Acting, in the generic case, with  $\text{ad}\hat{\mathcal{R}}_{zz}$  on (3.27), we directly get a generic case of (3.29). Acting on it once more with  $\text{ad}\hat{\mathcal{R}}_{xx}$ , we get a generic case of (3.30). Finally, acting successively two times with  $\text{ad}\hat{\mathcal{R}}_{zz}$  on the latter identity, we get a generic case of (3.28).

Non generic cases  $x = z$  of (3.28), (3.29) and (3.30) are also easily obtained by acting on corresponding generic cases with  $\text{ad}\hat{\mathcal{M}}_\beta$ , where  $\beta$  is a path from  $z$  to  $x$ . This operation shifts point  $x$  to  $z$ .  $\square$

### Remarks:

1. As will be seen later, the real and the imaginary parts of generators of the observable algebra are, in any physical representation, unbounded self-adjoint operators. Writing bilinear relations for unbounded operators is, in general, meaningless. However, as will be seen in the sequel, part of the observable algebra  $\mathcal{O}(\Lambda)$  is generated from a certain Lie subalgebra of the Lie algebra defined by formulae (3.12) – (3.21). We will show that non-degenerate representations of  $\mathcal{O}(\Lambda)$  are given by unitary representations of the corresponding Lie group (or, equivalently, from *integrable* representations of the Lie algebra). Commutation relations (3.12) – (3.21), together with *integrability* of the representation, imply that in equations (3.8), (3.26), (3.27), (3.28) and (3.30) we always multiply *strongly commuting* observables. Therefore, the products used in these equations will be always unambiguously defined. The same argument applies also to the right-hand-side of equation (3.29) and to its left-hand-side, provided  $x \neq z$ . The only problem could come from the left-hand-side of (3.29), for  $x = z$ , because the two elements  $\hat{\mathcal{L}}_\gamma$  and  $\hat{\mathcal{M}}_{\gamma'}$  do not commute.

Let us discuss this point in more detail: Without any loss of generality, we may limit ourselves to the case  $\gamma' = \gamma^{-1}$ . Observe that we have

$$\begin{aligned} \hat{\mathcal{L}}_\gamma \hat{\mathcal{M}}_{\gamma'} &= \left( \text{Re}\hat{\mathcal{L}}_\gamma + i\text{Im}\hat{\mathcal{L}}_\gamma \right) \cdot \left( \text{Re}\hat{\mathcal{M}}_{\gamma'} + i\text{Im}\hat{\mathcal{M}}_{\gamma'} \right) \\ &= \text{Re}\hat{\mathcal{L}}_\gamma \text{Re}\hat{\mathcal{M}}_{\gamma'} - \text{Im}\hat{\mathcal{L}}_\gamma \text{Im}\hat{\mathcal{M}}_{\gamma'} \\ &\quad + i \left( \text{Re}\hat{\mathcal{L}}_\gamma \text{Im}\hat{\mathcal{M}}_{\gamma'} + \text{Im}\hat{\mathcal{L}}_\gamma \text{Re}\hat{\mathcal{M}}_{\gamma'} \right) . \end{aligned} \tag{3.33}$$

It follows immediately from the commutation relations that  $\text{Re}\hat{\mathcal{L}}_\gamma$  commutes with  $\text{Im}\hat{\mathcal{M}}_{\gamma'}$ . Similarly,  $\text{Re}\hat{\mathcal{M}}_{\gamma'}$  commutes with  $\text{Im}\hat{\mathcal{L}}_\gamma$ . This



means, that the imaginary part of the above expression is unambiguously defined as a product of commuting observables. To give a meaning also to its real part we take the unambiguously defined identity

$$\hat{\mathcal{L}}_\gamma \hat{\mathcal{L}}_{\gamma'} = \hat{\mathcal{L}}_{y\gamma} \hat{\mathcal{L}}_{\gamma\gamma'} , \quad (3.34)$$

and act on both its sides with  $\text{ad}\hat{\mathcal{M}}_\beta$ , where  $\beta$  is a path from  $z$  to  $x$ . As a result we get precisely the real part of expression (3.33), which is, therefore, *unambiguously* defined by the adjoint action of self-adjoint operators on self-adjoint operators. This shows that the above relations of the observable algebra can be meaningfully formulated on the level of unbounded operators.

2. Applying successively operators  $\text{ad}\hat{\mathcal{L}}$  to identity (3.26), as we did in the above proof, we may produce a lot of new identities. Each of them could also have been used as a defining relation instead of (3.26). In the Lemma, we have listed only those identities, which will be used in the sequel.
3. By point 3 of the above Lemma, the observable  $\hat{\mathcal{Z}}$  defines a superselection rule. Therefore, the physical Hilbert space is a direct sum of charge superselection sectors,

$$\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha ,$$

on which  $\hat{\mathcal{Z}}$  acts as  $\mathcal{Z}_\alpha \hat{1}$ . Due to definition (3.25), the same is true for the global charge  $\hat{Q}$  and, moreover,  $\frac{1}{e}Q_\alpha = \mathcal{Z}_\alpha - N$ . As will be shown later,  $\mathcal{Z}_\alpha$  may assume only integer values which proves that also  $\frac{1}{e}Q_\alpha$  is integer.

### 3.2 Generating the observable algebra from the tree data

A convenient way to solve relations between generators is to choose a *tree*, i. e. to choose a unique path connecting any pair of lattice sites. More precisely, a tree is a pair  $(x_0, \mathcal{T})$ , where  $x_0$  is a distinguished lattice site (called *root*) and  $\mathcal{T}$  is a set of lattice links such that for any lattice site  $x$  there is exactly one path from  $x_0$  to  $x$ , with links belonging to  $\mathcal{T}$ . Suppose, we have chosen a tree. Then, for any pair  $(x, y)$  of lattice sites, there is a unique

along tree path from  $x$  to  $y$ . Denote by  $\hat{\mathcal{P}}_{x,y}$  the generator  $\hat{\mathcal{P}}_\gamma$  corresponding to this path. Due to equation (3.8), the remaining generators  $\hat{\mathcal{P}}$  may be expressed in terms of those, provided we know the magnetic field  $\hat{B}$ . To choose independent quantities among the electromagnetic generators, take any *off tree* link  $(x, x + \hat{k})$ , and denote by  $\hat{\mathcal{B}}_{x,x+\hat{k}}$  the magnetic flux through the surface spanned by the closed path composed of the link  $(x, x + \hat{k})$  and the unique *on-tree* path from  $x + \hat{k}$  to  $x$ . It is easily seen that, solving the Bianchi identities (3.7), starting from the root and moving outside, we can reconstruct all the magnetic fluxes  $\hat{B}_{x;\hat{k},\hat{l}}$  from  $\hat{\mathcal{B}}_{x,x+\hat{k}}$ . Among electric fields we may also choose only those which correspond to *off tree* links as independent quantities. Indeed, solving the Gauss law (3.6) for  $\hat{E}$ , starting from the boundary and moving towards the root, with local charges  $\hat{q}_x$  expressed in terms generators  $\hat{\mathcal{P}}_{x,y}$  and the boundary data  $\hat{E}_{x,\infty}$  given, we can reconstruct all the remaining elements  $\hat{E}_{x,x+\hat{k}}$ . This way we get the following

**Proposition 3.4.** *For a given tree  $\mathcal{T}$ , the set of tree data*

$$(\hat{\mathcal{P}}_{x,y}, \hat{\mathcal{B}}_{x,x+\hat{k}}, \hat{E}_{x,x+\hat{k}}), (x, x + \hat{k}) \notin \mathcal{T},$$

*constitute a complete set of generators of  $\mathcal{O}(\Lambda)$ .*

For further details of the proof, we refer the reader to the (completely analogous) proof for the case of QED with fermions, contained in [10].

