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**Application of a result of Ioffe/Tichomirov to multidimensional control problems of Dieudonné-Rashevsky type**

(revised version: May 2002)

by


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Preprint no.: 43 2002

# Application of a result of Ioffe/Tichomirov to multidimensional control problems of Dieudonné-Rashevsky type

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## 1. Introduction.

**a) Abstract control problems with mixed restrictions.** We start to investigate an abstract control problem $(\mathrm{P})_0$ with mixed restrictions:

$$(\mathrm{P})_0 \quad F(x,u) \longrightarrow \mathrm{Min}\,! \tag{1.1}$$
$$\text{subject to } (x,u) \in \mathrm{X} \times \mathrm{U} \text{ with} \tag{1.2}$$
$$G(x,u) = \mathfrak{o}_{\mathrm{Y}}\,; \tag{1.3}$$
$$H_l(x,u) \leqslant 0\,,\ \ 1 \leqslant l \leqslant w\,. \tag{1.4}$$

Together with $(\mathrm{P})_0$, we study a "relaxed" problem $(\widetilde{\mathrm{P}})_0$ arising from a formal extension of the original control set U:

$$(\widetilde{\mathrm{P}})_0 \quad F(x,u) \longrightarrow \mathrm{Min}\,! \tag{2.1}$$
$$\text{subject to } (x,u) \in \mathrm{X} \times \widetilde{\mathrm{U}} \text{ with} \tag{2.2}$$
$$G(x,u) = \mathfrak{o}_{\mathrm{Y}}\,; \tag{2.3}$$
$$H_l(x,u) \leqslant 0\,,\ \ 1 \leqslant l \leqslant w\,. \tag{2.4}$$

Let X and Y be Banach spaces; U and $\widetilde{\mathrm{U}}$ with $\mathrm{U} \subseteq \widetilde{\mathrm{U}}$ — closed subsets of some linear topological space; $F$, $G$ and $H_l$ — mappings with $F\colon\ \mathrm{X} \times \widetilde{\mathrm{U}} \to \mathbb{R}$, $G\colon\ \mathrm{X} \times \widetilde{\mathrm{U}} \to \mathrm{Y}$, $H_l\colon\ \mathrm{X} \times \widetilde{\mathrm{U}} \to \mathbb{R}$.

**b) Local relaxability.** Following [ 4 : p. 201, Theorem 1, Assumptions b) ] and [ 3 : p. 92, Definition 3.2 ] , we formulate the concept of (local) relaxability of the problem $(\mathrm{P})_0$.

**Definition 1.1.** We call the problem $(\mathrm{P})_0$ *relaxable at the feasible solution* $(x^*, u^*)$ *with respect to* $\widetilde{\mathrm{U}} \supseteq \mathrm{U}$ iff for any finite subset of feasible controls $\{\, u_1,\ \ldots\ ,\ u_r \,\} \subset \mathrm{U}$ and for each sufficiently small $\delta > 0$ there exist a number $\eta > 0$ and a mapping $v(x,\alpha)\colon\ \mathrm{X} \times \mathbb{R}^r_+ \to \widetilde{\mathrm{U}}$ such that the following conditions hold for all $x'$, $x'' \in \mathrm{B}(x^*, \eta) \subset \mathrm{X}$ and $\alpha'$, $\alpha'' \in \mathrm{B}(\mathfrak{o}_r, \eta) \cap \mathbb{R}^r_+$:

1) $v(x', \mathfrak{o}_r) = u^*\,;$

2) $$\Big\|\, G(x', v(x',\alpha')) - G(x'', v(x'',\alpha'')) - G_x(x^*,u^*)(x'-x'') - \sum_{s=1}^{r} (\alpha'_s - \alpha''_s)\,\big(\, G(x^*,u_s) - G(x^*,u^*)\,\big)\,\Big\|_{\mathrm{Y}} \leqslant \delta\,\Big(\, \|\, x' - x'' \,\|_{\mathrm{X}} + \sum_{s=1}^{r} |\, \alpha'_s - \alpha''_s \,|\,\Big)$$

