Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Neuman and second boundary value problems for Hessian and Gauss curvature flows

by

Knut Smoczyk and Oliver Schnürer

Preprint no.: 39 2002

NEUMANN AND SECOND BOUNDARY VALUE PROBLEMS FOR HESSIAN AND GAUSS CURVATURE FLOWS

OLIVER C. SCHNÜRER AND KNUT SMOCZYK

Abstract. We consider the flow of a strictly convex hypersurface driven by the Gauß curvature. For the Neumann boundary value problem and for the second boundary value problem we show that such a flow exists for all times and converges eventually to a solution of the prescribed Gauß curvature equation. We also discuss oblique boundary value problems and flows for Hessian equations.

Résumé. Nous considérons le flot d'une hypersurface strictement convexe piloté par la courbure de Gauß. Pour le problème à bord de Neumann et pour le deuxième problème à bord nous montrons qu'un tel flot existe pour tous les temps et converge ultimement vers une solution de l'équation de courbure de Gauß prescrite. Nous discutons aussi de problèmes à condition de bord oblique et de flots pour des équations hessiennes.

1. Introduction

This paper concerns – in its first part – the deformation of convex graphs over bounded, convex domains $\Omega\subset\mathbb{R}^n$, $n\ge 2$, with smooth boundary $\partial\Omega$ to convex graphs with prescribed Gauß curvature and Neumann boundary condition. More precisely, let u be a smooth strictly convex solution of

$$\begin{cases} \dot u = \Phi(\log\det(u_{ij})-\log f(x,u,Du)) & \text{in } \Omega\times[0,T),\\ u_\nu = \varphi(x,u) & \text{on } \partial\Omega\times[0,T),\\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

for a maximal time interval $[0,T)$, where $f,\varphi:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ are smooth functions, ν denotes the inner unit normal to $\partial\Omega$ and $u_0:\overline{\Omega}\to\mathbb{R}$, the initial value, is a smooth strictly convex function. Here $\Phi:\mathbb{R}\to\mathbb{R}$ is a smooth strictly increasing and concave function that vanishes at zero, i.e. Φ satisfies

$$\Phi(0)=0,\quad \Phi'>0,\quad \Phi''\le 0. \tag{1.2}$$

In the sequel we assume for simplicity $0\in\Omega$.

2000 *Mathematics Subject Classification.* Primary 53C44; Secondary 35K20, 53C42.

To guarantee shorttime existence for (1.1) and convergence to smooth graphs with prescribed Gauß curvature we have to assume several structure conditions. These are

$$\varphi_z \equiv \frac{\partial \varphi}{\partial z} \ge c_\varphi > 0, \tag{1.3}$$

$$f > 0 \quad \text{and} \quad f_z \ge 0. \tag{1.4}$$

Moreover, we will always either assume

$$\frac{f_z}{f} \ge c_f > 0 \tag{1.5}$$

or

$$\Phi\left(\log\det(u_0)_{ij} - \log f(x, u_0, Du_0)\right) \ge 0. \tag{1.6}$$

To guarantee smoothness up to $t = 0$ it is necessary to assume the following compatibility conditions to be fulfilled on the boundary $\partial\Omega$ for any $m \ge 0$

$$\left(\frac{d}{dt}\right)^m \left(\nu^i u_i - \varphi(x, u)\right)\big|_{t=0} = 0, \tag{1.7}$$

where time derivatives of u, u_i, $\ldots$ have to be substituted inductively by using $\dot{u} = \Phi$ and $u|_{t=0} = u_0$. Applying Theorem 5.3, p. 320 [12] and the implicit function theorem, we obtain smooth shorttime existence up to $t = 0$, see also [7].

During the flow, the smoothness of a solution guarantees that (1.7) is satisfied for any $m \ge 0$. So it is possible to extend a solution of the flow equation on a time interval $[0, T)$ to $[0, T]$ provided there are sufficient a priori estimates and then to $[0, T + \varepsilon)$ for a small $\varepsilon > 0$. In this way we obtain existence for all $t \ge 0$ from the a priori estimates. The same procedure works also for the other boundary conditions considered in this paper.

The main theorem for Neumann boundary conditions states

Theorem 1.1. *Assume that Ω is a bounded, strictly convex domain in $\mathbb{R}^n$, $n \ge 2$, with smooth boundary. Let $f, \varphi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, be smooth functions that satisfy (1.3)–(1.4). Let u_0 be a smooth, convex function that satisfies the compatibility conditions (1.7). Moreover, we assume that one of the conditions (1.5) or (1.6) is fulfilled. Then a smooth solution of (1.1) exists for all $t \ge 0$. As $t \to \infty$, the functions $u|_t$ smoothly converge to a smooth limit function u^∞ such that the graph of u^∞ satisfies the Neumann boundary value problem*

$$\begin{cases} \det(u^\infty_{ij}) = f(x, u^\infty, Du^\infty) & \text{in } \Omega, \\ u^\infty_\nu(x) = \varphi(x, u^\infty) & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where ν is the inward pointing unit normal of $\partial\Omega$. The rate of convergence is exponential provided (1.5) is satisfied.

When we assume condition (1.5), we obtain – by using (1.7) only for $m = 0$ – a solution of (1.1) which is smooth only for $t > 0$ and the rate of convergence is exponential only in time intervals $[\varepsilon, \infty)$, $\varepsilon > 0$.

In the case when condition (1.6) holds, we need only (1.7) for $m = 0, 1$ to obtain a solution of (1.1). Here u approaches u_0 for $t \to 0$ only up to its fourth derivatives, where time derivatives have to be counted twice.

In both cases, all the other claims of Theorem 1.1 remain unchanged.

Remark 1.2. If we consider for a smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ the evolution equation

$$\dot{u} = \Psi(\log \det u_{ij}, \log f)$$

and assume natural structure conditions, i. e. concavity of Ψ, $\Psi_1 > 0$ and $\Psi(x, x) = 0 \quad \forall\, x$, then we prove in Lemma C.1 that there exists $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi' > 0$, $\Phi'' \le 0$ such that Ψ has the following simpler form

$$\Psi(x, y) = \Phi(x - y).$$

Example 1.3. For $\Phi(x) = x$, our ansatz yields the logarithmic Gauß curvature flow

$$\dot{u} = \log \det u_{ij} - \log g(x, u) - \tfrac{n+2}{2} \log\left(1 + |Du|^2\right),$$

more precisely, the "vertical" velocity equals the difference of the logarithms of the actual and the prescribed Gauß curvature. Another interesting example is given by $\Phi(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, which gives the flow equation

$$\dot{u} = 1 - \left(\frac{f(x, u, Du)}{\det u_{ij}}\right)^{\lambda}.$$

In a second part, we consider the second boundary value problem for Hessian flow equations, more precisely, we solve the initial value problem

$$\begin{cases} \dot{u} &= \log F(D^2 u) - \log g(x, u, Du) & \text{in } \Omega \times [0, T), \\ Du(\Omega) &= \Omega^*, & \\ u|_{t=0} &= u_0 & \text{in } \Omega, \end{cases} \tag{1.9}$$

on a maximal time interval $[0, T)$, $T > 0$. We assume that $\Omega, \Omega^* \subset \mathbb{R}^n$, $n \ge 2$, are smooth strictly convex domains, $u_0 : \overline{\Omega} \to \mathbb{R}$ is a smooth strictly convex function, $Du_0(\Omega) = \Omega^*$ (= 0-th compatibility condition), $g : \overline{\Omega} \times \mathbb{R} \times \overline{\Omega^*} \to \mathbb{R}$ is a smooth positive function such that $g_z > 0$. F is a Hessian function of the class $\left(\tilde{K}^\star\right)$, for a precise definition we refer to Definition 5.1. Here we remark only, that the class of Hessian functions considered includes especially $F(D^2 u) = \det D^2 u$. We will show that a smooth strictly convex solution of (1.9) exists for all times, i. e. $T = \infty$, and converges smoothly to a solution u^∞ of the elliptic second boundary value problem

$$\begin{cases} F(D^2 u^\infty) &= g(x, u^\infty, Du^\infty) & \text{in } \Omega, \\ Du^\infty(\Omega) &= \Omega^*, & \end{cases} \tag{1.10}$$

when some structure conditions are fulfilled. The asymptotic behavior of g is given by

$$\begin{aligned} g(x,z,p) &\to \infty \text{ as } z \to \infty, \\ g(x,z,p) &\to 0 \quad \text{as } z \to -\infty, \end{aligned} \tag{1.11}$$

uniformly for $(x,p) \in \overline{\Omega} \times \overline{\Omega^*}$. Furthermore we will always assume that there holds either

$$\frac{g_z}{g} \geq c_g > 0 \tag{1.12}$$

or

$$\begin{cases} 0 \;\leq\; F(D^2 u_0) - \log g(x, u_0, Du_0) & \text{in } \Omega, \\ \quad\;\; \text{1st compatibility condition} & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where the inequality means that u_0 is a subsolution. We remark that the boundary condition $Du(\Omega) = \Omega^*$ is equivalent to $h(Du) = 0$ on $\partial\Omega$ for smooth strictly convex functions u, where $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth strictly concave function such that $h|_{\partial\Omega^*} = 0$ and $|\nabla h| = 1$ on $\partial\Omega^*$. For the second boundary value problem the compatibility conditions read as follows

$$\left(\frac{d}{dt}\right)^m h(Du)|_{t=0} = 0, \quad m \in \mathbb{N}, \tag{1.14}$$

where derivatives of u have to be replaced as above.

For the second boundary value problem, we obtain the following main theorem.

Theorem 1.4. *Assume that Ω, Ω^*, g, u_0 and F are as assumed above and either (i) (1.12) or (ii) (1.13) are satisfied. Then there exists a smooth strictly convex function $u : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ of (1.9), i. e. $T = \infty$, and u converges smoothly to a solution u^∞ of (1.10) as $t \to \infty$. Furthermore, u is continuous up to its (i) second/(ii) fourth derivatives at $t = 0$, where time derivatives have to be counted twice, and (i) gives exponential convergence $u \to u^\infty$ for $t \in [\varepsilon, \infty)$, $\varepsilon > 0$. If (1.14) is fulfilled for all $m \in \mathbb{N}$, then u is smooth in $[0, T)$ and (i) gives exponential convergence to u^∞ in $[0, \infty)$.*

This result extends to Hessian quotient equations as follows.

Theorem 1.5. *Theorem 1.4 holds also for $F = S_{n,k}$, $1 \leq k \leq n-1$, when g happens to be independent of the gradient of u, where $S_{n,k}(D^2u)$ is the quotient of the n-th and the k-th elementary symmetric polynomial of the eigenvalues of D^2u.*

Notation 1.6. Indices denote partial derivatives or vector components and are lifted and lowered with respect to δ_{ij} except for $\left(u^{ij}\right)$ that denotes the inverse of (u_{ij}). Indices z and p_i denote partial derivatives with respect to the argument used for the function u and for its gradient, respectively, dots refer to time derivatives. We use the Einstein summation convention and sum over repeated Latin indices from 1 to n. For a vector ν we use

$u_\nu \equiv u_i \nu^i$ with obvious generalizations to other quantities. We use c to denote a positive and already estimated constant. Its value may change from line to line if necessary. We point out that the inequalities remain valid when c is enlarged. A function $u : \Omega \times [0, T) \to \mathbb{R}$ is called (strictly) convex, if $u(\cdot, t)$ is (strictly) convex for every time $t \in [0, T)$. A function $u : \Omega \to \mathbb{R}$ is called strictly convex, if the eigenvalues of its Hessian are positive. This definition extends to hypersurfaces and sets by using their principal curvatures. Finally, we use

$$\hat{f} = \log f$$

to denote the logarithm of a function f.

