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Abstract

This article provides the complete bifurcation analysis of the Rankine-Hugoniot equations for compressible magnetohydrodynamics (MHD) in the case of a perfect gas. Particular scaling properties of the perfect-gas equation of state are used to reduce the number of bifurcation parameters. The smaller number, together with a novel choice, of these parameters results in a detailed picture of the global situation which is distinctly sharper than the one implied by previous literature. The description includes statements about the location, topology, and dimensions of various regimes, corresponding to different combinations of possible shock waves of given type, in dependence of the adiabatic exponent of the gas. The analysis is a crucial step for our detailed study of existence and bifurcation of viscous profiles for intermediate MHD shock waves that will be presented in a separate paper.

Key words: shock waves, compressible magnetohydrodynamics, Rankine-Hugoniot conditions, shock structure

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1 Introduction

Quasilinear systems of second order conservation laws in one space dimension can be written as

$$w_t + f(w)_x = (D(w)w_x)_x, \quad (1)$$

where $w = w(x, t) : \mathbf{R} \times [0, \infty) \rightarrow \mathcal{W}$ represents the unknown vector of d conservative variables, defined in an open subset $\mathcal{W} \subset \mathbf{R}^d$. The function $f : \mathcal{W} \rightarrow \mathbf{R}^d$ stands for the flux and $D : \mathcal{W} \rightarrow \mathbf{R}^{d \times d}$ for the dissipation matrix.

If the dissipative terms are neglected, i.e. D being replaced by 0, (1) turns into a first order hyperbolic PDE

$$w_t + f(w)_x = 0. \quad (2)$$

Prototype solutions of hyperbolic conservations laws (2) are examples of shock waves of the form

$$W(x, t) = \begin{cases} w^- & : x - st < 0, \\ w^+ & : x - st > 0. \end{cases} \quad (3)$$

In order to constitute a weak solution of (2) the states $w^\pm \in \mathcal{W}$ and the speed $s \in \mathbf{R}$ are supposed to satisfy the Rankine–Hugoniot relation:

$$f(w^+) - f(w^-) = s(w^+ - w^-).$$

In other words, for $(q, s) \in \mathbf{R}^{d+1}$, we look for w^- and w^+ that solve the (in general) nonlinear equation

$$f(w) - sw = q. \quad (4)$$

Our analysis will refer to q as the bifurcation parameter and classify the solution sets of (4) in their dependence on q . The parameterization by the *image* of the mapping $w \mapsto q$ gives a more consistent picture than the usual classification

$$\mathcal{H}(w^-; s) := \{w^+ \mid \exists s \in \mathbf{R} : f(w^+) - f(w^-) = s(w^+ - w^-)\}, \quad w^- \in \mathcal{W},$$

according to elements of the *preimage*.

If dissipation is not neglected one considers viscous profiles as typical solutions. A viscous profile, for (q, s) given and w^\pm solving (4), is a solution of the boundary value problem

$$D(\phi)\dot{\phi} = f(\phi) - s\phi - q \quad \phi(\pm\infty) = w^\pm. \quad (5)$$

Provided a viscous profile exists, the function $(x, t) \mapsto \phi(x - st)$ solves (1) and represents a smooth counterpart of the shock wave W from (3).

Depending on the flux f and the dissipation matrix D the dynamics of the ODE-system (5) with parameters (q, s) can become very complicated. The first step to analyze its dynamics is to obtain a complete picture on the existence and bifurcation of equilibria. Solutions of (4) are equilibria of (5). While the meaning of (1) and (2) is more transparent if the matrix D is at least invertible, it is also consistent to consider degenerate dissipation matrices as we will do now.

This paper will focus on the analysis of the Rankine–Hugoniot relation (4) in case of the classical compressible MHD equations. We consider plane-wave solutions in three space dimensions, i.e. solutions depending only on one space variable, say x . These are governed by the following equations of form (1).

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p + \tfrac{1}{2} |\mathbf{b}|^2)_x &= \zeta v_{xx}, \\ (\rho \mathbf{w})_t + (\rho v \mathbf{w} - a \mathbf{b})_x &= \mu \mathbf{w}_{xx} \\ \mathbf{b}_t + (v \mathbf{b} - a \mathbf{w})_x &= \nu \mathbf{b}_{xx}, \\ \mathcal{E}_t + (v(\mathcal{E} + p + \tfrac{1}{2} |\mathbf{b}|^2 - \tfrac{1}{2} a^2) - a \mathbf{w} \cdot \mathbf{b})_x &= (\zeta v v_x + \mu \mathbf{w} \cdot \mathbf{w}_x + \\ &\quad \nu \mathbf{b} \cdot \mathbf{b}_x + \kappa \theta_x)_x. \end{aligned} \quad (6)$$

Here $\rho > 0$ denotes density, $(v, \mathbf{w}) = (v, w^1, w^2) \in \mathbf{R}^3$ the vector of longitudinal and transverse velocity, $(a, \mathbf{b}) = (a, b^1, b^2) \in \mathbf{R}^3$ the magnetic field, and $\mathcal{E} > 0$ the total energy. Furthermore $p > 0$ stands for the pressure and $\theta > 0$ for the temperature. With internal energy $\epsilon > 0$ the total energy is given by

$$\mathcal{E} = \rho(\epsilon + \tfrac{1}{2}(v^2 + |\mathbf{w}|^2)) + \tfrac{1}{2}(a^2 + |\mathbf{b}|^2). \quad (7)$$

To close the system we assume our fluid to be a (thermically and calorically) **perfect gas**, i.e.

$$p = \rho R \theta, \quad \epsilon = c_V \theta, \quad (8)$$

where $R, c_V > 0$ are constants that denote the gas constant and the specific heat at constant specific volume satisfying

$$1 < 1 + R/c_V \equiv \gamma < 2. \quad (9)$$

γ is the adiabatic coefficient. Concerning dissipative effects the constants $\zeta, \mu > 0$ are positive combinations of the first and second fluid viscosity while $\nu, \kappa > 0$ are the coefficients of electrical resistivity and heat conduction. Since the divergence of the magnetic field is assumed to vanish a is only a constant parameter.

With $q = (m, j, \tilde{\mathbf{c}}, \mathbf{c}, e) \in \mathbf{R}^7$ and $s = 0$ the Rankine-Hugoniot equations (4) for (6) read

$$\begin{aligned} \rho v &= m, \\ mv + p + \frac{1}{2}|\mathbf{b}|^2 &= j, \\ m\mathbf{w} - a\mathbf{b} &= \tilde{\mathbf{c}}, \\ v\mathbf{b} - a\mathbf{w} &= \mathbf{c}, \\ mc_V\theta + \frac{m}{2}(v^2 + |\mathbf{w}|^2) + vp + v|\mathbf{b}|^2 - a\mathbf{w} \cdot \mathbf{b} &= e, \end{aligned} \quad (10)$$

while the profile ODE (5) takes the form

$$\begin{aligned} \rho v &= m, \\ \zeta \frac{dv}{dx} &= mv + p + \frac{1}{2}|\mathbf{b}|^2 - j, \\ \mu \frac{d\mathbf{w}}{dx} &= m\mathbf{w} - a\mathbf{b} - \tilde{\mathbf{c}}, \\ \nu \frac{d\mathbf{b}}{dx} &= v\mathbf{b} - a\mathbf{w} - \mathbf{c}, \\ \kappa \frac{d\theta}{dx} &= c_V m\theta - \frac{m}{2}(v^2 + |\mathbf{w}|^2) - \frac{1}{2}v|\mathbf{b}|^2 + a\mathbf{b} \cdot \mathbf{w} + vj + \mathbf{w} \cdot \tilde{\mathbf{c}} + \mathbf{b} \cdot \mathbf{c} - e. \end{aligned} \quad (11)$$

Both (10) and (11) are written in variables differing from the conserved quantities $\rho, \rho v, \rho\mathbf{w}, \mathbf{b}, \mathcal{E}$ since this is more convenient in the case considered here. The solutions of (10) are precisely the equilibria of (11). The fact that the first equation of (11) is algebraic instead of differential, reflects the degeneracy of the dissipation matrix D . Note that the assumption $s = 0$ can be made w.l.o.g., due to Galilean invariance.