Obviously, the tree data inherit commutation relations from Definition 3.2. In particular, from point 4 of this definition, we read off the following commutation relations:

$$[\hat{\mathcal{B}}_{x,x+\hat{k}}, \hat{E}_{y,y+\hat{l}}] = i\hbar \delta_{(x,y)} \delta_{(\hat{k},\hat{l})} \mathbb{1}, \quad (3.35)$$

where  $(x, x + \hat{k}) \notin \mathcal{T}$ ,  $(y, y + \hat{l}) \notin \mathcal{T}$ . Hence, the independent electromagnetic fields fulfill the canonical commutation relations. The corresponding associative algebra, generated by these fields, will be denoted by  $\mathcal{O}_{\mathcal{T}}^{em}$ . Moreover, for  $(z, z + \hat{k}) \notin \mathcal{T}$  we have:

$$[\hat{\mathcal{P}}_{xy}, \hat{\mathcal{B}}_{z,z+\hat{k}}] = 0, \quad (3.36)$$

$$[\hat{\mathcal{P}}_{xy}, \hat{E}_{z,z+\hat{k}}] = 0, \quad (3.37)$$

which means that  $\mathcal{O}_{\mathcal{T}}^{em}$  commutes with the subalgebra  $\mathcal{O}_{\mathcal{T}}^{mat} \subset \mathcal{O}(\Lambda)$ , spanned by the matter field generators  $\hat{\mathcal{P}}_{xy}$ . This fact, together with Proposition 3.4,

implies that the observable algebra decomposes as follows:

$$\mathcal{O}(\Lambda) = \mathcal{O}_{\mathcal{T}}^{em} \otimes \mathcal{O}_{\mathcal{T}}^{mat} . \quad (3.38)$$

We know already from point 5 of Proposition 3.1, respectively from point 5 of Definition 3.2, that the algebra  $\mathcal{O}_{\mathcal{T}}^{mat}$  is generated by a Lie algebra. The tree data inherit, of course, a Lie algebra structure, which we are now going to describe. For this purpose, let us label all the lattice sites by integers  $k = 1, \dots, N$ . (In what follows, it does not matter, which one among the points  $x_k$  coincides with the previously chosen tree root  $x_0$ .) To simplify notation we shall write  $\hat{\mathcal{P}}_{kl}$  instead of  $\hat{\mathcal{P}}_{x_k, x_l}$ . Then, from point 5 of Definition 3.2, we have the following commutation relations:

$$\frac{1}{i}[\hat{\mathcal{L}}_{kl}, \hat{\mathcal{L}}_{mn}] = 0 \quad (3.39)$$

$$\frac{1}{i}[\hat{\mathcal{L}}_{kl}, \hat{\mathcal{M}}_{mn}] = 2\hbar\delta_{kn}\hat{\mathcal{L}}_{ml} \quad (3.40)$$

$$\frac{1}{i}[\hat{\mathcal{L}}_{kl}, \hat{\mathcal{N}}_{mn}] = 2\hbar\delta_{ml}\hat{\mathcal{L}}_{kn} \quad (3.41)$$

$$\frac{1}{i}[\hat{\mathcal{L}}_{kl}, \hat{\mathcal{R}}_{mn}] = 2\hbar(\delta_{kn}\hat{\mathcal{N}}_{ml} + \delta_{ml}\hat{\mathcal{M}}_{kn}) \quad (3.42)$$

$$\frac{1}{i}[\hat{\mathcal{M}}_{kl}, \hat{\mathcal{M}}_{mn}] = 2\hbar(\delta_{kn}\hat{\mathcal{M}}_{ml} - \delta_{ml}\hat{\mathcal{M}}_{kn}) \quad (3.43)$$

$$\frac{1}{i}[\hat{\mathcal{M}}_{kl}, \hat{\mathcal{N}}_{mn}] = 0 \quad (3.44)$$

$$\frac{1}{i}[\hat{\mathcal{M}}_{kl}, \hat{\mathcal{R}}_{mn}] = 2\hbar\delta_{kn}\hat{\mathcal{R}}_{ml} \quad (3.45)$$

$$\frac{1}{i}[\hat{\mathcal{N}}_{kl}, \hat{\mathcal{N}}_{mn}] = 2\hbar(\delta_{ml}\hat{\mathcal{N}}_{kn} - \delta_{kn}\hat{\mathcal{N}}_{ml}) \quad (3.46)$$

$$\frac{1}{i}[\hat{\mathcal{N}}_{kl}, \hat{\mathcal{R}}_{mn}] = 2\hbar\delta_{ml}\hat{\mathcal{R}}_{kn} \quad (3.47)$$

$$\frac{1}{i}[\hat{\mathcal{R}}_{kl}, \hat{\mathcal{R}}_{mn}] = 0 \quad (3.48)$$

Moreover, from point 6 of this definition, we have:

$$(\hat{\mathcal{L}}_{kl})^* = \hat{\mathcal{L}}_{lk}, (\hat{\mathcal{M}}_{kl})^* = \hat{\mathcal{N}}_{lk}, (\hat{\mathcal{N}}_{kl})^* = \hat{\mathcal{M}}_{lk}, (\hat{\mathcal{R}}_{kl})^* = \hat{\mathcal{R}}_{lk}. \quad (3.49)$$

These relations define a complex Lie algebra, denoted by  $\tilde{\mathfrak{g}}_{\mathcal{T}}^{mat}$ , with Lie bracket  $\frac{1}{i}[\cdot, \cdot]$  and with conjugation “ $*$ ”, given by (3.49). Consider now the Lie algebra  $gl(2N, \mathbb{C})$  (with ordinary Lie bracket  $[A, B] = AB - BA$ ), and with conjugation “ $*$ ” defined as follows:

$$A^* := -\mathbb{1}_{(N,N)} A^\dagger \mathbb{1}_{(N,N)}, \quad A \in gl(2N, \mathbb{C}). \quad (3.50)$$

Here “ $\dagger$ ” denotes Hermitian conjugation and

$$\mathbb{1}_{(N,N)} := \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix},$$

with  $\mathbb{1}_N$  being the unit  $(N \times N)$ -matrix.

**Lemma 3.5.** *The mapping*

$$F : \widetilde{\mathfrak{g}}_{\mathcal{T}}^{mat} \longrightarrow gl(2N, \mathbb{C}) ,$$

defined by

$$F(\hat{\mathcal{L}}_{kl}) := \frac{1}{i} \begin{pmatrix} -E_{kl} & iE_{kl} \\ iE_{kl} & E_{kl} \end{pmatrix} \quad F(\hat{\mathcal{M}}_{kl}) := \frac{\hbar}{i} \begin{pmatrix} -iE_{kl} & E_{kl} \\ -E_{kl} & -iE_{kl} \end{pmatrix} \quad (3.51)$$

$$F(\hat{\mathcal{N}}_{kl}) := \frac{\hbar}{i} \begin{pmatrix} iE_{kl} & E_{kl} \\ -E_{kl} & iE_{kl} \end{pmatrix} \quad F(\hat{\mathcal{R}}_{kl}) := \frac{\hbar^2}{i} \begin{pmatrix} -E_{kl} & -iE_{kl} \\ -iE_{kl} & E_{kl} \end{pmatrix} , \quad (3.52)$$

with  $(E_{kl})$  being the canonical basis of  $gl(N, \mathbb{C})$ , is an isomorphism of complex Lie algebras with conjugation,

$$\begin{aligned} F(\tfrac{1}{i}[\mathcal{X}, \mathcal{Y}]) &= [F(\mathcal{X}), F(\mathcal{Y})] , \\ F(\mathcal{X}^*) &= F(\mathcal{X})^* , \end{aligned}$$

for  $\mathcal{X}, \mathcal{Y} \in \widetilde{\mathfrak{g}}_{\mathcal{T}}^{mat}$ .

**Proof:** by a lengthy, but simple calculation, which we leave to the reader.  $\square$

In what follows, we shall often omit writing  $F$  and identify  $F(\hat{\mathcal{P}})$  with  $\hat{\mathcal{P}}$ .

We denote the real form of  $\widetilde{\mathfrak{g}}_{\mathcal{T}}^{mat}$ , corresponding to “\*” by  $\mathfrak{g}_{\mathcal{T}}^{mat}$ . The elements of this real Lie algebra are physical observables, spanned by the self-adjoint elements  $(\text{Re}\hat{\mathcal{P}}_{kl}, \text{Im}\hat{\mathcal{P}}_{kl})$ . Under the above isomorphism,  $\mathfrak{g}_{\mathcal{T}}^{mat}$  is identified with the real form

$$u(N, N) := \{A \in gl(2N, \mathbb{C}) : A^* = A\}$$

of  $gl(2N, \mathbb{C})$ , corresponding to the conjugation defined by (3.50). Exponentiating  $u(N, N)$  we obtain the corresponding connected Lie group  $U(N, N)$ , consisting of those linear transformations of  $\mathbb{C}^{2N}$  which preserve the hermitian form defined by  $\mathbb{1}_{(N, N)}$ :

$$U(N, N) := \{U \in \text{Mat}_{2N \times 2N}(\mathbb{C}) : U^\dagger \mathbb{1}_{(N, N)} U = \mathbb{1}_{(N, N)}\} .$$

As we will see, non-degenerate representations of  $\mathcal{O}_{\mathcal{T}}^{mat}$  will be given by *integrable* representations of  $\mathfrak{g}_{\mathcal{T}}^{mat} \cong u(N, N)$ , this means unitary representations of the group  $U(N, N)$ .

Observe that due to identification (3.51) we have

$$\hat{\mathcal{Z}} = \frac{1}{\hbar} \sum_k \text{Im}(\hat{\mathcal{M}}_{kk}) = i \mathbb{1}_{(2N)} . \quad (3.53)$$

This element generates the center  $u(1)$  of  $u(N, N)$ . The restriction of any *non-degenerate* representation of the observable algebra to the center can be integrated to a representation of the subgroup  $\{\exp(i\tau\mathbb{1}_{(2N)}), \tau \in \mathbb{R}^1\} \cong U(1)$ . This implies that the spectrum of  $\hat{\mathcal{Z}}$  must be integer. We conclude that the spectrum of the charge operator  $\frac{\hat{Q}}{e}$ , defined by formula (3.25) must be integer, too.