3) $$F(x', v(x',\alpha')) - \Big(\, \sum_{s=1}^{r} \alpha'_s\, F(x',u_s) + \big(1 - \sum_{s=1}^{r} \alpha'_s\,\big)\, F(x',u^*)\Big) \leqslant \delta\,\Big(\, \|\, x' - x^* \,\|_{\mathrm{X}} + \sum_{s=1}^{r} |\, \alpha'_s \,|\,\Big)$$

4) $$H_l(x', v(x',\alpha')) - \Big(\, \sum_{s=1}^{r} \alpha'_s\, H_l(x',u_s) + \big(1 - \sum_{s=1}^{r} \alpha'_s\,\big)\, H_l(x',u^*)\Big) \leqslant \delta\,\Big(\, \|\, x' - x^* \,\|_{\mathrm{X}} + \sum_{s=1}^{r} |\, \alpha'_s \,|\,\Big)\,,\quad 1 \leqslant l \leqslant w\,.$$

Moreover, we will say that conditions 3) resp. 4) hold in sharpened form 3)′ resp. 4)′ if one can take the absolute value on the left-hand side of the corresponding inequalities:

$$3)' \quad \Big| F(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s F(x', u_s) + \big(1 - \sum_{s=1}^{r} \alpha'_s\big) F(x', u^*) \Big) \Big| \leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s | \Big)$$

$$4)' \quad \Big| H_l(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s H_l(x', u_s) + \big(1 - \sum_{s=1}^{r} \alpha'_s\big) H_l(x', u^*) \Big) \Big| \leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s | \Big), \quad 1 \leqslant l \leqslant w.$$

**c) Statement of Ioffe/Tichomirov's extremal principle.** Necessary first-order optimality conditions for problems $(\mathrm{P})_0$ are given by Ioffe/Tichomirov's "extremal principle for locally convex problems" [4: p. 201 f., Theorem 1]. We state this theorem in two versions. At first, it is reformulated as a theorem about comparison of the minimal values of problems $(\mathrm{P})_0$ and $(\widetilde{\mathrm{P}})_0$ (Theorem 1.2.). In case of their coincidence, the principle reads as separation theorem for convex sets (Theorem 1.3.) (cf. in particular [4: p. 203]), yielding a Lagrange multiplier rule.

**Theorem 1.2. (Comparison of minimal values of $(\mathrm{P})_0$ and $(\widetilde{\mathrm{P}})_0$)** *Assume that $(\mathrm{P})_0$ together with its global minimizer $(x^*, u^*)$ satisfies the following conditions:*

*1) $F$, $G$ and $H_l$ are continuous in all variables; $F$ and $H_l$ are Fréchet differentiable in $x$ while $G$ is in $x$ even continuously differentiable. The minimal value is zero: $F(x^*, u^*) = 0$.*

*2) The problem $(\mathrm{P})_0$ is relaxable in the sense of Definition 1.1. at $(x^*, u^*)$ with respect to $\widetilde{\mathrm{U}}$.*

*3) The range $\mathrm{Im}\, G_x(x^*, u^*) \subseteq \mathrm{Y}$ forms in $\mathrm{Y}$ a subspace with finite codimension.*

*4) $\mathfrak{o}_{\mathrm{Y}}$ is contained in the interior of the set $\mathrm{Im}\, G_x(x^*, u^*) + \mathrm{co}\,\{ G(x^*, u) \mid u \in \mathrm{U} \}$.*