Let us give an outline of the paper. In Section 2 we study the Rankine-Hugoniot equation (10) and the profile ODE (11) regarding their scaling properties. It turns out that five of the eight parameters a, q can be set to fixed

numbers. This simplification makes possible a complete bifurcation analysis for (10) in Sections 3 4, and 5. The main results of this paper concerning the bifurcation analysis are collected in Section 6, in particular in Theorems 8, 9, 10, and 11. The theorems provide detailed statements on the existence, number, and types of solutions of (10) for each value of the (rescaled) parameter q . At the same time it is derived what is the sign of the quantities density, pressure, and temperature so that unphysical rest points can be ruled out. Finally we discuss all regimes and illustrate them by numerous pictures in Section 7.

Closing the introduction we would like to stress that the ODE-system (11) has been the object of many old and recent investigations. For instance, consider [G59,C70,KL61,CS75,H84,MH90,W90,F91,FS95]. (Non)Existence of viscous profiles has been proven for a lot of scenarios but is up to now not completely understood for so-called intermediate shock waves. The existence of the latter makes the ODE system an instance of the complicate cases mentioned above. Note that there is nothing comparable in pure gas dynamics. Parts of our analysis can be viewed as reproducing well-known results that can be found in even standard textbooks [A63,C70]. A systematic global bifurcation analysis for the solutions of (10) as performed here is however not available in previous literature. The two crucial ingredients to our approach are (i): the parameterization by q (cf. discussion after (4)), and (ii): the particular scaling properties of the perfect-gas equation of state.

In a forthcoming paper we will present results on the local and global dynamics of (11) by means of numerical methods which have been introduced in [FR99,FR02]. The results will cover, in particular, cases which could not be treated analytically up to now and seem hardly to be accessible with present techniques. We will rely essentially on the analysis of the Rankine–Hugoniot locus which is the subject of the paper presented here.

2 Scaling properties

The number of parameters in (10) and (11) can be reduced essentially by using certain invariances. Neglecting the degenerate cases of $a = 0$ and $m = 0$ we will show that we can set without loss of generality

$$a = 1, \tilde{\mathbf{c}} = 0, \mathbf{c} = (c, 0)^T, \text{ for } c \geq 0, \text{ and } m = 1.$$

We note that –except the last one– all settings are well known and will be derived only for the sake of completeness. However, the setting $m = 1$, up to our knowledge, cannot be found in literature and leads to an interesting

rescaling of the resistivity coefficient ν .

We start by demonstrating that we can assume $a = 1$ in (11). Therefore set

$$\begin{aligned} v &= a\bar{v}, \mathbf{w} = a\bar{\mathbf{w}}, m = a\bar{m}, p = a^2\bar{p}, \mathbf{b} = a\bar{\mathbf{b}}, \tilde{\mathbf{c}} = a^2\bar{\tilde{\mathbf{c}}}, \\ \mathbf{c} &= a^2\bar{\mathbf{c}}, \theta = a^2\bar{\theta}, e = a^3\bar{e}. \end{aligned}$$

Note that the scaling leads to $\bar{p} = R\rho\bar{\theta}$. We observe, after re-denoting $\bar{v}, \bar{\mathbf{w}}, \bar{m}, \bar{p}, \bar{\mathbf{b}}, \bar{\tilde{\mathbf{c}}}, \bar{\mathbf{c}}, \bar{\theta}, \bar{e}$ again by $v, \mathbf{w}, m, p, \mathbf{b}, \tilde{\mathbf{c}}, \mathbf{c}, \theta, e$, that (11) becomes

$$\begin{aligned} \rho v &= m \\ \zeta \dot{v} &= mv + p + \frac{1}{2} |\mathbf{b}|^2 - j, \\ \mu \dot{\mathbf{w}} &= m\mathbf{w} - \mathbf{b} - \tilde{\mathbf{c}}, \\ \nu \dot{\mathbf{b}} &= v\mathbf{b} - \mathbf{w} - \mathbf{c}, \\ \kappa \dot{\theta} &= c_V m \theta - \frac{m}{2}(v^2 + |\mathbf{w}|^2) - \frac{1}{2}v|\mathbf{b}|^2 + \mathbf{b} \cdot \mathbf{w} + vj + \mathbf{w} \cdot \tilde{\mathbf{c}} + \mathbf{b} \cdot \mathbf{c} - e, \end{aligned} \tag{12}$$

where we used $\dot{} \equiv d/dx$. Replacing now

$$\mathbf{w} \text{ by } \bar{\mathbf{w}} + \frac{\tilde{\mathbf{c}}}{m}, \mathbf{c} \text{ by } \bar{\mathbf{c}} - \frac{\tilde{\mathbf{c}}}{m}, e \text{ by } \bar{e} + \frac{|\tilde{\mathbf{c}}|^2}{2m},$$

and re-denoting $\bar{\mathbf{w}}, \bar{\mathbf{c}}, \bar{e}$ as $\mathbf{w}, \mathbf{c}, e$, we see that –instead of (12)– we may henceforth work with

$$\begin{aligned} \rho v &= m \\ \zeta \dot{v} &= mv + p + \frac{1}{2} |\mathbf{b}|^2 - j, \\ \mu \dot{\mathbf{w}} &= m\mathbf{w} - \mathbf{b}, \\ \nu \dot{\mathbf{b}} &= v\mathbf{b} - \mathbf{w} - \mathbf{c}, \\ \kappa \dot{\theta} &= c_V m \theta - \frac{m}{2}(v^2 + |\mathbf{w}|^2) - \frac{1}{2}v|\mathbf{b}|^2 + \mathbf{b} \cdot \mathbf{w} + vj + \mathbf{b} \cdot \mathbf{c} - e. \end{aligned} \tag{13}$$

Now, letting

$$\rho = m^2 \bar{\rho}, v = \frac{\bar{v}}{m}, \mathbf{w} = \frac{\bar{\mathbf{w}}}{m}, \mathbf{c} = \frac{\bar{\mathbf{c}}}{m}, \theta = \frac{\bar{\theta}}{m^2}, e = \frac{\bar{e}}{m^2}, \nu = \frac{\bar{\nu}}{m^2}, x = \frac{\bar{x}}{m},$$

and re-denoting $\bar{\rho}, \bar{v}, \bar{\mathbf{w}}, \bar{\mathbf{c}}, \bar{\theta}, \bar{e}, \bar{\nu}, \bar{x}$ as $\rho, v, \mathbf{w}, \mathbf{c}, \theta, e, \nu, x$, we see that we can assume that $m = 1$ which leads to the following final form

$$\begin{aligned}
\rho v &= 1, \\
\zeta \dot{v} &= v + p + \frac{1}{2} |\mathbf{b}|^2 - j, \\
\mu \dot{\mathbf{w}} &= \mathbf{w} - \mathbf{b}, \\
\nu \dot{\mathbf{b}} &= v\mathbf{b} - \mathbf{w} - \mathbf{c}, \\
\kappa \dot{\theta} &= c_v \theta - \frac{1}{2} (|\mathbf{w}|^2 - 2\mathbf{b} \cdot \mathbf{w} + v|\mathbf{b}|^2) - \frac{v^2}{2} + jv + \mathbf{b} \cdot \mathbf{c} - e.
\end{aligned} \tag{14}$$

Using exactly the same transformations for the (more simple) system (10) results in the following problem. We seek for a state vector

$$w = (\rho, v, \mathbf{w}, \mathbf{b}, p) \tag{15}$$

such that

$$\begin{aligned}
\rho v &= 1, \\
v + p + \frac{1}{2} |\mathbf{b}|^2 &= j, \\
\mathbf{w} - \mathbf{b} &= 0, \\
v\mathbf{b} - \mathbf{w} &= \mathbf{c}, \\
\frac{\gamma}{\gamma-1} vp + \frac{1}{2} (v^2 + |\mathbf{w}|^2) + v|\mathbf{b}|^2 - \mathbf{w} \cdot \mathbf{b} &= e.
\end{aligned} \tag{16}$$

Note that the scaling for v, θ and (16) are consistent with the ideal gas law (8) for the pressure. The pressure satisfies (in final variables)

$$p = \rho R \theta = \frac{R \theta}{v}, \tag{17}$$

which has been used in (16) to write the temperature in terms of the pressure. Finally we recognize, for \mathbf{c} rotating, that solutions of (16) or (12) are invariant up to the same rotation with respect to \mathbf{w} and \mathbf{b} . Therefore we will choose for \mathbf{c} without loss of generality:

$$\mathbf{c} = (c, 0)^T, \quad c \geq 0. \tag{18}$$

Finally, to analyze the Rankine–Hugoniot system (11) it suffices to consider (13). Slightly abusing the notation from Section 1, it is therefore our task to

look for

$$w = (\rho, v, \mathbf{w}, \mathbf{b}, p) \in \mathcal{W} = \mathbf{R}_{>0} \times \mathbf{R}^5 \times \mathbf{R}_{>0}$$

satisfying (16).