Observe that equations (3.27) – (3.30) (implied by axiom (3.26)) may be rewritten in the following form: For any triple  $x_i, x_j, x_k$  of lattice points we have:

$$\hat{\mathcal{L}}_{ij}\hat{\mathcal{L}}_{jk} = \hat{\mathcal{L}}_{jj}\hat{\mathcal{L}}_{ik} , \quad (3.54)$$

$$\hat{\mathcal{R}}_{ij}\hat{\mathcal{R}}_{jk} = \hat{\mathcal{R}}_{jj}\hat{\mathcal{R}}_{ik} . \quad (3.55)$$

If, moreover,  $j \neq k$ , then we have:

$$\hat{\mathcal{L}}_{ij}\hat{\mathcal{M}}_{jk} = \hat{\mathcal{L}}_{jj}\hat{\mathcal{M}}_{ik} . \quad (3.56)$$

If, moreover,  $i \neq j$ , then we have:

$$\hat{\mathcal{N}}_{ij}\hat{\mathcal{M}}_{jk} = \hat{\mathcal{L}}_{jj}\hat{\mathcal{R}}_{ik} = \hat{\mathcal{R}}_{ik}\hat{\mathcal{L}}_{jj} . \quad (3.57)$$

### 3.3 Functional analytic structure

Now we are able to endow the observable algebra with a functional analytic structure. Recall that  $\mathcal{O}(\Lambda)$  is the tensor product of  $\mathcal{O}_T^{em}$  and  $\mathcal{O}_T^{mat}$ , see (3.38). It turns out that both components can be endowed with the structure of a  $C^*$ -algebra generated by unbounded elements in the sense of Woronowicz, see [17].

Since  $\mathcal{O}_T^{mat}$  is generated by the Lie algebra  $u(N, N)$ , we take the  $C^*$ -algebra  $C^*(U(N, N))$  and factorize it with respect to the ideal generated by relations (3.54) – (3.57). (In fact, it is sufficient to impose only one of these relations: the proof of Lemma 3.3 shows that one of these identities implies the remaining ones.)

The  $C^*$ -algebra  $C^*(G)$  of any locally compact group may be obtained as the  $C^*$ -completion of the group algebra of  $G$ . By definition, the latter is the space  $L^1(G)$  of integrable functions on the group, with convolution providing the product structure. The completion is taken with respect to the following norm:

$$\|f\| := \sup_{\pi} \|\pi(f)\| , \quad (3.58)$$

where  $f \in L^1(G)$ , the supremum is taken over all representations  $\pi$  of the group and  $\pi(f)$  denotes the operator obtained by smearing the representation over the group with the function  $f$ , see [14], [18]. We note that  $C^*(G)$  is one of the classes of examples considered by Woronowicz. It is generated by any basis of the Lie algebra of  $G$ , see [16] and [17]. We stress that these generators are not elements of  $C^*(G)$ , they are only *affiliated* in the  $C^*$ -sense.

Thus, take the algebra  $C^*(U(N, N))$ . To impose relations (3.54) – (3.57) on generators (more precisely, their real and imaginary parts) we multiply them from both sides by elements of  $C^*(U(N, N))$  belonging to the common dense domain of the generators (e. g. the so called *smooth* elements, corresponding to functions of the class  $C_0^\infty(U(N, N)) \subset L^1(U(N, N))$ ). This way we generate a double-sided ideal  $\mathcal{J} \subset C^*(U(N, N))$ . We define the matter algebra  $\mathcal{O}_{\mathcal{T}}^{mat}$  as the quotient:

$$\mathcal{O}_{\mathcal{T}}^{mat} \cong C^*(U(N, N))/\mathcal{J}. \quad (3.59)$$

It is worthwhile to notice that the same ideal may be obtained as the space of elements whose norm vanishes, if we replace (3.58) by the supremum over only those representations, which fulfill additional identities (3.54) – (3.57). Thus,  $\mathcal{O}_{\mathcal{T}}^{mat}$  could be defined also as the completion of  $L^1(U(N, N))$  with respect to the latter norm (cf. [16], [17]).

Next, we endow the electromagnetic component  $\mathcal{O}_{\mathcal{T}}^{em}$  with a  $C^*$ -structure. Again, the theory of Woronowicz can be applied. By the von Neumann theorem, all irreducible representations of the (electromagnetic) canonical commutation relations (3.35) are isomorphic to the Schrödinger representation. We take as  $\mathcal{O}_{\mathcal{T}}^{em}$  the  $C^*$ -algebra  $CB(H)$  of compact operators acting on the Hilbert space  $H$  of this representation. Here, no additional identities have to be imposed. Again, the generators  $\hat{B}_{x,x+\hat{k}}$  and  $\hat{E}_{y,y+\hat{l}}$  are affiliated with the algebra.

Finally, the  $C^*$ -algebra  $\mathcal{O}(\Lambda)$  is, by definition, the *minimal* tensor product of  $\mathcal{O}_{\mathcal{T}}^{mat}$  with  $\mathcal{O}_{\mathcal{T}}^{em}$ . Then, for elements  $A$ , affiliated with  $\mathcal{O}_{\mathcal{T}}^{mat}$ , and  $B$ , affiliated with  $\mathcal{O}_{\mathcal{T}}^{em}$ ,  $A \otimes B$  is affiliated with  $\mathcal{O}(\Lambda)$ .

**Remarks:**

1. We stress that  $\mathcal{O}(\Lambda)$  is a  $C^*$ -algebra without identity. On the abstract level, the element  $\hat{1}$  appearing in formulae (3.24) and (3.25) has to be understood as the identity in the multiplier algebra  $M(\mathcal{O}(\Lambda))$ , which is a subalgebra of the set of affiliated elements.

2. As noticed by Woronowicz [15], there exists an abstract definition of  $C^*(G)$ , which applies to any topological group  $G$ , not necessarily being a Lie group. More precisely, a  $C^*$ -algebra  $A$  and a homomorphism  $\alpha : G \rightarrow M(A)$  of  $G$  into unitary elements of the multiplier algebra  $M(A)$  is called a  $C^*$ -algebra of  $G$  and denoted  $C^*(G)$  if the pair  $(A, \alpha)$  is universal in the following sense: for any other such pair  $(B, \beta)$ , there exists a morphism  $\varphi \in \text{Mor}(A, B)$ , such that  $\beta = \varphi \circ \alpha$  (for the definition of  $\text{Mor}(A, B)$  see [17]). An effective construction in case of a locally compact group consists in taking the above mentioned  $C^*$ -completion of the group algebra of  $G$ .

## 4 Representations of the Observable Algebra and the Charge Superselection Structure

We are going to construct all faithful, irreducible and non-degenerate representations of the observable algebra  $\mathcal{O}(\Lambda)$ . Concerning the electromagnetic part, the von Neumann theorem guarantees uniqueness of representations. As for the matter part, there is a one-to-one correspondence (c.f. [14], [18]) between non-degenerate representations of the algebra  $C^*(U(N, N))$  with unitary representations of the Lie group  $U(N, N)$ . Thus, by Definition 3.59, the non-degenerate representations of  $\mathcal{O}_T^{mat}$  are given by those faithful, irreducible and integrable representations of the Lie algebra  $\mathfrak{g}_T^{mat}$ , which respect the additional relations (3.54) – (3.57). To find them, we shall further reduce  $\mathfrak{g}_T^{mat}$  to a certain Lie subalgebra  $\mathfrak{h}_T^{mat} \subset \mathfrak{g}_T^{mat}$ , such that it also generates  $\mathcal{O}_T^{mat}$  and that the enveloping algebra of  $\mathfrak{h}_T^{mat}$  does not inherit any identities from the relations defining the ideal  $\mathcal{J}$ . We will show that, given a representation of  $\mathfrak{h}_T^{mat}$  of the above type, relations (3.54) – (3.57) enable us to construct a unique representation of  $\mathcal{O}_T^{mat}$ , for a given value of total charge. Moreover, every representations of  $\mathcal{O}_T^{mat}$  is obtained this way.

To define the Lie subalgebra  $\mathfrak{h}_T^{mat}$ , we fix one lattice point, say  $x_N$ , and take only those  $\hat{\mathcal{L}}$ 's and  $\hat{\mathcal{M}}$ 's, which start at this point. More precisely, we denote:

$$Q_k := \hat{\mathcal{L}}_{Nk} , \tag{4.1}$$

$$P_k := \hat{\mathcal{M}}_{Nk} , \tag{4.2}$$

$$R := \hat{\mathcal{L}}_{NN} , \tag{4.3}$$

$$K := \text{Re} (\hat{\mathcal{M}}_{NN}) , \tag{4.4}$$

for  $k = 1, \dots, N-1$ . Observe that  $P$  and  $Q$  are normal,  $[P, P^*] = [Q, Q^*] = 0$ , whereas  $R$  and  $K$  are hermitean,  $R^* = R$ ,  $K^* = K$ . Denoting  $Q_k = q_k^1 + iq_k^2$  and  $P_k = p_k^1 + ip_k^2$ , with  $q_k^a$  and  $p_k^a$  being self-adjoint, we read off from relations (3.39) – (3.48) the following commutation relations:

$$\frac{1}{i} [q_k^a, \frac{1}{\hbar} p_l^b] = R \delta_{kl} \delta^{ab}, \quad (4.5)$$

$$\frac{1}{i} [R, \frac{1}{\hbar} K] = R, \quad (4.6)$$

$$\frac{1}{i} [q_k^a, \frac{1}{\hbar} K] = q_k^a, \quad (4.7)$$

$$\frac{1}{i} [p_k^a, \frac{1}{\hbar} K] = p_k^a, \quad (4.8)$$

whereas the remaining commutators vanish.