We briefly discuss the relation of our result with the existing literature. In [6] smooth, compact, strictly convex and rotationally symmetric hypersurfaces in $\mathbb{R}^3$ have been deformed by its Gauß curvature to round points. The Gauß curvature flow

$$\frac{d}{dt} F = -K\nu$$

for smooth embeddings F of hyperspheres in $\mathbb{R}^{n+1}$ has been the subject in [1]. For the n-th root of K this flow has been considered in [4]. In [3, 8] the authors use flow equations to prove existence theorems for closed hypersurfaces of prescribed curvature. For Gauß curvature flows strict convexity is an essential assumption because then the flow becomes strictly parabolic. In addition the degenerate Gauß curvature flow with flat sides has been investigated in [5]. There are also several papers about curvature flows with Dirichlet boundary condition, we only mention [11]. The elliptic version of our flow equations (1.1), (1.9) has been explored in [14, 17, 18, 19] by using the continuity method, see also [16] for a related problem. Some of the techniques used there will be applied in our paper as well.

The organization of our paper is as follows: In the first part, we study flow equations subject to prescribed Neumann boundary values. In section 2 we prove uniform estimates for $|\dot{u}|$. This will be used in section 3 to derive C^0-estimates. C^1-estimates then follow from [14]. As a consequence we will obtain a uniform positive lower bound for $\det u_{ij}$. In section 4 we derive C^2-estimates and in section 10 we mention how to obtain Hölder regularity for the second derivatives of u and prove Theorem 1.1. In a second part we consider the second boundary value problem. In section 5 we introduce Hessian functions and a dual problem, next, we prove the strict obliqueness of our boundary condition. After the estimates for $\dot{u}$ and u in section 7, we give a quantitative version of our obliqueness result. In section 9 we establish C^2-estimates and in section 10 we prove Theorem 1.4. As far as the second boundary value problem is concerned, we will use methods of [17, 19] without mentioning this explicitly there. In the appendix we state generalizations to oblique boundary value problems for Hessian equations and indicate how to obtain the result for Hessian quotient equations. We remark that our results are parabolic versions of [14, 17, 18], so our results can be considered as alternative existence proofs using parabolic methods.

This paper has been finished as the second author visited the program "Nonlinear Partial Differential Equations" of the Isaac Newton Institute for Mathematical Sciences, Cambridge, in 2001. He wants to thank the organizers for their invitation to this wonderful and very stimulating location. Both authors thank Claus Gerhardt for useful suggestions and discussions. In addition, the authors are indebted to Jürgen Jost for his support at the Max Planck Institute for Mathematics in the Sciences where part of this paper has been written.

2. $\dot{u}$-estimates

For a constant λ we define the function

$$r := e^{\lambda t}(\dot{u})^2.$$

An easy computation shows that (1.1) implies the following evolution equation for r

$$\dot{r} = \Phi' u^{ij} r_{ij} - 2e^{\lambda t}\Phi' u^{ij}\dot{u}_i\dot{u}_j - \Phi'\frac{f_{p_i}}{f}r_i + \left(\lambda - 2\Phi'\frac{f_z}{f}\right)r. \tag{2.1}$$

Lemma 2.1. *As long as a smooth convex solution of (1.1) exists we obtain the estimate*

$$\min\left\{\min_{t=0}\dot{u}, 0\right\} \le \dot{u} \le \max\left\{\max_{t=0}\dot{u}, 0\right\}.$$

Proof. If $(\dot{u})^2$ admits a positive local maximum in $x \in \partial\Omega$ for a positive time, then we differentiate the Neumann boundary condition and obtain from (1.3)

$$\left((\dot{u})^2\right)_\nu = 2(\dot{u})^2\varphi_z > 0$$

which contradicts the maximality of $(\dot{u})^2$ at x. Now we choose $\lambda = 0$ in (2.1) and get

$$\frac{d}{dt}(\dot{u})^2 \le \Phi' u^{ij}\left((\dot{u})^2\right)_{ij} - \Phi'\frac{f_{p_i}}{f}\left((\dot{u})^2\right)_i - 2\Phi'\frac{f_z}{f}(\dot{u})^2.$$

So we obtain from (1.2) and (1.4) that a positive increasing local maximum of $(\dot{u})^2$ on $\overline{\Omega} \times [0, t_0]$ cannot occur at an interior point of Ω for any time $0 < t_0 < T$. $\square$

Corollary 2.2. *As long as a smooth convex solution of (1.1) exists we get a positive lower bound for Φ', $\frac{1}{c_\Phi} > \Phi' > c_\Phi > 0$.*

Proof. This follows immediately as Lemma 2.1, $\Phi(0) = 0$ and the strict monotonicity of Φ give a bound for the argument of Φ. □

Lemma 2.3. *As long as a smooth convex solution of (1.1) exists we obtain the estimate*

$$\min\left\{\min_{t=0} \dot{u}, 0\right\} \leq \dot{u}e^{\lambda t} \leq \max\left\{\max_{t=0} \dot{u}, 0\right\}$$

for $\lambda \leq c_f c_\Phi$ provided (1.5) is fulfilled.

Proof. This statement follows from Corollary 2.2 and a proof similar to the proof of Lemma 2.1. □

Lemma 2.4. *A solution of our flow (1.1) satisfies $\dot{u} > 0$ or equivalently $\Phi > 0$ for $t > 0$ if $0 \not\equiv \dot{u} \geq 0$ for $t = 0$.*

Proof. Differentiating the flow equation yields

$$\ddot{u} = \Phi' u^{ij} \dot{u}_{ij} - \Phi' \left(\hat{f}_z \dot{u} + \hat{f}_{p_i} \dot{u}_i\right), \tag{2.2}$$

thus

$$\frac{d}{dt}\left(\dot{u}e^{\lambda t}\right) = \Phi' u^{ij} \left(\dot{u}e^{\lambda t}\right)_{ij} - \Phi' \left(\hat{f}_z \dot{u}e^{\lambda t} + \hat{f}_{p_i} \left(\dot{u}e^{\lambda t}\right)_i\right) + \lambda \dot{u}e^{\lambda t}. \tag{2.3}$$

We fix $t_0 > 0$ and a constant $\lambda > 0$ such that $\lambda > \Phi' \hat{f}_z$ for $(x,t) \in \overline{\Omega} \times [0, t_0]$. From (2.3) and the strong parabolic maximum principle we see that $\dot{u}e^{\lambda t}$ has to vanish identically if it vanishes in $(x_0, t) \in \Omega \times (0, t_0)$, contradicting $\dot{u} \not\equiv 0$ for $t = 0$. If $\dot{u}e^{\lambda t} = 0$ for $x_0 \in \partial\Omega$ the Neumann boundary condition implies

$$\left(\dot{u}e^{\lambda t}\right)_\nu = \varphi_z \left(\dot{u}e^{\lambda t}\right) = 0,$$

but this is impossible in view of the Hopf lemma applied to (2.3) because $\lambda > \Phi' \hat{f}_z$. □

Remark 2.5. The constant λ in the previous proof depends on t_0. It can be chosen independent of t_0, if $\Phi' \hat{f}_z$ is uniformly bounded above and this is true, if u is bounded in C^1.

3. C^0- and C^1-estimates

Remark 3.1. The strict convexity of u and the fact that $\varphi(\cdot, z) \to \infty$ uniformly as $z \to \infty$ imply that u is uniformly a priori bounded from above as $u_\nu = \varphi(x, u)$ on $\partial\Omega$.

Lemma 3.2. *Under the assumptions of Lemma 2.3 we have the following lower bound for u*

$$u \geq \min_{t=0} u + \frac{1}{\lambda} \min\left\{\min_{t=0} \dot{u}, 0\right\}$$

for all $0 < \lambda \le c_\Phi c_f$.

Proof. This easily follows from Lemma 2.3

$$\begin{aligned} u(x,t) &= u(x,0) + \int\limits_0^t \dot{u}(x,\tau)d\tau \\ &\ge u(x,0) + \min\left\{\min_{t=0} \dot{u}, 0\right\} \int\limits_0^t e^{-\lambda\tau} d\tau \\ &\ge \min_{t=0} u + \frac{1}{\lambda} \min\left\{\min_{t=0} \dot{u}, 0\right\}. \end{aligned}$$

□

Lemma 3.3 (C^1-estimates)**.** *For a smooth and convex solution u of the flow equation (1.1), the gradient of u remains bounded during the evolution.*

Proof. This follows from the C^0-estimates obtained so far and Theorem 2.2 in [14]. □

Remark 3.4. As long as a smooth solution u of our flow equation (1.1) exists and $\log\det u_{ij}$ remains bounded, u remains strictly convex provided u_0 is strictly convex. The quantity $\log\det u_{ij}$, however, stays bounded as both the argument of Φ (see Lemma 2.1, Corollary 2.2) and $\log f$ are estimated. Finally, $\log f$ remains bounded as $|u|_1$ is a priori bounded.

4. C^2-estimates

4.1. **Preliminary results.** We use ν for the inner unit normal of $\partial\Omega$ and τ for a direction tangential to $\partial\Omega$.

Lemma 4.1 (Mixed C^2-estimates at the boundary)**.** *Let u be a solution of our flow equation (1.1). Then the absolute value of $u_{\tau\nu}$ remains a priori bounded on $\partial\Omega$ during the evolution.*

Proof. We represent $\partial\Omega$ locally as $\operatorname{graph}\omega$ over its tangent plane at a fixed point $x_0 \in \partial\Omega$ such that locally $\Omega = \{(x^n, \hat{x}) : x^n > \omega(\hat{x})\}$. We differentiate the Neumann boundary condition

$$\nu^i(\hat{x})u_i(\hat{x}, \omega(\hat{x})) = \varphi((\hat{x}, \omega(\hat{x})), u(\hat{x}, \omega(\hat{x}))), \quad \hat{x} \in \mathbb{R}^{n-1},$$

with respect to $\hat{x}^j$, $1 \le j \le n-1$,

$$\nu^i_j u_i + \nu^i u_{ij} + \nu^i u_{in}\omega_j = \varphi_j + \varphi_n\omega_j + \varphi_z u_j + \varphi_z u_n \omega_j$$

and obtain at $x_0 \equiv (\hat{x}_0, \omega(\hat{x}_0)) \in \partial\Omega$ a bound for $\nu^i u_{ij}$ in view of the C^1-estimates and $D\omega(\hat{x}_0) = 0$. Multiplying with τ^j gives the result. We remark that it is only possible to multiply the equation with a tangential vector as the differentiation with respect to $\hat{x}^j$ and so also j correspond to tangential directions. □

Lemma 4.2 (Double normal C^2-estimates at the boundary)**.** *For any solution of the flow equation (1.1) the absolute value of $u_{\nu\nu}$ is a priori bounded from above on $\partial\Omega$. ($u_{\nu\nu} > 0$ also follows from the strict convexity of a solution.)*

Proof. We use methods known from the Dirichlet problem [15], where more details can be found and assume the same geometric situation as in the proof of Lemma 4.1 with $x_0 \in \partial\Omega$. From (1.1) we obtain

$$\dot{u}_k = \Phi' u^{ij} u_{ijk} - \Phi'(\hat{f}_k + \hat{f}_z u_k + \hat{f}_{p_i} u_{ki})$$

and define therefore

$$Lw := \dot{w} - \Phi' u^{ij} w_{ij} + \Phi' \hat{f}_{p_i} w_i,$$

where we evaluate the terms by using the function u. From the definition of L it is easy to see that for appropriate extensions of ν and φ

$$\left| L\left(\nu^k u_k - \varphi(x,u)\right)\right| \le c\cdot\left(1 + \operatorname{tr} u^{ij}\right),$$

where - here and in the following - c is an a priori bounded positive constant that may change its value as necessary. We define $\Omega_\delta := \Omega \cap B_\delta(x_0)$ for $\delta > 0$ sufficiently small and set

$$\vartheta := d - \mu d^2$$

for $\mu \gg 1$ sufficiently large where d denotes the distance from $\partial\Omega$. We will show that $L\vartheta \ge \frac{\varepsilon}{3}\Phi' \operatorname{tr} u^{ij}$ for a small constant $\varepsilon > 0$ (depending only on a positive lower bound for the principal curvatures of $\partial\Omega$) in Ω_δ.

