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Abstract

Steiner symmetrization is known not to increase perimeter of sets in $\mathbb{R}^n$. The sets whose perimeter is preserved under this symmetrization are characterized in the present paper.

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1 Introduction and Main Results

Steiner symmetrization, one of the simplest and most powerful symmetrization processes ever introduced in analysis, is a classical and very well-known device, which has seen a number of remarkable applications to problems of geometric and functional nature. Its importance stems from the fact that, besides preserving Lebesgue measure, it acts monotonically on several geometric and analytic quantities associated with subsets of $\mathbb{R}^n$. Among these, perimeter certainly holds a prominent position. Actually, the proof of the isoperimetric property of the ball was the original motivation for Steiner to introduce his symmetrization in [17].
The main property of perimeter in connection with Steiner symmetrization is that if E is any set of finite perimeter $P(E)$ in $\mathbb{R}^n$, $n \geq 2$, and H is any hyperplane, then also its Steiner

symmetral E^s about H is of finite perimeter, and

$$P(E^s) \le P(E) \ . \tag{1.1}$$

Recall that E^s is a set enjoying the property that its intersection with any straight line L orthogonal to H is a segment, symmetric about H, whose length equals the (1-dimensional) measure of $L \cap E$. More precisely, let us label the points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as $x = (x', y)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y = x_n$, assume, without loss of generality, that $H = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$, and set

$$E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\} \qquad \text{for } x' \in \mathbb{R}^{n-1}, \tag{1.2}$$

$$\ell(x') = \mathcal{L}^1(E_{x'}) \qquad \text{for } x' \in \mathbb{R}^{n-1}, \tag{1.3}$$

and

$$\pi(E)^+ = \{x' \in \mathbb{R}^{n-1} : \ell(x') > 0\} \ ,$$

where $\mathcal{L}^m$ denotes the *outer Lebesgue measure* in $\mathbb{R}^m$. Then E^s can be defined as

$$E^s = \{(x', y) \in \mathbb{R}^n : x' \in \pi(E)^+, \ |y| \le \ell(x')/2\} \ . \tag{1.4}$$

The objective of the present paper is to investigate on the cases of equality in (1.1). Namely, we address ourselves to the problem of characterizing those sets of finite perimeter E which satisfy

$$P(E^s) = P(E) \ . \tag{1.5}$$

Partial are the available results about this problem. It is classical, and not difficult to see by elementary considerations, that if E is convex and fulfills (1.5), then it is equivalent to E^s (up to translations along the y-axis). On the other hand, as far as we know, the best result in the literature concerning a general set of finite perimeter $E \subset \mathbb{R}^n$ satisfying (1.5), states that its section $E_{x'}$ is equivalent to a segment for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$ (see [18]). Our first theorem strengthens this result on establishing the symmetry of the generalized inner normal $\nu^E = (\nu_1^E, \dots, \nu_{n-1}^E, \nu_y^E)$ to E, which is well defined at each point of the reduced boundary $\partial^* E$ of E.

Theorem 1.1 *Let E be any set of finite perimeter in $\mathbb{R}^n$, $n \ge 2$, satisfying* (1.5). *Then either E is equivalent to $\mathbb{R}^n$, or $\mathcal{L}^n(E) < \infty$ and for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$*

$$E_{x'} \textit{ is equivalent to a segment, say } (y_1(x'), y_2(x')), \tag{1.6}$$

and

$$(\nu_1^E, \dots, \nu_{n-1}^E, \nu_y^E)(x', y_1(x')) = (\nu_E^1, \dots, \nu_{n-1}^E, -\nu_y^E)(x', y_2(x')) \ . \tag{1.7}$$

Conditions (1.6)–(1.7) might seem sufficient to conclude about the symmetry of E. However, this is not the case. In fact, the equivalence of E and E^s cannot be inferred under the sole assumption (1.5), as shown by the following simple examples.
Consider, for instance, the two-dimensional situation depicted in Figure 1.

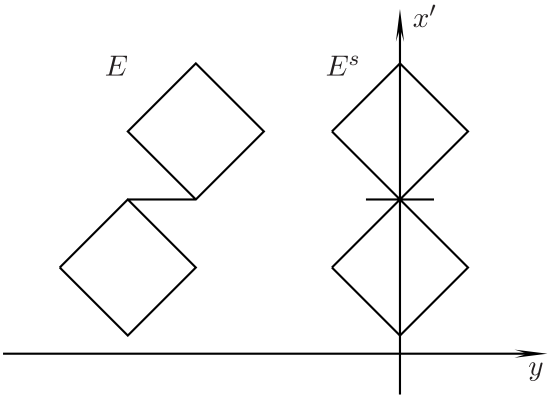

FIGURE 1.

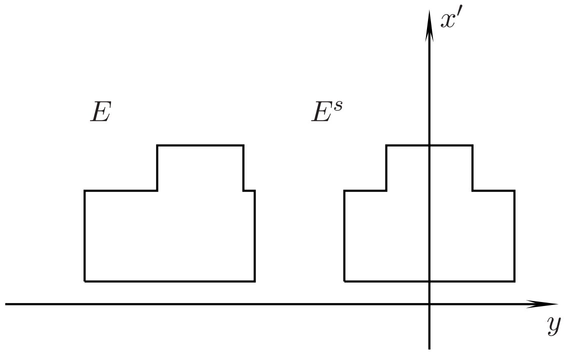

FIGURE 2.

Obviously, $P(E) = P(E^s)$, but E is not equivalent to any translate of E^s. The point in this example is that E^s (and E) fails to be connected in a proper sense for the present setting (although both E and E^s are connected from a strictly topological point of view).
The same phenomenon can occur also under different circumstances. Indeed, in the example of Figure 2 both E and E^s are connected in any reasonable sense, but again (1.5) holds without E being equivalent to any translate of E^s. What comes into play now is the fact that $\partial^* E^s$ (and $\partial^* E^s$) contains straight segments, parallel to the y-axis, whose projection on the line $\{(x', 0) : x' \in \mathbb{R}\}$ is an inner point of $\pi(E)^+$.
Let us stress, however, that preventing $\partial^* E^s$ and $\partial^* E$ from containing segments of this kind is not yet sufficient to ensure the symmetry of E. With regard to this, take, as an example,

$$E = \{(x', y) \in \mathbb{R}^2 : \, |x'| < 1, \; -2c(|x'|) \leq y \leq c(|x'|)\} \, ,$$

where $c : [0, 1] \to [0, 1]$ is the decreasing Cantor-Vitali function satisfying $c(1) = 0$ and $c(0) = 1$. Since c has bounded variation in $(0, 1)$, then E is a set of finite perimeter and, since the derivative of c vanishes $\mathcal{L}^1$-a.e., then $P(E) = 10$ (Theorem B, Section 2). It is easily verified that

$$E^s = \{(x', y) \in \mathbb{R}^2 : \, |x'| < 1, \; |y| \leq 3c(|x'|)/2\} \, .$$

Thus, $P(E^s) = 10$ as well, but E is not equivalent to any translate of E^s. Loosely speaking, in the situation at hand both $\partial^* E^s$ and $\partial^* E$ contain uncountably many infinitesimal segments parallel to the y-axis having total positive length.

In view of these results and examples, the problem arises of finding minimal additional assumptions to (1.5) ensuring the equivalence (up to translations) of E and E^s. These are elucidated in Theorem 1.3 below, which also provides a local symmetry result for E on any cylinder parallel to the y-axis having the form $\Omega \times \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^{n-1}$. Two are the relevant additional assumptions involved in that theorem, and both of them concern just E^s (compare with subsequent Remark 1.4).
To begin with, as illustrated by the last two examples, non negligible flat parts of $\partial^* E^s$ along the y-axis in $\Omega \times \mathbb{R}$ have to be excluded. This condition can be properly formulated by

requiring that

$$
\mathcal{H}^{n-1}\big(\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\} \cap (\Omega \times \mathbb{R})\big) = 0 \,. \tag{1.8}
$$

Hereafter, $\mathcal{H}^m$ stands for the m-dimensional Hausdorff measure. Assumption (1.8), of geometric nature, turns out to be equivalent to the vanishing of the perimeter of E^s relative to cylinders, of zero Lebesgue measure, parallel to the y-axis. It is also equivalent to a third purely analytical condition, such as the membership in the Sobolev space $W^{1,1}(\Omega)$ of the function ℓ, which, in general, is just of bounded variation (Lemma 3.1, Section 3). Hence, one derives from (1.8) information about the set of points $x' \in \mathbb{R}^{n-1}$ where the Lebesgue representative $\tilde{\ell}$ of ℓ, classically given by

$$
\lim_{r \to 0} \frac{1}{\mathcal{L}^{n-1}(B_r(x'))} \int_{B_r(x')} |\ell(z) - \tilde{\ell}(x')| \, dz = 0 \,,
$$

is well defined. Here, $B_r(x')$ denotes the ball centered at x' and having radius r.

Proposition 1.2 *Let E be any set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$, such that E^s is not equivalent to $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^{n-1}$. Then the following conditions are equivalent:*

(i) $\mathcal{H}^{n-1}\big(\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\} \cap (\Omega \times \mathbb{R})\big) = 0$,

(ii) $P(E^s; B \times \mathbb{R}) = 0$ *for every Borel set* $B \subset \Omega$ *such that* $\mathcal{L}^{n-1}(B) = 0$*; here* $P(E^s; B \times \mathbb{R})$ *denotes the perimeter of* E^s *in* $B \times \mathbb{R}$;

(iii) $\ell \in W^{1,1}(\Omega)$.

In particular, if any of (i)-(iii) holds, then $\tilde{\ell}$ is defined and finite $\mathcal{H}^{n-2}$-a.e. in Ω.

The second hypothesis to be made on E^s is concerned with connectedness. An assumption of this kind is indispensable in view of the example in figure 1. This is a crucial point since, as already pointed out, standard topological notions are not appropriate. A suitable form of the assumption in question amounts to demanding that no (too large) subset of $E^s \cap (\Omega \times \mathbb{R})$ shrinks along the y-axis till to be contained in $\Omega \times \{0\}$. Precisely, we require that $\tilde{\ell}$ does not vanish in Ω, except at most on a $\mathcal{H}^{n-2}$-negligible set, or, equivalently, that

$$
\tilde{\ell}(x') > 0 \qquad \text{for } \mathcal{H}^{n-2}\text{-a.e. } x' \in \Omega \,. \tag{1.9}
$$

Notice that condition (1.9) is perfectly meaningful, owing to the last stated property in Proposition 1.2.

Theorem 1.3 *Let E be a set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$, satisfying* (1.5)*. Assume that* (1.8)–(1.9) *are fulfilled for some open subset Ω of $\mathbb{R}^{n-1}$. Then $E \cap (\Omega_\alpha \times \mathbb{R})$ is equivalent to a translate along the y-axis of $E^s \cap (\Omega_\alpha \times \mathbb{R})$ for each connected component Ω_α of Ω.*
In particular, if (1.8)–(1.9) *are satisfied for some connected open subset Ω of $\mathbb{R}^{n-1}$ such that $\mathcal{L}^{n-1}(\pi(E)^+ \setminus \Omega) = 0$, then E is equivalent to E^s (up to translations along the y-axis).*

Remark 1.4 A sufficient condition for (1.8) to hold for some open set $\Omega \subset \mathbb{R}^{n-1}$ is that an analogous condition on E, namely

$$\mathcal{H}^{n-1}\big(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (\Omega \times \mathbb{R})\big) = 0 \,, \tag{1.10}$$

be fulfilled (see Proposition 4.2). Let us notice that, conversely, any set of finite perimeter E, satisfying both (1.5) and (1.8), also satisfies (1.10) (see Proposition 4.2 again). On the other hand, if (1.5) is dropped, then (1.8) may hold without (1.10) being fulfilled, as shown by the simple example represented in Figure 3.