*Assume further that precisely the first $k \leqslant w$ of the mixed restrictions are active in $(x^*, u^*)$, i. e. $H_l(x^*, u^*) = 0$, $1 \leqslant l \leqslant k$, and $H_l(x^*, u^*) < 0$, $(k+1) \leqslant l \leqslant w$. Then the set $\mathrm{C} \subset \mathbb{R} \times \mathbb{R}^k \times \mathrm{Y}$ defined by $\mathrm{C} =$*

$$\left\{ \begin{pmatrix} \tau_0 + \langle\, F_x(x^*, u^*),\ x \,\rangle \\ \tau_l + \langle\, (H_l)_x(x^*, u^*),\ x \,\rangle \\ G_x(x^*, u^*)\, x \end{pmatrix} \;\middle|\; \begin{matrix} \tau_0 \geqslant 0 \\ \tau_l \geqslant 0,\ 1 \leqslant l \leqslant k \\ x \in \mathrm{X} \end{matrix} \right\} + \mathrm{co} \left\{ \begin{pmatrix} F(x^*, u) - F(x^*, u^*) \\ H_l(x^*, u) - H_l(x^*, u^*) \\ G(x^*, u) - G(x^*, u^*) \end{pmatrix} \;\middle|\; u \in \mathrm{U} \right\} \tag{3}$$

*is convex with nonempty interior. Consider the cone*

$$\mathrm{R} = \left\{ \begin{pmatrix} -\varrho_0 \\ -\varrho_l \\ \mathfrak{o}_{\mathrm{Y}} \end{pmatrix} \;\middle|\; \begin{matrix} \varrho\ > 0 \\ \varrho_l > 0,\ 1 \leqslant l \leqslant k \end{matrix} \right\} \subset \mathbb{R} \times \mathbb{R}^k \times \mathrm{Y}. \tag{4}$$

*If the intersection $\mathrm{C} \cap \mathrm{R}$ is nonempty then it holds $\inf (\widetilde{\mathrm{P}})_0 < \inf (\mathrm{P})_0$.* □

**Remarks.** 1) The theorem was originally stated for the case $\mathrm{U} = \widetilde{\mathrm{U}}$ and under little more general assumptions: It still holds for "strong local minimizers" $(x^*, u^*)$ [4: p. 201 above] and functionals $F$ und $H_l$ being "regularly locally convex in $x$" at the point $x^*$ [4: p. 188].

2) If the equality $\inf (\mathrm{P})_0 = \inf (\widetilde{\mathrm{P}})_0$ holds then the intersection $\mathrm{C} \cap \mathrm{R}$ is empty, and there exists a hyperplane separating properly the convex sets C und R. (This case will occur, in particular, if condition 2) can be fulfilled with $\widetilde{\mathrm{U}} = \mathrm{U}$.) Then we arrive at a set of first-order necessary optimality conditions collected in the following theorem.

**Theorem 1.3. (Extremal principle for $(\mathrm{P})_0$)** *Let the problem $(\mathrm{P})_0$ together with its global minimizer $(x^*, u^*)$ satisfy all conditions 1) – 4) of Theorem 1.2., and let $\inf (\mathrm{P})_0 = \inf (\widetilde{\mathrm{P}})_0$ hold. Then there exists a* *nontrivial collection of multipliers* $\lambda_0 > 0,\ \lambda_1 \geqslant 0,\ \ldots\ ,\ \lambda_w \geqslant 0$ *and* $y^* \in \mathrm{Y}^*$*, satisfying together with* $(x^*, u^*)$ *the maximum condition* $(\mathcal{M})$*, the canonical equation* $(\mathcal{K})$ *as well as the complementarity conditions* $(\mathcal{C})$*:*

$$(\mathcal{M}):\quad \lambda_0 \big( F(x^*, u) - F(x^*, u^*) \big) + \sum_{l=1}^{w} \lambda_l \big( H_l(x^*, u) - H_l(x^*, u^*) \big) + \langle\, y^*\,,\, G(x^*, u) - G(x^*, u^*) \,\rangle \geqslant 0 \qquad \forall\, u \in \mathrm{U}\,;$$

$$(\mathcal{K}):\quad \lambda_0 \,\langle\, F_x(x^*, u^*)\,,\, x \,\rangle + \sum_{l=1}^{w} \lambda_l \,\langle\, (H_l)_x(x^*, u^*)\,,\, x \,\rangle + \langle\, y^*\,,\, G_x(x^*, u^*)\, x \,\rangle = 0 \quad \forall\, x \in \mathrm{X}\,;$$

$$(\mathcal{C}):\quad \lambda_l\, H_l(x^*, u^*) = 0\,,\ \ 1 \leqslant l \leqslant w\,.$$