$$\begin{aligned} L\vartheta &= -\Phi' u^{ij} d_{ij} + 2\mu\Phi' u^{ij} d_i d_j + 2\mu \Phi' u^{ij} d d_{ij} \\ &\quad + \Phi' \hat{f}_{p_i}(d_i - 2\mu d d_i) \\ &\ge -\Phi' u^{ij} d_{ij} + 2\mu\Phi' u^{ij} d_i d_j - c\mu d\left(1 + \operatorname{tr} u^{ij}\right) - c. \end{aligned}$$

We use the strict convexity of $\partial\Omega$, $d_i \approx \delta_{in}$, $|u^{kl}| \le \operatorname{tr} u^{ij}$, $1 \le k, l \le n$, and the inequality for arithmetic and geometric means

$$\begin{aligned} L\vartheta &\ge \varepsilon\,\Phi' \operatorname{tr} u^{ij} + \Phi' \mu u^{nn} - c\mu\delta\left(1 + \operatorname{tr} u^{ij}\right) - c \\ &\ge \Phi' \frac{n}{3}\left(\det u^{ij}\right)^{\frac{1}{n}} \cdot \varepsilon^{\frac{n-1}{n}} \cdot \mu^{\frac{1}{n}} + \frac{2}{3}\varepsilon\,\Phi' \operatorname{tr} u^{ij} \\ &\quad - c\mu\delta\left(1 + \operatorname{tr} u^{ij}\right) - c. \end{aligned} \tag{4.1}$$

As $\det u^{ij}$ is a priori bounded from below by a positive constant in view of

$$\det u^{ij} = (\det u_{ij})^{-1} = \exp\left(-\hat{f} - \Phi^{-1}(\dot{u})\right),$$

we may choose μ so large that the first term in (4.1) is greater than $c+1$. For $\delta \le \frac{1}{c\mu}\min\left\{1, \frac{1}{3}\varepsilon\right\}$ we get

$$L\vartheta \ge \frac{1}{3}\varepsilon\,\Phi'\operatorname{tr} u^{ij}$$

and furthermore $\vartheta \ge 0$ on $\partial\Omega_\delta$ if we choose δ smaller if necessary.

For constants A, $B > 0$ consider the function

$$\Theta := A\vartheta + B|x-x_0|^2 \pm (\nu^i u_i - \varphi(x,u)) + l,$$

where l is an affine linear function such that $\Theta \ge 0$ for $t=0$ and $l(x_0)=0$. We fix $B \gg 1$, get $\Theta \ge 0$ on $\partial\Omega_\delta$, and deduce for $A \gg B$ that $L\Theta \ge 0$ as $\operatorname{tr} u^{ij}$ is bounded from below by a positive constant. The maximum principle yields $\Theta \ge 0$ in Ω_δ. As $\Theta(x_0)=0$ we have $\Theta_\nu(x_0) \ge 0$ which in turn gives immediately $|u_{\nu\nu}| \le c$. □

Remark 4.3. From Section 3 and the uniform estimates for $\dot u$ we get for a fixed positive constant μ_0

$$\min\{\det u_{ij}, f\} \ge \mu_0 > 0.$$

According to [14] we obtain unique convex solutions $\psi_\rho \in C^2\left(\overline{\Omega}\right)$ for $0 \le \rho \le 1$ of the boundary value problem

$$\begin{cases} \det\psi_{ij} &=& \frac{1}{2}\mu_0 & \text{in } \Omega,\\ \psi_\nu &=& \varphi(x,\psi+\rho|x|^2) - 2\rho\langle x,\nu\rangle & \text{on } \partial\Omega \end{cases}$$

such that $|\psi_\rho|_{2,\overline{\Omega}} \le c$ and $\psi_{ij} \ge \lambda\delta_{ij}$ for positive constants independent of ρ. Fix $\rho > 0$ sufficiently small such that $\overline{\psi}_\rho = \psi_\rho + \rho|x|^2$ satisfies

$$\det\overline{\psi}_{ij} < \mu_0 \quad \text{in } \Omega,$$

where we dropped the index ρ as ρ is fixed now.

Lemma 4.4. *For $\overline{\psi}$ as constructed above, $u \le \overline{\psi}$ is valid during the evolution.*

Proof. The function $\overline{\psi}$ satisfies the elliptic differential inequality

$$\begin{cases} \det\overline{\psi}_{ij} &<& \mu_0 & \text{in } \Omega,\\ \overline{\psi}_\nu &=& \varphi\left(x,\overline{\psi}\right) & \text{on } \partial\Omega \end{cases}$$

and the parabolic differential inequality

$$\begin{cases} \dot{\overline{\psi}} &>& \Phi\left(\log\det\overline{\psi}_{ij} - \log\mu_0\right) & \text{in } \Omega\times[0,T),\\ \overline{\psi}_\nu &=& \varphi\left(x,\overline{\psi}\right) & \text{on } \partial\Omega\times[0,T) \end{cases}$$

as $\overline{\psi}$ is independent of t, so $\dot{\overline{\psi}} = 0$. Furthermore we have the following elliptic differential inequality

$$\begin{cases} \det(u_0)_{ij} &\ge& \mu_0 & \text{in } \Omega,\\ (u_0)_\nu &=& \varphi(x,u_0) & \text{on } \partial\Omega \end{cases}$$

and the parabolic differential inequality

$$\begin{cases} \dot{u} &= \Phi\left(\log\det u_{ij} - \hat{f}\right) \\ &\le \Phi\left(\log\det u_{ij} - \log\mu_0\right) \quad \text{in } \Omega\times[0,T), \\ u_\nu &= \varphi(x,u) \quad\quad\quad\quad\quad\quad\quad \text{on } \partial\Omega\times[0,T). \end{cases}$$

We combine the elliptic differential inequalities and obtain by the mean value theorem with a positive definite matrix a^{ij} and a positive function C

$$\begin{cases} a^{ij}\left(u_0-\overline{\psi}\right)_{ij} > 0 & \text{in } \Omega, \\ \left(u_0-\overline{\psi}\right)_\nu = C\cdot\left(u_0-\overline{\psi}\right) & \text{on } \partial\Omega, \end{cases}$$

thus we obtain $u = u_0 \le \overline{\psi}$ for $t=0$ in view of the elliptic maximum principle. From the parabolic differential inequalities we get

$$\begin{cases} \dot{u}-\dot{\overline{\psi}} < \tilde{a}^{ij}\left(u-\overline{\psi}\right)_{ij} & \text{in } \Omega\times[0,T), \\ \left(u-\overline{\psi}\right)_\nu = \tilde{C}\cdot\left(u-\overline{\psi}\right) & \text{on } \partial\Omega\times[0,T), \end{cases}$$

so the parabolic maximum principle gives $u\le\overline{\psi}$ for all $t\ge 0$. □

Corollary 4.5. *For ψ as constructed above there exists a positive constant δ_0 such that*

$$(\psi-u)_\nu \ge \delta_0 > 0.$$

Proof. As $u\le\overline{\psi}$ we deduce from the Neumann boundary condition

$$\overline{\psi}_\nu - u_\nu = \varphi\left(x,\overline{\psi}\right)-\varphi(x,u) = \int\limits_0^1 \varphi_z\left(x,\tau\overline{\psi}+(1-\tau)u\right)\,d\tau\cdot\left(\overline{\psi}-u\right),$$

so $\left(\overline{\psi}-u\right)_\nu\ge 0$, and furthermore

$$(\psi-u)_\nu = \left(\overline{\psi}-\rho|x|^2-u\right)_\nu \ge -2\rho\langle x,\nu\rangle \ge \delta_0 > 0$$

as Ω is strictly convex and $0\in\Omega$. □

4.2. **Interior estimates.** To establish a priori C^2-estimates everywhere, we proceed as in [14]. For the reader's convenience, however, we repeat the argument given there modified for the parabolic case. We may take T slightly smaller than the maximal time interval for which a solution exists. We define for $(x,\xi,t)\in\overline{\Omega}\times S^{n-1}\times[0,T]$

$$W(x,\xi,t) := \log w + \beta\left(u^i u_i + M(\psi-u)\right)$$

where

$$\begin{aligned} w(x,\xi,t) &= u_{\xi\xi} - 2\langle\xi,\nu\rangle\left(\xi^i-\langle\xi,\nu\rangle\nu^i\right)\cdot\left(\varphi_i+\varphi_z u_i - u_k\nu^k_i\right) \\ &\equiv u_{\xi\xi}+a^k u_k + b, \end{aligned}$$

and ν is a smooth extension of the inner unit normal to $\partial\Omega$ that vanishes outside a tubular neighborhood of $\partial\Omega$; a^k and b depend only on x and u.

Lemma 4.6 (Interior C^2-estimates)**.** *For a solution of the flow equation (1.1), W attains its maximum over $\overline{\Omega} \times S^{n-1} \times [0,T]$ at a boundary point, i. e. in $\partial\Omega \times S^{n-1} \times [0,T]$, provided $\beta \gg M \gg 1$ are chosen large enough or $\left|D^2 u\right|$ is a priori bounded by a constant determined by the C^2-norm of u_0 and known or estimated quantities.*

Remark 4.7. More precisely we assume for the maximum of W $w \geq 1$, $c \leq \Phi' \frac{\lambda}{2} \operatorname{tr} u^{ij}$, see (4.7), and furthermore (4.10), (4.11), where ε is determined just above (4.13) and β is determined directly below (4.13). This gives a possibility to calculate an upper bound of $\left|D^2 u\right|$ in view of the above a priori estimates, if the maximum of W is attained in $\Omega \times S^{n-1} \times (0,T]$.

Proof of Lemma 4.6. We assume that W attains its maximum in the point $(x_0, \xi_0, t_0) \in \Omega \times S^{n-1} \times (0,T]$ (but later on we write again ξ for simplicity) and w is positive in a neighborhood of x_0, so we calculate there

$$\begin{aligned} W_i &= \frac{w_i}{w} + 2\beta u^k u_{ki} + \beta M(\psi - u)_i, \\ W_{ij} &= \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2\beta u^k_j u_{ki} + 2\beta u^k u_{kij} + \beta M(\psi - u)_{ij}, \\ \dot{W} &= \frac{\dot{w}}{w} + 2\beta u^k \dot{u}_k + \beta M(\dot{\psi} - \dot{u}). \end{aligned}$$

We differentiate the flow equation twice

$$\dot{u}_i = \Phi' u^{kl} u_{kli} - \Phi' D_i \hat{f}, \tag{4.2}$$

$$\begin{aligned} \dot{u}_{\xi\xi} &= \Phi' u^{ij} u_{ij\xi\xi} - \Phi' u^{ik} u^{jl} u_{ij\xi} u_{kl\xi} - \Phi' D_{\xi\xi} \hat{f} \\ &\quad + \Phi'' \left(u^{ij} u_{ij\xi}\right)^2 - 2\Phi'' u^{ij} u_{ij\xi} D_\xi \hat{f} + \Phi'' \left(D_\xi \hat{f}\right)^2 \\ &\leq \Phi' u^{ij} u_{ij\xi\xi} - \Phi' u^{ik} u^{jl} u_{ij\xi} u_{kl\xi} - \Phi' D_{\xi\xi} \hat{f}, \end{aligned} \tag{4.3}$$

where we have used the concavity of Φ. $D.$ indicates that the chain rule has not yet been applied to the respective terms.