We conclude the section by pointing out two important properties of the scaling by m .

For the viscous equations (6), (11), (14) the rescaling of m involves also one of the dissipation coefficients, namely the electrical resistivity ν . Even if a regime with extremely small/large resistivity ν (compared with the size of ζ, μ, κ) does not exist in nature it makes sense to study the ODE (11) in this regime since the mass flux m can vary arbitrarily.

To perform the rescaling $m = 1$, the linearity of the functions $p = p(\rho, \theta)$ and $\epsilon = \epsilon(\rho, \theta)$ with respect to density and temperature was essential. For example, a Van–der–Waals fluid, would not allow such scaling. The possibility of scaling the Rankine–Hugoniot equations and the profile ODE by m reflects a fundamental invariance property that the Euler/Navier–Stokes/MHD equations have precisely in the case of a perfect gas.

3 Bifurcation loci for the case $c > 0$

The goal of this and the next two sections is to obtain a complete picture about the existence, number, and types of states $w \in \mathcal{W}$ that satisfy (16) for fixed $(c, j, e) \in \mathbf{R}_{\geq 0} \times \mathbf{R}^2$. For the analysis, we first allow also nonpositive values of pressure p and temperature θ . Thereafter, we identify the regimes with positive pressure and temperature. We distinguish between the two cases $c > 0$ and $c = 0$, treating $c > 0$ in this and the next section, and the case $c = 0$ in Section 5.

Consider the full Rankine–Hugoniot conditions as given in (16). Since $\mathbf{b} = \mathbf{w}$ the choice $c > 0$ implies $v \neq 1$ and thus $b^1 = w^1 \neq 0$ respectively $b^2 = w^2 = 0$. For notational simplicity we define $b = b^1$ from now on.

Thus the study of the full system (16) can be reduced to the study of

$$\begin{aligned} (v-1)b &= c, \\ v + p + \frac{1}{2}b^2 &= j, \\ \frac{\gamma}{\gamma-1}vp + \frac{1}{2}(b^2 + v^2) + (v-1)b^2 &= e, \end{aligned} \tag{19}$$

and by resolving the first equation in (19) with respect to b we obtain

$$\begin{aligned} v + p + \frac{1}{2} \frac{c^2}{(v-1)^2} &= j, \\ \frac{\gamma}{\gamma-1} vp + \frac{1}{2} v^2 + \frac{1}{2} \frac{c^2}{(v-1)^2} + \frac{c^2}{v-1} &= e. \end{aligned} \quad (20)$$

The set of values (v, p) satisfying the first equation in (20) for fixed j is usually called the Rayleigh curve. Resolving the first two equations in (19) with respect to v and p , finally leads to the study of the zeros of the fourth order polynomial π with

$$\begin{aligned} \pi(b) &= b^4 + c(2 - \gamma)b^3 + (\gamma + 1 - 2\gamma j + 2(\gamma - 1)e)b^2 \\ &\quad + 2c(\gamma + 1 - \gamma j)b + (\gamma + 1)c^2. \end{aligned} \quad (21)$$

Consequently (16) can have up to four solutions. The bifurcation scenario can be completely analyzed from the study of π but, essentially, we will make use of the subsequent mapping motivated by (20)

$$g_c : \begin{cases} \mathbf{R}_{\geq 0} \setminus \{1\} \times \mathbf{R}_{\geq 0} \rightarrow & \mathbf{R}^2 \\ (v, p) & \mapsto \begin{pmatrix} J_c(v, p) \\ E_c(v, p) \end{pmatrix}, \end{cases} \quad (22)$$

where the functions J_c, E_c are defined by

$$\begin{aligned} J_c(v, p) &= v + p + \frac{1}{2} \frac{c^2}{(v-1)^2}, \\ E_c(v, p) &= \frac{\gamma}{\gamma-1} vp + \frac{1}{2} v^2 + \frac{1}{2} \frac{c^2}{(v-1)^2} + \frac{c^2}{v-1}. \end{aligned}$$

For the set of critical values of g_c (i.e. the set $\{g_c(v, p) \mid |Dg_c(v, p)| = 0\}$ there is (in the generic case) a change in the number of zeros of π or –equivalently– the number of solutions of (16). Vice versa each change of the number of solutions of (16) respectively number of zeros of π defines a critical value for g_c . The determinant of Dg_c

$$\begin{aligned} \det(Dg_c(v, p)) &= \left| \begin{pmatrix} 1 - \frac{c^2}{(v-1)^3} & 1 \\ \frac{\gamma}{\gamma-1} p + v - \frac{c^2}{(v-1)^3} - \frac{c^2}{(v-1)^2} & \frac{\gamma}{\gamma-1} v \end{pmatrix} \right| \\ &= \frac{1}{\gamma-1} \left(v \left(1 - \frac{c^2}{(v-1)^3} \right) - \gamma p \right) \end{aligned}$$

vanishes if and only if

$$p = p_*(v) \equiv \frac{1}{\gamma} v \left(1 - \frac{c^2}{(v-1)^3} \right).$$

Note that we have for p_*

$$p_*(v) > 0 \Leftrightarrow v \in (0, 1) \cup (1 + c^{\frac{2}{3}}, \infty) \text{ and } p_*(v) = 0 \Leftrightarrow v = 0, 1 + c^{\frac{2}{3}}. \quad (23)$$

We define the curves Γ_c^s, Γ_c^f and the sets $\mathcal{D}_c^f, \mathcal{D}_c^s$ by

$$\begin{aligned} \Gamma_c^s &= g_c(\{(v, p) \mid p = p_*(v), 0 \leq v < 1\}), \quad \mathcal{D}_c^s = (c^2/2, \infty) \times (-c^2/2, \infty), \\ \Gamma_c^f &= g_c(\{(v, p) \mid p = p_*(v), v \geq 1 + c^{2/3}\}), \quad \mathcal{D}_c^f = (j_c^*, \infty) \times (e_c^*, \infty), \end{aligned}$$

where

$$(j_c^*, e_c^*) = (1 + \frac{3}{2}c^{2/3}, \frac{1}{2} + \frac{3}{2}c^{2/3} + \frac{3}{2}c^{4/3}). \quad (24)$$

Lemma 1 [Properties of Γ_c^s, Γ_c^f]

(i) The functions J_c and E_c satisfy:

$$\begin{aligned} \frac{d}{dv} J_c(v, p_*(v)) &> 0 \quad (v \in [0, 1) \text{ or } v \in [1 + c^{2/3}, \infty)), \\ \frac{d}{dv} E_c(v, p_*(v)) &\geq 0 \quad (v \in [0, 1) \text{ or } v \in [1 + c^{2/3}, \infty)). \end{aligned}$$

In the last line equality holds if and only if $v = 0$. The curves Γ_c^s, Γ_c^f are graphs of monotone increasing functions in the (j, e) -plane.

- (ii) The curve Γ_c^s behaves for $v \rightarrow 1^-$ like the function $j \mapsto \frac{\gamma}{\gamma+1}j + \mathcal{O}(j^{2/3})$ for $j \rightarrow \infty$. The curve Γ_c^f approaches for $v \rightarrow \infty$ the graph of the function $j \mapsto (\gamma^2/2(\gamma^2 - 1))j^2$.
- (iii) $\Gamma_c^s \subseteq \bar{\mathcal{D}}_c^s, \Gamma_c^f \subseteq \bar{\mathcal{D}}_c^f$.

PROOF. The first assertion follows from $p_*(v) \geq 0$ for $v \in [0, 1) \cup [1 + c^{2/3}, \infty)$.

From the definition of p_* we obtain

$$p'_*(v) = \frac{1}{\gamma} \left(p_*(v) + \frac{3c^2v}{(v-1)^4} \right) > 0 \quad (v \in (0, 1) \text{ or } v \in [1 + c^{2/3}, \infty)). \quad (25)$$

Consequently we have for $v \in [0, 1)$ and $v \in [1 + c^{2/3}, \infty)$

$$\frac{d}{dv} J_c(v, p_*(v)) = 1 + p'_*(v) - \frac{c^2}{(v-1)^3} > 0. \quad (26)$$

Furthermore,

$$\frac{d}{dv} E_c(v, p_*(v)) = \frac{\gamma}{\gamma-1} (p_*(v) + v p'_*(v)) + v - \frac{c^2}{(v-1)^3} - \frac{c^2}{(v-1)^2}.$$

This leads by (23), (25), and $v \in [0, 1)$ to

$$\frac{d}{dv} E_c(v, p_*(v)) \geq v - \frac{c^2}{(v-1)^3} - \frac{c^2}{(v-1)^2} \geq \frac{c^2 v}{(1-v)^3} \geq 0,$$

where equality holds exactly for $v = 0$. Analogously, for $v \in [1 + c^{2/3}, \infty)$,

$$\frac{d}{dv} E_c(v, p_*(v)) > v - \frac{c^2}{(v-1)^3} - \frac{c^2}{(v-1)^2} \geq 1 + c^{2/3} - 1 - c^{2/3} = 0.$$

(i) is proven. (ii) and (iii) are direct consequences of the definition of Γ_c^f , Γ_c^s . \square

To illustrate the curves Γ_c^f , Γ_c^s we refer to the left-hand diagram in Figure 1.