Observe that formulae (4.5) are the commutation relations of the real Heisenberg Lie algebra  $\mathcal{H}_{2(N-1)}$ , generated by  $2(N-1)$  canonically conjugate pairs  $(q_j^a, \frac{1}{\hbar} p_j^a)$  and with center generated by  $R$ . The 1-dimensional Lie algebra generated by  $\frac{1}{\hbar} K$  acts by (4.6) – (4.8) on  $\mathcal{H}_{2(N-1)}$ , endowing  $\mathfrak{h}_T^{mat}$  with the structure of a semidirect sum,

$$\mathfrak{h}_T^{mat} = \mathbb{R}^1 \oplus_s \mathcal{H}_{2(N-1)}. \quad (4.9)$$

Observe that the enveloping algebra spanned by  $\mathfrak{h}_T^{mat}$  does not inherit any identities from the defining relations (3.54) – (3.57), indeed.

## 4.1 Representations of the Lie algebra $\mathfrak{h}_T^{mat}$

**Theorem 4.1.** *There are exactly two faithful, irreducible and integrable representations of the Lie algebra  $\mathfrak{h}_T^{mat}$ . They are both defined on the Hilbert space  $\mathcal{H} := L^2(\mathbb{C}^{N-1} \times \mathbb{R}^1)$ , and are given by the following formulae:*

$$(Q_k \Psi)(z_1, \dots, z_{N-1}, \lambda) = \pm e^\lambda z_k \Psi(z_1, \dots, z_{N-1}, \lambda), \quad (4.10)$$

$$(P_k \Psi)(z_1, \dots, z_{N-1}, \lambda) = e^\lambda \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} \Psi(z_1, \dots, z_{N-1}, \lambda), \quad (4.11)$$

$$(R \Psi)(z_1, \dots, z_{N-1}, \lambda) = \pm e^{2\lambda} \Psi(z_1, \dots, z_{N-1}, \lambda), \quad (4.12)$$

$$(K \Psi)(z_1, \dots, z_{N-1}, \lambda) = \frac{\hbar}{i} \frac{\partial}{\partial \lambda} \Psi(z_1, \dots, z_{N-1}, \lambda), \quad (4.13)$$

where  $\Psi \in \mathcal{H}$  and  $(z_1, \dots, z_{N-1}, \lambda) \in \mathbb{C}^{N-1} \times \mathbb{R}^1$ .

**Proof:** We use the following matrix representation of  $\mathfrak{h}_T^{mat}$ :

$$X(k, r, \mathbf{p}, \mathbf{q}) := \begin{pmatrix} k & \mathbf{p} & r \\ 0 & 0 & \mathbf{q} \\ 0 & 0 & -k \end{pmatrix}, \quad (4.14)$$



where  $\mathbf{q}^T, \mathbf{p} \in \mathbb{R}^{2(N-1)}$ ,  $k, r \in \mathbb{R}^1$ , and the generators are identified as follows:

$$\begin{aligned}\frac{1}{\hbar}K &= X(1, 0, 0, 0), \\ R &= X(0, 1, 0, 0), \\ \frac{1}{\hbar}p_j^1 &= X(0, 0, \delta_j^i, 0), \\ \frac{1}{\hbar}p_j^2 &= X(0, 0, \delta_{2(N-1)+j}^i, 0), \\ q_k^1 &= X(0, 0, 0, \delta_j^i), \\ q_k^2 &= X(0, 0, 0, \delta_{2(N-1)+j}^i).\end{aligned}$$

Exponentiating representation (4.14), we obtain the following matrix representation of the connected, simply connected Lie group  $H_{\mathcal{T}}^{mat}$ , corresponding to  $\mathfrak{h}_{\mathcal{T}}^{mat}$ ,

$$g(u, c, \mathbf{a}, \mathbf{b}) = \begin{pmatrix} u & \mathbf{a} & c \\ 0 & \mathbf{1} & \mathbf{b} \\ 0 & 0 & u^{-1} \end{pmatrix}, \quad (4.15)$$

where  $\mathbf{a}^T, \mathbf{b} \in \mathbb{R}^{2(N-1)}$ ,  $c \in \mathbb{R}^1$  and  $u \in \mathbb{R}_+$ . For  $u = 1$ , this formula yields the standard matrix representation of the  $(4(N-1) + 1)$ -dimensional real Heisenberg group  $H_{2(N-1)}$ . On the other hand, for  $\mathbf{a} = 0 = \mathbf{b}$  and  $c = 0$ , it yields the real (multiplicative) group in 1 dimension. Thus,  $H_{\mathcal{T}}^{mat}$  coincides with the following semidirect product:

$$H_{\mathcal{T}}^{mat} = \mathbb{R}_+ \times_s H_{2(N-1)}. \quad (4.16)$$

Since  $H_{2(N-1)}$  is a closed normal subgroup of  $H_{\mathcal{T}}^{mat}$ , we can apply the method of induced representations, see [18]. The faithful, unitary, irreducible representations of  $H_{2(N-1)}$  are labeled by  $t \in \mathbb{R}_* = \mathbb{R}^1 \setminus 0$ ,

$$(U_t(g(1, c, \mathbf{a}, \mathbf{b}))f)(\mathbf{x}) = e^{it(\mathbf{b}\mathbf{x}+c)} f(\mathbf{x} + \mathbf{a}), \quad (4.17)$$

with  $f \in L^2(\mathbb{R}^{2(N-1)})$ , see [18]. Any two representations labeled by  $t$  and  $t'$ , with  $t \neq t'$ , are unitarily inequivalent. We have to find the orbits of the action of  $H_{\mathcal{T}}^{mat}$  on the space  $\hat{H}_{2(N-1)}$  of equivalence classes of unitary irreducible representations of  $H_{2(N-1)}$ . This action reduces to the action of  $\mathbb{R}_+$ ,

$$\hat{H}_{2(N-1)} \times \mathbb{R}_+ \ni (U_t, g(u, 0, 0, 0)) \rightarrow U_t \circ \text{Ad}g(u, 0, 0, 0) \in \hat{H}_{2(N-1)}.$$

Using formula (4.17) it can be shown that – up to unitary equivalence – we have

$$U_t \circ \text{Ad}g(u, 0, 0, 0) = U_{u^2 t}.$$

Thus, there are two orbits,  $\mathcal{O}_+ = \{t \in \mathbb{R}^1 : t > 0\}$  and  $\mathcal{O}_- = \{t \in \mathbb{R}^1 : t < 0\}$ . For both orbits, the stabilizer of each point on the orbit is conjugate to  $H_{2(N-1)}$ , and  $\mathbb{R}_+$  acts transitively on each orbit. Consequently, there are two equivalence classes of faithful, unitary, irreducible representations  $U_\pm$  of  $H_{\mathcal{T}}^{mat}$ , induced from the representations  $U_t$ , defined on the spaces

$$\mathcal{H}_\pm = L^2(\mathcal{O}_\pm, L^2(\mathbb{R}^{2(N-1)}))$$

of functions on  $\mathcal{O}_\pm$  with values in the representation space  $L^2(\mathbb{R}^{2(N-1)})$  of  $H_{2(N-1)}$ . Identifying both orbits with  $\mathbb{R}^1$ , by putting  $t = \pm e^{2\lambda}$ , we have  $\mathcal{H}_\pm \cong L^2(\mathbb{R}^{2(N-1)+1})$ , and a simple calculation yields the following induced representations:

$$(U_\pm(g(u, c, \mathbf{a}, \mathbf{b}))f)(\mathbf{x}, \lambda) = e^{\pm i e^{2\lambda} u(e^{-\lambda} \mathbf{b} \mathbf{x} + c)} f(\mathbf{x} + e^\lambda \mathbf{a}, \lambda + \ln u). \quad (4.18)$$

Differentiating these representations along the one parameter subgroups generated by the above defined basis elements of  $\mathcal{H}_{\mathcal{T}}^{mat}$  and identifying  $\mathbb{R}^{2(N-1)}$  with  $\mathbb{C}^{(N-1)}$ , we obtain exactly formulae (4.11) – (4.13).  $\square$

Observe that the transformation

$$I(\hat{\mathcal{L}}_{xy}) = -\hat{\mathcal{R}}_{xy}, \quad (4.19)$$

$$I(\hat{\mathcal{R}}_{xy}) = -\hat{\mathcal{L}}_{xy}, \quad (4.20)$$

$$I(\hat{\mathcal{M}}_{xy}) = \hat{\mathcal{N}}_{xy}, \quad (4.21)$$

$$I(\hat{\mathcal{N}}_{xy}) = \hat{\mathcal{M}}_{xy}, \quad (4.22)$$

preserves the defining relations (3.12) – (3.28) and, therefore, generates an automorphism of  $\mathcal{O}_{\mathcal{T}}^{mat}$ . This automorphism intertwines the two representations given by Theorem 4.1. Thus, it is sufficient to consider the positive representation only, because the other one is equivalent to one obtained from the positive representation by relabeling of the elements of  $\mathcal{O}_{\mathcal{T}}^{mat}$ . Such a relabeling changes, however, the physical meaning of some observables. This is, in particular, true for the electric charge. Indeed, the sign of  $\text{Im}(\hat{\mathcal{M}}_{xx})$

changes under the transformation  $I$ . Consequently, the definition (3.24) of the electric charge, would have to be supplemented by the sign of the representation. Therefore, we choose the positive representation once for ever. This choice implies the positivity of all the elements  $\hat{\mathcal{L}}_{xx}$  and  $\hat{\mathcal{R}}_{xx}$ .