As $|u^k u_{k\nu}|$ is bounded on $\partial\Omega$ we may fix M such that

$$M\delta_0 \geq 2 \left.|u^k u_{k\nu}|\right|_{\partial\Omega}, \tag{4.4}$$

where we use δ_0 as introduced in Corollary 4.5.

Now we restrict our attention to the point where the maximum is attained. We have there $W_i = 0$, $W_{ij} \leq 0$, $\dot{W} \geq 0$ and $\Phi' > 0$, so we get

$$\begin{aligned} 0 &\leq \dot{W} - \Phi' u^{ij} W_{ij} \\ &\leq \frac{\dot{w}}{w} - \Phi' \frac{1}{w} u^{ij} w_{ij} + \Phi' \frac{1}{w^2} u^{ij} w_i w_j \\ &\quad + 2\beta u^k \dot{u}_k - 2\beta \Phi' \Delta u - 2\beta \Phi' u^k u^{ij} u_{ijk} - \lambda\beta M \Phi' \operatorname{tr} u^{ij} + c\beta M \end{aligned} \tag{4.5}$$

with $\lambda > 0$ as in Remark 4.3. We remark that c also depends on Φ'. From (4.2) and (4.3) we get

$$\begin{aligned} \dot{w} - \Phi' u^{ij} w_{ij} &\le \dot{u}_{\xi\xi} + a^k \dot{u}_k - \Phi' u^{ij} u_{\xi\xi ij} - \Phi' a^k u^{ij} u_{kij} + c \cdot \left(1 + \operatorname{tr} u^{ij}\right) \\ &\le -\Phi' u^{ik} u^{jl} u_{ij\xi} u_{kl\xi} - \Phi' D_{\xi\xi} \hat{f} \\ &\quad + c \cdot \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right). \end{aligned} \tag{4.6}$$

We assume now that W is large in its maximum, more precisely $c \le \Phi' \frac{\lambda}{2} \operatorname{tr} u^{ij}$, combine (4.5) and (4.6) and take (4.2) into account

$$\begin{aligned} 0 &\le -\Phi' u^{ik} u^{jl} u_{ij\xi} u_{kl\xi} - \Phi' D_{\xi\xi} \hat{f} + \Phi' \frac{1}{w} u^{ij} w_i w_j - 2\beta w \Phi' u^k D_k \hat{f} \\ &\quad - 2\beta w \Phi' \Delta u - \frac{\lambda}{2} \beta M w \Phi' \operatorname{tr} u^{ij} + c \cdot \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right). \end{aligned} \tag{4.7}$$

We consider the quantity $u^{ij} w_i w_j$ separately and use Young's inequality for $0 < \varepsilon < 1$ to be fixed later

$$\begin{aligned} u^{ij} w_i w_j &= u^{ij} (u_{\xi\xi i} + D_i a^k u_k + D_i b + a^k u_{ki}) \cdot \\ &\quad \cdot (u_{\xi\xi j} + D_j a^l u_l + D_j b + a^l u_{lj}) \\ &\equiv u^{ij} (u_{\xi\xi i} + B_i + a^k u_{ki})(u_{\xi\xi j} + B_j + a^l u_{lj}) \\ &\le (1+\varepsilon) u^{ij} u_{\xi\xi i} u_{\xi\xi j} + \frac{2}{\varepsilon} u^{ij} B_i B_j + \frac{2}{\varepsilon} u^{ij} u_{kj} a^k u_{li} a^l \\ &\quad + u^{ij} B_i B_j + 2 B_i a^i + a^i a^j u_{ij} \\ &\le (1+\varepsilon) u^{ij} u_{\xi\xi i} u_{\xi\xi j} + \frac{c}{\varepsilon} \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right). \end{aligned} \tag{4.8}$$

On the other hand we get in view of $W_i = 0$

$$u^{ij} w_i w_j \le c \beta^2 w^2 \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right), \tag{4.9}$$

where c depends on the constant M fixed above.

We assume that $u_{\xi\xi}$ and the greatest eigenvalue of u_{ij} at x_0, $u_{\eta\eta}$, are nearly as large as w, more precisely

$$0 < \frac{1}{1+\varepsilon} \le \frac{u_{\eta\eta}}{w} \le 1 + \varepsilon, \tag{4.10}$$

and for later use

$$1 \le u_{\xi\xi}, \quad 1 \le \left|D^2 u\right|, \quad \frac{1}{2} \left|D^2 u\right| \le w \le 2 \left|D^2 u\right|, \tag{4.11}$$

so we get for $0 < \varepsilon \ll 1$ in view of (4.8) and (4.9)

$$\begin{aligned} \frac{1}{w} u^{ij} w_i w_j &= (1 - 3\varepsilon) \frac{1}{w} u^{ij} w_i w_j + 3\varepsilon \frac{1}{w} u^{ij} w_i w_j \qquad (4.12) \\ &\le u^{\eta\eta} u^{ij} u_{\xi\xi i} u_{\xi\xi j} \\ &\quad + \frac{c}{\varepsilon w} \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right) + c \varepsilon \beta^2 w \left(1 + \left|D^2 u\right| + \operatorname{tr} u^{ij}\right). \end{aligned}$$

We calculate for $-D_{\xi\xi}\hat{f}-2\beta w u^k D_k\hat{f}$ in view of $W_i=0$

$$\begin{aligned} -\Phi' D_{\xi\xi}\hat{f}-2\beta\Phi' w u^k D_k\hat{f} \quad &\leq \quad -\Phi'\hat{f}_{p_k}u_{k\xi\xi}-2\beta\Phi' w u^k\hat{f}_{p_i}u_{ik} \\ &\qquad +c\beta\left(1+\left|D^2u\right|\right)+c\left(1+|D^2u|^2\right) \\ &\leq \quad c\beta\left(1+\left|D^2u\right|\right)+c\left(1+\left|D^2u\right|^2\right), \end{aligned}$$

where it is important to notice that the $2\beta w\hat{f}_{p_i}u^k u_{ki}$-terms cancel. We plug this estimate and (4.12) in (4.7)

$$\begin{aligned} 0 \quad \leq \quad & -\Phi' u^{ik}u^{jl}u_{ij\xi}u_{kl\xi}+\Phi' u^{\eta\eta}u^{ij}u_{\xi\xi i}u_{\xi\xi j}-2\beta w\Phi'\Delta u-\frac{\lambda}{2}w\beta M\Phi'\operatorname{tr}u^{ij} \\ & +\frac{c}{\varepsilon w}\left(1+\left|D^2u\right|+\operatorname{tr}u^{ij}\right)+c\varepsilon\beta^2 w\left(1+\left|D^2u\right|+\operatorname{tr}u^{ij}\right) \\ & +c\beta\left(1+\left|D^2u\right|\right)+c\left(1+\left|D^2u\right|^2+\operatorname{tr}u^{ij}\right). \end{aligned}$$

The sum of the first two terms is known to be nonpositive, see e. g. [14]. We choose $\varepsilon=\frac{1}{\beta^2}$, so we obtain

$$\begin{aligned} 0 \quad \leq \quad & -2\Phi'\beta\left|D^2u\right|^2-\frac{\lambda}{2}\left|D^2u\right|\beta M\Phi'\operatorname{tr}u^{ij} \\ & +c\beta^2\left(\left|D^2u\right|+\operatorname{tr}u^{ij}\right)+c\left|D^2u\right|\cdot\left(\left|D^2u\right|+\operatorname{tr}u^{ij}\right). \end{aligned} \tag{4.13}$$

If we fix β sufficiently large, it is easy to see that $\left|D^2u\right|(x_0,t_0)$ has to be a priori bounded by a constant. □

4.3. **Remaining boundary estimates.** The proof of the tangential C^2-estimates at the boundary can be carried out as in [14]. There, however, the authors only mention that this estimate can be obtained similar as at the beginning of Section 3 there. So we repeat the argument for readers not familiar with [14].

Before stating the lemma we wish to point out that it is in general not true that ξ_0 is a direction tangential to $\partial\Omega$ when W attains its maximum at $(x_0,\xi_0,t_0)\in\partial\Omega\times S^{n-1}\times(0,T]$.

Lemma 4.8. *The second derivatives of a solution u of our flow equation (1.1) are a priori bounded in $\overline{\Omega}\times[0,T]$.*

Proof. In view of Lemma 4.2 and Lemma 4.6 we may assume without loss of generality that W attains its maximum at a point $(x,\xi,t)\in\partial\Omega\times S^{n-1}\times(0,T)$ with $\xi\neq\nu$ and distinguish two cases.

(i) tangential: If ξ is tangential to $\partial\Omega$, we differentiate

$$\nu^i u_i=\varphi(x,u)$$

with respect to tangential directions under the assumptions stated in the proof of Lemma 4.1 and get in view of $D\omega(\hat{x}_0) = 0$

$$\begin{aligned} & \nu^i_{\xi\xi} u_i + 2\nu^i_\xi u_{i\xi} + \nu^i u_{i\xi\xi} + \nu^i u_{in} \omega_{\xi\xi} \\ = \; & \varphi_{\xi\xi} + \varphi_n \omega_{\xi\xi} + 2\varphi_{z\xi} u_\xi + \varphi_{zz} u_\xi u_\xi + \varphi_z u_{\xi\xi} + \varphi_z u_n \omega_{\xi\xi}, \end{aligned}$$

so we obtain

$$\begin{aligned} u_{\nu\xi\xi} &\geq -2\nu^i_\xi u_{i\xi} + \varphi_z u_{\xi\xi} - c \\ &\geq \varphi_z u_{\xi\xi} - c \end{aligned} \tag{4.14}$$

as $\partial\Omega$ is strictly convex. On the other hand the maximality of W at x gives $0 \geq W_\nu$,

$$0 \geq u_{\xi\xi\nu} - c + w\beta \left(2u^i u_{i\nu} + M(\psi - u)_\nu\right)$$

and furthermore using (4.4) and Corollary 4.5

$$0 \geq u_{\xi\xi\nu} - c,$$

so we obtain in view of (4.14) and $\varphi_z \geq c_\varphi > 0$ the desired estimate $u_{\xi\xi} \leq c$.

(ii) non-tangential: If ξ is neither tangential nor normal we need the tricky choice of w in [14]. We find $0 < \alpha < 1$ and a tangential direction τ such that

$$\xi = \alpha\tau + \sqrt{1 - \alpha^2}\nu.$$

We rewrite w as

$$\begin{aligned} w(x, \xi) &= u_{\xi\xi} - 2\alpha\sqrt{1 - \alpha^2}\tau^i(\varphi_i + \varphi_z u_i - u_k \nu^k_i) \\ &= u_{\xi\xi} - 2\alpha\sqrt{1 - \alpha^2} u_{\tau\nu} \end{aligned}$$

in view of the differentiated Neumann boundary condition, so we see that

$$\begin{aligned} u_{\xi\xi} &= \alpha^2 u_{\tau\tau} + (1 - \alpha^2) u_{\nu\nu} + 2\alpha\sqrt{1 - \alpha^2} u_{\tau\nu} \\ &= \alpha^2 u_{\tau\tau} + (1 - \alpha^2) u_{\nu\nu} - w(x, \xi) + u_{\xi\xi} \end{aligned}$$

and obtain in view of the maximality of W and the fact that $W - \log w$ is independent of ξ and $w(x, \tau) = u_{\tau\tau}$, $w(x, \nu) = u_{\nu\nu}$

$$\begin{aligned} w(x, \tau) &\leq w(x, \xi), \\ w(x, \xi) &= \alpha^2 u_{\tau\tau} + (1 - \alpha^2) u_{\nu\nu} \\ &= \alpha^2 w(x, \tau) + (1 - \alpha^2) w(x, \nu) \\ &\leq \alpha^2 w(x, \xi) + (1 - \alpha^2) w(x, \nu). \end{aligned}$$

Therefore $w(x, \xi) \leq w(x, \nu)$ gives the upper bound $u_{\xi\xi} \leq c$ proving the statement. $\square$

In the following sections we consider the second boundary value problem. In section 10 we will come back to Neumann boundary conditions. Sections 5 to 9 will not be used for the proof of Theorem 1.1.