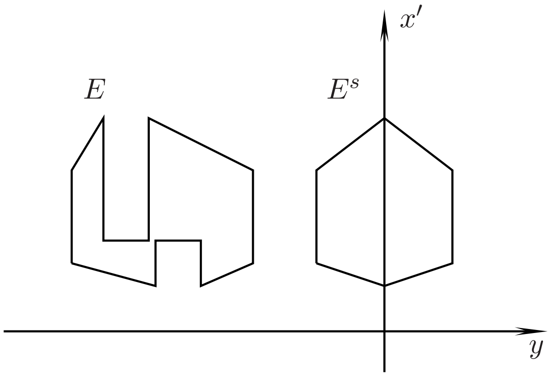

FIGURE 3.

Remark 1.5 Any convex body E satisfies (1.8)–(1.9) when Ω equals the interior of $\pi(E)^+$, an open convex set equivalent to $\pi(E)^+$. Thus, the aforementioned result for convex bodies is recovered by Theorem 1.3.

Remark 1.6 Condition (1.9) is automatically fulfilled, with $\Omega = E^s \cap \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$, if E is any open set. Thus, any bounded open set E of finite perimeter satisfying (1.5) is certainly equivalent to a translate of E^s, provided that $\pi(E)^+$ is connected and

$$\mathcal{H}^{n-1}\big(\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\} \cap (\pi(E)^+ \times \mathbb{R})\big) = 0 \,.$$

Proofs of Theorems 1.1 and 1.3 are presented in Sections 2 and 3, respectively. Like other known characterizations of equality cases in geometric and integral inequalities involving symmetries or symmetrizations (see e.g. [2], [4], [6], [7], [8], [9], [10], [15], [16]), the issues discussed in these theorems hide quite subtle matters. Their treatment calls for a careful analysis exploiting delicate tools from geometric measure theory. The material from this theory coming into play in our proofs is collected in Section 2.

2 Background

The definitions contained in this section are basic to geometric measure theory, and are recalled mainly to fix notations. Part of the results are special instances of very general theorems, appearing in certain cases only in [13], which are probably known only to specialists in the field; other results are more standard, but are stated here in a form suitable for our applications.

Let E be any subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. The *upper* and *lower density* of E at x are defined by

$$\overline{D}(E, x) = \limsup_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \qquad \text{and} \qquad \underline{D}(E, x) = \liminf_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))}$$

respectively. If $\overline{D}(E, x)$ and $\underline{D}(E, x)$ agree, then their common value is called the *density* of E at x and is denoted by $D(E, x)$. Note that $\overline{D}(E, \cdot)$ and $\underline{D}(E, \cdot)$ are always Borel functions, even if E is not Lebesgue measurable. Hence, for each $\alpha \in [0, 1]$,

$$E^\alpha = \{x \in \mathbb{R}^n : D(E, x) = \alpha\}$$

is a Borel set. The *essential boundary* of E, defined as

$$\partial^M E = \mathbb{R}^n \setminus (E^0 \cup (\mathbb{R}^n \setminus E)^0) ,$$

is also a Borel set. Obviously, if E is Lebesgue measurable, then $\partial^M E = \mathbb{R}^n \setminus (E^0 \cup E^1)$. As a straightforward consequence of the definition of essential boundary, we have that, if E and F are subsets of $\mathbb{R}^n$, then

$$\partial^M (E \cup F) \cup \partial^M (E \cap F) \subset \partial^M E \cup \partial^M F . \tag{2.1}$$

Let f be any real-valued function in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. The *approximate upper* and *lower* limit of f at x are defined as

$$f_+(x) = \inf\{t : D(\{f > t\}, x) = 0\} \qquad \text{and} \qquad f_-(x) = \sup\{t : D(\{f < t\}, x) = 0\} ,$$

respectively. The function f is said to be *approximately continuous* at x if $f_-(x)$ and $f_+(x)$ are equal and finite; the common value of $f_-(x)$ and $f_+(x)$ at a point of approximate continuity x is called the *approximate limit* of f at x and is denoted by $\overline{f}(x)$.

Let U be an open subset of $\mathbb{R}^n$. A function $f \in L^1(U)$ is called of *bounded variation* if its distributional gradient Df is an $\mathbb{R}^n$-valued Radon measure in U and the total variation $|Df|$ of Df is finite in U. The space of functions of bounded variation in U is called $BV(U)$. The space $BV_{\rm loc}(U)$ is defined accordingly. Given $f \in BV(U)$, the absolutely continuous part and the singular part of Df with respect to the Lebesgue measure are denoted by $D^a f$ and $D^s f$, respectively; moreover, ∇f stands for the density of $D^a f$ with respect to $\mathcal{L}^n$. Therefore, the Sobolev space $W^{1,1}(U)$ (resp. $W^{1,1}_{\rm loc}(U)$) can be identified with the subspace of those functions of $BV(U)$ ($BV_{\rm loc}(U)$) such that $D^s f = 0$. In particular, since $D^s f$ is concentrated

in a negligible set with respect to $\mathcal{L}^n$, then $f \in W^{1,1}(U)$ if and only if $|Df|(A) = 0$ for every Borel subset A of U, with $\mathcal{L}^n(A) = 0$.
The following result deals with the Lebesgue points of Sobolev functions (see [12, Section 4.8]).

Theorem A *Let U be an open subset of $\mathbb{R}^n$, and let $f \in W^{1,1}_{\mathrm{loc}}(U)$. Then there exists a Borel set N, with $\mathcal{H}^{n-1}(N) = 0$, such that f is approximately continuous at every $x \in U \setminus N$. Furthermore,*

$$\overline{f}(x) = \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} f(z)\, dz \qquad \text{for every } x \in U \setminus N\,. \tag{2.2}$$

Let E be a measurable subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^n$. Then E is said to be of *finite perimeter* in U if $D\chi_E$ is a vector-valued Radon measure in U having finite total variation; moreover, the perimeter of E in U is given by

$$P(E;U) = |D\chi_E|(U)\,. \tag{2.3}$$

The abridged notation $P(E)$ will be used for $P(E;\mathbb{R}^n)$. For any Borel subset A of U, the perimeter $P(E;A)$ of E in A is defined as $P(E;A) = |D\chi_E|(A)$. Notice that, if E is a set of finite perimeter in U, then $\chi_E \in BV_{\mathrm{loc}}(U)$; if, in addition, $\mathcal{L}^n(E\cap U) < \infty$, then $\chi_E \in BV(U)$. Given a set E of finite perimeter in U, and denoted by $D_i\chi_E$, $i = 1,\dots,n$, the components of $D\chi_E$, we have

$$\int_E \frac{\partial \varphi}{\partial x_i}\, dx = -\int_U \varphi\, dD_i\chi_E \qquad i = 1,\dots,n\,, \tag{2.4}$$

for every $\varphi \in C^1_0(U)$. Functions of bounded variation and sets of finite perimeter are related by the following result (see [14, Chap. 4, Sec. 1.5, Theorem 1, and Chap. 4, Sec. 2.4, Theorem 4]).

Theorem B *Let Ω be an open bounded subset of $\mathbb{R}^{n-1}$ and let $u \in L^1(\Omega)$. Then the subgraph of u, defined as*

$$\mathcal{S}_u = \{(x', y) \in \Omega \times \mathbb{R} : y < u(x')\}\,, \tag{2.5}$$

is a set of finite perimeter in $\Omega \times \mathbb{R}$ if and only if $u \in BV(\Omega)$. Moreover, in this case,

$$P(\mathcal{S}_u; B \times \mathbb{R}) = \int_B \sqrt{1 + |\nabla u|^2}\, dx' + |D^s u|(B) \tag{2.6}$$

for every Borel set $B \subset \Omega$.

Let E be a set of finite perimeter in an open subset U of $\mathbb{R}^n$. Then we denote by ν_i^E, $i = 1,\dots,n$, the derivative of the measure $D_i\chi_E$ with respect to $|D\chi_E|$. Thus

$$\nu_i^E(x) = \lim_{r\to 0} \frac{D_i\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \qquad i = 1,\dots,n\,, \tag{2.7}$$

at every $x \in U$ such that the indicated limit exists. The *reduced boundary* $\partial^* E$ of E is the set of all points $x \in U$ such that the vector $\nu^E(x) = (\nu_1^E(x), \dots, \nu_n^E(x))$ exists and $|\nu^E(x)| = 1$. The vector $\nu^E(x)$ is called the *generalized inner normal* to E at x. The reduced boundary of any set of finite perimeter E is a $(n-1)$-rectifiable set, and

$$D\chi_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E \tag{2.8}$$

(see [1, Theorem 3.59]). Equality (2.8) implies that

$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \tag{2.9}$$

and that

$$|D_i \chi_E| = |\nu_i^E| \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad i = 1, \dots, n\ . \tag{2.10}$$

Every point $x \in \partial^* E$ is a Lebesgue point for ν^E with respect to the measure $|D\chi_E|$ ([1, Remark 3.55]). Hence,

$$|\nu_i^E(x)| = \lim_{r \to 0} \frac{|D_i \chi_E|(B_r(x))}{|D\chi_E|(B_r(x))} \qquad \text{for every } \ x \in \partial^* E\ . \tag{2.11}$$

From the fact that the approximate tangent plane at any point $x \in \partial^* E$ is orthogonal to $\nu^E(x)$ ([1, Theorem 3.59]), and from the locality of the approximate tangent plane ([1, Remark 2.87]), we immediately get the following result.

Theorem C *Let E and F be sets of finite perimeter in $\mathbb{R}^n$. Then*

$$\nu^E(x) = \pm \nu^F(x) \qquad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \partial^* E \cap \partial^* F\ .$$

If E is a measurable set in $\mathbb{R}^n$, the jump set J_{χ_E} of the function χ_E is defined as the set of those points $x \in \mathbb{R}^n$ for which a unit vector $n^E(x)$ exists such that

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^n(B_r^+(x; n^E(x)))} \int_{B_r^+(x; n^E(x))} \chi_E(z)\, dz = 1$$

and

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^n(B_r^-(x; n^E(x)))} \int_{B_r^-(x; n^E(x))} \chi_E(z)\, dz = 0\ ,$$

where $B_r^{\pm}(x; n^E(x)) = \{z \in B_r(x) :\ \langle z - x, n^E(x) \rangle \gtrless 0\}$.
The inclusion relations among the various notions of boundary of a set of finite perimeter are clarified by the following result due to Federer (see [1, Theorem 3.61 and Remark 3.68]).

Theorem D *Let U be an open subset of $\mathbb{R}^n$ and let E be a set of finite perimeter in U. Then*

$$\partial^* E \subset J_{\chi_E} \subset E^{1/2} \subset \partial^M E\ .$$

Moreover,

$$\mathcal{H}^{n-1}((\partial^M E \setminus \partial^* E) \cap U) = 0\,.$$

Equation (2.9) and Theorem D ensure that, if E is a set of finite perimeter in the open set U, then $\mathcal{H}^{n-1}(\partial^M E \cap U)$ equals $P(E;U)$, and hence it is finite. A much deeper result by Federer ([13, Theorem 4.5.11]) tells us that the converse is also true.

Theorem E *Let U be an open set in $\mathbb{R}^n$ and let E be any subset of U. If $\mathcal{H}^{n-1}(\partial^M E \cap U) < \infty$, then E is Lebesgue measurable and of finite perimeter in U.*

Theorem F below is a consequence of the coarea formula for rectifiable sets in $\mathbb{R}^n$ (see [1, (2.72)]), and of the orthogonality between the generalized inner normal and the approximate tangent plane at any point $x \in \partial^* E$. In what follows, the n-th component of ν^E will be denoted by ν_y^E.