It is interesting to observe that we could postulate the positivity of only one of them, say  $\hat{\mathcal{L}}_{x_0x_0}$ , as an additional axiom for the observable algebra. Indeed, identity (3.26) implies

$$\hat{\mathcal{L}}_{x_0x_0} \cdot \hat{\mathcal{L}}_{yy} = \hat{\mathcal{L}}_{x_0y} \cdot \hat{\mathcal{L}}_{yx_0} = \hat{\mathcal{L}}_{x_0y} \cdot \left( \hat{\mathcal{L}}_{x_0y} \right)^* > 0 .$$

Similarly, we get for  $x \neq y$

$$\hat{\mathcal{R}}_{xx} \cdot \hat{\mathcal{L}}_{yy} = \hat{\mathcal{N}}_{xy} \cdot \hat{\mathcal{M}}_{yx} = \hat{\mathcal{N}}_{xy} \cdot \left( \hat{\mathcal{N}}_{xy} \right)^* > 0 .$$

Hence, choosing the positive sign of  $\hat{\mathcal{L}}_{x_0x_0}$  we obtain positivity for all of them.

## 4.2 Local charge operators

**Lemma 4.2.** *In the positive representation we have*

$$(\hat{q}_{x_k} \Psi)(z_1, \dots, z_{N-1}, \lambda) = \frac{e}{i} \frac{\partial}{\partial \arg z_k} \Psi(z_1, \dots, z_{N-1}, \lambda) , \quad (4.23)$$

for  $k = 1, \dots, N-1$ .

**Proof:** Due to identity

$$Q_k^* P_k = \hat{\mathcal{L}}_{kN} \hat{\mathcal{M}}_{Nk} = \hat{\mathcal{M}}_{kk} \hat{\mathcal{L}}_{NN} = R \hat{\mathcal{M}}_{kk} ,$$

and to formulae (4.10) – (4.13), we have:

$$R \hat{\mathcal{M}}_{kk} = R \bar{z}_k \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} . \quad (4.24)$$

The operator  $R$  is strictly positive and, hence, we may skip it on both sides of the equation. This way we obtain:

$$\hat{\mathcal{M}}_{kk} = \bar{z}_k \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} . \quad (4.25)$$

The imaginary part of this operator gives us (4.23). □

**Remark :** Deriving (4.25) from (4.24), we have, in fact, divided both sides of the equation by  $R = \hat{\mathcal{L}}_{NN}$ . Such an operation is, of course, meaningless on the level of the abstract algebra  $\mathcal{O}_T^{mat}$ . On the level of representations, however, we have proved that  $R$  cannot be a divisor of zero. Hence, this operation is fully justified.

**Corollary 4.3.** *In every representation of  $\mathcal{O}_T^{mat}$ , the local charge operators  $\hat{q}_{x_k}$  take integer eigenvalues (in units of the elementary charge) only. Thus, the spectrum of each  $\hat{q}_{x_k}$  is given by*

$$\text{Sp}(\hat{q}_{x_k}) = e\mathbb{N} .$$

### 4.3 Constructing representations of $\mathcal{O}_T^{mat}$ from representations of $\mathfrak{h}_T^{mat}$

We will show that each irreducible, faithful and non-degenerate representation of  $\mathcal{O}_T^{mat}$ , assigned to a given tree, is uniquely generated by a corresponding representation of its Lie subalgebra  $\mathfrak{h}_T^{mat}$ , provided the total electric charge  $Q$  carried by the matter field is fixed.

Suppose that a representation of  $\mathcal{O}_T^{mat}$  is given. Choosing instead of  $x_N$  any other reference point  $x_i$ , we obtain a family  $\mathfrak{h}_T^{mat}(x_i)$  of Lie subalgebras. The restrictions of the above representation to these subalgebras are all given by Theorem 4.1. In particular, for  $x_1$ , the corresponding Lie subalgebra  $\mathfrak{h}_T^{mat}(x_1)$  is generated by the following observables:

$$\tilde{Q}_k := \hat{\mathcal{L}}_{1k} , \tag{4.26}$$

$$\tilde{P}_k := \hat{\mathcal{M}}_{1k} , \tag{4.27}$$

$$\tilde{R} := \hat{\mathcal{L}}_{11} , \tag{4.28}$$

$$\tilde{K} := \text{Re} (\hat{\mathcal{M}}_{11}) , \tag{4.29}$$

for  $k = 2, \dots, N$ . The corresponding positive representation is equivalent to the following one:

$$(\tilde{Q}_k \tilde{\Psi})(\mu, w_2, \dots, w_N) = e^\mu w_k \tilde{\Psi}(\mu, w_2, \dots, w_N) , \tag{4.30}$$

$$(\tilde{P}_k \tilde{\Psi})(\mu, w_2, \dots, w_N) = e^\mu \frac{2\hbar}{i} \frac{\partial}{\partial w_k} \tilde{\Psi}(\mu, w_2, \dots, w_N) , \tag{4.31}$$

$$(\tilde{R} \tilde{\Psi})(\mu, w_2, \dots, w_N) = e^{2\mu} \tilde{\Psi}(\mu, w_2, \dots, w_N) , \tag{4.32}$$

$$(\tilde{K} \tilde{\Psi})(\mu, w_2, \dots, w_N) = \frac{\hbar}{i} \frac{\partial}{\partial \mu} \tilde{\Psi}(\mu, w_2, \dots, w_N) , \tag{4.33}$$

with  $\tilde{\Psi} \in \tilde{\mathcal{H}} \cong L^2(\mathbb{R}^1 \times \mathbb{C}^{N-1})$ . We stress that if we dealt with the restrictions of an arbitrary representation of the Lie algebra  $\mathfrak{g}_T^{mat}$  to the two subalgebras under consideration, then these restrictions would be completely independent. Here, the additional constraints (3.27) – (3.28) imply a relation between the two Lie algebra representations, which enables us to express the wave function  $\tilde{\Psi}$  in terms of the wave function  $\Psi$  *uniquely*. For this purpose, take the polar decomposition of  $w_N$ ,

$$w_N = e^\lambda \xi ,$$

where  $|\xi| = 1$  and  $e^\lambda = |w_N|$ . For any  $\Psi \in \mathcal{H}$ , define the following function  $\Phi \in \tilde{\mathcal{H}}$ :

$$\Phi(\mu, w_2, \dots, w_N) := e^{\mu-\lambda} \Psi(e^\mu \xi^{-1}, w_2 \xi^{-1}, \dots, w_{N-1} \xi^{-1}, \lambda) . \quad (4.34)$$

It is easy to check that this formula defines an isometric isomorphism from  $\mathcal{H}$  to  $\tilde{\mathcal{H}}$ . (The factor in front of  $\Psi$  is necessary to convert the radial measure  $|z_1|d|z_1|$  coming from the two dimensional measure  $dz_1 d\bar{z}_1$  into the measure  $d\mu$  and the measure  $d\lambda$  into the radial measure  $|w_N|d|w_N|$ , coming from the two dimensional measure  $dw_N d\bar{w}_N$ .) Hence, the transformation from  $\Phi$  to  $\tilde{\Psi}$  must be unitary, too. We are going to prove that this transformation is of a special type, consisting in multiplication by a phase factor only.

**Lemma 4.4.** *Given a faithful, irreducible, positive and non-degenerate representation of  $\mathcal{O}_T^{mat}$ , the wave functions  $\tilde{\Psi}$  and  $\Phi$  differ by a phase factor depending only upon the phase  $\xi$  of  $w_N$ . More precisely, for any quantum state  $\Psi$  we have:*

$$\tilde{\Psi} = \xi^{\frac{Q}{e}} \Phi , \quad (4.35)$$

where  $Q$  is the total charge carried by the representation under consideration.