**Proof.** By Theorem 1.2., the equality $\inf(\mathrm{P})_0 = \inf(\widetilde{\mathrm{P}})_0$ implies the existence of a hyperplane $\big(\lambda_0, \lambda_1,\ \ldots\ , \lambda_k,\, y^*\big) \in \big(\mathbb{R} \times \mathbb{R}^k \times \mathrm{Y}\big)^*$ which separates properly the convex sets C and R defined above. This leads to the variational inequality

$$\lambda_0 \Big[ \tau_0 + \langle\, F_x(x^*, u^*)\,,\, x \,\rangle + \sum_{s=1}^{r} \alpha_s \big( F(x^*, u_s) - F(x^*, u^*) \big) \Big] + \sum_{l=1}^{k} \lambda_l \Big[ \tau_l + \langle\, (H_l)_x(x^*, u^*)\,,\, x \,\rangle$$
$$+ \sum_{s=1}^{r} \alpha_s \big( H_l(x^*, u_s) - H_l(x^*, u^*) \big) \Big] + \Big\langle\, y^*\,,\, G_x(x^*, u^*)\, x + \sum_{s=1}^{r} \alpha_s \big( G(x^*, u) - G(x^*, u^*) \big) \Big\rangle$$
$$\geqslant -\lambda_0\, \varrho_0 - \sum_{l=1}^{k} \lambda_l\, \varrho_l \tag{5}$$

for all $\tau_0 \geqslant 0$, $\tau_l \geqslant 0$, $x \in \mathrm{X}$, $r \geqslant 1$, $\alpha_s \geqslant 0$ with $\sum_{s=1}^{r} \alpha_s = 1$, $u_s \in \mathrm{U}$, $\varrho_0 > 0$, $\varrho_l > 0$, $1 \leqslant l \leqslant k$, $1 \leqslant s \leqslant r$. The conditions $\lambda_0 \geqslant 0$, $\lambda_l \geqslant 0$, $1 \leqslant l \leqslant w$, $(\mathcal{M})$ and $(\mathcal{K})$ are immediate consequences of this inequality if the collection of multipliers is completed by $\lambda_l = 0$ for $(k+1) \leqslant l \leqslant w$. Then $(\mathcal{C})$ holds too. As in [ 4 : p. 90 f. ] , the occurence of the regular case $\lambda_0 > 0$ can be derived from condition 4) of Theorem 1.2. □

**d) Outline and aim of the paper.** Multidimensional control problems of Dieudonné-Rashevsky type are characterized by the presence of first-order PDE restrictions of the shape $\partial x_i(t)/\partial t_j = g_{ij}(t, x(t), u(t))$. Obviuosly, Theorems 1.2. and 1.3. cannot be applied immediately to this kind of problems since $\mathrm{Im}\, G_x(x^*, u^*)$ is of infinite codimension in the corresponding function spaces. In Section 2 we will show how to overcome this principal difficulty: We replace $(\mathrm{P})_0$ by an enlarged problem which is related closely enough with the original one but satisfies the assumption about codimension. This procedure leads to an $\varepsilon$-extremal principle for the original problem. In Section 3, this result is applied to Dieudonné-Rashevsky type problems. In our general framework, the $\varepsilon$-optimality conditions known before for this kind of problems are reproduced. Additionally, we observe as a new condition that the multiplier $y^*$ associated with the equality restrictions must annihilate a specific subspace $\widetilde{\mathrm{Y}} \subset \mathrm{Y}$ .