Theorem F *Let E be a subset of finite perimeter in $\mathbb{R}^n$ and let g be any Borel function from $\mathbb{R}^n$ into $[0,+\infty]$. Then*

$$\int_{\partial^* E} g(x)|\nu_y^E(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dx' \int_{(\partial^* E)_{x'}} g(x',y)\, d\mathcal{H}^0(y)\,. \tag{2.12}$$

A version of a result by Vol'pert ([19]) on restrictions of characteristic functions of sets of finite perimeter E is contained in the next theorem. In the statement, χ_E^* will denote the precise representative of χ_E, defined as

$$\chi_E^*(x) := \begin{cases} \overline{\chi}_E(x) & \text{if } x \in E^0 \cup E^1 \\ 0 & \text{if } x \in \partial^M E \setminus J_{\chi_E} \\ \dfrac{1}{2} & \text{if } x \in J_{\chi_E}\,. \end{cases}$$

Theorem G *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,*

$$E_{x'} \text{ has finite perimeter in } \mathbb{R} \text{ and } \chi_E^*(x',\cdot) = \chi_E(x',\cdot)\ \mathcal{L}^1\text{-a.e. in } E_{x'}\,; \tag{2.13}$$

$$(\partial^M E)_{x'} = (\partial^* E)_{x'} = \partial^*(E_{x'}) = \partial^M(E_{x'})\,; \tag{2.14}$$

$$\nu_y^E(x',t) \neq 0 \text{ for every } t \text{ such that } (x',t) \in \partial^* E\,; \tag{2.15}$$

$$\begin{cases} \lim\limits_{y\to t^+} \chi_E^*(x',y) = 1, & \lim\limits_{y\to t^-} \chi_E^*(x',y) = 0 \quad \text{if } \nu_y^E(x',t) > 0 \\ \lim\limits_{y\to t^+} \chi_E^*(x',y) = 0, & \lim\limits_{y\to t^-} \chi_E^*(x',y) = 1 \quad \text{if } \nu_y^E(x',t) < 0\,. \end{cases} \tag{2.16}$$

In particular, a Borel set $G_E \subseteq \pi(E)^+$ exists such that $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$ and (2.13)–(2.16) are fulfilled for every $x' \in G_E$.

Proof. Assertion (2.13) follows from Theorem 3.108 of [1] applied to the function χ_E. The same theorem also tells us that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,

$$(J_{\chi_E})_{x'} = J_{\chi_{E_{x'}}} \, , \tag{2.17}$$

$$\nu_y^E(x', t) \neq 0 \text{ for every } t \text{ such that } (x', t) \in J_{\chi_E} \, , \tag{2.18}$$

$$\text{equations (2.16) hold for every } t \text{ such that } (x', t) \in J_{\chi_E} \, . \tag{2.19}$$

Since, by Theorem D, $\mathcal{H}^{n-1}(\partial^M E \setminus J_{\chi_E}) = \mathcal{H}^{n-1}(J_{\chi_E} \setminus \partial^* E) = 0$, then, owing to Lemma 2.95 of [1],

$$(\partial^M E)_{x'} = (J_{\chi_E})_{x'} = (\partial^* E)_{x'} \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1} \, . \tag{2.20}$$

By (2.18)–(2.19), the last equation implies (2.15)–(2.16). Moreover, since any set of finite perimeter in $\mathbb{R}$ is equivalent to a finite union of disjoint intervals, then $\partial^M(E_{x'}) = J_{\chi_{E_{x'}}} = \partial^*(E_{x'})$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. Thus (2.14) follows from (2.17) and (2.20). □

We conclude this section with two results which are consequences of Theorem 2.10.45 and of Theorem 2.10.25 of [13], respectively.

Theorem H *Let m be a nonnegative integer. Then there exists a positive constant $c(m)$, depending only on m, such that if X is any subset of $\mathbb{R}^{n-1}$ with $\mathcal{H}^m(X) < \infty$ and Y is a Lebesgue measurable subset of $\mathbb{R}$, then*

$$\frac{1}{c(m)} \mathcal{H}^{m+1}(X \times Y) \leq \mathcal{H}^m(X) \mathcal{L}^1(Y) \leq c(m) \mathcal{H}^{m+1}(X \times Y) \, .$$

The next statement involves the projection of a set $E \subset \mathbb{R}^n$ into the hyperplane $\{(x', 0) : x' \in \mathbb{R}^{n-1}\}$, defined as

$$\pi(E) = \{x' \in \mathbb{R}^{n-1} : \text{ there exists } y \in \mathbb{R} \text{ such that } (x', y) \in E\}; .$$

Theorem I *Let m be a nonnegative integer and let E be any subset of $\mathbb{R}^n$. If $\mathcal{H}^m(\pi(E)) > 0$ and $\mathcal{L}^1(E_{x'}) > 0$ for $\mathcal{H}^m$-a.e. $x' \in \pi(E)$, then $\mathcal{H}^{m+1}(E) > 0$.*

3 Proof of Theorem 1.1

The first part of this section is devoted to a study of the function ℓ. As a preliminary step, we prove a relation between $D\ell$ and $D\chi_E$ (Lemma 3.1), which, in particular, entails that $\ell \in BV(\mathbb{R}^{n-1})$. A basic ingredient in our approach to Theorem 1.1 is then established in Lemma 3.2, where a formula for $\nabla \ell$, of possible independent interest, is found in terms of the generalized inner normal to E.

Lemma 3.1 *Let E be any set of finite perimeter in $\mathbb{R}^n$. Then either $\ell(x') = \infty$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, or $\ell(x') < \infty$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ and $\mathcal{L}^n(E) < \infty$. Moreover, in the latter case, $\ell \in BV(\mathbb{R}^{n-1})$ and*

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\, dD_i\ell(x') = \int_{\mathbb{R}^n} \varphi(x')\, dD_i\chi_E(x), \qquad i = 1, \dots, n-1, \tag{3.1}$$

for any bounded Borel function φ in $\mathbb{R}^{n-1}$. In particular,

$$|D\ell|(B) \le |D\chi_E|(B \times \mathbb{R}) \tag{3.2}$$

for every Borel set $B \subset \mathbb{R}^{n-1}$.

Proof. If ℓ were infinite in a subset of $\mathbb{R}^{n-1}$ of positive Lebesgue measure, and finite in another subset of positive measure, then both E and $\mathbb{R}^n \setminus E$ would have infinite measure. This is impossible, since E is of finite perimeter (see e.g. [1, Theorem 3.46]). Thus ℓ is either $\mathcal{L}^{n-1}$-a.e. infinite in $\mathbb{R}^{n-1}$, or it is $\mathcal{L}^{n-1}$-a.e. finite. Let us focus on the latter case. Since $\mathcal{L}^n(\mathbb{R}^n \setminus E) = \infty$ in this case, then $\mathcal{L}^n(E) < \infty$. Now, let $\varphi \in C^1_0(\mathbb{R}^{n-1})$ and let $\{\psi_j\}_{j\in\mathbb{N}}$ be any sequence in $C^1_0(\mathbb{R})$, satisfying $0 \le \psi_j(y) \le 1$ for $y \in \mathbb{R}$ and $j \in \mathbb{N}$, and such that $\lim_{j\to\infty} \psi_j(y) = 1$ for every $y \in \mathbb{R}$. Fix any $i \in \{1, \dots, n-1\}$. Then, by the dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \frac{\partial\varphi}{\partial x_i}(x')\ell(x')\, dx' &= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} \frac{\partial\varphi}{\partial x_i}(x')\chi_E(x', y)\, dy \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial x_i}(x')\psi_j(y)\chi_E(x', y)\, dx'dy \\ &= -\lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(x')\psi_j(y)\, dD_i\chi_E = -\int_{\mathbb{R}^n} \varphi(x')dD_i\chi_E\, .\end{aligned} \tag{3.3}$$

On taking the supremum in (3.3) as φ ranges among all functions in $C^1_0(\mathbb{R}^{n-1})$ with $\|\varphi\|_\infty \le 1$, and making use of the fact that $\chi_E \in BV(\mathbb{R}^n)$, we conclude that $\ell \in BV(\mathbb{R}^{n-1})$. Equation (3.1) holds for every $\varphi \in C^1_0(\mathbb{R}^{n-1})$ as a straightforward consequence of (3.3); by density, it also holds for every bounded Borel function φ. Finally, inequality (3.2) easily follows from (3.1). □

Lemma 3.2 *Let E be a set of finite perimeter in $\mathbb{R}^n$ having finite measure. Then*

$$\frac{\partial\ell}{\partial x_i}(x') = \int_{(\partial^* E)_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|}\, d\mathcal{H}^0(y), \qquad i = 1, \dots, n-1\,, \tag{3.4}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$.

Remark 3.3 An application of Lemma 3.2 and of (2.14) to E^s yields, in particular,

$$\frac{\partial\ell}{\partial x_i}(x') = 2\left(\frac{\nu_i^{E^s}(x', \cdot)}{|\nu_y^{E^s}(x', \cdot)|}\right)\Bigg|_{(\partial^* E^s)_{x'}} = 2\frac{\nu_i^{E^s}(x', \frac{1}{2}\ell(x'))}{|\nu_y^{E^s}(x', \frac{1}{2}\ell(x'))|} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \pi(E)^+\, . \tag{3.5}$$

Proof of Lemma 3.2. Let G_E be the set given by Theorem G. Obviously, we may assume that $\ell(x') < \infty$ for every $x' \in G_E$. By (2.7), (2.11) and (2.15), we have that

$$\frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} = \lim_{r \to 0} \frac{D_i \chi_E(B_r(x', y))}{|D_y \chi_E|(B_r(x', y))} \tag{3.6}$$

for every $x' \in G_E$ and every y such that $(x', y) \in \partial^* E$. Hence, by Besicovitch differentiation theorem (see e.g. [1, Theorem 2.22])

$$D_i \chi_E \llcorner (G_E \times \mathbb{R}) = \frac{\nu_i^E}{|\nu_y^E|} |D_y \chi_E| \llcorner (G_E \times \mathbb{R}) \, . \tag{3.7}$$

Now, let g be any function in $C_0(\mathbb{R}^{n-1})$, and set $\varphi(x') = g(x') \chi_{G_E}(x')$. From (3.1) and (3.7) one gets

$$\begin{aligned} \int_{G_E} g(x') \, dD_i \ell &= \int_{\mathbb{R}^n} g(x') \chi_{G_E}(x') \, dD_i \chi_E = \int_{G_E \times \mathbb{R}} g(x') \, dD_i \chi_E \\ &= \int_{G_E \times \mathbb{R}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} g(x') \, d|D_y \chi_E| \, . \end{aligned} \tag{3.8}$$

Moreover, by (2.10) and Theorem F,

$$\begin{aligned} \int_{G_E \times \mathbb{R}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} g(x') \, d|D_y \chi_E| &= \int_{\partial^* E \cap (G_E \times \mathbb{R})} g(x') \nu_i^E(x', y) \, d\mathcal{H}^{n-1} \\ &= \int_{G_E} g(x') \, dx' \int_{(\partial^* E)_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} \, d\mathcal{H}^0(y) \, . \end{aligned} \tag{3.9}$$

Combining (3.8), (3.9) yields

$$\int_{G_E} g(x') \, dD_i \ell = \int_{G_E} g(x') \, dx' \int_{(\partial^* E)_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} \, d\mathcal{H}^0(y) \, . \tag{3.10}$$

Hence, owing to the arbitrariness of g,

$$D_i \ell \llcorner G_E = \left(\int_{(\partial^* E)_{x'}} \frac{\nu_i^E}{|\nu_y^E|} \, d\mathcal{H}^0(y) \right) \mathcal{L}^{n-1} \llcorner G_E \, .$$

The conclusion follows, since $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$. □

We now turn to a local version of inequality (1.1), which will be needed both in the proof of Theorem 1.1 and in that of Theorem 1.3. Even not explicitly stated, such a result is contained in [18]. Here, we give a somewhat different proof relying on formula (3.4).