5. Legendre transformation and Hessian functions

We introduce some classes of Hessian functions similar to [8, 15]. A slightly different class of Hessian functions is considered in [17].

Let $\Gamma_+ \subset \mathbb{R}^n$ be the open positive cone and $F \in C^\infty(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right)$ a symmetric function satisfying the condition

$$F_i = \frac{\partial F}{\partial \lambda^i} > 0;$$

then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices $Sym^+(n)$, for, let $(u_{ij}) \in Sym^+(n)$ with eigenvalues λ_i, $1 \le i \le n$, then define F on $Sym^+(n)$ by

$$F(u_{ij}) = F(\lambda_i).$$

We have $F \in C^\infty(Sym^+) \cap C^0\left(\overline{Sym^+}\right)$. If we define

$$F^{ij} = \frac{\partial F}{\partial u_{ij}},$$

then we get in an appropriate coordinate system

$$F^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \lambda_i}\left|\xi^i\right|^2 \quad \forall\, \xi \in \mathbb{R}^n,$$

and F^{ij} is diagonal, if u_{ij} is diagonal. We define furthermore

$$F^{ij,kl} = \frac{\partial^2 F}{\partial u_{ij}\partial u_{kl}}.$$

Definition 5.1. A Hessian function F is said to be of the class (K), if

$$F \in C^\infty(\Gamma_+) \cap C^0\left(\overline{\Gamma}_+\right), \tag{5.1}$$

$$F \text{ is symmetric}, \tag{5.2}$$

$$F \text{ is positive homogeneous of degree } d_0 > 0,$$

$$F_i = \frac{\partial F}{\partial \lambda_i} > 0 \quad \text{in } \Gamma_+, \tag{5.3}$$

$$F|_{\partial\Gamma_+} = 0, \tag{5.4}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}\left(F^{ij}\eta_{ij}\right)^2 - F^{ik}\tilde{u}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym,$$

where $\left(\tilde{u}^{ij}\right)$ denotes the inverse of (u_{ij}), or, equivalently, if we set $\hat{F} = \log F$,

$$\hat{F}^{ij,kl}\eta_{ij}\eta_{kl} \le -\hat{F}^{ik}\tilde{u}^{jl}\eta_{ij}\eta_{kl} \quad \forall\, \eta \in Sym,$$

where F is evaluated at (u_{ij}).

If F satisfies

$$\exists\, \varepsilon_0 > 0: \quad \varepsilon_0 FH \equiv \varepsilon_0 F \operatorname{tr} u_i^j \le F^{ij} u_{ik} u_j^k$$

for any $(u_{ij}) \in Sym^+$, where the index is lifted by means of the Kronecker-Delta, then we indicate this by using an additional star, $F \in (K^\star)$.

The class of Hessian functions F which fulfill, instead of the homogeneity condition, the following weaker assumption

$$\exists\, \delta_0 > 0: \quad 0 < \frac{1}{\delta_0} F \le \sum_i F_i \lambda_i \le \delta_0 F$$

is denoted by an additional tilde, $F \in \left(\tilde{K}\right)$ or $F \in \left(\tilde{K}^\star\right)$.

A Hessian function F which satisfies for any $\varepsilon > 0$

$$F(\varepsilon, \ldots, \varepsilon, R) \to +\infty, \quad \text{as } R \to +\infty,$$

or equivalently

$$F(1, \ldots, 1, R) \to +\infty, \quad \text{as } R \to +\infty,$$

in the homogeneous case, a condition similar to an assumption in [2], is said to be of the class (CNS).

Example 5.2. We mention examples of Hessian functions of the class $\left(\tilde{K}^\star\right)$ as given in [8, 15].

Let H_k be the k-th elementary symmetric polynomials,

$$\begin{aligned} H_k(\lambda_i) &:= \sum_{1 \le i_1 < \ldots < i_k \le n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_k}, \quad 1 \le k \le n, \\ \sigma_k &:= (H_k)^{\frac{1}{k}} \end{aligned} \tag{5.5}$$

the respective Hessian functions homogeneous of degree 1 and define furthermore

$$\tilde{\sigma}_k(\lambda_i) := \frac{1}{\sigma_k\left(\lambda_i^{-1}\right)} \equiv (S_{n,n-k})^{\frac{1}{k}}.$$

The functions $S_{n,k}$ belong to the class (K) for $1 \le k \le n-1$ and H_n belongs to the class $(K^\star)$.

Furthermore, see [8],

$$F := H_n^{a_0} \cdot \prod_{i=1}^{N} F_{(i)}^{a_i}, \quad a_i > 0 \tag{5.6}$$

belongs to the class $\left(\tilde{K}^\star\right)$ provided $F_{(i)} \in (\tilde{K})$, and we may even allow $F_{(i)} \neq 0$ on $\partial\Gamma_+$.

An additional construction gives inhomogeneous examples [15]. Let F be as in (5.6), $\eta \in C^\infty(\mathbb{R}_{\geq 0})$ and $c_\eta > 0$ such that

$$0 < \frac{1}{c_\eta} \leq \eta \leq c_\eta, \quad \eta' \leq 0,$$

then

$$\tilde{F}(\lambda_i) := F\left(\exp\left(\int_1^{\lambda_i} \frac{\eta(\tau)}{\tau}\, d\tau\right)\right)$$

belongs to the class $\left(\tilde{K}^\star\right)$.

Important properties of the class $\left(\tilde{K}^\star\right)$ for the a priori estimates of the second derivatives of u at the boundary are stated in the following lemmata.

Lemma 5.3. *Let $F \in \left(\tilde{K}^\star\right)$, then for fixed $\varepsilon > 0$*

$$F(\varepsilon, \ldots, \varepsilon, R) \to \infty \quad \text{as } R \to \infty,$$

i. e. $\left(\tilde{K}^\star\right) \subset (\tilde{K}) \cap (CNS)$, moreover, when $F \in \left(\tilde{K}\right) \cap (CNS)$, $0 < \frac{1}{c} \leq F \leq c$, and

$$0 < \lambda_1 \leq \ldots \leq \lambda_n,$$

then the following three conditions are equivalent

$$\lambda_1 \to 0, \quad \lambda_n \to \infty, \quad \operatorname{tr} F^{ij} \to \infty.$$

Proof. We refer to [15]. □

For the dual functions we have a similar lemma.

Lemma 5.4. *Let $F \in \left(\tilde{K}^\star\right)$,*

$$0 < \lambda_1 \leq \ldots \leq \lambda_n,$$

and $0 < \frac{1}{c} \leq F \leq c$. Then the following three conditions are equivalent

$$\lambda_1 \to 0, \quad \lambda_n \to \infty, \quad \operatorname{tr} F^{*ij} \to \infty,$$

where F^* is defined by

$$F^*(\lambda_i) = \frac{1}{F\left(\frac{1}{\lambda_i}\right)}.$$

Proof. We have $F_1 \ge \ldots \ge F_n > 0$, see [9, 17], so we get in view of the definition of F^*

$$F_i^*(\lambda_1, \ldots, \lambda_n) = \frac{F_i\left(\frac{1}{\lambda_i}\right)}{F^2} \cdot \frac{1}{\lambda_i^2}.$$

Thus $F_1^* \to \infty$ as $\lambda_1 \to 0$ gives the result and $\lambda_n \to \infty$ forces $\lambda_1 \to 0$ in view of Lemma 5.3. To get $\operatorname{tr} F^{*ij} \to \infty$, λ has to leave any compact subset of Γ_+. $\square$

Lemma 5.5. *Let $F \in \left(\tilde{K}\right) \cap (CNS)$. Then F^* as defined above satisfies (5.1), (5.2), (5.3), (5.4) and $F^* \in (CNS)$. For $F = (S_{n,k})^{\frac{1}{n-k}}$, $1 \le k \le n-1$, and obviously, see Lemma 5.3, also for $F \in \left(\tilde{K}\right) \cap (CNS)$ we have for any $\varepsilon > 0$*

$$\sum_i F_i \lambda_i^2 \le (c(\varepsilon) + \varepsilon \cdot |\lambda|) \cdot \sum_i F_i, \tag{5.7}$$

provided $0 < \frac{1}{c} \le F \le c$.

Proof. See [17]. $\square$

Instead of Lemma 5.4 we get the following weaker result for $S_{n,n-k}^*$.

Lemma 5.6. *Let $F = (S_{n,n-k})^* = H_k$, $1 \le k \le n-1$, and assume $0 < \frac{1}{c} \le F \le c$ and*

$$0 < \lambda_1 \le \ldots \le \lambda_n,$$

then

$$F_1 \ge \ldots \ge F_n > 0.$$

Moreover, at least one of the following conditions is fulfilled

$$F_n \ge \frac{1}{c} > 0 \tag{5.8}$$

or

$$\operatorname{tr} F^{ij} \to \infty.$$

Proof. The first inequality and the case $k = 1$ are obviously true. If $0 < \frac{1}{c} \le \lambda_{n-k+1} \le \ldots \le \lambda_n$, then $F_n \ge \frac{1}{c} > 0$. If $\lambda_{n-k+1} \to 0$, then $\lambda_n \to \infty$ as

$\lambda_{n-k+1}\cdot\ldots\cdot\lambda_n \ge \frac{1}{c}\cdot H_k \ge \frac{1}{c} > 0$, so

$$\begin{aligned}(H_k)_1 \quad &\ge \quad \ldots \ge (H_k)_{n-k+1} \ge \lambda_{n-k+2}\cdot\ldots\cdot\lambda_n = \frac{1}{\lambda_{n-k+1}}\cdot\lambda_{n-k+1}\cdot\ldots\cdot\lambda_n\\ &\ge \quad \frac{1}{\lambda_{n-k+1}}\cdot\frac{1}{c}\cdot H_k \ge \frac{1}{\lambda_{n-k+1}}\cdot\frac{1}{c}\to\infty.\end{aligned}$$

□

By direct calculations we obtain the following

Lemma 5.7. *If u is a strictly convex C^2-solution of (1.9), then the Legendre transform of u, $u^*:\Omega^*\times[0,T)\to\mathbb{R}$, defined by*

$$u^*(y,t) := x^i u_i(x,t) - u(x,t) \equiv x^i y_i - u, \quad y^i = u^i(x,t)$$

satisfies the evolution equation

$$\begin{cases} \dot{u}^* &= \hat{F}^*(D^2u^*) - \log g^*(y,u^*,Du^*) \quad \text{in } \Omega^*\times[0,T),\\ Du^*(\Omega^*) &= \Omega,\\ u^*|_{t_0} &= u_0^* \quad\quad \text{in } \Omega^*,\end{cases}$$

where u_0^* is defined similarly as u^*,

$$F^*(\lambda_i) = \frac{1}{F\left(\frac{1}{\lambda_i}\right)},$$

and

$$g^*(y,z^*,q^*) := \frac{1}{g(q^*, y^i q_i^* - z^*, y)}$$

and the time derivative of u^* is taken with y fixed.