**e) Notations.** $C^{k,n}(\Omega)$, $L_p^n(\Omega)$ and $W_p^{k,n}(\Omega)$ $(1 \leqslant p \leqslant \infty)$ denote the spaces of $n$-dimensional vector functions on $\Omega$ whose components are $k$-times continuously differentiable resp. belong to $L_p(\Omega)$ or to the Sobolev space of $L_p(\Omega)$-functions with weak derivatives up to $k^{th}$ order in $L_p(\Omega)$. Instead of $C^{k,1}(\Omega)$, we write shorter $C^k(\Omega)$. Functions of the space $C_\circ^{k,n}(\Omega)$ are additionally subjected to a zero boundary condition. For the classical as well as for the weak partial derivatives of $x_i$ by $t_j$ we use the same notation: $x_{i\,;t_j}$ . $\delta_v$ denotes the Dirac measure supported on $v$. A family $\mu = \{\, \mu_t \mid t \in \Omega \,\}$ of probability measures $\mu_t \in rca\,(\mathrm{K}\,,\, \mathfrak{B}_{\mathrm{K}}\,)$ acting on the $\sigma$-algebra $\mathfrak{B}_{\mathrm{K}}$ of the Borel sets of K is called (a representative of) a generalized control on K iff for any continuous function $f \in C^0(\Omega \times \mathrm{K})$ the function $h_f :\ \Omega \times \mathrm{K} \to \mathbb{R}$ with $h_f(t) = \int_{\mathrm{K}} f(t, v)\, d\mu_t(v)$ is measurable [ 2 : p. 23 ] . The generalized controls form equivalence classes by identifying two families $\mu'$,

$\mu''$ iff $\mu'_t \equiv \mu''_t$ a. e. on $\Omega$. The set of all generalized controls is denoted by $\mathfrak{Y}_{\mathrm{K}}$. The set $\mathfrak{Y}_{\mathrm{K}}$ is convex [2: p. 25] and, assuming compactness of K, sequentially compact in an appropriate topology [6: p. 391, Theorem 4]. The abbreviation "$(\forall)\, t \in \mathrm{A}$" reads as: "for almost all $t$ from A" resp. "for all $t$ from A except some Lebesgue null set". Finally, $\mathfrak{o}$ denotes the zero element resp. the zero function of the actual space.

## 2. Enlargement of the original problem and its consequences.

**a) Statement of problems $(\mathrm{P})_\varepsilon$ and $(\widetilde{\mathrm{P}})_\varepsilon$.** Let the abstract problems $(\mathrm{P})_0$ and $(\widetilde{\mathrm{P}})_0$ under the assumptions of Sect. 1. a) be given. Assume further that the minimal value of $(\mathrm{P})_0$ is finite: $(-\infty) < \inf(\mathrm{P})_0 < (+\infty)$. Choosing a number $\varepsilon > 0$ and a closed subspace $\widetilde{\mathrm{Y}} \subseteq \mathrm{Y}$, we state the following enlarged problems $(\mathrm{P})_\varepsilon$ and $(\widetilde{\mathrm{P}})_\varepsilon$:

$$(\mathrm{P})_\varepsilon: \quad \widetilde{F}(x,z,u) := F(x,u) - \inf(\mathrm{P})_0 \longrightarrow \mathrm{Min}! \qquad \widetilde{F}: \mathrm{X} \times \widetilde{\mathrm{Y}} \times \mathrm{U} \to \mathbb{R}\,; \tag{6.1}$$

$$\widetilde{G}(x,z,u) := G(x,u) + \varepsilon\, z = \mathfrak{o}_{\mathrm{Y}}\,; \qquad \widetilde{G}: \mathrm{X} \times \widetilde{\mathrm{Y}} \times \mathrm{U} \to \mathrm{Y}\,; \tag{6.2}$$

$$\widetilde{H}_l(x,z,u) := H_l(x,u) \leqslant 0\,; \qquad \widetilde{H}_l: \mathrm{X} \times \widetilde{\mathrm{Y}} \times \mathrm{U} \to \mathbb{R}\,,\ 1 \leqslant l \leqslant w\,; \tag{6.3}$$

$$\widetilde{H}_{w+1}(x,z,u) := -\varepsilon\,\big(F(x,u) - \inf(\mathrm{P})_0\big) \leqslant 0\,; \qquad \widetilde{H}_{w+1}: \mathrm{X} \times \widetilde{\mathrm{Y}} \times \mathrm{U} \to \mathbb{R}\,. \tag{6.4}$$

In completely analogous manner, $(\widetilde{\mathrm{P}})_\varepsilon$ is defined after replacing U by $\widetilde{\mathrm{U}}$ everywhere in (6.1) – (6.4). For $\varepsilon = 0$, one comes back, up to a constant in the cost functionals, to the original problems.