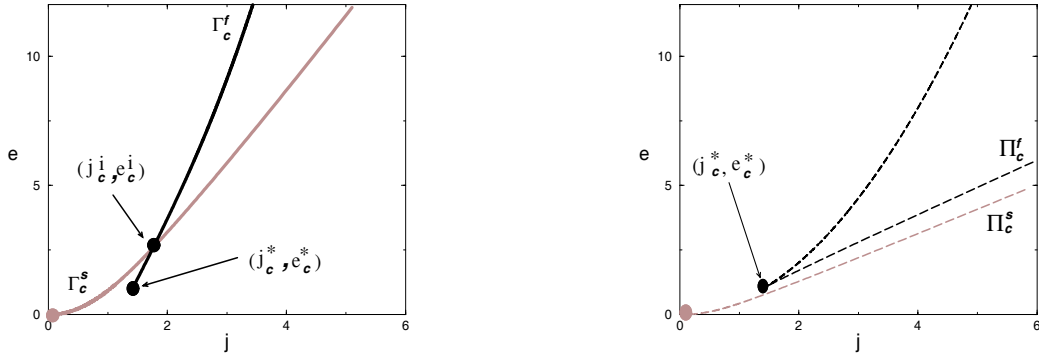


Fig. 1. The curves Γ_c^f, Γ_c^s (left) and Π_c^f, Π_c^s (right) for $c = 0.15$ and $\gamma = 7/5$.

According to the definition of the two curves Γ_c^s, Γ_c^f we define

Definition 2 A state vector $w \in \mathbf{R}^7$ that satisfies the Rankine–Hugoniot equations (16) with $0 < v < 1$ ($v > 1 + c^{2/3}$) is called a **slow (fast) state**.

Note that, for $c > 0$, there are no solutions with $v = 1$ or $b = 0$. The next lemma shows that the curves Γ_c^s, Γ_c^f indeed are bifurcation loci for the Rankine–Hugoniot relation.

Lemma 3 *Let $c > 0$.*

(i) *For $(j, e) \in \bar{\mathcal{D}}_c^f$, the Rankine Hugoniot relation (16) is satisfied by*

$$\left\{ \begin{array}{c} \text{no} \\ \text{one} \\ \text{two} \end{array} \right\} \text{ fast state}(s) \left\{ \begin{array}{c} \\ w_{0=1} \\ w_0, w_1 \text{ with } v_0 > v_1 \end{array} \right\} \text{ iff } (j, e) \text{ is } \left\{ \begin{array}{c} \text{left to} \\ \text{lying on} \\ \text{right to} \end{array} \right\}$$

the curve Γ_c^f , and, for $(j, e) \in \bar{\mathcal{D}}_c^s$, the Rankine Hugoniot relation (16) is satisfied by

$$\left\{ \begin{array}{c} \text{no} \\ \text{one} \\ \text{two} \end{array} \right\} \text{ slow state}(s) \left\{ \begin{array}{c} \\ w_{2=3} \\ w_2, w_3 \text{ with } v_2 > v_3 \end{array} \right\} \text{ iff } (j, e) \text{ is } \left\{ \begin{array}{c} \text{left to} \\ \text{lying on} \\ \text{right to} \end{array} \right\}$$

the curve Γ_c^s .

(ii) *The curves Γ_c^s and Γ_c^f intersect exactly in one point (j_c^i, e_c^i) with*

$$\begin{aligned} j_c^i &= \sqrt{\gamma+1} \frac{2-\gamma}{2\gamma} c + 1 + \frac{1}{\gamma}, \\ e_c^i &= \frac{(2-\gamma)^2}{8(\gamma-1)} c^2 - \frac{\gamma\sqrt{\gamma+1}}{2(\gamma-1)} c + \frac{\gamma+1}{2(\gamma-1)}. \end{aligned} \tag{27}$$

In this case we have exactly one solution $w_{2=3}$ of slow type and one solution $w_{0=1}$ of fast type.

PROOF. We show that there exists a vector (j, e) such that π has two positive and two negative zeros, and another vector (j, e) such that π has no zeros. (i) then follows from the definition of Γ_c^f, Γ_c^s as the range of the critical values of g_c and the fact that a bifurcation for g_c coincides with a bifurcation for π . Fix $e = -(2k-1)/2$. From Lemma 1 we know that $(j, -(2k-1)/2)$ is right to Γ_c^f, Γ_c^s for j big enough. Since $\pi(\pm b) \rightarrow \infty$ for $b \rightarrow \infty$ and $\pi(0) = (\gamma+1)c^2 > 0$ it suffices to show that there exists a positive and a negative $b \in \mathbf{R}$ with $\pi(b) < 0$ for j big enough. An easy calculation shows that this holds in particular for $b = c$ and $b = -2c$.

To show that there exists (j, e) such that π has no solutions choose $j = (\gamma+1)/\gamma$ and let e big enough so that we are left of Γ_c^f, Γ_c^s (Lemma 1). Then we have

$$\pi(b) - (\gamma+1)c^2 = b^2(b^2 + c(2-\gamma)b + 2(\gamma-1)e - \gamma - 1).$$

The right hand side is positive for e big enough. This implies $\pi(b) > 0$.

From (i) we know that the curves Γ_c^f, Γ_c^s define the locus where the number of solution of (16) changes. Consequently in an intersection point we have exactly one solution of slow and one of fast type. To prove that there is only one point of intersection we show that there is only one pair (j_c^i, e_c^i) such that π has one positive and one negative solution.

If we denote the solutions by $-b_n, b_p > 0$ we obtain

$$\begin{aligned}\pi(b) &= (b - b_n)^2(b - b_p)^2 \\ &= b^4 - 2(b_p + b_n)b^3 + (b_n^2 + 4b_nb_p + b_p^2)b^2 - 2b_nb_p(b_n + b_p)b + b_n^2b_p^2,\end{aligned}$$

which implies by (21) and $b_1b_2 < 0$ the following relations for j_c^i, e_c^i :

$$\begin{aligned}2c(\gamma + 1 - \gamma j) &= -\sqrt{\gamma + 1}c^2(2 - \gamma), \\ \gamma + 1 - 2\gamma j + 2(\gamma - 1)e &= \frac{1}{4}c^2(2 - \gamma)^2 - 2\sqrt{\gamma + 1}c.\end{aligned}$$

The solution of this linear system is unique, with solutions (j_c^i, e_c^i) as given in (27). \square

4 Zero pressure loci for $c > 0$

The preceding lemma states that there can be up to four different solutions w of the Rankine–Hugoniot equations (which of course is evident also from the fact that π is a fourth order polynomial). Following classical work [G59, KL61] these have been ordered such that the v -component decreases with the index:

$$v_0 > v_1 > v_2 > v_3. \tag{28}$$

According to this numbering it can be proven that

$$S(v_0, \theta_0) < S(v_1, \theta_1) < S(v_2, \theta_2) < S(v_3, \theta_3) \tag{29}$$

holds with $S(v, \theta) = v^{\gamma-1}\theta$. For the proof of this observation and its relations to the (physical) entropy associated to the MHD system we refer to [FS95].