Lemma 3.4 *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then*

$$P(E^s; B \times \mathbb{R}) \le P(E; B \times \mathbb{R}) \tag{3.11}$$

for every Borel set $B \subset \mathbb{R}^{n-1}$.

Our proof of Lemma 3.4 requires the following preliminary result.

Lemma 3.5 *Let E be any set of finite perimeter in $\mathbb{R}^n$ having finite measure. Then*

$$P(E^s; B\times\mathbb{R}) \le |D\ell|(B) + |D_y\chi_{E^s}|(B\times\mathbb{R}) \tag{3.12}$$

for every Borel set $B\subset\mathbb{R}^{n-1}$.

Proof. The present proof is related to certain arguments used in [18]. Let $\{\ell_j\}_{j\in\mathbb{N}}$ be a sequence of nonnegative functions from $C_0^1(\mathbb{R}^{n-1})$ such that $\ell_j\to\ell$ $\mathcal{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$ and $|D\ell_j|\to|D\ell|$ weakly* in the sense of measures. Moreover, denote by E_j^s the set defined as in (1.4) with ℓ replaced by ℓ_j. Fix any open set $\Omega\subset\mathbb{R}^{n-1}$ and let $f=(f_1,\dots,f_n)\in C_0^1(\Omega\times\mathbb{R},\mathbb{R}^n)$. Then standard results on the differentiation of integrals enable us to write

$$\begin{aligned}\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\mathrm{div}f\,dx &= \int_\Omega dx'\int_{-\ell_j(x')/2}^{\ell_j(x')/2}\mathrm{div}f\,dy + \int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial f_n}{\partial y}\,dx\\ &= -\frac{1}{2}\int_{\pi(\mathrm{supp}f)}\sum_{i=1}^{n-1}\Big[f_i\Big(x',\frac{\ell_j(x')}{2}\Big)-f_i\Big(x',-\frac{\ell_j(x')}{2}\Big)\Big]\frac{\partial\ell_j}{\partial x_i}\,dx' + \int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial f_n}{\partial y}\,dx\,.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\mathrm{div}f\,dx \le& \int_{\pi(\mathrm{supp}f)}\sqrt{\sum_{i=1}^{n-1}\Big[\frac{1}{2}\Big(f_i\Big(x',\frac{\ell_j(x')}{2}\Big)-f_i\Big(x',-\frac{\ell_j(x')}{2}\Big)\Big)\Big]^2}\,|\nabla\ell_j|\,dx'\\ &+\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial f_n}{\partial y}\,dx\,.\end{aligned} \tag{3.13}$$

If $\|f\|_\infty\le 1$, we deduce from (3.13) that

$$\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\mathrm{div}f\,dx \le |D\ell_j|(\pi(\mathrm{supp}f)) + \int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial f_n}{\partial y}\,dx\,. \tag{3.14}$$

Since $\chi_{E_j^s}\to\chi_{E^s}$ $\mathcal{L}^n$-a.e. and $\pi(\mathrm{supp}f)$ is a compact subset of Ω, then taking the lim sup in (3.14) as j goes to ∞ yields

$$\begin{aligned}\int_{\Omega\times\mathbb{R}}\chi_{E^s}\mathrm{div}f\,dx &\le |D\ell|(\pi(\mathrm{supp}f)) + \int_{\Omega\times\mathbb{R}}\chi_{E^s}\frac{\partial f_n}{\partial y}\,dx\\ &\le |D\ell|(\Omega) + |D_y\chi_E|(\Omega\times\mathbb{R})\,.\end{aligned} \tag{3.15}$$

Inequality (3.15) implies that (3.12) holds whenever B is an open set, and hence also when B is any Borel set. □

Proof of Lemma 3.4. If $\ell=\infty$ $\mathcal{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$, then E^s is equivalent to $\mathbb{R}^n$; hence $P(E^s;B\times\mathbb{R})=0$ for every Borel set $B\subset\mathbb{R}^{n-1}$ and (3.11) is trivially satisfied. Thus, by Lemma 3.1, we may assume that $\ell<\infty$ $\mathcal{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$. Let G_E and G_{E^s} be the sets

associated with E and E^s, respectively, as in Theorem G. Let B a Borel subset of $\mathbb{R}^{n-1}$. We shall prove inequality (3.11) when either $B \subset \mathbb{R}^{n-1} \setminus G_{E^s}$ or $B \subset G_{E^s}$. The general case then follows on splitting B into $B \setminus G_{E^s}$ and $B \cap G_{E^s}$.
Assume first that $B \subset \mathbb{R}^{n-1} \setminus G_{E^s}$. Combining (3.12) and (3.2) gives

$$P(E^s; B \times \mathbb{R}) \le P(E; B \times \mathbb{R}) + |D_y \chi_{E^s}|(B \times \mathbb{R}) \,. \tag{3.16}$$

By (2.10), Theorem F and (2.14),

$$|D_y \chi_{E^s}|(B \times \mathbb{R}) = \int_{\partial^* E^s \cap (B \times \mathbb{R})} |\nu_y^{E^s}| \, d\mathcal{H}^{n-1} = \int_B \mathcal{H}^0((\partial^* E^s)_{x'}) \, dx' = \int_B \mathcal{H}^0(\partial^* (E^s)_{x'}) \, dx' \,.$$

Since $\mathcal{L}^{n-1}(\pi(E)^+ \cap B) = 0$, then the last integral equals $\int_{(\mathbb{R}^{n-1} \setminus \pi(E)^+) \cap B} \mathcal{H}^0(\partial^*(E^s)_{x'}) dx'$, and hence vanishes. Thus, (3.11) is a consequence of (3.16).
Suppose now that $B \subset G_{E^s}$. We have

$$\begin{aligned} P(E^s; B \times \mathbb{R}) &= \int_{\partial^* E^s \cap (B \times \mathbb{R})} d\mathcal{H}^{n-1} = \int_B dx' \int_{(\partial^* E^s)_{x'}} \frac{d\mathcal{H}^0(y)}{|\nu_y^{E^s}(x', y)|} \\ &= \int_{G_E \cap B} dx' \int_{(\partial^* E^s)_{x'}} \frac{d\mathcal{H}^0(y)}{|\nu_y^{E^s}(x', y)|} = \int_{G_E \cap B} dx' \int_{(\partial^* E^s)_{x'}} \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\nu_i^{E^s}(x', y)}{\nu_y^{E^s}(x', y)} \right)^2} \, d\mathcal{H}^0(y), \end{aligned} \tag{3.17}$$

where the first equality is due to (2.9), the second to Theorem F (which we may apply since we are assuming that $B \subset G_{E^s}$), the third to the fact that $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$, and the fourth to the fact that ν^{E^s} is a unit vector. By (3.5) and by property (2.14) for E^s

$$\begin{aligned} &\int_{G_E \cap B} dx' \int_{(\partial^* E^s)_{x'}} \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\nu_i^{E^s}(x', y)}{\nu_y^{E^s}(x', y)} \right)^2} \, d\mathcal{H}^0(y) \\ &\quad = \int_{G_E \cap B} dx' \int_{\partial^* (E^s)_{x'}} 2 \sqrt{1 + \frac{1}{4} |\nabla \ell(x')|^2} \, d\mathcal{H}^0(y) = \int_{G_E \cap B} \sqrt{4 + |\nabla \ell(x')|^2} \, dx' \,. \end{aligned} \tag{3.18}$$

Owing to the isoperimetric inequality in $\mathbb{R}$ and to (3.4) and (2.14), the last integral does not exceed

$$\int_{G_E \cap B} \sqrt{ \left(\int_{\partial^* (E_{x'})} d\mathcal{H}^0 \right)^2 + \sum_{i=1}^{n-1} \left(\int_{\partial^* (E_{x'})} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} \, d\mathcal{H}^0(y) \right)^2 } \, dx' \,,$$

an expression which, by Minkowski integral inequality, is in turn smaller than or equal to

$$\int_{G_E \cap B} dx' \int_{\partial^* (E_{x'})} \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\nu_i^E(x', y)}{\nu_y^E(x', y)} \right)^2} \, d\mathcal{H}^0(y) \,.$$

An analogous chain of equalities as in (3.17) yields

$$\int_{G_E\cap B} dx' \int_{\partial^*(E_{x'})} \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^E(x',y)}{\nu_y^E(x',y)}\right)^2}\, d\mathcal{H}^0(y) = P(E;(G_E\cap B)\times\mathbb{R})\,.$$

Since obviously $P(E;(G_E\cap B)\times\mathbb{R}) \le P(E;B\times\mathbb{R})$, inequality (3.11) follows. □

Proof of Theorem 1.1. If $\ell = \infty$ $\mathcal{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$, then E^s is equivalent to $\mathbb{R}^n$ and $P(E^s) = 0$. Therefore, E is equivalent to $\mathbb{R}^n$ (and hence to E^s), otherwise $P(E) > 0$, thus contradicting (1.5). Assume now that ℓ is not infinite $\mathcal{L}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$. Then, by Lemma 3.1, $\mathcal{L}^n(E) < \infty$. Equality (1.5) and inequality (3.11) imply that

$$P(E^s;B\times\mathbb{R}) = P(E;B\times\mathbb{R}) \tag{3.19}$$

for every Borel set $B \subset \mathbb{R}^{n-1}$. Let G_E and G_{E^s} be the sets associated with E and E^s, respectively, as in Theorem G. Then $\mathcal{L}^{n-1}(\pi(E)^+ \setminus (G_E\cap G_{E^s})) = 0$, and the same steps as in the proof of Lemma 3.4 yield

$$\begin{aligned}
P(E^s;(G_E\cap G_{E^s})\times\mathbb{R}) &= \int_{G_E\cap G_{E^s}} dx' \int_{(\partial^* E^s)_{x'}} \frac{d\mathcal{H}^0(y)}{|\nu_y^{E^s}(x',y)|} \\
&= \int_{G_E\cap G_{E^s}} dx' \int_{(\partial^* E^s)_{x'}} \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^{E^s}(x',y)}{\nu_y^{E^s}(x',y)}\right)^2}\, d\mathcal{H}^0(y) = \int_{G_E\cap G_{E^s}} \sqrt{4+|\nabla\ell(x')|^2}\, dx' \\
&\le \int_{G_E\cap G_{E^s}} \sqrt{\left(\int_{\partial^*(E_{x'})} d\mathcal{H}^0\right)^2 + \sum_{i=1}^{n-1}\left(\int_{\partial^*(E_{x'})} \frac{\nu_i^E(x',y)}{|\nu_y^E(x',y)|}\, d\mathcal{H}^0(y)\right)^2}\, dx' \\
&\le \int_{G_E\cap G_{E^s}} dx' \int_{\partial^*(E_{x'})} \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^E(x',y)}{\nu_y^E(x',y)}\right)^2}\, d\mathcal{H}^0(y) \\
&= \int_{G_E\cap G_{E^s}} dx' \int_{(\partial^* E)_{x'}} \frac{d\mathcal{H}^0(y)}{|\nu_y^E(x',y)|} = P(E;(G_E\cap G_{E^s})\times\mathbb{R})\,.
\end{aligned} \tag{3.20}$$