6. Strict obliqueness

Lemma 6.1. *As long as a solution as in Theorem 1.4 exists, our boundary condition is strictly oblique, i. e.*

$$\langle \nu(x), \nu^*(Du(x,t))\rangle > 0, \quad x\in\partial\Omega, \tag{6.1}$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively.

Proof. To prove (6.1) we use

$$\nu^i(x)\cdot\nu_i^*(Du(x,t)) = \nu^i\cdot h_{p_i}(Du(x,t)).$$

As $h(Du)$ is positive in Ω and vanishes on $\partial\Omega$, we get on $\partial\Omega$ for τ orthogonal to ν

$$h_{p_k}u_{k\tau} = 0, \quad h_{p_k}u_{k\nu}\ge 0. \tag{6.2}$$

Thus we see from

$$h_{p_k}\nu^k = h_{p_k}u_{ki}u^{ij}\nu_j = h_{p_k}u_{k\nu}\cdot u^{\nu\nu}\ge 0 \tag{6.3}$$

that the quantity whose positivity we wish to show is at least nonnegative. We compute in view of (6.2) and (6.3) on $\partial\Omega$

$$\begin{aligned}\left(h_{p_k}\nu^k\right)^2 &= u^{\nu\nu}h_{p_k}u_{k\nu}u^{\nu\nu}u_{\nu l}h_{p_l}\\ &= u^{\nu\nu}h_{p_k}u_{ki}u^{ij}u_{jl}h_{p_l}\\ &= u^{\nu\nu}u_{kl}h_{p_k}h_{p_l},\end{aligned}$$

so we deduce the positivity of the quantity considered. □

7. $\dot{u}$- and C^0-estimates

Remark 7.1 ($\dot{u}$-estimates). The results of section 2 hold also for the flow (1.9), as in both cases, the flow equation is parabolic and the boundary condition is strictly oblique.

If condition (1.12) is fulfilled, uniform C^0-a priori estimates follow immediately by integrating the estimate in Lemma 2.3, see also Lemma 3.2. In the case of condition (1.13), the positivity of $\dot{u}$, Lemma 2.4, gives a lower bound for u. So it remains to establish an upper bound for u in the case $\dot{u} \ge 0$.

Lemma 7.2. *A solution u of our flow equation (1.9) is uniformly bounded.*

Proof. The concavity of $\hat{F}(\cdot)$ gives the estimate

$$\hat{F}(D^2u) \le \hat{F}^{ij}(1, \dots, 1)(u_{ij} - \delta_{ij}) + \hat{F}(1, \dots, 1) \le c \cdot \Delta u + c.$$

For $0 < t_1 < t_2$ we integrate the flow equation and estimate in view of the inequality above, the divergence theorem and $|Du| \le c$ ($Du(\Omega) = \Omega^*$)

$$\begin{aligned}\int\limits_{t_1}^{t_2}\int\limits_{\Omega} \log g(x,u,Du) &\le c\cdot\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\Delta u + c\cdot(t_2-t_1) - \int\limits_{\Omega}\left(u|_{t_2} - u|_{t_1}\right)\\ &\le c\cdot\int\limits_{t_1}^{t_2}\int\limits_{\partial\Omega}|Du| + c\cdot(t_2-t_1) - \int\limits_{\Omega}\left(u|_{t_2} - u|_{t_1}\right)\\ &\le c\cdot(t_2-t_1) - \int\limits_{\Omega}\left(u|_{t_2} - u|_{t_1}\right). \end{aligned}\tag{7.1}$$

The boundedness of Du and the convexity of Ω yield the estimate

$$|u(x_1,t) - u(x_2,t)| \le c_{\Omega,\Omega^*} \quad \forall\, x_1,\, x_2 \in \Omega \quad \forall\, t > 0. \tag{7.2}$$

So we obtain from (7.1) for any $x \in \Omega$

$$\frac{1}{|\Omega|}\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\log g(x,u,Du) \le c + c\cdot(t_2-t_1) - u(x,t_2) + u(x,t_1). \tag{7.3}$$

Now we fix $T > 0$ and assume that

$$u(x_0, T) = \max_{\overline{\Omega}\times[0,T]} u =: M > \max\left\{2 \max_{\overline{\Omega}} u_0, 0\right\}.$$

We choose $t \in (0, T)$ maximal such that $u(x_0, t) = \frac{M}{2}$. From the monotonicity of g and (7.3) we get the estimate

$$\begin{aligned}\frac{M}{2} &= u(x_0, T) - u(x_0, t)\\ &\le c + c\cdot(T-t) - (T-t)\cdot \inf_{x\in\Omega}\inf_{p\in\Omega^*} \log g\left(x, \frac{M}{2} - c_{\Omega,\Omega^*}, p\right)\end{aligned}$$

and after rearranging

$$\frac{\frac{M}{2} - c}{T-t} \le c - \inf_{x\in\Omega}\inf_{p\in\Omega^*} \log g\left(x, \frac{M}{2} - c_{\Omega,\Omega^*}, p\right).$$

For $M \to \infty$ the left-hand side of this inequality remains positive, whereas the right-hand side tends to $-\infty$ in view of (1.11), so M is a priori bounded proving the lemma. □

Corollary 7.3. *During the evolution, $\hat{F}(D^2u)$ is a priori bounded from above and from below.*

Proof. This follows from $|Du| \le c$ and from the flow equation. □

8. Strict obliqueness estimates

The following lemma establishes a uniform lower bound for the quantity whose positivity we proved in Lemma 6.1.

Lemma 8.1. *During the evolution (1.9), we have the strict obliqueness estimate*

$$\langle \nu(x), \nu^*(Du(x,t))\rangle \ge \frac{1}{c} > 0, \quad x \in \partial\Omega, \tag{8.1}$$

where ν and ν^* denote the inner unit normals of Ω and Ω^*, respectively. The positive lower bound is independent of t.

Proof. We assume that a solution of our flow equation exists for a time interval $(0, T]$ and prove an estimate for $h_{p_k}\nu^k$ during this time interval which is independent of T. To establish a positive lower bound, we choose $(x_0, t_0) \in \partial\Omega\times[0, T]$ such that $h_{p_k}\nu^k$ is minimal there. As we have a positive lower bound for $h_{p_k}\nu^k$ on $\partial\Omega \times \{0\}$, we may assume that $t_0 > 0$. Further on, we may assume that $\nu(x_0) = e_n$ and extend ν smoothly to a tubular neighborhood of $\partial\Omega$ such that in the matrix sense

$$D_k\nu^l \equiv \nu^l_k \le -\frac{1}{c_1}\delta^l_k \tag{8.2}$$

there for a positive constant c_1. For a positive constant A to be chosen we define

$$v = h_{p_k}\nu^k + Ah(Du).$$

The function $v|_{\partial\Omega\times(0,T]}$ attains its minimum over $\partial\Omega\times(0,T]$ in (x_0,t_0), so we deduce there

$$\begin{aligned} 0 &= v_r = h_{p_n p_k}u_{kr} + h_{p_k}\nu^k_r + Ah_{p_k}u_{kr}, \quad 1\le r\le n-1, && (8.3)\\ 0 &\ge \dot v. && (8.4)\end{aligned}$$

We assume for a moment that there holds

$$v_n(x_0,t_0) \ge -c(A), \tag{8.5}$$

show that this estimate yields a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$ and prove (8.5) afterwards. Then the lemma follows from the calculations in the proof of Lemma 6.1 and from a positive lower bound for $u^{\nu\nu}$.

We rewrite (8.5) as

$$h_{p_n p_l}u_{ln} + h_{p_k}\nu^k_n + Ah_{p_k}u_{kn} \ge -c(A).$$

Multiplying this with h_{p_n} and adding (8.3) multiplied with h_{p_r} we obtain at (x_0,t_0)

$$Au_{kl}h_{p_k}h_{p_l} \ge -c(A)h_{p_n} - h_{p_k}\nu^k_l h_{p_l} - h_{p_k}h_{p_n p_l}u_{lk}.$$

Using (6.2), the concavity of h and (8.2), we obtain at x_0

$$Au_{kl}h_{p_k}h_{p_l} \ge -c(A)h_{p_n} + \frac{1}{c_1}$$

as $|\nabla h| = 1$ on $\partial\Omega^*$. We may assume that the right-hand side of the inequality above is positive as otherwise $h_{p_n} = h_{p_k}\nu^k$ is bounded from below. Thus we deduce a positive lower bound for $u_{kl}h_{p_k}h_{p_l}$.

We now sketch the proof of (8.5). There is another slightly shorter proof of this inequality obtained by constructing a barrier in a tubular neighborhood of $\partial\Omega$ avoiding the term $|x-x_0|^2$ below, but we prefer the following proof as it uses only local properties of the involved quantities. As for a similar proof with more details we refer to Lemma 4.2. Direct calculations using (1.9) give

$$\begin{aligned} Lv &\le \hat F^{ij}u_{li}u_{jm}\nu^k h_{p_k p_l p_m} + A\cdot \hat F^{ij}u_{ki}u_{lj}h_{p_k p_l} + c(A)\cdot \operatorname{tr}\hat F^{ij}\\ &\le c(A)\cdot\operatorname{tr}\hat F^{ij}\end{aligned}$$

for A sufficiently large and

$$Lw := -\dot w + \hat F^{ij}w_{ij} - \hat g_{p_i}w_i.$$

We wish to mention that this definition differs from the definition of L in Lemma 4.2 by a sign. As Ω is strictly convex, there exist $\mu \gg 1$ and $\varepsilon > 0$ such that for $\vartheta := d - \mu d^2$, where $d = \operatorname{dist}(\cdot,\partial\Omega)$, we have near $\partial\Omega$ in view of Lemma 5.3

$$L\vartheta \le -\varepsilon\cdot\operatorname{tr}\hat F^{ij}. \tag{8.6}$$

We consider ϑ only in $\Omega_\delta := \Omega \cap B_\delta(x_0)$, where $\delta > 0$ is chosen so small that ϑ is smooth and nonnegative there and the above inequality holds. As v is bounded and attains its minimum over $\partial\Omega \times [0,T]$ in (x_0, t_0) we find $C \gg B \gg 1$ and an affine linear function l with $l(x_0) = 0$ such that the function

$$\Theta := C \cdot \vartheta + B \cdot |x - x_0|^2 + v - v(x_0, t_0) + l$$

satisfies

$$\begin{cases} \Theta & \geq & 0 & \text{on } (\partial\Omega_\delta \times [0,T]) \cup (\Omega_\delta \times \{0\}), \\ L\Theta & \leq & 0 & \text{in } \Omega_\delta \times [0,T]. \end{cases}$$

Thus the maximum principle gives

$$(C \cdot \vartheta + v + l)_n(x_0, t_0) \geq 0$$

as the function $C \cdot \vartheta + B \cdot |x - x_0|^2 + v - v(x_0, t_0) + l$ vanishes in (x_0, t_0). This shows inequality (8.5).