Finally we have to check whether the states are admissible, i.e. whether the density, pressure, and temperature are positive. The definition of g_c (22) shows

that the density is positive if and only if $(j, e) \in \bar{\mathcal{D}}_c^s$. Moreover pressure and temperature have always the same sign by (17). Therefore, it suffices to analyze the sign of the pressure. The inequalities (28), (29), and the gas law (17) lead to

$$p_0 < p_1 < p_2 < p_3. \quad (30)$$

We consider again the mapping g_c and search for points $(j, e) \in \bar{\mathcal{D}}_c^s$ resp. $\bar{\mathcal{D}}_c^f$ where the pressure of the solutions can change sign. We define the associated sets Π_c^s, Π_c^f by

$$\Pi_c^s = g_c(\{(v, p) \mid p = 0, 0 < v < 1\}),$$

$$\Pi_c^f = g_c(\{(v, p) \mid p = 0, v > 1\}).$$

We summarize the basic properties of Π_c^s, Π_c^f in

Lemma 4(i) *The functions J_c and E_c satisfy:*

$$\begin{aligned} \frac{d}{dv} J_c(v, 0), \frac{d}{dv} E_c(v, 0) &> 0 & \forall v \in (0, 1), \\ \frac{d}{dv} J_c(v, 0), \frac{d}{dv} E_c(v, 0) &< 0 & \forall v \in (1, v^*), \\ \frac{d}{dv} J_c(v, 0), \frac{d}{dv} E_c(v, 0) &= 0 & v = v^*, \\ \frac{d}{dv} J_c(v, 0), \frac{d}{dv} E_c(v, 0) &> 0 & \forall v \in (v^*, \infty), \end{aligned}$$

i.e. the curve Π_c^s is the graph of a monotone increasing function in the (j, e) -plane. Π_c^f is the graph of a monotone decreasing function for $1 < v < v^ = 1 + c^{2/3}$. It has a singularity of cusp type for $v = v^*$ in (j_c^*, e_c^*) and is monotone increasing for $v > v^*$. The straight line tangent to the cusp is given by $j \mapsto (1 + c^{2/3})j - 1/2 - c^{2/3}$.*

- (ii) *For $v \rightarrow \infty$ the curve Π_c^f tends to the graph of the function $j \mapsto \frac{1}{2}j^2$, and, for $v \rightarrow 1^+$, to the graph of a function of type $j \mapsto j + \mathcal{O}(j^{1/2})$.*
- (iii) $\Pi_c^s \subseteq \bar{\mathcal{D}}_c^s, \Pi_c^f \subseteq \bar{\mathcal{D}}_c^f$.

PROOF. The statements of (i) follow from

$$\frac{d}{dv} J_c(v, 0) = 1 - \frac{c^2}{(v-1)^3}, \quad \frac{d}{dv} E_c(v, 0) = v - \frac{c^2}{(v-1)^3} - \frac{c^2}{(v-1)^2},$$

and straightforward calculations. To analyze the behaviour of Π_c^f in $v = v^*$ we compute the Taylor expansions of $J_c(v, 0), E_c(v, 0)$ in v^* :

$$J_c(v, 0) = j_c^* + \frac{3}{2}c^{-2/3}(v - v^*)^2 + \dots,$$

$$E_c(v, 0) = e_c^* + \frac{3}{2}(1 + c^{-2/3})(v - v^*)^2 + \dots.$$

Consequently we have a cusp with tangential slope

$$\lim_{v \rightarrow v^*} \frac{E(v, 0) - e_c^*}{J(v, 0) - j_c^*} = 1 + c^{2/3}.$$

Statement (ii) and (iii) are obvious by the definition of Π_c^f, Π_c^s . \square

Note that the cusp point (j_c^*, e_c^*) coincides with the "starting point" of Γ_c^f . An example of the curves Π_c^s, Π_c^f is displayed in Figure 1. The next lemma shows that the pressure vanishes for values in Π_c^s, Π_c^f generically, i.e the pressure changes sign when (j, e) passes the curves transversally.

Lemma 5 *Let $c > 0$ and $(j, e) \in \bar{\mathcal{D}}_c^s$.*

(i) *If (16) admits two slow states w_2, w_3 and (j, e) is*

$$\left\{ \begin{array}{c} \text{above} \\ \text{on} \\ \text{below} \end{array} \right\} \text{ the curve } \Pi_c^s \text{ we have } \left\{ \begin{array}{c} p_2, p_3 > 0 \\ p_2 = 0, p_3 > 0 \\ p_2 < 0, p_3 > 0. \end{array} \right\}$$

(ii) *If (16) admits two fast states w_0, w_1 and (j, e) is*

$$\left\{ \begin{array}{c} \text{above} \\ \text{on} \\ \text{below} \end{array} \right\} \text{ the part of } \Pi_c^f \text{ with } v > 1 + c^{2/3} \text{ we have } \left\{ \begin{array}{c} p_0 > 0 \\ p_0 = 0, \\ p_0 < 0. \end{array} \right\}$$

(iii) *If (16) admits two fast states w_0, w_1 and (j, e) is*

$$\left\{ \begin{array}{c} \text{above} \\ \text{on} \\ \text{below} \end{array} \right\} \text{ the part of } \Pi_c^f \text{ with } 1 < v < 1 + c^{2/3} \text{ we have } \left\{ \begin{array}{c} p_1 > 0 \\ p_1 = 0 \\ p_1 < 0. \end{array} \right\}$$

- (iv) Π_c^s and Π_c^f have no joint point. Π_c^s is located below Π_c^f in the (j, e) -plane for $j > j_c^*$.
- (v) The curves Π_c^s and Γ_c^s touch only for $v = 0$ in $(c^2/2, -c^2/2)$. Π_c^s is located below Γ_c^s in the (j, e) -plane. Π_c^f is located below Γ_c^f . The curves touch for $v = 1 + c^{2/3}$ in (j_c^*, e_c^*) .

PROOF. First, we collect some properties of the polynomial π and the pressure. Consider π for parameters $e = e^1, e^2 \geq -c^2/2$ and fixed values of $c > 0$ and $j \geq c^2/2$. The b -components of slow states w_0, w_1 satisfy $b_0 < b_1$ (cf. Lemma 3 and $b = c/(v-1)$). Since b_0, b_1 are zeros of π , and since only the coefficient in front of b^2 in π depends linearly on e we deduce

$$e^1 < (>) e^2 \Rightarrow b_0(e^1) > (<) b_0(e^2), \quad b_1(e^1) < (>) b_1(e^2). \quad (31)$$

Analogous arguments for w_2 and $(j, e^1), (j, e^2) \in \bar{D}_c^f$ lead to

$$e^1 < (>) e^2 \Rightarrow b_2(e^1) > (<) b_2(e^1). \quad (32)$$

For a state $w \in \mathbf{R}^7$ solving (16), the pressure satisfies the equation

$$p = \wp(b) \equiv -\frac{1}{2}b^2 - \frac{c}{b} + j - 1, \quad (33)$$

where \wp is independent of e and is characterized by (cf. Figure 2)

- (a) The function \wp is strictly increasing for $b < 0$ with $\wp(b) \rightarrow \infty$ ($\wp(b) \rightarrow -\infty$) for $b \rightarrow 0^-$ ($b \rightarrow -\infty$),
- (b) For $b > 0$ the function \wp is concave with maximum in $b_{max} = c^{1/3}$, $\wp(b_{max}) = j - (1 + 3c^{2/3}/2)$. We have $\wp(b) \rightarrow -\infty$ for $b \rightarrow 0^+$ and $b \rightarrow \infty$.

Now, for fixed $\bar{j} > j_c^*$, let $S = \{(\bar{j}, e) \mid e > -c^2/2\}$. It suffices to prove (i),(ii),(iii) for $(j, e) \in S$.

If we have $\bar{j} > j_c^*$, Lemma 4(ii) shows that S intersects the upper part of Π_c^f , the lower part of Π_c^f , and Π_c^s exactly once. Consider the intersection point with the upper part of Π_c^f where the pressure of at least one fast state is zero.

(30) tells us that this is w_0 . The monotonicity of \wp in a small neighbourhood of its negative zero(cf. (b)), the fact that \wp is independent of e , and (31) show that p_0 changes sign when crossing the upper part of Π_c^f . p_0 will stay negative when decreasing e further which follows again from

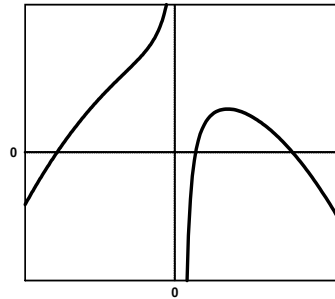


Fig. 2. Graph of \wp .

(31) and \wp being independent of e .

Therefore we have $p_1 = 0$ for the intersection point of S with the lower part of Π_c^s . For $j > c^2/2$ the same arguments and (a) show that p_2 changes sign when crossing Π_c^s while $p_3 > 0$ always.

The statement (iv) follows directly from the inequalities (30).

The curves Γ_c^s and Γ_c^f can take the same value in the (j, e) -plane iff $v = 0$ since p_* vanishes iff $v = 0$ for $v \in [0, 1)$ (cf. (23)). Recall that p_* defines the values of the pressure on Γ_c^s and $p = 0$ on Π_c^s by definition. From $p'_* > 0$ and (i) we conclude that Γ_c^s is above Π_c^s in the (j, e) -plane.