On applying (3.19) with $B = G_E\cap G_{E^s}$, we infer that both inequalities in (3.20) must hold as equalities. The former of these equalities entails that $\mathcal{H}^0(\partial^*(E_{x'})) = 2$ for $\mathcal{L}^{n-1}$-a.e. $x' \in G_E\cap G_{E^s}$, whence $E_{x'}$ is equivalent to some segment $(y_1(x'), y_2(x'))$ for $\mathcal{L}^{n-1}$-a.e. $x' \in G_E\cap G_{E^s}$. The latter implies that $\dfrac{\nu_i^E(x',y_1(x'))}{|\nu_y^E(x',y_1(x'))|} = \dfrac{\nu_i^E(x',y_2(x'))}{|\nu_y^E(x',y_2(x'))|}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in G_E\cap G_{E^s}$; hence, since ν^E is a unit vector, then $\nu_i^E(x',y_1(x')) = \nu_i^E(x',y_2(x'))$, $i = 1,\dots,n-1$, and $|\nu_y^E(x',y_1(x'))| = |\nu_y^E(x',y_2(x'))|$ for the same values of $x' \in G_E\cap G_{E^s}$. Let us now fix any such x'. From (2.13) we get $\lim_{y\to y_1(x')^+}\chi^*(x',y) = 1$, $\lim_{y\to y_2(x')^-}\chi^*(x',y) = 1$. Thus, by (2.16), one necessarily has $\nu_y^E(x',y_1(x')) > 0$ and $\nu_y^E(x',y_2(x')) < 0$. Hence $\nu_y^E(x',y_1(x')) = -\nu_y^E(x',y_2(x'))$. The proof is complete. □

4 Proof of Theorem 1.3

The present section is organized as follows. We begin with the proof of Proposition 1.2, concerning conditions equivalent to (1.8), and with a further result, described in Proposition 4.2, relating assumption (1.8) on E^s with its counterpart (1.10) on E. A decisive technical step towards Theorem 1.3 is accomplished in subsequent Lemma 4.3, whose proof is split in two parts. The core of the argument is contained in the first part, dealing with sets E which are bounded, or more generally, bounded in the direction y; via suitable truncations, such an assumption is removed in the second part, and is replaced by the weaker condition (4.9) appearing in the statement. With Lemma 4.3 in place, even in the special case enucleated in the first part of its proof, Theorem 1.3 follows quite easily when E is a bounded set. For the reader's convenience, we present the proof of this case separately, just after Lemma 4.3. The general case is treated in the last part of the section, and requires an extra reflection argument, which enables us to restrict our attention to those sets that, besides (1.8)–(1.9), satisfy the additional assumption (4.9) of Lemma 4.3. The relevant reflection process can be regarded as a special case of the so called polarization about hyperplanes. Polarization techniques were used in [3] and [11]; a closer study on this subject has been carried out in [5]. The properties of use for our purposes are summarized in Lemma 4.4. Some of them (in a weaker, but yet sufficient form) could be derived from results of [5]. For completeness, we present a complete proof of this lemma which rests on the methods of this paper.

Lemma 4.1 *Let E be any set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$, and let A be any Borel subset of $\mathbb{R}^{n-1}$. Then*

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (A \times \mathbb{R})) = 0 \tag{4.1}$$

if and only if

$$P(E; B \times \mathbb{R}) = 0 \text{ for each Borel subset } B \text{ of } A \text{ such that } \mathcal{L}^{n-1}(B) = 0\,. \tag{4.2}$$

Proof. Assume that (4.1) is in force. Let B be any Borel subset of A with $\mathcal{L}^{n-1}(B) = 0$. Then

$$\begin{aligned} P(E; B \times \mathbb{R}) = \int_{\partial^* E \cap (B \times \mathbb{R})} d\mathcal{H}^{n-1} &= \int_{\partial^* E} \frac{1}{|\nu_y^E(x)|} \chi_{\{\nu_y^E \neq 0\} \cap (B \times \mathbb{R})}(x) |\nu_y^E(x)| d\mathcal{H}^{n-1}(x) \\ &\quad + \int_{\partial^* E} \chi_{\{\nu_y^E = 0\} \cap (B \times \mathbb{R})}(x) d\mathcal{H}^{n-1}(x) \\ &= \int_B dx' \int_{(\partial^* E)_{x'}} \frac{\chi_{\{\nu_y^E \neq 0\}}(x', t)}{|\nu_y^E(x', t)|} d\mathcal{H}^0(t) + \mathcal{H}^{n-1}(\{\nu_y^E = 0\} \cap (B \times \mathbb{R}))\,. \end{aligned} \tag{4.3}$$

Notice that we made use of (2.9) in the first equality and of (2.12) in the third. Now, the last integral vanishes, since $\mathcal{L}^{n-1}(B) = 0$; moreover, $\mathcal{H}^{n-1}(\{\nu_y^E = 0\} \cap (B \times \mathbb{R})) = 0$, by (4.1). Hence (4.2) follows.

Conversely, suppose that (4.2) is fulfilled. Let G_E be the set given by Theorem G. Since $\mathcal{L}^{n-1}(A \setminus G_E) = 0$, then, by (4.2),

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (A \times \mathbb{R})) \leq \mathcal{H}^{n-1}(\partial^* E \cap [(A \setminus G_E) \times \mathbb{R}]) = P(E; (A \setminus G_E) \times \mathbb{R}) = 0\,.$$

□

Proof of Proposition 1.2. The equivalence of (i) and (ii) is nothing but a special case of Lemma 4.1, when $E = E^s$ and $A = \Omega$.
Let us show that (ii) implies (iii). By Lemma 3.1, $\ell \in BV(\Omega)$. Moreover, by inequality (3.2), and by (ii), $|D\ell|(B) = 0$ for every Borel subset B of Ω such that $\mathcal{L}^{n-1}(B) = 0$. Hence, $\ell \in W^{1,1}(\Omega)$, and (iii) follows.
Assume now that $\ell \in W^{1,1}(\Omega)$. Set

$$F_1 = \{(x', y) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, y < -\ell(x')/2\}, \quad F_2 = \{(x', y) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, y > \ell(x')/2\} .$$

Let B be any Borel subset of Ω. Then

$$P(E^s; B\times\mathbb{R}) = P(\mathbb{R}^n \setminus E^s; B\times\mathbb{R}) \le P(F_1; B\times\mathbb{R}) + P(F_2; B\times\mathbb{R}) = 2P(F_1; B\times\mathbb{R}) , \tag{4.4}$$

where the inequality is an immediate consequence of the fact that $\mathbb{R}^n \setminus E^s$ is equivalent to $F_1 \cup F_2$. Since $\ell \in W^{1,1}(\Omega)$, then by (2.6)

$$P(F_1; B \times \mathbb{R}) = \int_B \sqrt{1 + \frac{1}{4}|\nabla \ell|^2} dx' . \tag{4.5}$$

Combining (4.4)–(4.5) yields $P(E^s; B \times \mathbb{R}) = 0$ whenever $\mathcal{L}^{n-1}(B) = 0$, and hence (ii) holds.
□

Proposition 4.2 *Let E be any set of finite perimeter in $\mathbb{R}^n$ and let A be any Borel subset of $\mathbb{R}^{n-1}$. If*

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (A \times \mathbb{R})) = 0 , \tag{4.6}$$

then

$$\mathcal{H}^{n-1}(\{x \in \partial^* E^s : \nu_y^{E^s}(x) = 0\} \cap (A \times \mathbb{R})) = 0 . \tag{4.7}$$

Conversely, if E satisfies $P(E^s) = P(E)$ and (4.7) *holds, then* (4.6) *holds as well.*

Proof. Assume that (4.6) is fulfilled. Then, by Lemma 4.1, $P(E; B \times \mathbb{R}) = 0$ for every Borel subset B of A with $\mathcal{L}^{n-1}(B) = 0$. Thus by inequality (3.11), $P(E^s; B \times \mathbb{R}) = 0$ as well. Hence (4.7) follows, owing to Lemma 4.1 applied to E^s.
Suppose now that (4.7) is fulfilled and that $P(E^s) = P(E)$. Then by Lemma 3.4,

$$P(E^s; B \times \mathbb{R}) = P(E; B \times \mathbb{R}) \tag{4.8}$$

for every Borel set B in $\mathbb{R}^{n-1}$. The same argument as above, with (3.11) replaced by (4.8), tell us that (4.7) implies (4.6).
□

Lemma 4.3 *Let Ω be an open bounded set subset of $\mathbb{R}^{n-1}$ and let E be a set of finite perimeter in $\Omega \times \mathbb{R}$ having the property that there exist functions $y_1, y_2 : \Omega \to \mathbb{R}$ such that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega$, $y_1(x') \le y_2(x')$ and $E_{x'}$ is equivalent to $(y_1(x'), y_2(x'))$. Assume that* (1.10) *and* (1.9) *are fulfilled and that*

$$y_1(x') \le k \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \Omega , \tag{4.9}$$

for some constant $k \in \mathbb{R}$. Then $y_1, y_2 \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and

$$P(E; \Omega \times \mathbb{R}) = \sum_{i=1}^{2} \int_\Omega \sqrt{1 + |\nabla y_i|^2} dx' . \tag{4.10}$$

Proof of Lemma 4.3.
Part I Here we prove the statement with (4.9) replaced by the stronger assumption that

$$-k \le y_1(x') \le y_2(x') \le k \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \Omega , \tag{4.11}$$

for some $k > 0$.
On replacing, if necessary, E by an equivalent set, we may assume, without loss of generality, that (4.11) holds for every $x' \in \Omega$ and that

$$E \cap (\Omega \times \mathbb{R}) = \{(x', y) : x' \in \Omega,\ y_1(x') \le y \le y_2(x')\} .$$

Let us set

$$A_1 = \{(x', y) : x' \in \Omega,\ \ y < y_1(x')\} \quad \text{and} \quad A_2 = \{(x', y) : x' \in \Omega,\ \ y > y_2(x')\} . \tag{4.12}$$

We shall prove that A_1 and A_2 are sets of finite perimeter in $\Omega \times \mathbb{R}$ and that

$$P(E; \Omega \times \mathbb{R}) = \sum_{i=1}^{2} P(A_i; \Omega \times \mathbb{R}) . \tag{4.13}$$

Owing to Theorem E, in order to prove that A_2 is of finite perimeter in $\Omega \times \mathbb{R}$, it suffices to show that

$$\mathcal{H}^{n-1}((\partial^M A_2 \setminus \partial^M E) \cap (\Omega \times \mathbb{R})) = 0 . \tag{4.14}$$

Assume, by contradiction, that (4.14) is false; namely,

$$\mathcal{H}^{n-1}((\partial^M A_2 \setminus \partial^M E) \cap (\Omega \times \mathbb{R})) > 0 . \tag{4.15}$$

Let us set

$$Z = \{x \in \Omega \times \mathbb{R} : \overline{D}(A_i, x) > 0,\ i = 1, 2\} . \tag{4.16}$$

Then we claim that

$$(\partial^M A_2 \setminus \partial^M E) \cap (\Omega \times \mathbb{R}) \subset Z = \partial^M A_1 \cap \partial^M A_2 \cap (\Omega \times \mathbb{R}) \,. \tag{4.17}$$