**Proof:** For  $k = 2, \dots, N-1$ , we have in the first representation:

$$R\bar{z}_1 z_k = Q_1^* Q_k = \hat{\mathcal{L}}_{1N} \hat{\mathcal{L}}_{Nk} = \hat{\mathcal{L}}_{NN} \hat{\mathcal{L}}_{1k} = R\tilde{Q}_k , \quad (4.36)$$

which, due to positivity of  $R$  implies  $\tilde{Q}_k = \bar{z}_1 z_k$ . Thus,  $\tilde{Q}_k$  acts on  $\Phi$  in the same way as on  $\tilde{\Psi}$ :

$$\tilde{Q}_k = \bar{z}_1 z_k = e^\mu \cdot \xi \cdot w_k \cdot \xi^{-1} = e^\mu w_k .$$

For  $k = N$ , this also holds true:

$$\tilde{Q}_N = \hat{\mathcal{L}}_{1N} = \left( \hat{\mathcal{L}}_{N1} \right)^* = Q_1^* = \bar{z}_1 e^\lambda = e^\mu \xi e^\lambda = e^\mu w_N .$$

Moreover, in the first representation we have :

$$R\tilde{R} = R\hat{\mathcal{L}}_{11} = \hat{\mathcal{L}}_{NN}\hat{\mathcal{L}}_{11} = \left( \hat{\mathcal{L}}_{N1} \right)^* \hat{\mathcal{L}}_{N1} = Q_1^* Q_1 = e^{2\lambda} \bar{z}_1 z_1 = R e^{2\mu} ,$$

and, consequently,  $\tilde{R} = e^{2\mu}$ . We conclude that the “position operators”  $(\tilde{R}, \tilde{Q}_k)$  of the second representation act in the same way on  $\Phi$  as on  $\tilde{\Psi}$ . This implies that  $\Phi$  and  $\tilde{\Psi}$  may differ only by a phase factor,

$$\tilde{\Psi} = \exp(if)\Phi , \quad (4.37)$$

where  $f$  is real. Next, we prove that the phase factor  $f$  does not depend upon variables  $\mu$ ,  $|w_N|$  and  $w_k$ , for  $k = 2, \dots, N-1$ . For this purpose, observe that in the first representation we have

$$R\hat{\mathcal{M}}_{11} = \hat{\mathcal{L}}_{NN}\hat{\mathcal{M}}_{11} = \left( \hat{\mathcal{L}}_{N1} \right)^* \hat{\mathcal{M}}_{N1} = Q_1^* P_1 = R\bar{z}_1 \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_1} ,$$

and, consequently

$$\hat{\mathcal{M}}_{11} = \bar{z}_1 \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_1} . \quad (4.38)$$

The real part of this operator applied to  $\Phi$  gives

$$\tilde{K} = \frac{\hbar}{i} \frac{\partial}{\partial \mu} , \quad (4.39)$$

which coincides with  $\tilde{K}$  acting on  $\tilde{\Psi}$ . We conclude that the phase factor cannot depend upon  $\mu$ . To prove that it does not depend upon  $w_k$ , for  $k = 2, \dots, N-1$ , neither, observe that in the first representation we have

$$R\tilde{P}_k = \hat{\mathcal{L}}_{NN}\hat{\mathcal{M}}_{1k} = \left( \hat{\mathcal{L}}_{N1} \right)^* \hat{\mathcal{M}}_{Nk} = Q_1^* P_k = R\bar{z}_1 \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} ,$$

and, consequently,

$$\tilde{P}_k = \bar{z}_1 \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} . \quad (4.40)$$

When applied to  $\Phi$  it yields

$$\tilde{P}_k = e^\mu \xi \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{z}_k} = e^\mu \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{w}_k}, \quad (4.41)$$

exactly as in the  $\tilde{\Psi}$ -representation. This implies that  $f$  does not depend upon  $w_k$ . Finally, to prove the independence of  $f$  from  $|w_N|$  observe that in the second representation holds

$$\tilde{R}\hat{\mathcal{M}}_{NN} = \hat{\mathcal{L}}_{11}\hat{\mathcal{M}}_{NN} = \left(\hat{\mathcal{L}}_{1N}\right)^* \hat{\mathcal{M}}_{1N} = \tilde{Q}_N^* \tilde{P}_N = \tilde{R}\bar{w}_N \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{w}_k},$$

and, consequently,

$$\hat{\mathcal{M}}_{NN} = \bar{w}_N \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{w}_N}. \quad (4.42)$$

The real part of this operator applied to  $\tilde{\Psi}$  yields

$$\begin{aligned} K &= \text{Re}(\hat{\mathcal{M}}_{NN}) = \text{Re}\left(\bar{w}_N \cdot \frac{2\hbar}{i} \frac{\partial}{\partial \bar{w}_N}\right) = \frac{\hbar}{i} \frac{\partial}{\partial |w_N|} |w_N| \\ &= \frac{1}{|w_N|} \frac{\hbar}{i} \frac{\partial}{\partial \ln |w_N|} |w_N|. \end{aligned} \quad (4.43)$$

On the other hand, in the first representation  $K$  acts on  $\Psi$  as  $\frac{\hbar}{i} \frac{\partial}{\partial \lambda}$ . Transformed to the representation in terms of  $\Phi$  it gives precisely (4.43). This implies that the phase factor  $f$  may depend only upon the phase of  $w_N$ ,  $f = f(\xi)$ . To prove the specific form of  $f$  in equation (4.35), we differentiate  $\Phi$  in formula (4.34) and get

$$\frac{e}{i} \frac{\partial}{\partial \arg w_N} \Phi = \frac{e}{i} \frac{\partial}{\partial \arg \xi} \Phi = -\frac{e}{i} \sum_{k=1}^{N-1} \frac{\partial}{\partial \arg z_k} \Psi. \quad (4.44)$$

Thus, by Lemma 4.2 we have:

$$\frac{e}{i} \frac{\partial}{\partial \arg w_N} \Phi = -\sum_{k=1}^{N-1} \hat{q}_{x_k} \Phi = (\hat{q}_{x_N} - Q) \Phi. \quad (4.45)$$

On the other hand, in the second representation we have:

$$\hat{q}_{x_N} \tilde{\Psi} = \frac{e}{i} \frac{\partial}{\partial \arg w_N} \tilde{\Psi} = \frac{e}{i} \frac{\partial}{\partial \arg \xi} (\exp(if) \Phi) \quad (4.46)$$

$$= \exp(if) \left( e \frac{\partial f}{\partial \arg \xi} \Phi + (\hat{q}_{x_N} - Q) \Phi \right), \quad (4.47)$$

which implies

$$\frac{\partial f}{\partial \arg \xi} = \frac{Q}{e} . \quad (4.48)$$

□

As a consequence we get the following

**Theorem 4.5.** *Given a value  $Q \in e\mathbb{N}$  of the total electric charge carried by the matter field, there exists exactly one faithful, irreducible, positive and non-degenerate representation of the algebra  $\mathcal{O}_T^{mat}$ , assigned to a given tree. This representation is uniquely generated by its restriction to the Lie subalgebra  $\mathfrak{h}_T^{mat}$ . For different values of  $Q$  these representations are inequivalent. There is no non-degenerate representation of  $\mathcal{O}_T^{mat}$  corresponding to  $Q \notin e\mathbb{N}$ .*

**Proof:** Choosing different points  $x_k$ , together with the corresponding Lie subalgebras  $\mathfrak{h}_T^{mat}(x_k)$ , and transforming between different representations, we may uniquely calculate elements  $\hat{\mathcal{L}}_{kl}$ ,  $\hat{\mathcal{M}}_{kl}$  and  $\hat{\mathcal{N}}_{kl} = \left(\hat{\mathcal{M}}_{lk}\right)^*$ , for  $k \neq l$ , in a single, fixed representation, say the first one, described by the wave function  $\Psi$ . Then, we can represent also all the “ultralocal” observables  $\hat{\mathcal{L}}_{kk}$ ,  $\hat{\mathcal{M}}_{kk}$  and  $\hat{\mathcal{N}}_{kk} = \left(\hat{\mathcal{M}}_{kk}\right)^*$  in this fixed representation. Indeed, in the representation based on  $x_k$  we have  $\hat{\mathcal{L}}_{kk} = R$ , whereas

$$\hat{\mathcal{M}}_{kk} = K + i\hbar \left( \frac{\hat{q}_{x_k}}{e} + \mathbb{1} \right) \quad (4.49)$$

according to formulae (4.4) and (3.24). Finally, we find the representation of the observables  $\hat{\mathcal{R}}_{kl}$  by choosing any  $x_n$ , different from both  $x_k$  and  $x_l$ , and using formula

$$\hat{\mathcal{L}}_{nn} \hat{\mathcal{R}}_{kl} = \hat{\mathcal{N}}_{kn} \hat{\mathcal{M}}_{nl} . \quad (4.50)$$

□

We denote the irreducible representation space corresponding to eigenvalue  $Z$  of the total charge  $\frac{1}{e}\hat{Q}$  by  $\mathcal{H}_T^{mat}(Z)$ .