This is the first part of (v), the second follows analogously. \square

We note that the zeros of the function \wp in the proof of Lemma 5 give natural bounds on the b -components of the states w_0, w_1, w_2 with positive density and pressure. The b -component of states w_3 with positive density is trivially estimated from below by $-c$ by the first line in (19). These bounds might be useful to derive quantitative results on the global dynamics of the ODE-system (14).

We now show that there exists a critical value \bar{c} such that

$$c = \bar{c} \Leftrightarrow j_c^* = j_c^i, e_c^* = e_c^i,$$

or in words: for $c = \bar{c}$ the (unique) intersection point of Γ_c^f and Γ_c^s coincides with the starting point of Γ_c^f . When enforcing $j_c^* = j_c^i$ we obtain from (24) and (27) in terms of $d = c^{1/3}$:

$$d^3 - \frac{3\gamma}{\sqrt{\gamma+1}(2-\gamma)}d^2 + \frac{2}{\sqrt{\gamma+1}(2-\gamma)} = 0.$$

The polynomial has two positive zeros for $1 < \gamma < 2$. Only the smallest zero can be $\bar{d} = \bar{c}^{1/3}$. To see that, assume that for the smallest zero we do not have $e_c^* = e_c^i$. Then Γ_c^f has vertical tangent somewhere. This is not possible by Lemma 1. A tedious but straightforward calculation shows that \bar{d} is given by

$$\bar{d} = -\frac{\gamma}{(2-\gamma)\sqrt{\gamma+1}} \left(-1 + \cos\left(\frac{\phi}{3}\right) - \sqrt{3} \sin\left(\frac{\phi}{3}\right) \right),$$

$$\phi = \arg(-(4-3\gamma^2) + (2-\gamma)(\gamma+2)\sqrt{\gamma+1}\sqrt{\gamma-1}i).$$

It can be checked that this formula implies $e_c^* = e_c^i$, thus we have found \bar{c} .

5 Bifurcation loci and zero pressure loci for the case $c = 0$

This case was already discussed in [F87]. In view of the Rankine Hugoniot equations (16) the condition $c = 0$ implies $\mathbf{b} = 0$ or $v = 1$.

Definition 6 *States $w \in \mathbf{R}^7$ that satisfy (16) with $\mathbf{b} = 0$ are called **gasdynamic** while states satisfying (16) with $v = 1$ are **Alfvén-type** states.*

For $\mathbf{b} = 0$ the system (16) simplifies to a (2×2) -system for (v, p)

$$\begin{aligned} v + p &= j, \\ \frac{\gamma}{\gamma - 1}vp + \frac{1}{2}v^2 &= e, \end{aligned} \tag{34}$$

while $v = 1$ leads to a (2×2) -system for $(|\mathbf{b}|^2, p)$

$$\begin{aligned} 1 + p + \frac{1}{2}|\mathbf{b}|^2 &= j, \\ \frac{\gamma}{\gamma - 1}p + \frac{1}{2} + \frac{1}{2}|\mathbf{b}|^2 &= e. \end{aligned} \tag{35}$$

In analogy to the definition of the critical loci in Sections 3, 4 we define with $g = g(j) = \frac{\gamma^2}{2(\gamma^2 - 1)}j^2$ and $\alpha = \alpha(j) = \frac{\gamma}{\gamma - 1}j - \frac{\gamma + 1}{2(\gamma - 1)}$ the curves $\Gamma^g, \Gamma^{\mathcal{A}}, \Pi^g, \Pi^{\mathcal{A}}$ by

$$\Gamma^g = \{(j, e) \in \mathbf{R}_{\geq 0}^2 \mid e = g(j)\},$$

$$\Gamma^{\mathcal{A}} = \{(j, e) \in [1, \infty) \times [1/2, \infty) \mid e = \alpha(j)\},$$

and

$$\Pi^g = \{(j, e) \in \mathbf{R}_{\geq 0}^2 \mid e = j^2/2\},$$

$$\Pi^{\mathcal{A}} = \{(j, e) \in [1, \infty) \times [1/2, \infty) \mid e = j - 1/2\}.$$

Straightforward calculations imply

Lemma 7 *Let $c = 0$.*

(i) *For $(j, e) \in \mathbf{R}_{\geq 0}^2$, the Rankine–Hugoniot relation (16) is satisfied by*

$$\left\{ \begin{array}{c} \text{no} \\ \text{one} \\ \text{two} \end{array} \right\} \text{ state}(s) \text{ of gasdynamic type } \left\{ \begin{array}{c} w_{0=3} \\ w_0, w_3 \end{array} \right\} \text{ iff } (j, e) \text{ is } \left\{ \begin{array}{c} \text{left to} \\ \text{lying on} \\ \text{right to} \end{array} \right\}$$

the curve Γ^g .

Furthermore the vector $(\mathbf{b}_{0/3}, v_{0/3}, p_{0/3})$ is given by

$$\left(0, 0, \frac{\gamma}{\gamma+1}j \pm \sqrt{2\frac{\gamma-1}{\gamma+1}(g(j)-e)}, \frac{2\gamma+1}{\gamma+1}j \mp \sqrt{2\frac{\gamma-1}{\gamma+1}(g(j)-e)} \right),$$

and

$$(\mathbf{b}_{0=3}, v_{0=3}, p_{0=3}) = \left(0, 0, \frac{\gamma}{\gamma+1}j, \frac{1}{\gamma+1}j \right).$$

(ii) *For $(j, e) \in [1, \infty) \times [1/2, \infty)$, the Rankine–Hugoniot relation (16) is satisfied by*

$$\left\{ \begin{array}{c} \text{no} \\ \text{one} \\ \text{a set } \mathcal{A} \text{ of} \end{array} \right\} \text{ state}(s) \text{ of Alfvén-type iff } (j, e) \text{ is } \left\{ \begin{array}{c} \text{left to} \\ \text{lying on} \\ \text{right to} \end{array} \right\}$$

the curve $\Gamma^{\mathcal{A}}$. The states $w_{\mathcal{A}} \in \mathcal{A}$ satisfy for $r = \sqrt{2(\gamma-1)(\alpha(j)-e)}$

$$v_{\mathcal{A}} = 1, |\mathbf{b}_{\mathcal{A}}| = |\mathbf{w}_{\mathcal{A}}| = r, p_{\mathcal{A}} = (\gamma-1)(e - j + \frac{1}{2}).$$

(iii) *The states $w_3, w_{0=3}$ are always in \mathcal{W} .*

If the state w_0 exists we have $p_0 < 0$, $(p_0 = 0)$, $[p_0 > 0]$, iff (j, e) is left to (lying on) [right to] the curve Π^g .

If states $w_{\mathcal{A}}$ exist we have $p_{\mathcal{A}} < 0$, $(p_{\mathcal{A}} = 0)$, $[p_{\mathcal{A}} > 0]$, iff (j, e) is left to (lying on) [right to] the curve $\Pi^{\mathcal{A}}$.

6 Complete bifurcation scenario — theorems

In the preceding sections we determined the bifurcation and zero pressure loci for the Rankine–Hugoniot system (16) with the free parameters c, j, e . In this and the final section we collect main features of the bifurcation scenario into

a complete picture obtained in the (j, e) -plane for each value of $c \geq 0$. We start with the theorem concerning the position of the critical loci.

Theorem 8 *Let $c > 0$.*

Then there exist curves $\Gamma_c^f, \Gamma_c^s, \Pi_c^s, \Pi_c^f$ in the (j, e) -plane such that

- (i) *for $(j, e) \in \Gamma_c^f$, the two fast states w_0, w_1 exist and coincide,*
- (ii) *for $(j, e) \in \Gamma_c^s$, the two slow states w_2, w_3 exist and coincide,*
- (iii) *Π_c^f has two branches forming a cusp, and, for $(j, e) \in \Pi_c^f$, w_0 exists with $p_0 = 0$ on one branch and w_1 exists with $p_1 = 0$ on the other branch,*
- (iv) *for $(j, e) \in \Pi_c^s$, w_2 exists with $p_2 = 0$,*

and

- (v) *Γ_c^f starts in the same point where Π_c^f has the cusp,*
- (vi) *the curves Γ_c^f and Γ_c^s intersect in exactly one point for $c < \bar{c}$, and do not intersect for $c > \bar{c}$.*

PROOF. This is a direct consequence of the Lemmata 1, 3, 4, 5. \square

The second theorem covers the situation for regular values in the (j, e) -plane.