The equality in (4.17) is an easy consequence of the definition of essential boundary. As for the inclusion, observe that if $x \in \partial^M A_2 \setminus \partial^M E$, then $x \in E^0 \cup E^1$. But $x \notin E^1$, since, otherwise, $D(A_2, x) = 0$, and this is impossible, inasmuch as $x \in \partial^M A_2$. Thus, necessarily $x \in E^0$. Since $x \in \partial^M A_2$, then $\overline{D}(A_2, x) > 0$. We also have $\overline{D}(A_1, x) > 0$. Actually, if $D(A_1, x) = 0$ and $x \in E^0$, then $D((\Omega \times \mathbb{R}) \setminus A_2, x) = 0$, and this contradicts the fact that $x \in \partial^M A_2$.
Assumption (4.15) and the inclusion in (4.17) imply that

$$\mathcal{H}^{n-1}(Z) > 0 \,. \tag{4.18}$$

Hence, by Theorem H,

$$\mathcal{H}^{n-2}(\pi(Z)) > 0 \,. \tag{4.19}$$

Now, assumption (1.10) implies (1.8), by Proposition 4.2. Thus, $\ell \in W^{1,1}(\Omega)$, owing to Proposition 1.2, whence $\ell_-(x') = \overline{\ell}(x') = \tilde{\ell}(x')$ for $\mathcal{H}^{n-2}$-a.e. $x' \in \Omega$ by Theorem A. Consequently, on setting

$$X = \{x' \in \pi(Z) : \ell_-(x') > 0\} \,,$$

we deduce from (4.19) and (1.9) that

$$\mathcal{H}^{n-2}(X) > 0 \,. \tag{4.20}$$

A contradiction will be reached if we show that two real-valued functions z_1, z_2 in X exist such that $z_1(x') < z_2(x')$ and

$$\{x'\} \times (z_1(x'), z_2(x')) \subset \partial^M E \tag{4.21}$$

for every $x' \in X$. Indeed, since, by Theorem G, $\mathcal{H}^0((\partial^M E)_{x'}) < \infty$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega$, then (4.21) implies that $\mathcal{L}^{n-1}(X) = 0$. On the other hand, inequality (4.20) entails, via Theorem I, that $\mathcal{H}^{n-1}\Big(\bigcup_{x' \in X} \{x'\} \times (z_1(x'), z_2(x'))\Big) > 0$, whence, by (4.21), $P(E; X \times \mathbb{R}) = \mathcal{H}^{n-1}(\partial^M E \cap (X \times \mathbb{R})) > 0$. This contradicts assumption (1.8), owing to Proposition 1.2.
Our task in now to exhibit a couple of functions z_1 and z_2 as above. Fixed any $x' \in X$, let y be any real number such that $(x', y) \in Z$, and set $x = (x', y)$. We shall construct $z_1(x')$ and $z_2(x')$ in such a way that $y \le z_1(x')$. Given any $\delta > 0$, we denote by $C^n(x, \delta)$ the (open) cube in $\mathbb{R}^n$, centered at x, having sides of length δ; consistently, we set $C^{n-1}(x', \delta) = \pi(C^n(x, \delta))$. First, it is not difficult to see that, if $\overline{x}$ is any point of the form $\overline{x} = (x', \overline{y})$, with $\overline{y} > y$, then

$$\overline{D}(\mathbb{R}^n \setminus E, \overline{x}) > 0 \,. \tag{4.22}$$

Actually,

$$\mathcal{L}^n(C^n(\overline{x}, \delta) \cap A_2) \ge \mathcal{L}^n(C^n(x, \delta) \cap A_2) \,.$$

Hence, since $\overline{D}(A_2, x) > 0$, then $\limsup_{\delta\to 0} \delta^{-n}\mathcal{L}^n(C^n(\overline{x}, \delta) \cap A_2) > 0$, and, obviously, $\overline{D}(A_2, \overline{x}) > 0$. The last inequality implies (4.22).
Next, since $x \in Z$, then $\overline{D}(A_1, x) > 0$. Consequently $\limsup_{\delta\to 0} \delta^{-n}\mathcal{L}^n(C^n(x, \delta) \cap A_1) > 0$. Therefore a positive number $\tau > 0$ and a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ exist such $\delta_i > 0$ for $i \in \mathbb{N}$, $\lim_{i\to+\infty} \delta_i = 0$ and

$$\mathcal{L}^n(C^n(x, \delta_i) \cap A_1) > \tau\delta_i^n \qquad \text{for } i \in \mathbb{N}\ . \tag{4.23}$$

Inequality (4.23) and the inclusion

$$C^n(x, \delta_i) \cap A_1 \subset \{z' \in C^{n-1}(x', \delta_i) : \, y_1(z') > y - \delta_i/2\} \times (y - \delta_i/2, y + \delta_i/2) \quad \text{for } i \in \mathbb{N}$$

ensure that

$$\mathcal{L}^{n-1}(\{z' \in C^{n-1}(x', \delta_i) : \, y_1(z') > y - \delta_i/2\}) > \tau\delta_i^{n-1} \qquad \text{for } i \in \mathbb{N}. \tag{4.24}$$

Recall that we are denoting by $\mathcal{L}^{n-1}$ the outer Lebesgue measure in $\mathbb{R}^{n-1}$: indeed, at this stage, the set appearing on the left-hand side of (4.24) is not known to be Lebesgue measurable yet. Fix any $t \in (0, \ell_-(x'))$. Since $D(\{\ell \le t\}, x') = 0$, then

$$\mathcal{L}^{n-1}(\{z' \in C^{n-1}(x', \delta) : \, \ell(z') \le t\}) < \frac{\tau}{2}\delta^{n-1}\ , \tag{4.25}$$

provided that $\delta > 0$ is sufficiently small. On setting

$$Y_i = \{z' \in C^{n-1}(x', \delta_i) : \, \ell(z') > t \;\text{ and }\; y_1(z') > y - t/3\}\ , \tag{4.26}$$

we deduce from (4.25)–(4.26) that

$$\mathcal{L}^{n-1}(Y_i) > \frac{\tau\delta_i^{n-1}}{2} \tag{4.27}$$

if i is sufficiently large. Let us define $y_j = y + t(j-1)/3$ and

$$Y_{i,j} = \{z' \in Y_i : \, \{z'\} \times [y_j, y_{j+1}] \subset E\}$$

for $j \in \mathbb{N}$, and let us call $j_{\max}$ the largest $j \in \mathbb{N}$ not exceeding $3(k-y)/t$. Since $y_2(z') - y_1(z') = \ell(z') > t$ for $z' \in Y_i$, then

$$Y_i = \bigcup_{j=1}^{j_{\max}} Y_{i,j}\ .$$

Thus, by (4.27), for any sufficiently large i there exists $j_i \in \{1, \dots, j_{\max}\}$ such that

$$\mathcal{L}^{n-1}(Y_{i,j_i}) > \frac{\tau\delta_i^{n-1}}{2j_{\max}}\ .$$

Hence, an infinite subset I of $\mathbb{N}$ and an index $j_0 \in \{1, \dots, j_{\max}\}$ exist such that

$$\mathcal{L}^{n-1}(Y_{i,j_0}) > \frac{\tau\delta_i^{n-1}}{2j_{\max}} \qquad \text{for every } \; i \in I\ . \tag{4.28}$$

If $\overline{x} \in \{x'\} \times (y_{j_0}, y_{j_0+1})$ and i is a sufficiently large index from I, then

$$C^n(\overline{x}, \delta_i) \cap E \supset Y_{i,j_0} \times (\overline{y} - \delta_i/2, \overline{y} + \delta_i/2) .$$

Thus, from inequality (4.28) and Theorem H we infer that there exists a positive constant γ, depending only on n such that

$$\frac{\mathcal{L}^n(C^n(\overline{x}, \delta_i) \cap E)}{\delta_i^n} \geq \gamma \frac{\mathcal{L}^{n-1}(Y_{i,j_0})}{\delta_i^{n-1}} \geq \frac{\gamma\tau}{2j_{\max}} \tag{4.29}$$

provided that i belongs to I and is large enough. Inequality (4.29) implies that

$$\overline{D}(E, \overline{x}) > 0 \qquad \text{for any } \overline{x} \in \{x'\} \times (y_{j_0}, y_{j_0+1}) . \tag{4.30}$$

Inequalities (4.22) and (4.30) tell us that

$$\{x'\} \times (y_{j_0}, y_{j_0+1}) \subset \partial^M E .$$

Hence, (4.21) follows, with $z_1(x') = y_{j_0}$ and $z_2(x') = y_{j_0+1}$. The fact that A_2 is of finite perimeter in $\Omega \times \mathbb{R}$ is fully proved.
Since $A_1 = (\Omega \times \mathbb{R}) \setminus (E \cup A_2)$, then, by (2.1),

$$\begin{aligned} \partial^M A_1 \cap (\Omega \times \mathbb{R}) &= \partial^M (E \cup A_2) \cap (\Omega \times \mathbb{R}) \subset (\partial^M E \cup \partial^M A_2) \cap (\Omega \times \mathbb{R}) \\ &= (\partial^M E \cap (\Omega \times \mathbb{R})) \cup [(\partial^M A_2 \setminus \partial^M E) \cap (\Omega \times \mathbb{R})] . \end{aligned}$$

Thus, by (4.14),

$$\mathcal{H}^{n-1}((\partial^M A_1 \setminus \partial^M E) \cap (\Omega \times \mathbb{R})) = 0 \tag{4.31}$$

and

$$\mathcal{H}^{n-1}(\partial^M A_1 \cap (\Omega \times \mathbb{R})) < \infty . \tag{4.32}$$

Hence, also A_1 is of finite perimeter in $\Omega \times \mathbb{R}$, thanks to Theorem E.
Now, we have

$$\begin{aligned} &\mathcal{H}^{n-1}((\partial^M A_1 \cup \partial^M A_2) \cap (\Omega \times \mathbb{R})) \leq \mathcal{H}^{n-1}(\partial^M E \cap (\Omega \times \mathbb{R})) \\ &\qquad = \mathcal{H}^{n-1}(\partial^M (A_1 \cup A_2) \cap (\Omega \times \mathbb{R})) \leq \mathcal{H}^{n-1}((\partial^M A_1 \cup \partial^M A_2) \cap (\Omega \times \mathbb{R})) , \end{aligned}$$

where the first inequality is due to (4.14) and (4.31) and the last inequality to (2.1). Consequently,

$$\mathcal{H}^{n-1}((\partial^M A_1 \cup \partial^M A_2) \cap (\Omega \times \mathbb{R})) = \mathcal{H}^{n-1}(\partial^M E \cap (\Omega \times \mathbb{R})) . \tag{4.33}$$

On the other hand, the contradiction argument which has led to (4.14) tells us that, in fact, $\mathcal{H}^{n-1}(Z) = 0$, whence, by (4.17),

$$\mathcal{H}^{n-1}((\partial^M A_1 \cap \partial^M A_2) \cap (\Omega \times \mathbb{R})) = 0 . \tag{4.34}$$

From (4.34) one easily deduces that

$$\mathcal{H}^{n-1}((\partial^M A_1 \cup \partial^M A_2) \cap (\Omega \times \mathbb{R})) = \sum_{i=1}^{2} \mathcal{H}^{n-1}(\partial^M A_i \cap (\Omega \times \mathbb{R})) \,. \tag{4.35}$$

Combining (4.33) and (4.35) yields

$$\begin{aligned} P(E; \Omega \times \mathbb{R}) = \mathcal{H}^{n-1}(\partial^M E \cap (\Omega \times \mathbb{R})) &= \sum_{i=1}^{2} \mathcal{H}^{n-1}(\partial^M A_i \cap (\Omega \times \mathbb{R})) \\ &= \sum_{i=1}^{2} P(A_i; \Omega \times \mathbb{R}) \,. \end{aligned} \tag{4.36}$$

Since, A_1 and A_2 are, in particular, measurable sets, then y_1 and y_2 are measurable functions in Ω. Under assumption (4.11), this fact immediately yields that $y_1, y_2 \in L^1(\Omega)$. Hence, by Theorem B, the functions $y_1, y_2 \in BV(\Omega)$. Furthermore, if B is any Borel subset of Ω with $\mathcal{L}^{n-1}(B) = 0$, then

$$\begin{aligned} |Dy_i|(B) = |D^s y_i|(B) = P(A_i; B \times \mathbb{R}) &= \mathcal{H}^{n-1}(\partial^M A_i \cap (B \times \mathbb{R})) \\ \le \mathcal{H}^{n-1}(\partial^M E \cap (B \times \mathbb{R})) &= P(E; B \times \mathbb{R}) = 0 \end{aligned} \tag{4.37}$$

for $i = 1, 2$. Notice that the first equality in (4.37) holds since $\mathcal{L}^{n-1}(B) = 0$, the second holds by (2.6), the last one is a consequence of assumption (1.10) and of Lemma 4.1, and the inequality is due either to (4.31) or to (4.14), according to whether $i = 1$ or $i = 2$. From (4.37) we infer that $y_1, y_2 \in W^{1,1}(\Omega)$ and hence, by (2.6), that

$$P(A_i; \Omega \times \mathbb{R}) = \int_\Omega \sqrt{1 + |\nabla y_i|^2}\, dx', \qquad i = 1, 2 \,. \tag{4.38}$$

Equation (4.10) follows from (4.36) and (4.38).