Similar to the argument above we extend ν^* smoothly to a tubular neighborhood of $\partial\Omega^*$ such that $\nu^{*k}_i \leq -\frac{1}{c}\delta^k_i$ in the matrix sense and take h^* as a smooth strictly concave function such that $\{h^* = 0\} = \partial\Omega$ and $|Dh^*| = 1$ on $\partial\Omega$. We define

$$v^* = h^*_{q_k}(Du^*)\nu^{*k} + Ah^*(Du^*)$$

and a linear operator by

$$L^* w := -\dot{w} + \hat{F}^{*ij} w_{ij} - \hat{g}^*_{q_i} w_i.$$

As before we obtain that $v^*|_{\partial\Omega\times[0,T]}$ is positive. We fix $T > 0$ and assume that $v^*|_{\partial\Omega\times[0,T]}$ attains its minimum in (y_0, t_0). As we wish to establish a positive lower bound for v^* we may assume that $t_0 > 0$. By calculations as above – using Lemma 5.4 – we obtain in (y_0, t_0) an inequality of the form

$$Au^*_{kl} h^*_{q_k} h^*_{q_l} \geq -c(A) h^*_{q_k} \nu^{*k} - \nu^{*l}_k h^*_{q_k} h^*_{q_l}. \tag{8.7}$$

Since $h^*_{q_k}\nu^{*k} = \langle \nu^*, \nu\rangle$, we may assume again that this quantity is small. The second term on the right-hand side is bounded below by a positive constant in view of the convexity of Ω^* and $|Dh^*| = 1$ on $\partial\Omega^*$, so we deduce $u^*_{kl} h^*_{q_k} h^*_{q_l} \geq \frac{1}{c} > 0$. Using $u^*_{kl} = u^{kl}$ and $h^*_{q_k} = \nu^k$ we obtain a positive lower bound for $u^{\nu\nu}$ completing the strict obliqueness estimate. □

9. C^2-estimates

For convenience we use the notation $h_{p_k}(Du) = \beta^k$. We state the following estimates on $\partial\Omega$ obtained by differentiating the boundary condition

$$u_{\tau\beta} = 0, \quad u_{\nu\beta} \geq 0 \tag{9.1}$$

where τ and ν denote a tangential vector and the inner unit normal, respectively, see also (6.2). The estimates in this section are valid for any $\varepsilon > 0$ if ε is not fixed explicitly. Thus multiplying a term of the form $c(\varepsilon) + \varepsilon \cdot M$

with a constant yields again a term of the form $c(\varepsilon)+\varepsilon\cdot M$. We obtain the following

Lemma 9.1. *A solution of our flow equation (1.9) in a time interval* $[0,T]$ *satisfies for all* $\varepsilon>0$

$$u_{\beta\beta}\le c(\varepsilon)+\varepsilon\cdot M \quad \text{in } \Omega, \tag{9.2}$$

where $M:=\sup\limits_{\overline{\Omega}\times[0,T]}|D^2u|$.

Proof. We set $H=h(Du)$,

$$Lw:=-\dot{w}+\hat{F}^{ij}w_{ij}-\hat{g}_{p_i}w_i$$

and compute the differential inequality

$$LH\ge-(c(\varepsilon)+\varepsilon M)\cdot\operatorname{tr}\hat{F}^{ij}$$

where we have used Lemma 5.5 and the boundedness of $\hat{F}(D^2u)$. Applying the maximum principle as in Lemma 8.1 to the function $-A\cdot(c(\varepsilon)+\varepsilon\cdot M)\cdot\vartheta-B\cdot|x-x_0|^2+H+l$ with ϑ, l as in Lemma 8.1, $A\gg B\gg 1$ sufficiently large positive constants and $x_0\in\partial\Omega$, we obtain

$$u_{\beta\nu}\le c(\varepsilon)+\varepsilon\cdot M \quad \text{on } \partial\Omega. \tag{9.3}$$

We remark that ε in (8.6) is fixed and is not related to ε used here. In view of $u_{\beta\tau}=0$ on $\partial\Omega$ and the strict obliqueness estimate or by using the maximum principle as above for β instead of ν, the claimed inequality follows. □

As to the interior second derivative estimates we recall from [10]

Lemma 9.2. *For a solution of our flow equation in a time interval* $[0,T]$ *we have the estimate*

$$\sup_{\Omega\times[0,T]}|D^2u|\le c+\sup_{\partial\Omega\times[0,T]}|D^2u|+\sup_{\Omega\times\{0\}}|D^2u|.$$

Proof. Similar computations as in [10] in the elliptic case – under the assumption that

$$\overline{\Omega}\times[0,T]\times S^{n-1}\ni(x,t,\xi)\mapsto\gamma\cdot|Du(x,t)|^2+\log u_{\xi\xi}(x,t)$$

attains its maximum in $\Omega\times(0,T]\times S^{n-1}$ for γ sufficiently large – give the above bound. We remark, that we used $F\in\left(\tilde{F}^\star\right)$ and not only $F\in\left(\tilde{K}\right)\cap(CNS)$. □

Up to now we control $u_{\beta\tau}(=0)$, $u_{\beta\beta}$ and we have an interior estimate for the second derivatives of u. In the following lemma we bound double tangential derivatives at the boundary. This completes the C^2-a priori estimates.

Lemma 9.3. *For a solution of our flow equation in a time interval* $[0,T]$ *the second tangential derivatives at the boundary are a priori uniformly bounded.*

Proof. We may assume

$$\sup_{\partial\Omega\times[0,T]}\ \sup_{|\tau|=1,\,\langle\tau,\nu\rangle=0} u_{\tau\tau}=u_{11}(x_0,t_0) \tag{9.4}$$

where $x_0\in\partial\Omega$, $t_0\in(0,T]$ and furthermore that $\nu=e_n$ is the inner unit normal at $x_0\in\partial\Omega$. At a boundary point we decompose any direction ξ, i. e. a vector $\xi\in\mathbb{R}^n$ such that $|\xi|=1$,

$$\xi=\tau(\xi)+\frac{\langle\nu,\xi\rangle}{\langle\beta,\nu\rangle}\beta,$$

where

$$\tau(\xi)=\xi-\langle\nu,\xi\rangle\nu-\frac{\langle\nu,\xi\rangle}{\langle\beta,\nu\rangle}\beta^T,\quad \beta^T=\beta-\langle\beta,\nu\rangle\nu,$$

and obtain the estimate

$$|\tau(\xi)|^2\le 1+c\cdot\langle\nu,\xi\rangle^2-2\langle\nu,\xi\rangle\frac{\langle\beta^T,\xi\rangle}{\langle\beta,\nu\rangle}. \tag{9.5}$$

We set $\tau:=\tau(e_1)$ and obtain on $\partial\Omega$ in view of the estimates (9.1), (9.2), (9.4) and (9.5) above

$$u_{11}\le\left(1+c\cdot\langle\nu,e_1\rangle^2-2\frac{\langle\nu,e_1\rangle\cdot\langle\beta^T,e_1\rangle}{\langle\beta,\nu\rangle}\right)\cdot u_{11}(x_0,t_0)+(c(\varepsilon)+\varepsilon\cdot M)\langle\nu,e_1\rangle^2.$$

Before we proceed, we establish an estimate for the quantity M introduced in Lemma 9.1. Lemma 9.2 gives

$$M\le c+\sup_{\partial\Omega\times[0,T]}|D^2u|, \tag{9.6}$$

where the supremum also includes non-tangential directions. For a direction ξ we obtain in view of $u_{\beta\tau}=0$ on $\partial\Omega$, (9.4) and (9.2)

$$\begin{aligned} u_{\xi\xi} &\le u_{\tau(\xi)\tau(\xi)}+c\cdot u_{\beta\beta}\\ &\le c\cdot u_{11}(x_0,t_0)+c(\varepsilon)+\varepsilon\cdot M. \end{aligned}$$

Combining this inequality for $\varepsilon>0$ small enough with (9.6) we get

$$M\le c\cdot(1+u_{11}(x_0,t_0)). \tag{9.7}$$

We dropped ε as it was fixed sufficiently small to get this inequality and will be fixed differently later-on.

We may assume in view of (9.7) for the rest of the proof that $u_{11}(x_0,t_0)\ge 1$ and for

$$w:=\frac{u_{11}}{u_{11}(x_0,t_0)}+2\frac{\langle\nu,e_1\rangle\cdot\langle\beta^T,e_1\rangle}{\langle\beta,\nu\rangle}$$

we obtain – by using (9.7) – the estimate

$$w\le 1+c(\varepsilon)|x'|^2\quad\text{on }\partial\Omega\text{ near }x_0,$$

where $x' \equiv (x^1, \ldots, x^{n-1})$, and we get furthermore $w \le c(\varepsilon)$ everywhere on $\partial\Omega$. We consider $2\frac{\langle \nu, e_1\rangle \cdot \langle \beta^T, e_1\rangle}{\langle \beta, \nu\rangle}$ as a known function depending on (x, Du), use the flow equation, and obtain in Ω by direct calculation

$$-\dot{w} + \hat{F}^{ij} w_{ij} - \hat{g}_{p_i} w_i \ge -c \cdot (c(\varepsilon) + \varepsilon \cdot M) \operatorname{tr} \hat{F}^{ij}.$$

Thus the maximum principle gives with a barrier function as constructed above

$$u_{11\beta}(x_0, t_0) \le (c(\varepsilon) + \varepsilon \cdot M)\, u_{11}(x_0, t_0). \tag{9.8}$$

Differentiating the boundary condition twice in the direction e_1 we obtain at (x_0, t_0)

$$h_{p_k p_l} u_{k1} u_{l1} + u_{\beta 11} + u_{\beta n} \omega_{11} = 0,$$

where ω is a function such that locally $\partial\Omega$ is represented as graph ω over its tangent plane at x_0. Combining this equality with (9.8) and (9.3), we obtain in the non-trivial case $u_{11}(x_0, t_0) \ge c(\partial\Omega)$ which we will assume in the following

$$(c(\varepsilon) + \varepsilon \cdot M) \cdot u_{11}(x_0, t_0) + h_{p_k p_l} u_{k1} u_{l1} \ge 0. \tag{9.9}$$

Inequality (9.7) and the uniform concavity of h yield

$$(c(\varepsilon) + \varepsilon \cdot u_{11}(x_0, t_0)) \cdot u_{11}(x_0, t_0) \ge \frac{1}{c}(u_{11}(x_0, t_0))^2.$$

We now fix $\varepsilon > 0$ sufficiently small and get a bound for $u_{11}(x_0, t_0)$. □

10. Proof of the main theorems

We return to the case of a Neumann boundary value problem.

From the uniform C^2-estimates for u and the uniform C^0-estimates for $\dot{u} = \Phi$ we obtain that u remains uniformly convex and we conclude that the flow operator is uniformly parabolic and concave. So we can apply the results of chapter 14 in [13] to obtain uniform $C^{2,\alpha}$-estimates for u, with a small positive constant α. Then standard Schauder estimates [12] imply uniform bounds in C^k, for all $k \ge 0$. It follows that a smooth solution of (1.1) exists for all $t \ge 0$. We then need the following Lemma.

Lemma 10.1. *If a solution of the flow equation (1.1) exists for all $t \ge 0$ and either (1.5) or (1.6) are satisfied, then the flow converges to a solution of the Neumann problem i. e.*

$$\lim_{t \to \infty} u(x, t) =: u^\infty(x)$$

exists and

$$\begin{cases} u^\infty_\nu = \varphi(x, u^\infty) & \text{on } \partial\Omega, \\ \det u^\infty_{ij} = f(x, u^\infty, Du^\infty) & \text{in } \Omega. \end{cases}$$

Moreover, $u(t, \cdot) \to u^\infty$ smoothly. If (1.5) holds, then the convergence is exponentially fast in any C^k-norm, $k \ge 0$.

Proof. First, we assume that (1.6) is fulfilled. We may assume $\dot{u}(0,\cdot)\not\equiv 0$ and proceed as in [8]. Integrating the flow equation gives

$$u(t,x)-u(0,x)=\int\limits_0^t \Phi.$$

The left-hand side is uniformly bounded in view of the C^0-estimates. As $\log\det u_{ij}-\hat{f}$ is non-negative we find $t_k=t_k(x)\to\infty$ such that

$$\left(\log\det(D^2u)-\hat{f}(x,u,Du)\right)\Big|_{t=t_k}\to 0. \tag{10.1}$$

On the other hand, $u(x,\cdot)$ is monotone, so $\lim\limits_{t\to\infty}u(x,t)=:u^\infty(x)$ exists and is smooth in view of our a priori estimates. Dini's theorem and interpolation inequalities of the form

$$||D\tilde{u}||\le c||\tilde{u}||\cdot(||D^2\tilde{u}||+||D\tilde{u}||),$$

for $\tilde{u}=u-u^\infty$, where $||\cdot||$ denotes the sup-norm, yield smooth convergence $u\to u^\infty$. Thus we conclude in view of (10.1) that u^∞ is a smooth solution of the stationary problem (1.8).