Theorem 9 *Let $c \in (0, \bar{c})$. The set $\mathcal{D}_c^s \cup \mathcal{D}_c^f \setminus \{\Gamma_c^f, \Gamma_c^s, \Pi_c^f, \Pi_c^s\}$ is partitioned into the bounded open set*

- (i) *\mathbf{I}_c where four states w_0, w_1, w_2, w_3 with $p_0, p_1, p_2, p_3 > 0$ exist, with boundaries $\Gamma_c^f, \Gamma_c^s, \Pi_c^f$,*

and the unbounded open sets

- (ii) *A_c where no (fast or slow) states exist, with boundaries Γ_c^f, Γ_c^s ,*
- (iii) *B_c where states w_0, w_1 with $p_0, p_1 > 0$ exist, with boundaries $\Gamma_c^f, \Gamma_c^s, \Pi_c^f$,*
- (iv) *C_c where states w_0, w_1 with $-p_0, p_1 > 0$ exist, with boundaries Γ_c^s, Π_c^f ,*
- (v) *D_c where states w_0, w_1, w_2, w_3 with $-p_0, p_1, p_2, p_3 > 0$ exist, with boundaries Γ_c^s, Π_c^f ,*
- (vi) *E_c where either four states w_0, w_1, w_2, w_3 with $-p_0, -p_1, p_2, p_3 > 0$ or states w_2, w_3 with $p_2, p_3 > 0$ exist, with boundaries $\Gamma_c^f, \Gamma_c^s, \Pi_c^f, \Pi_c^s$,*
- (vii) *F_c where either four states w_0, w_1, w_2, w_3 with $-p_0, -p_1, -p_2, p_3 > 0$ or states w_2, w_3 with $-p_2, p_3 > 0$ exist, with boundary Π_c^s .*

Let $c = \bar{c}$. Then the set $\mathcal{D}_c^s \cup \mathcal{D}_c^f \setminus \{\Gamma_c^f, \Gamma_c^s, \Pi_c^f, \Pi_c^s\}$ is partitioned into unbounded open sets A_c, \dots, F_c . The sets A_c, B_c, C_c, D_c, F_c satisfy (ii), (iii), (iv), (v), (vii), while E_c satisfies (vi) concerning the properties of the states but now has boundaries $\Gamma_c^s, \Pi_c^f, \Pi_c^s$.

Let $c > \bar{c}$. Then the set $\mathcal{D}_c^s \cup \mathcal{D}_c^f \setminus \{\Gamma_c^f, \Gamma_c^s, \Pi_c^f, \Pi_c^s\}$ is partitioned into unbounded open sets A_c, \dots, F_c . The sets B_c, C_c, D_c, F_c satisfy (iii), (iv), (v), (vii), while A_c satisfies (ii) concerning the properties of the states but now has boundaries $\Gamma_c^f, \Gamma_c^s, \Pi_c^f$ and E_c satisfies (vi) concerning the properties of the states but now has boundaries $\Gamma_c^s, \Pi_c^f, \Pi_c^s$.

PROOF. The proof follows easily from the asymptotic behavior of the curves $\Gamma_c^f, \Gamma_c^s, \Pi_c^f, \Pi_c^s$, their relative position in the (j, e) -plane, and the (non)existence of intersections as stated in the Lemmata 1, 3, 4, 5. \square

We want to stress the fact that the bounded set \mathbf{I}_c where all **four** states w_0, w_1, w_2, w_3 with **positive** pressure exist occurs for $c \in (0, \bar{c})$ only.

To complete our analysis we now turn to the case $c = 0$. Although some statements are trivial we add them to clarify the comparison with the case $c \in \{0, \bar{c}\}$ as stated in Theorems 8, 9 above.

Theorem 10 *Let $c = 0$. Then there exist curves $\Gamma_0^g, \Gamma_0^A, \Pi_0^g, \Pi_0^A$ in the (j, e) -plane such that*

- (i) *for $(j, e) \in \Gamma^g$, the two states w_0, w_3 of gasdynamic type exist and coincide,*
- (ii) *for $(j, e) \in \Gamma^A$, one and only one state of Alfvén-type w_A exists,*
- (iii) *for $(j, e) \in \Pi^g$, w_0 exists with $p_0 = 0$,*
- (iv) *for $(j, e) \in \Pi^A$, a ring of states w_A of Alfvén-type exists with $p_A = 0$,*

and

- (v) *Γ^A and Π^A start in the same point which also lies on Π^g ,*
- (vi) *the curves Γ^g and Γ^A intersect in exactly one point.*

PROOF. It follows from Lemma 7.

Theorem 11 *Let $c = 0$. The set $\mathbf{R}_{>0}^2 \setminus \{\Gamma^g, \Gamma^A, \Pi^g, \Pi^A\}$ is partitioned into the bounded open set*

- (i) \mathbf{I}_0 *where two states w_0, w_3 of gasdynamic type and a ring of states w_A of Alfvén-type with $p_0, p_3, p_A > 0$ exist, with boundaries Γ^A, Π^g ,*

and the open sets

- (ii) A_0 *where no states (of gasdynamic or Alfvén-type) exist, with boundary Γ^g ,*
- (iii) B_0 *where states of gasdynamic type w_0, w_3 with $p_0, p_3 > 0$ exist, with boundaries $\Gamma^g, \Gamma^A, \Pi^g$,*

- (iv) C_0 where states of gasdynamic type w_0, w_3 with $-p_0, p_3 > 0$ exist, with boundaries Π^g, Γ^A ,
- (v) D_0 where two states w_0, w_3 of gasdynamic type and a ring of states w_A of Alfvén-type with $-p_0, p_3, p_A > 0$ exist, with boundaries Π^g, Γ^A, Π^A ,
- (vi) E_0 where two states w_0, w_3 of gasdynamic type with $p_0, p_3 > 0$ exist, with boundaries $\Gamma^g, \Gamma^A, \Pi^g$,
- (vii) F_0 where either two states w_0, w_3 of gasdynamic type and a ring of states w_A of Alfvén-type with $-p_0, p_3, -p_A > 0$ or two states w_0, w_3 of gasdynamic type with $p_0, -p_3 > 0$ exist, with boundaries Π^g, Π_c^A .

While A_0, B_0, C_0, D_0, F_0 are unbounded E_0 is bounded.

PROOF. It follows also from Lemma 7.

7 Complete bifurcation scenario — diagrams

To illustrate the four cases $c < \bar{c}$, $c = \bar{c}$, $c > \bar{c}$, and $c = 0$ we display the curves and sets from Theorems 8, 9 10, 11.

In Figure 3 the partition of $\mathcal{D}_c^s \cup \mathcal{D}_c^f$ is shown for the case $c = 0.15 \in (0, \bar{c})$, $\gamma = 7/5$. In particular, for $c \in (0, \bar{c})$, there is the bounded region \mathbf{I}_c which allows for all four states w_0, \dots, w_3 having positive density, temperature, and pressure. Figure 4 shows the partition of $\mathcal{D}_c^s \cup \mathcal{D}_c^f$ for the cases $c = \bar{c}$ and $c > \bar{c}$.

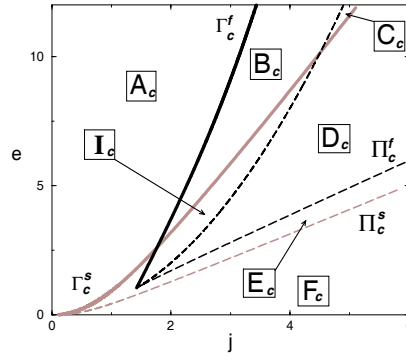


Fig. 3. The curves $\Gamma_{0.15}^s, \Gamma_{0.15}^f, \Pi_{0.15}^f, \Pi_{0.15}^s$ and the partition of $\mathcal{D}_{0.15}^s \cup \mathcal{D}_{0.15}^f$ into the sets $A_{0.15}, \dots, F_{0.15}$ and $\mathbf{I}_{0.15}$.

We discuss in more detail the possible configurations of the states w_0, \dots, w_3 in the case $c = 0.15 \in (0, \bar{c})$ and $(j, e) \in I_{0.15}, A_{0.15}, \dots, F_{0.15}$.