Part II Here, we remove the assumption (4.11). This will be accomplished in steps.
Step 1 Suppose that (4.9) is replaced by

$$y_2(x') \le k \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \Omega \,. \tag{4.39}$$

Then A_1 and A_2 are sets of finite perimeter in $\Omega \times \mathbb{R}$; moreover, (4.14), (4.31) and (4.34) hold, and

$$P(E; \Omega \times \mathbb{R}) = \sum_{i=1}^{2} P(A_i; \Omega \times \mathbb{R}) \,. \tag{4.40}$$

The proof is the same as in Part I. Actually, an inspection of that proof reveals that the inequality $-k \le y_1(x')$, appearing in (4.11), does not play any role in the argument leading to the conclusions of the present step.

Step 2 If E is any set as in the statement, then A_1 and A_2 are sets of finite perimeter. For any fixed $h > k$, set $E_h = E \cap \{y \leq h\}$. Then, by (2.1),

$$\partial^M E_h \subset \partial^M E \cup \{y = h\} \,. \tag{4.41}$$

Inclusion (4.41) ensures that E_h is of finite perimeter in $\Omega \times \mathbb{R}$, by Theorem E. The same inclusion, via an application of Lemma 4.1, tells us that condition (1.10) is fulfilled also with E replaced by E_h. Furthermore, since

$$E_h \cap (\Omega \times \mathbb{R}) = \{(x', y) : \, x' \in \Omega, \; y_1(x') \leq y \leq y_2^h(x')\} \,, \tag{4.42}$$

where $y_2^h(x') = \min\{h, y_2(x')\}$, then $\mathcal{L}^1((E_h)_{x'}) \geq \min\{h - k, \ell(x')\}$. Thus, assumption (1.9) is satisfied with E replaced by E_h as well. On setting

$$A_2^h = \{(x', y) : \, x' \in \Omega, \; y > y_2^h(x')\}$$

and applying Step 1 to E_h, one gets that A_1 is of finite perimeter, and that A_2^h is of finite perimeter for every $h > k$. Furthermore, by (4.40)-(4.41),

$$P(A_2^h; \Omega \times \mathbb{R}) \leq P(E_h; \Omega \times \mathbb{R}) \leq P(E; \Omega \times \mathbb{R}) + \mathcal{L}^{n-1}(\Omega) \qquad \text{for } \; h > k \,. \tag{4.43}$$

Since $\chi_{A_2^h} \to \chi_{A_2}$ in $L^1_{\rm loc}(\Omega \times \mathbb{R})$ as $h \to +\infty$, then estimate (4.43) and the lower semicontinuity of perimeter entail that A_2 is of finite perimeter in $\Omega \times \mathbb{R}$.

Step 3 Under the additional assumption (4.39), the conclusions of the lemma hold.
We begin by showing that $y_1, y_2 \in BV_{\rm loc}(\Omega)$. Given any positive number h, define $A_1^h = A_1 \cup \{y < -h\}$. Thus,

$$A_1^h \cap (\Omega \times \mathbb{R}) = \{(x', y) : \, x' \in \Omega, \; y < y_1^h(x')\} \,,$$

where $y_1^h(x') = \max\{y_1(x'), -h\}$. Since A_1 is of finite perimeter in $\Omega \times \mathbb{R}$ by Step 2, and since

$$\partial^M A_1^h \subset \partial^M A_1 \cup \{y = -h\} \tag{4.44}$$

by (2.1), then A_1^h is a set of finite perimeter in $\Omega \times \mathbb{R}$. Moreover, (4.31) and (1.10) ensure, via Lemma 4.1, that

$$P(A_1; B \times \mathbb{R}) = 0 \tag{4.45}$$

for every Borel set $B \subset \Omega$ such that $\mathcal{L}^{n-1}(B) = 0$. Inclusion (4.44) and equality (4.45) easily imply that

$$P(A_1^h; B \times \mathbb{R}) = 0 \tag{4.46}$$

for every Borel set $B \subset \Omega$ with $\mathcal{L}^{n-1}(B) = 0$. Hence, the same argument as that at the end of Part I tells us that $y_1^h \in W^{1,1}(\Omega)$. Furthermore, by (2.6) and by (4.44),

$$\int_\Omega |\nabla y_1^h| \, dx' \leq P(A_1^h; \Omega \times \mathbb{R}) \leq P(A_1; \Omega \times \mathbb{R}) + \mathcal{L}^{n-1}(\Omega) \,. \tag{4.47}$$

Now, let ω be any connected open subset of Ω having a Lipschitz boundary. Obviously, there exist positive constants $h_0 \geq k$ and a such that

$$\mathcal{L}^{n-1}(\{-h_0 < y_1^h(x')\} \cap \omega) \geq a \qquad \text{for } \ h > h_0 \, . \tag{4.48}$$

A form of the Poincaré inequality (see e.g. [20], Chap. 4) ensures that a constant C, depending only on n, ω, and a exists such that

$$\int_\omega |u|\, dx' \leq C \int_\omega |\nabla u|\, dx' \tag{4.49}$$

for every $u \in W^{1,1}(\omega)$ satisfying $\mathcal{L}^{n-1}(\{u = 0\}) \geq a$. Applying (4.49) with $u(x') = \min\{y_1^h(x') + h_0, 0\}$, and making use of (4.48) and of the fact that $y_1^h \leq k \leq h_0$ yield, via (4.47),

$$\int_\omega |y_1^h|\, dx' \leq CP(A_1; \Omega \times \mathbb{R}) + (C + 2h_0)\mathcal{L}^{n-1}(\Omega) \, . \tag{4.50}$$

On passing to the limit as $h \to +\infty$ in (4.50) one gets $y_1 \in L^1(\omega)$, whence, by Theorem B, $y_1 \in BV(\omega)$. Equation (4.45) then gives $y_1 \in W^{1,1}(\omega)$. Clearly also $y_2 \in W^{1,1}(\omega)$, inasmuch as $y_2 = y_1 + \ell$ and $\ell \in W^{1,1}(\Omega)$. Hence, $y_1, y_2 \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Owing to (4.40) applied with Ω replaced by any open subset Ω' satisfying $\overline{\Omega'} \subset \Omega$, we have

$$P(E; \Omega' \times \mathbb{R}) = \sum_{i=1}^{2} P(A_i; \Omega' \times \mathbb{R}) = \sum_{i=1}^{2} \int_{\Omega'} \sqrt{1 + |\nabla y_i|^2}\, dx' \, . \tag{4.51}$$

On approximating Ω from inside by an increasing sequence of open subsets, equation (4.10) follows from (4.51).

Step 4 Conclusion.
On applying Step 3 to the set E_h defined in Step 2, one deduces that $y_1 \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Hence, $y_2 = y_1 + \ell \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Thus, only (4.10) remains to be proved. By Step 1 applied to E_h, $\mathcal{H}^{n-1}((\partial^M A_1 \setminus \partial^M E_h) \cap (\Omega \times \mathbb{R})) = 0$, $\mathcal{H}^{n-1}((\partial^M A_2^h \setminus \partial^M E_h) \cap (\Omega \times \mathbb{R})) = 0$ and $\mathcal{H}^{n-1}(\partial^M A_1 \cap \partial^M A_2^h \cap (\Omega \times \mathbb{R})) = 0$. Thereby, for every Borel set B with $\overline{B} \subset \Omega$, we have

$$\begin{aligned} P(E_h; B \times \mathbb{R}) &= \mathcal{H}^{n-1}(\partial^M A_1 \cap (B \times \mathbb{R})) + \mathcal{H}^{n-1}(\partial^M A_2^h \cap (B \times \mathbb{R})) \\ &= P(A_1; B \times \mathbb{R}) + P(A_2^h; B \times \mathbb{R}) \\ &= \int_B \sqrt{1 + |\nabla y_1|^2}\, dx' + \int_B \sqrt{1 + |\nabla y_2^h|^2}\, dx' \, . \end{aligned} \tag{4.52}$$

Clearly, equation (4.52) continues to hold for any Borel set $B \subset \Omega$, as an approximation argument for B by an increasing sequence of Borel sets converging to B from inside shows. On applying (4.52) with $B = \{x' \in \Omega : y_2(x') < h\}$ and observing that $\partial^M E_h \cap (\{y_2 < h\} \times \mathbb{R}) = \partial^M E \cap (\{y_2 < h\} \times \mathbb{R})$ yields

$$P(E; \{y_2 < h\} \times \mathbb{R}) = \int_{\{y_2 < h\}} \sqrt{1 + |\nabla y_1|^2}\, dx' + \int_{\{y_2 < h\}} \sqrt{1 + |\nabla y_2|^2}\, dx' \, . \tag{4.53}$$

Notice that here we have made use of the fact that $\nabla y_2^h = \nabla y_2 \chi_{\{y_2 < h\}}$ $\mathcal{L}^{n-1}$-a.e. in Ω. Equation (4.10) follows from (4.53), by letting h go to infinity. $\square$

Proof of Theorem 1.3 – Case of bounded sets. Let E be a bounded set of finite perimeter satisfying (1.5) and satisfying (1.8)–(1.9) for some open subset Ω of $\mathbb{R}^{n-1}$. By Theorem 1.1, there exist functions $y_1, y_2 : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $y_1(x') \le y_2(x')$ and $E_{x'}$ is equivalent to $(y_1(x'), y_2(x'))$. Observe that, since we are assuming that E is bounded, then condition (4.11) is certainly fulfilled. Moreover, assumption (1.8) and Proposition 4.2 ensure that (1.10) is satisfied. Thus, all the hypotheses of Lemma 4.3 are in force (even in the more stringent form appearing in Part I of its proof). Therefore, $y_1, y_2 \in W^{1,1}(\Omega)$. Furthermore, (4.11) holds with Ω replaced by any of its bounded open subset, and hence also for Ω. On the other hand, since, by definition,

$$(E^s)_{x'} = \big[-\ell(x')/2, \ell(x')/2\big] \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \Omega ,$$

then, owing to Lemma 4.3 applied to E^s, we have

$$P(E^s; \Omega \times \mathbb{R}) = 2 \int_\Omega \sqrt{1 + \frac{1}{4}|\nabla \ell|^2}\, dx' . \tag{4.54}$$

Since $\ell(x') = y_2(x') - y_1(x')$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega$, and since the function $\sqrt{1+(\cdot)^2}$ is convex, then (4.10) and (4.54) yield

$$\begin{aligned} P(E^s; \Omega \times \mathbb{R}) &= 2 \int_\Omega \sqrt{1 + \Big|\frac{\nabla(y_2 - y_1)}{2}\Big|^2}\, dx' \\ &\le \int_\Omega \sqrt{1 + |\nabla y_2|^2}\, dx' + \int_\Omega \sqrt{1 + |\nabla y_1|^2}\, dx' = P(E; \Omega \times \mathbb{R}) . \end{aligned} \tag{4.55}$$