## 4.4 Charge superselection structure

Now we use the fact that, for a given tree, the observable algebra decomposes,

$$\mathcal{O}(\Lambda) = \mathcal{O}_T^{em} \otimes \mathcal{O}_T^{mat} , \quad (4.51)$$



see (3.38). But the electromagnetic part  $\mathcal{O}_{\mathcal{T}}^{em}$  is generated by a finite number of canonically conjugate pairs. Thus, its integrable representations are (up to unitary equivalence) unique, due to the von Neumann theorem [8]. Thus, the faithful, irreducible and non-degenerate representations of  $\mathcal{O}(\Lambda)$  are labeled by the irreducible charge sectors of the matter field part  $\mathcal{O}_{\mathcal{T}}^{mat}$ , as described in the previous subsection. For a given tree, we get the physical Hilbert space as a direct sum of charge superselection sectors,

$$\mathcal{H}(\Lambda) = \bigoplus_Z \{ \mathcal{H}_{\mathcal{T}}^{em} \otimes \mathcal{H}_{\mathcal{T}}^{mat}(Z) \} . \quad (4.52)$$

Finally, it can be easily shown that a different choice of the tree induces a similar decomposition of  $\mathcal{O}(\Lambda)$  as above and the two decompositions are related to each other via an isomorphism of the corresponding electromagnetic observable algebras.

## 5 Discussion

As we have seen, the fact that the observable algebra is generated by a certain Lie algebra, is extremely helpful for the classification of its irreducible representations. It should be also helpful for constructing the thermodynamic limit, because for taking the limit  $N \rightarrow \infty$  in the generating Lie algebra there seems to exist appropriate mathematical tools for studying the resulting representations, see [20]. However, it is doubtful, whether the classification of charge superselection sectors obtained here will carry over to the thermodynamic limit in a straightforward way. One should rather expect that these considerations will have to be supplemented by a discussion of field dynamics. For this purpose we define the lattice version of the field Hamiltonian (2.3) by the following procedure. In formulae (3.2) – (3.5) we put  $\gamma = (x, x + \hat{k})$  and use the substitution

$$\exp\left(ig \int_x^{x+\hat{k}} \hat{A}\right) \simeq \hat{\mathbb{1}} + iagA_k , \quad (5.53)$$

$$\hat{\phi}_{x+\hat{k}} \simeq \hat{\phi}_x + a\partial_k \hat{\phi}_x . \quad (5.54)$$

Consequently, we obtain the following approximation:

$$|\vec{D}\phi(x)|^2 \simeq \frac{1}{a^2} (\hat{\mathcal{L}}_{xx})^{-1} \sum_{\hat{k}} \left( \hat{\mathcal{L}}_{(x,x+\hat{k})} - \hat{\mathcal{L}}_{xx} \right)^* \left( \hat{\mathcal{L}}_{(x,x+\hat{k})} - \hat{\mathcal{L}}_{xx} \right) . \quad (5.55)$$

This leads to the lattice version of the Hamiltonian (2.3):

$$\hat{H} = \hat{H}_{el-mag} + \hat{H}_{matter} + \hat{H}_{int} , \quad (5.56)$$

where the electromagnetic, matter and interaction parts of the Hamiltonian are given by:

$$\hat{H}_{el-mag} = \frac{a}{2} \sum_{(x, x+\hat{k})} \left( \hat{E}_{x, x+\hat{k}} \right)^2 + \frac{a}{2} \sum_{x; \hat{k}, \hat{l}} \left( \hat{B}_{x; \hat{k}, \hat{l}} \right)^2 , \quad (5.57)$$

$$\hat{H}_{matter} = a^3 \sum_x \left( \frac{1}{2} \hat{\mathcal{R}}_{xx} + V(\hat{\mathcal{L}}_{xx}) \right) , \quad (5.58)$$

$$\hat{H}_{int} = \frac{a}{2} \sum_{x, \hat{k}} (\hat{\mathcal{L}}_{xx})^{-1} \left( \hat{\mathcal{L}}_{(x, x+\hat{k})} - \hat{\mathcal{L}}_{xx} \right)^* \left( \hat{\mathcal{L}}_{(x, x+\hat{k})} - \hat{\mathcal{L}}_{xx} \right) . \quad (5.59)$$

Given a finite lattice, we have thus approximated the field by a quantum-mechanical system whose dynamics is governed by the positively defined Hamiltonian<sup>1</sup> (5.56). Its ground state will be treated as a finite-lattice-approximation of the vacuum. Suppose that this vacuum state has been found. Then, applying the strategy outlined in [10], one should find the vacuum in the thermodynamic limit as a projective limit of the above vacuum states corresponding to finite lattices. Using this vacuum, the physically admissible representations of the observable algebra (in the thermodynamic limit) could be singled out via the GNS-construction. To find the “true” vacuum will be, of course, extremely difficult, but may be one can find an approximation, which is “much better” than the perturbative vacuum. Then one can hope to get deeper insight into nonperturbative aspects of this model. The same remark applies to one- or multi-particle states.

Of course, the ultra-violet limit of the theory (obtained by sending the lattice spacing  $a$  to zero) is, probably, much more difficult to investigate. But, in principle, the strategy outlined in [10] applies also here.

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<sup>1</sup>When considering different boundary conditions, an additional surface term in the Hamiltonian will be necessary.

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## Appendix

### A A Non-integrable Representation Carrying Non-integer Charge

Quantization of charge is due to integrability of the representations of the Lie algebra generating the matter field part of the observable algebra. Below, we construct a weak (non-integrable) representation, which carries non-integer charge. At the same time we construct an interesting example of weakly commuting operators, which do not commute strongly (cf. [13]).

We limit ourselves to a single lattice point ( $N = 1$ ) and multiply the wave function by the phase factor  $\xi^c = \exp(ic \cdot \arg z)$ ,  $c \in \mathbb{N}$ , as in formula (4.35). After this operation the spectrum of the charge operator

$$\hat{j} := \frac{\hat{q}}{e} = \frac{1}{i} \frac{\partial}{\partial \arg z} , \quad (\text{A.1})$$

gets shifted by the value  $c$ . It is interesting that a similar “shift” can be also defined for any real number  $c \in \mathbb{R}$ . Of course, the result of such a “shift” cannot be unitarily equivalent to the original operator  $\hat{j}$ . As will be seen in the sequel, this leads to a simple example of canonical commutation relations, which are fulfilled only in the weak sense: momenta  $\hat{p}_1$  and  $\hat{p}_2$  do not commute strongly or, in other words,  $\hat{P} = \hat{p}_1 + i\hat{p}_2$  is not normal.

Consider, therefore, the Hilbert space  $\mathcal{H} = L^2(\mathbb{C})$  and define the following two groups of unitary transformations:

$$(\hat{U}_1(t)\Psi)(z) = \frac{1}{\varphi_1(z)}(\varphi_1(z + t\hbar)\Psi(z + t\hbar)) , \quad (\text{A.2})$$

$$(\hat{U}_2(s)\Psi)(z) = \frac{1}{\varphi_2(z)}(\varphi_2(z + is\hbar)\Psi(z + is\hbar)) , \quad (\text{A.3})$$

where  $\varphi_1$  (respectively  $\varphi_2$ ) is a phase factor obtained from the multivalued function

$$\varphi(z) := \exp(ic \cdot \arg z) ,$$

by the cut along the real (respectively the imaginary) positive half-axis. Denote by  $\hat{p}_a$ ,  $a = 1, 2$ , the self-adjoint generators of these groups:

$$\hat{U}_a(t) = \exp(itp_a) .$$

These operators may be easily calculated on their common, dense domain  $D$ , consisting of those functions, which are smooth and vanish identically in a neighbourhood of the two cuts. Indeed, for  $\Psi \in D$ , a straightforward calculation gives:

$$(\hat{p}_1\Psi)(z) = \frac{1}{\varphi_1(z)} \frac{\hbar}{i} \frac{\partial}{\partial \operatorname{Re} z} (\varphi_1(z)\Psi(z)) , \quad (\text{A.4})$$

$$(\hat{p}_2\Psi)(z) = \frac{1}{\varphi_2(z)} \frac{\hbar}{i} \frac{\partial}{\partial \operatorname{Im} z} (\varphi_2(z)\Psi(z)) . \quad (\text{A.5})$$

This immediately implies weak commutation:

$$\hat{p}_1\hat{p}_2\Psi = \hat{p}_2\hat{p}_1\Psi .$$

However, the commutation relations are not satisfied in the strong sense, because the groups  $\hat{U}_1$  and  $\hat{U}_2$  do not commute. Indeed, take any smooth function  $\Psi$  with support contained in the square  $S = \{-2 \leq \operatorname{Re} z, \operatorname{Im} z \leq -1\}$ . It is easy to check that we have

$$\hat{U}_2(3)\hat{U}_1(3)\Psi = \exp(2\pi ic)\hat{U}_1(3)\hat{U}_2(3)\Psi , \quad (\text{A.6})$$

and, whence, we obtain the strong commutation only for  $c \in \mathbb{N}$ . A simple geometric interpretation of this result consists in interpreting the multivalued function  $\Phi = \varphi \cdot \Psi$  as a function defined on the Riemann surface  $\mathcal{R}$  of the logarithm,

$$\mathcal{R} = \{(|z|, \arg z) : |z| \in \mathbb{R}_+, \arg z \in \mathbb{R}^1\} ,$$

and satisfying the condition:

$$\Phi(|z|, \arg z + 2\pi) = \exp(2\pi ic)\Phi(|z|, \arg z) .$$

These functions form a Hilbert space  $\tilde{\mathcal{H}}$  with scalar product defined by integration over any closed set  $D \subset \mathcal{R}$ , which covers almost the whole space  $\mathbb{C}$  only once. The mapping  $\mathcal{H} \ni \Psi \rightarrow \Phi := \varphi\Psi \in \tilde{\mathcal{H}}$  is an isomorphism of Hilbert spaces. When interpreted in terms of  $\tilde{\mathcal{H}}$ , the operators  $\hat{U}_2(3)\hat{U}_1(3)$

and  $\hat{U}_1(3)\hat{U}_2(3)$  describe dragging  $\Phi$  along the spiral surface  $\mathcal{R}$  with respect to the two opposite helicities.