In case (1.5) we use the a priori bounds for all C^k-norms and $\dot{u}\to 0$, see Lemma 2.3, to get smooth convergence to $u^\infty(x)$. Again by Lemma 2.3 we conclude

$$||u-u^\infty||<c_0e^{-\lambda_0 t}$$

for constants $\lambda_0>0$, $c_0>0$. Then we apply interpolation inequalities as above to $\tilde{u}=u-u^\infty$ and derive

$$||u-u^\infty||_k<c_ke^{-\lambda_k t}$$

for any $k\ge 0$ and positive constants λ_k, c_k. Clearly, u^∞ is a smooth solution of the stationary problem (1.8). □

Remark 10.2. By an iteration method applied to the interpolation inequality one can even show (Lemma C.2) that λ_k can be chosen independent of k.

Proof of Theorem 1.1 and Theorem 1.4. The a priori estimates obtained so far guarantee longtime existence for solutions of our flow equations, so the statement of Theorem 1.1 follows from Lemma 10.1 and the claim of Theorem 1.4 follows from a similar lemma. □

Appendix A. Oblique boundary value problems

A.1. **Flows solving the oblique boundary value problem for Hessian equations.** We get the following theorem for Hessian flow equations.

Theorem A.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth uniformly strictly convex domain, f a positive smooth function defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ with $f_z \geq 0$, let φ be a smooth function defined on $\overline{\Omega} \times \mathbb{R}$ with $\varphi_z > 0$ in $\partial\Omega \times \mathbb{R}$ and*

$$\varphi(x,z) \to \sigma\infty, \quad z \to \sigma\infty, \quad \sigma \in \{-1,1\},$$

uniformly in x. Let $F \in \left(\tilde{K}^\star\right)$ and β a smooth vectorfield on $\partial\Omega$ that is C^1-close to the inner unit normal ν as described in [18]*. Then the initial value problem for the parabolic boundary value problem*

$$\begin{cases} \dot{u} &= \log F\left(D^2 u\right) - \log f(x,u,Du) & \text{in } \Omega, \\ u_\beta &= \varphi(x,u) & \text{on } \partial\Omega \end{cases}$$

has a unique smooth strictly convex solution u and u converges smoothly to a solution of the Neumann boundary value problem

$$\begin{cases} F\left(D^2 u\right) &= f(x,u,Du) & \text{in } \Omega, \\ u_\beta &= \varphi(x,u) & \text{on } \partial\Omega \end{cases}$$

if we start with a smooth strictly convex subsolution $u_0 = u|_{t=0}$, i. e.

$$\begin{cases} 0 &\leq F\left(D^2 u_0\right) - f(x,u_0,Du_0) & \text{in } \Omega, \\ (u_0)_\beta &= \varphi(x,u_0) & \text{on } \partial\Omega \end{cases}$$

and assume compatibility conditions for $t=0$ as in our main theorem.

Proof. We sketch the proof which can be obtained by combining the proofs of [18] and of the corresponding Neumann boundary value problem above. The C^0-estimates follow from the maximum principle, C^1-estimates are stated in [14]. The crucial proof of the C^2-estimates is obtained as a combination of the proof above and the proof of [18], where the inequality for geometric and arithmetic means has to be avoided as in [15]. Instead we use Lemma 5.3. Higher regularity follows by standard theory and the considerations above give smooth convergence to a solution of the oblique boundary value problem for Hessian equations. ◻

Remark A.2. A similar result can be obtained in the case that (1.5) is fulfilled instead of $0 \leq F\left(D^2 u_0\right) - f(x,u_0,Du_0)$.

Again it is possible to modify the flow equation by introducing Φ as above.

It is also possible to obtain the existence proof for the elliptic oblique boundary value problem stated in Theorem A.1 by elliptic methods ([18]) as well as for $\beta = \nu$ by modifying the proof of [14].

A.2. **Nonconvex domains.** If Ω fails to be convex, then all the steps above work perfectly - provided $\varphi_z \geq c(\partial\Omega) > 0$ is sufficiently large - besides the a priori estimates for $u_{\nu\nu}$ that seem to be out of reach at the moment.

Appendix B. Hessian quotient equations

Proof of Theorem 1.5. We confine ourselves to the most important differences compared to the proof of Theorem 1.4.

Inequality (8.6) can be obtained by taking ϑ as a strictly concave function that vanishes on $\partial\Omega$, because $\hat{g}_{p_i} \equiv 0$. Lemma 5.6 is needed to obtain (8.7). The original function ϑ can be used here. Lemma 9.1 follows by using Lemma 5.5. Finally, to obtain Lemma 9.2, we have to use a positive lower bound for $\operatorname{tr} F^{ij}$. This bound follows from the concavity of $\hat{F}$ or from Lemma 5.6. Here it suffices to maximize

$$w := \log u_{\xi\xi} + \lambda \cdot |x|^2$$

for $\lambda \gg 1$. $\square$

Appendix C. Miscellaneous results

Lemma C.1. *Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth concave function such that $\Psi_1 > 0$ or $\Psi_2 < 0$ and $\Psi(x,x) = 0\ \forall\, x$. Then there exists $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi' > 0$, $\Phi'' \le 0$ such that*

$$\Psi(x,y) = \Phi(x-y).$$

Proof. The monotonicity of Ψ implies that $\Psi > 0$ in $\{x > y\}$, so the concavity of $\Psi|_{\{x-y=c\}}$, $c > 0$, gives that $\Psi|_{\{x-y=c\}}$ is constant. We fix $\alpha > 0$ and consider $\{(x,y) : \Psi(x,y) = -\alpha\}$. Our claim follows immediately if we show that $\{\Psi = -\alpha\}$ is a straight line. We consider only the case $\Psi_1 > 0$. As $\Psi|_{\{x\ge y\}}$ is constant along $\{x-y=c\}$ we see that $\Psi_1(x,x) =: \beta > 0$ is independent of x. Due to the concavity of Ψ we get thus

$$\Psi(x-\lambda, x) \le \Psi(x,x) - \Psi_1(x,x)\lambda = -\beta\lambda \quad \forall\, x$$

and deduce that there exists $c > 0$ such that

$$\{\Psi = -\alpha\} \subset \{-c < x-y < 0\}.$$

Using again the concavity of Ψ we see that $\{\Psi = -\alpha\}$ is a convex curve that can be represented as a graph over $\{x = 0\}$ due to $\Psi_1 > 0$. Such a curve in a strip as mentioned above, however, has to be a straight line. $\square$

Lemma C.2. *Let $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be a smooth function such that*

$$\|D^l u\|^2_{\Omega} \le C_l$$

for constants C_l independent of t. If there exist positive constants $c, \tilde{\lambda}$ such that

$$\|u\|^2_{\Omega} \le c e^{-\tilde{\lambda} t},$$

then for any $0 < \lambda < \tilde{\lambda}$ we can find positive constants c_l such that

$$\|D^l u\|^2_{\Omega} \le c_l e^{-\lambda t}.$$

Proof. Without loss of generality we assume $\tilde{\lambda} = 1$. We use interpolation inequalities of the form

$$\|Dv\|^2 \le c\|v\| \cdot (\|D^2v\| + \|Dv\|)$$

inductively. This induction gives the following sequence for $l,\, k \in \mathbb{N}$

$$a_{l,k} := \begin{cases} 1, & l = 0, \\ 0, & k = 0,\, l > 0, \\ \dfrac{a_{l-1,k-1} + a_{l+1,k-1}}{2}, & l > 0,\, k > 0, \end{cases}$$

where $a_{l,k}$ are exponents such that

$$\|D^l v\| \le C_{l,k} e^{-a_{l,k} t}$$

for positive constants $C_{l,k}$. Here k is the induction variable. We will prove that $a_{l,k} \to 1$ as $k \to \infty$. We have $0 \le a_{l,k} \le 1$, $a_{l,k} \ge a_{l+1,k}$, and $a_{l,k} \le a_{l,k+1}$ for all $l,\, k \ge 0$. Let $a_l := \lim\limits_{k\to\infty} a_{l,k}$ and observe that $a_l = \frac{a_{l-1}+a_{l+1}}{2}$ for $l \ge 1$, so we deduce

$$a_{m+l} = a_m - l \cdot (a_m - a_{m+1}).$$

As $1 \ge a_l \ge 0$, we see that $a_l = 1$ for all l. □

References

1. B. Andrews: Gauss curvature flow: The shape of the rolling stones. Invent. Math. **138** (1999), 151–161.
2. L. Caffarelli, L. Nirenberg, J. Spruck: Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces. Comm. Pure Appl. Math. **41** (1988), 47–70.
3. K.-S. Chou, X.-J. Wang: A logarithmic Gauss curvature flow and the Minkowski problem. Ann. Inst. Henri Poincaré. Analyse non linéaire **17**, 6 (2000) 733-751.
4. B. Chow: Deforming convex hypersurfaces by the nth root of the Gaussian curvature, J. Diff. Geom. **22**, no. 1 (1985), 117–138.
5. P. Daskalopoulos, R. Hamilton: The free boundary in the Gauss curvature flow with flat sides, J. reine angew. Math. **510** (1999), 187–227.
6. W. Firey: Shapes of worn stones, Mathematica **21** (1974), 1–11.
7. C. Gerhardt: Existenz für kleine Zeiten bei Neumann Randbedingungen, Lecture Notes.
8. C. Gerhardt: Hypersurfaces of prescribed curvature in Lorentzian manifolds, Indiana Univ. Math. J. **49** (2000), 1125-1153.
9. C. Gerhardt: Hypersurfaces of prescribed Weingarten curvature. Math. Z. **224** (1997), 167–194.
10. D. Gilbarg, N. S. Trudinger: Elliptic partial differential equations of second order. Second edition. Grundlehren der Mathematischen Wissenschaften, **224**. Springer-Verlag, Berlin, 1983. xiii+513 pp.
11. N. M. Ivochkina, O. A. Ladyženskaja: Estimation of the second derivatives on the boundary for surfaces evolving under the action of their principal curvatures. (Russian) Algebra i Analiz **9** (1997), 30–50, translation in St. Petersburg Math. J. **9** (1998), 199–217.

12. O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'zeva: Linear and quasilinear equations of parabolic type. (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. **23** American Mathematical Society, Providence, R.I. 1967 xi+648 pp.
13. G. M. Lieberman: Second order parabolic differential equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439 pp.
14. P.-L. Lions, N. S. Trudinger, J. I. E. Urbas: The Neumann problem for equations of Monge-Ampère type. Comm. Pure Appl. Math. **39** (1986), 539–563.
15. O. C. Schnürer: The Dirichlet problem for Weingarten hypersurfaces in Lorentz manifolds, to appear in Math. Z.
16. J. Urbas: Weingarten hypersurfaces with prescribed gradient image, preprint.
17. J. Urbas: The second boundary value problem for a class of Hessian equations, Comm. Partial Differential Equations **26** (2001), 859–882.
18. J. Urbas: Oblique boundary value problems for equations of Monge-Ampère type. Calc. Var. Partial Differential Equations **7** (1998), 19–39.
19. J. Urbas: On the second boundary value problem for equations of Monge-Ampère type. J. reine angew. Math. **487** (1997), 115–124.

Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany

E-mail address: `Oliver.Schnuerer@mis.mpg.de, Knut.Smoczyk@mis.mpg.de`