Combining (1.5) and Lemma 3.4 tells us that $P(E^s; \Omega \times \mathbb{R}) = P(E; \Omega \times \mathbb{R})$. Consequently, equality must hold in the inequality of (4.55). Since the function $\sqrt{1+(\cdot)^2}$ is strictly convex, this entails that $-\nabla y_1 = \nabla y_2$ $\mathcal{L}^{n-1}$-a.e. in Ω. Thus $y_1 + y_2 \in W^{1,1}_{\rm loc}(\Omega)$ and $\nabla(y_1 + y_2) = 0$ $\mathcal{L}^{n-1}$-a.e. in Ω. Hence, for any connected component Ω_α of Ω, there exists $c_\alpha \in \mathbb{R}$ such that $y_1 + y_2 = c_\alpha$ $\mathcal{L}^{n-1}$-a.e. in Ω_α (see e.g. [20, Corollary 2.1.9]). Clearly, the last equation implies that $E \cap (\Omega_\alpha \times \mathbb{R})$ is equivalent to a translate of $E^s \cap (\Omega_\alpha \times \mathbb{R})$ along the y-axis. Finally, if (1.8)–(1.9) are fulfilled for some connected open subset Ω of $\mathbb{R}^{n-1}$ such that

$$\mathcal{L}^{n-1}(\pi(E)^+ \setminus \Omega) = 0 , \tag{4.56}$$

then $E \cap (\Omega \times \mathbb{R})$ is equivalent to a translate of $E^s \cap (\Omega \times \mathbb{R})$ along the y-axis. Thus, E is equivalent to a translate of E^s along the y-axis, since $E \cap (\Omega \times \mathbb{R})$ and $E^s \cap (\Omega \times \mathbb{R})$ are equivalent to E and E^s, respectively, thanks to (4.56). □

Lemma 4.4 *Let Ω be an open subset of $\mathbb{R}^{n-1}$ and let E be a set of finite perimeter in $\Omega \times \mathbb{R}$ having the property that there exist two functions $y_1, y_2 : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $y_1(x') \le y_2(x')$ and $E_{x'}$ is equivalent to $(y_1(x'), y_2(x'))$. Assume that* (1.10) *and* (1.9) *are fulfilled. Given any $t \in \mathbb{R}$, set*

$$\hat{E}_{(t)} = \{(x', y) : x' \in \pi(E)^+, \max\{y_1(x') - t, t - y_2(x')\} \le y \le \max\{t - y_1(x'), y_2(x') - t\}\}. \tag{4.57}$$

Then

$$\mathcal{L}^n(\hat{E}_{(t)}) = \mathcal{L}^n(E) \tag{4.58}$$

and

$$P(\hat{E}_{(t)}; \Omega \times \mathbb{R}) = P(E; \Omega \times \mathbb{R}) \, . \tag{4.59}$$

Moreover, conditions (1.10) *and* (1.9) *are satisfied with E replaced by $\hat{E}_{(t)}$.*

Proof. We shall assume that $t = 0$, the other cases being completely analogous, and denote $\hat{E}_{(0)}$ simply by $\hat{E}$. As in the proof of Lemma 4.3, we assume also, without loss of generality, that $y_1(x') \le y_2(x')$ and that $E_{x'} = (y_1(x'), y_2(x'))$ for every $x' \in \mathbb{R}^{n-1}$. Definition 4.57 immediately tells us that $\mathcal{L}^1(\hat{E}_{x'}) = \mathcal{L}^1(E_{x'}) = \ell(x')$ for $x' \in \mathbb{R}^{n-1}$. Hence (4.58) holds, and (1.9) is trivially satisfied also with E replaced by $\hat{E}$.
Set $\tilde{E} = \{(x', y) : (x', -y) \in E\}$, and observe that

$$\hat{E} \text{ is equivalent to } [(E \cup \tilde{E}) \cap \{y \ge 0\}] \cup [(E \cap \tilde{E}) \cap \{y < 0\}] \, . \tag{4.60}$$

Thus, $\hat{E}$ is a set of finite perimeter in $\Omega \times \mathbb{R}$. Moreover, from (4.60) and (2.1) we infer that

$$\partial^M \hat{E} \subset \partial^M (E \cup \tilde{E}) \cup \partial^M (E \cap \tilde{E}) \cup \{y = 0\} \subset \partial^M E \cup \partial^M \tilde{E} \cup \{y = 0\} \, ,$$

whence condition (1.10) for $\hat{E}$ easily follows. Now we prove (4.59). Let G_E, $G_{\tilde{E}}$ and $G_{\hat{E}}$ be the sets associated with E, $\tilde{E}$ and $\hat{E}$, respectively, as in Theorem G. Clearly, $G_E = G_{\tilde{E}}$. Set $G = G_E \cap G_{\hat{E}} \cap \Omega$. Then

$$(\partial^* E)_{x'} = \partial^*(E_{x'}), \;\; (\partial^* \tilde{E})_{x'} = \partial^*(\tilde{E}_{x'}), \;\; (\partial^* \hat{E})_{x'} = \partial^*(\hat{E}_{x'}), \quad \text{for } x' \in G \, . \tag{4.61}$$

By the very definition of $\hat{E}$, either $\hat{E}_{x'} = E_{x'}$, or $\hat{E}_{x'} = \tilde{E}_{x'}$. Thus, equations (4.61) imply that

$$\text{either } (\partial^* \hat{E})_{x'} = (\partial^* E)_{x'}, \text{ or } (\partial^* \hat{E})_{x'} = (\partial^* \tilde{E})_{x'} \quad \text{for } x' \in G \, . \tag{4.62}$$

On the other hand, by Theorem C, there exists a set $N \subset \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(N) = 0$ and

$$\begin{cases} \nu^E(x) = \pm \nu^{\hat{E}}(x) & \text{if } x \in (\partial^* E \cap \partial^* \hat{E} \cap (\Omega \times \mathbb{R})) \setminus N \\ \nu^{\tilde{E}}(x) = \pm \nu^{\hat{E}}(x) & \text{if } x \in (\partial^* \tilde{E} \cap \partial^* \hat{E} \cap (\Omega \times \mathbb{R})) \setminus N \, . \end{cases} \tag{4.63}$$

Set $M = \pi(N)$. By ([1, Proposition 2.49(iv)]), $\mathcal{L}^{n-1}(M) = 0$. From (4.62)–(4.63) we have that

$$\begin{cases} \nu^{\hat{E}}(x', \cdot) = \pm \nu^E(x', \cdot) & \text{if } x' \in (G \setminus M) \cap \pi(\partial^* E \cap \partial^* \hat{E}) \\ \nu^{\hat{E}}(x', \cdot) = \pm \nu^{\tilde{E}}(x', \cdot) & \text{if } x' \in (G \setminus M) \cap \pi(\partial^* \tilde{E} \cap \partial^* \hat{E}) \, . \end{cases} \tag{4.64}$$

Moreover, by symmetry,

$$|\nu_y^E(x', y_i(x'))| = |\nu_y^{\tilde{E}}(x', -y_i(x'))| \qquad \text{for } x' \in G \, . \tag{4.65}$$

Hence,

$$P(\hat{E};\Omega\times\mathbb{R}) = \mathcal{H}^{n-1}(\partial^*\hat{E}\cap(\Omega\times\mathbb{R})) = \mathcal{H}^{n-1}(\partial^*\hat{E}\cap[(G\setminus M)\times\mathbb{R}])$$
$$= \int_{G\setminus M} dx' \int_{(\partial^*\hat{E})_{x'}} \frac{1}{|\nu_y^{\hat{E}}(x)|} d\mathcal{H}^0 = \int_{G\setminus M} dx' \int_{(\partial^* E)_{x'}} \frac{1}{|\nu_y^{E}(x)|} d\mathcal{H}^0 = P(E;\Omega\times\mathbb{R})\ , \tag{4.66}$$

where the second equality in (4.66) holds since $\mathcal{L}^{n-1}((\pi(E)^+\cap\Omega)\setminus(G\setminus M)) = 0$ and (1.10) is fulfilled with E replaced by $\hat{E}$, the third is an application of the coarea formula (2.12), the fourth is a consequence of (4.62), (4.64) and (4.65), and the last one is due to the first three equalities applied with $\hat{E}$ replaced by E. □

Proof of Theorem 1.3 - General case. There is no loss of generality in assuming that $\mathcal{L}^n(E) < \infty$, since otherwise E is equivalent to $\mathbb{R}^n$, by Theorem 1.1, and there is nothing to prove.

One can start as in the proof of the case where E is bounded and observe that, if there exists $k \in \mathbb{R}$ such that $y_1(x') \le k$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega$, then the assumptions of Lemma 4.3 are fulfilled, and the proof proceeds exactly as in that case. Obviously, the same argument, applied to $\tilde{E}$, yields the conclusion also under the assumption that $y_2(x') \ge k$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega$ and for some $k \in \mathbb{R}$.

In the general case, fix any $t \in \mathbb{R}$ and consider the set $\hat{E}_{(t)}$ defined as in (4.57). By (1.8) and Proposition 4.2, assumption (1.10) is fulfilled. Thus, by Lemma 4.4, assumptions (1.10) and (1.9) are fulfilled also with E replaced by $\hat{E}_{(t)}$. Thus $\hat{E}_{(t)}$ satisfies the same hypotheses as E, and enjoys the additional property that

$$(\hat{E}_{(t)})_{x'} = (\max\{y_1(x')-t, t-y_2(x')\}, \max\{t-y_1(x'), y_2(x')-t\}) \qquad \text{for } x' \in \pi(E)^+$$

with

$$\max\{t-y_1(x'), y_2(x')-t\} \ge 0\ .$$

Hence, $\hat{E}_{(t)}$ satisfies the assumptions of Lemma 4.3. Moreover, by (4.59),

$$P(\hat{E}_{(t)},\Omega\times\mathbb{R}^n) = P(E,\Omega\times\mathbb{R}^n) = P(E^s,\Omega\times\mathbb{R}^n) = P((\hat{E}_{(t)})^s,\Omega\times\mathbb{R}^n)\ . \tag{4.67}$$

Arguing as above, tells us that the conclusions of the theorem are true with E replaced by $\hat{E}_{(t)}$. Thus $\hat{E}_{(t)}\cap(\Omega_\alpha\times\mathbb{R})$ is equivalent to a translate of $E^s\cap(\Omega_\alpha\times\mathbb{R})$ along the y-axis, for every connected component Ω_α of Ω; namely there exists $c_{\alpha,t} \in \mathbb{R}$ such that

$$\max\{y_1(x')-t, t-y_2(x')\} + \max\{t-y_1(x'), y_2(x')-t\} = c_{\alpha,t} \ \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \Omega_\alpha. \tag{4.68}$$

Equation (4.68) implies that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \Omega_\alpha$,

$$\text{either } \ y_1(x') + y_2(x') - 2t = c_{\alpha,t}, \quad \text{or } \ y_1(x') + y_2(x') - 2t = -c_{\alpha,t}\ . \tag{4.69}$$

Choosing any two different values of t in (4.69) easily entails that $y_1(x') + y_2(x')$ must be constant $\mathcal{L}^{n-1}$-a.e. in Ω_α, whence $E\cap(\Omega_\alpha\times\mathbb{R})$ is equivalent to a translate of $E^s\cap(\Omega_\alpha\times\mathbb{R})$ along the y-axis. □

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