Now, consider the operator  $\hat{X} = \hat{x}_1 + i\hat{x}_2$ , where  $\hat{x}_1$  (resp.  $\hat{x}_2$ ) is the operator of multiplication by the real (resp. the imaginary) part of  $z$ . Formulae (A.2) and (A.3) imply that  $\hat{p}_1$  and  $\hat{x}_1$  satisfy the canonical commutation relations in the strong sense. The same is true for  $\hat{p}_2$  and the operator  $\hat{x}_2$ . Hence, formally  $\hat{P} = \hat{p}_1 + i\hat{p}_2$  and  $\hat{X} = \hat{x}_1 + i\hat{x}_2$  satisfy the commutation relations (2.11) for quantum mechanics of two degrees of freedom in the weak sense. But, these relations are not satisfied strongly, because the real and the imaginary parts of  $\hat{P}$ , although being self-adjoint, do not commute strongly. It is easy to show that the formal “charge operator” built from them,

$$\hat{q} := \text{Im} \left( \hat{X}^* \hat{P} \right) ,$$

is self-adjoint and has spectrum shifted by  $c$  with respect to the ordinary spectrum:

$$\text{Sp } \hat{q} = \{n + c : n \in \mathbb{N}\} .$$

We conclude that the operators  $\hat{\mathcal{L}} = \hat{X}^* \hat{X}$ ,  $\hat{\mathcal{M}} = \hat{X}^* \hat{P}$ ,  $\hat{\mathcal{N}} = \hat{P}^* \hat{X}$  and  $\hat{\mathcal{R}} = \hat{P}^* \hat{P}$  satisfy the axioms of the observable algebra weakly, but do not provide its strong (integrable) realization.

## B A Unified Description of the Bosonic and the Fermionic Case

Consider at each lattice point  $x_k$  the following bosonic annihilation and creation operators:

$$a_k := \frac{1}{\sqrt{2}} \left( \text{Re} \hat{\phi}_k + \frac{i}{\hbar} \text{Re} \hat{\pi}_k \right) , \quad (\text{B.1})$$

$$b_k := \frac{1}{\sqrt{2}} \left( \text{Im} \hat{\phi}_k + \frac{i}{\hbar} \text{Im} \hat{\pi}_k \right) , \quad (\text{B.2})$$

$$a_k^* := \frac{1}{\sqrt{2}} \left( \text{Re} \hat{\phi}_k - \frac{i}{\hbar} \text{Re} \hat{\pi}_k \right) , \quad (\text{B.3})$$

$$b_k^* := \frac{1}{\sqrt{2}} \left( \text{Im} \hat{\phi}_k - \frac{i}{\hbar} \text{Im} \hat{\pi}_k \right) . \quad (\text{B.4})$$

Then, take their combinations:

$$\chi_k := \frac{1}{\sqrt{2}} (a_k + ib_k) = \frac{1}{2} \left( \hat{\phi}_k + \frac{i}{\hbar} \hat{\pi}_k \right) , \quad (\text{B.5})$$

$$\varphi_k^* := \frac{1}{\sqrt{2}} (a_k^* + ib_k^*) = \frac{1}{2} \left( \hat{\phi}_k - \frac{i}{\hbar} \hat{\pi}_k \right) , \quad (\text{B.6})$$

$$\chi_k^* := \frac{1}{\sqrt{2}} (a_k^* - ib_k^*) = \frac{1}{2} \left( \hat{\phi}_k^* - \frac{i}{\hbar} \hat{\pi}_k^* \right) , \quad (\text{B.7})$$

$$\varphi_k := \frac{1}{\sqrt{2}} (a_k - ib_k) = \frac{1}{2} \left( \hat{\phi}_k^* + \frac{i}{\hbar} \hat{\pi}_k^* \right) . \quad (\text{B.8})$$

The entire information about the field algebra may be encoded in the following objects:

$$\psi_k = \begin{pmatrix} \chi_k \\ \varphi_k^* \end{pmatrix} , \quad \psi_k^* = \begin{pmatrix} \chi_k^* \\ \varphi_k \end{pmatrix} . \quad (\text{B.9})$$

In [10] and [11] we considered spinorial QED, where the matter field was described by a similar structure. The only difference was, that there both  $\chi$  and  $\varphi$  carried an additional spinorial index  $K = 1, 2$ , (but this only multiplies the number of degrees of freedom). These objects fulfill the canonical (anti)-commutation relations:

$$[\chi_k, \chi_l^*]_{\mp} = [\varphi_k, \varphi_l^*]_{\mp} = \delta_{kl} \hat{\mathbb{1}} , \quad (\text{B.10})$$

where the upper sign always applies to the bosonic and the lower to the fermionic case. The remaining (anti)-commutators vanish. Due to (2.2), under a gauge transformation the field  $\psi$  is multiplied by  $e^{-ig\lambda(x)}$ , whereas  $\psi^*$  is multiplied by  $e^{ig\lambda(x)}$ . Hence, as the observable algebra generators we may use the following gauge invariant combinations:

$$l_{ij} := \pm \varphi_i \exp(ig \int_{\gamma} \hat{A}) \varphi_j^* , \quad (\text{B.11})$$

$$r_{ij} := -\chi_i^* \exp(ig \int_{\gamma} \hat{A}) \chi_j , \quad (\text{B.12})$$

$$m_{ij} := i^{\epsilon} \varphi_i \exp(ig \int_{\gamma} \hat{A}) \chi_j , \quad (\text{B.13})$$

$$n_{ij} := i^{\epsilon} \chi_i^* \exp(ig \int_{\gamma} \hat{A}) \varphi_j^* . \quad (\text{B.14})$$

Here, by  $\epsilon$  we denote the number  $\epsilon := \frac{1 \mp 1}{2}$ , which equals 0 for the bosonic and 1 for the fermionic case. It is easy to check that these generators fulfill

the following *universal* commutation relations, the same for bosons as for fermions:

$$[l_{ij}, l_{kl}] = -\delta_{kj}l_{il} + \delta_{il}l_{kj} , \quad (\text{B.15})$$

$$[l_{ij}, m_{kl}] = -\delta_{kj}m_{il} , \quad (\text{B.16})$$

$$[l_{ij}, n_{kl}] = \delta_{il}n_{kj} , \quad (\text{B.17})$$

$$[l_{ij}, r_{kl}] = 0 , \quad (\text{B.18})$$

$$[m_{ij}, m_{kl}] = 0 , \quad (\text{B.19})$$

$$[m_{ij}, n_{kl}] = \delta_{kj}l_{il} - \delta_{il}r_{kj} , \quad (\text{B.20})$$

$$[m_{ij}, r_{kl}] = -\delta_{kj}m_{il} , \quad (\text{B.21})$$

$$[n_{ij}, n_{kl}] = 0 , \quad (\text{B.22})$$

$$[n_{ij}, r_{kl}] = \delta_{il}n_{kj} , \quad (\text{B.23})$$

$$[r_{ij}, r_{kl}] = -\delta_{kj}r_{il} + \delta_{il}r_{kj} . \quad (\text{B.24})$$

But the conjugation is different in both cases:

$$l_{ij}^* = l_{ji} , \quad r_{ij}^* = r_{ji} , \quad m_{ij}^* = \pm n_{ji} . \quad (\text{B.25})$$

It is easy to check that, under the matrix presentation (3.51) and (3.52), we have:

$$l_{kl} := \begin{pmatrix} iE_{kl} & 0 \\ 0 & 0 \end{pmatrix} \quad m_{kl} := i^\epsilon \begin{pmatrix} 0 & E_{kl} \\ 0 & 0 \end{pmatrix} \quad (\text{B.26})$$

$$n_{kl} := i^\epsilon \begin{pmatrix} 0 & 0 \\ E_{kl} & 0 \end{pmatrix} \quad r_{kl} := \begin{pmatrix} 0 & 0 \\ 0 & iE_{kl} \end{pmatrix} . \quad (\text{B.27})$$

In the bosonic case, the conjugation (B.25) implies (3.50) and, therefore, the algebra of self-adjoint observables is generated by  $u(N, N)$ . In the fermionic case, (B.25) implies  $A^* = -A^\dagger$  and the algebra of self-adjoint observables is generated by two copies of  $u(2N)$ , corresponding to two values of the spinorial index  $K$ . In both cases, the following formula for the total charge holds:

$$\frac{1}{e} Q = \sum l_{ii} + \sum r_{ii} - \mathbb{1} . \quad (\text{B.28})$$

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