Starting with $(j, e) \in A_{0.15}$ we have no solutions, i.e. the graphs of the relations

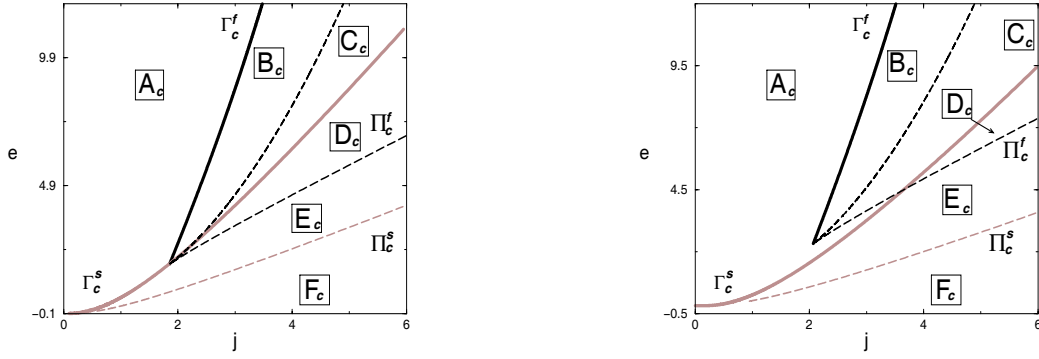


Fig. 4. The curves $\Gamma_c^s, \Gamma_c^f, \Pi_c^f, \Pi_c^s$ and the partition of $\mathcal{D}_c^s \cup \mathcal{D}_c^f$ into the sets A_c, \dots, F_c for $c = \bar{c}$ and $c = 0.6 > \bar{c}$, respectively.

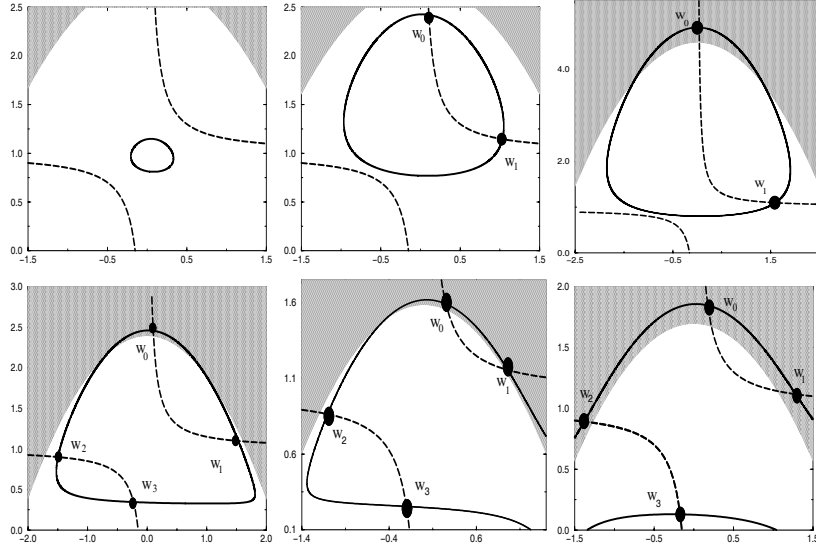


Fig. 5. The first row shows the location of the existing states together with the regions of positive (no background) and negative (grey background) pressure in the (b, v) -plane for $(j, e) \in A_{0.15}, B_{0.15}, C_{0.15}$; the second row shows the same for $(j, e) \in D_{0.15}, E_{0.15}, F_{0.15}$. The lines correspond to the graphs of the nullclines given by (36).

$$\begin{aligned} (v-1)b &= c, \\ v + \frac{\gamma-1}{\gamma v} \left(e - \frac{1}{2}v^2 - \frac{1}{2} \frac{c^2}{(v-1)^2} - \frac{c^2}{v-1} \right) &= j, \end{aligned} \quad (36)$$

derived from (19) by eliminating p , do not intersect in the (b, v) -plane. This can be seen in the upper left picture in Figure 5. Points (b, v) lying in the grey hatched area denote states which would have nonpositive pressure (when

being rest points). This area is bounded by the graph of the parabola

$$v = v(b) = -\frac{1}{2}b^2 + j.$$

For $(j, e) \in B_{0.15}$, we have two fast states w_0, w_1 with positive pressure whereof p_0 gets negative when (j, e) switches to $C_{0.15}$ (cf. second and third picture in the first row, Figure 5). Crossing $\Gamma_{0.15}^f$, also the lower branch of the hyperbola intersects with the graph of the other curve in two slow states w_2, w_3 . Then all four states exist, p_0 always being negative. For $(j, e) \in E_{0.15}$, also p_1 becomes negative and for $(j, e) \in F_{0.15}$ additionally p_2 is negative. All these scenarios are illustrated in the second row of Figure 5.

We focus on the particularly interesting cases $(j, e) \in \mathbf{I}_{0.15}$, $(j, e) \in \partial\mathbf{I}_{0.15}$. In the latter case the four states w_0, \dots, w_3 with positive pressure that exist for $(j, e) \in \mathbf{I}_{0.15}$, undergo local bifurcations. The different local bifurcations are displayed in Figure 6. For the boundary cases $(j, e) \in \partial\mathbf{I}_{0.15} \cap \Gamma_{0.15}^f$, $(j, e) \in \partial\mathbf{I}_{0.15} \cap \Gamma_{0.15}^s$, the two states w_0, w_1 collapse to $w_{0=1}$, respectively the two states w_2, w_3 collapse to $w_{2=3}$. If and only if $(j, e) = (j_c^i, e_c^i)$ we have two states $w_{0=1}$ and $w_{2=3}$.

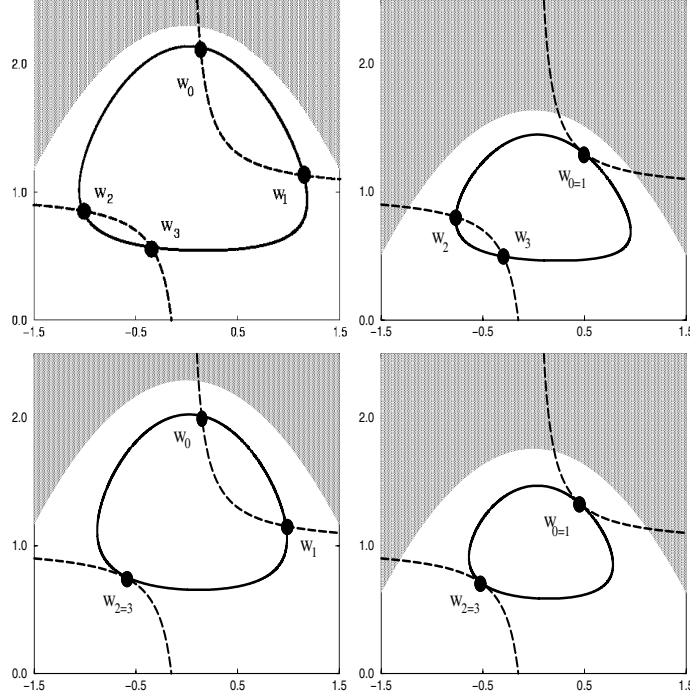


Fig. 6. The four pictures show the configuration of states with positive pressure in the bv -plane for $(j, e) \in \mathbf{I}_{0.15}$, $(j, e) \in \partial\mathbf{I}_{0.15} \cap \Gamma_{0.15}^f$, $(j, e) \in \partial\mathbf{I}_{0.15} \cap \Gamma_{0.15}^s$, $(j, e) = (j_c^i, e_c^i)$, respectively.

Finally, for the sake of illustration of Theorems 10, 11 treating the case $c = 0$ consider Figure 7.

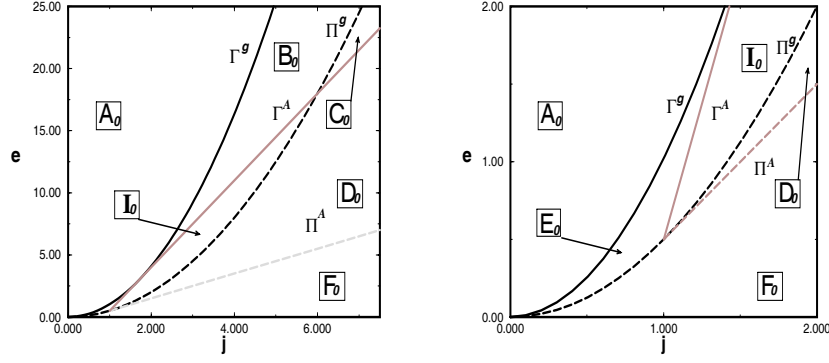


Fig. 7. The critical curves and the partition of the (j, e) -parameter space for the case $c = 0$. The right hand figure shows a zoom into the lower left corner of the left hand figure.

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