**Remark.** By presence of the additional mixed restriction $\widetilde{H}_{w+1}(x,z,u) \leqslant 0$, on the one hand the coincidence of the minimal values of $(\mathrm{P})_0$ and $(\mathrm{P})_\varepsilon$ is ensured, and on the other hand, the coincidence of the minimal values of $(\mathrm{P})_\varepsilon$ and $(\widetilde{\mathrm{P}})_\varepsilon$ is enforced.

**Theorem 2.1.** *Let a problem $(\mathrm{P})_0$ together with its global minimizer $(x^*, u^*)$ satisfy only the assumption 1) of Theorem 1.2. Then it follows: For every number $0 < \varepsilon < 1$ and every closed subspace $\widetilde{\mathrm{Y}} \subseteq \mathrm{Y}$ with $\mathrm{Im}\, G_x(x^*, u^*) + \widetilde{\mathrm{Y}} = \mathrm{Y}$, the triple $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ forms a global minimizer of $(\mathrm{P})_\varepsilon$, the minimal value of $(\mathrm{P})_\varepsilon$ ist zero, and the problem $(\mathrm{P})_\varepsilon$ satisfies together with $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ the assumptions 1), 3) and 4) of Theorem 1.2.*

**Proof.** By assumption 1), it holds for all feasible solutions $(x,u)$ of $(\mathrm{P})_0$: $F(x,u) \geqslant F(x^*,u^*) = 0$. Since $\widetilde{H}_{w+1}(x,z,u) = -\varepsilon\, F(x,u) \leqslant 0$ it follows:

$$(x,z,u) \text{ is feasible in in } (\mathrm{P})_\varepsilon \implies \widetilde{F}(x,z,u) = F(x,u) \geqslant 0\,; \tag{7.1}$$

$$(x,\mathfrak{o}_{\mathrm{Y}},u) \text{ is feasible in } (\mathrm{P})_\varepsilon \iff (x,u) \text{ is feasible in } (\mathrm{P})_0; \tag{7.2}$$

$$(x^*,\mathfrak{o}_{\mathrm{Y}},u^*) \text{ is a global minimizer of } (\mathrm{P})_\varepsilon \iff (x^*,u^*) \text{ is a global minimizer of } (\mathrm{P})_0; \tag{7.3}$$

$$\inf(\mathrm{P})_\varepsilon = \inf(\mathrm{P})_0 = 0\,. \tag{7.4}$$

The assumptions about continuity and differentiability of the data of $(\mathrm{P})_0$ carry over to $(\mathrm{P})_\varepsilon$. Thus assumption 1) of Theorem 1.2. is satisfied for $(\mathrm{P})_\varepsilon$. Further, we assumed above that it holds $\mathrm{Y} = \mathrm{Im}\, G_x(x^*,u^*) + \widetilde{\mathrm{Y}} = \mathrm{Im}\, \widetilde{G}_{(x,z)}(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ having codimension zero. Obviously, $(\mathrm{P})_\varepsilon$ thus satisfies assumptions 3) and 4) of Theorem 1.2. since $\mathfrak{o}_{\mathrm{Y}}$ is an interior point of $\mathrm{Im}\, \widetilde{G}_{(x,z)}(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*) + \mathrm{co}\,\big\{\, \widetilde{G}(x^*, \mathfrak{o}_{\mathrm{Y}}, u) \;\big|\; u \in \mathrm{U} \,\big\} = \mathrm{Y}\,.$ □

**Theorem 2.2.** *Let a problem $(\mathrm{P})_0$ together with its global minimizer $(x^*, u^*)$ satisfy only assumptions 1) and 2) of Theorem 1.2.: $(\mathrm{P})_0$ is relaxable at $(x^*, u^*)$ with respect to $\widetilde{\mathrm{U}}$. Assume further that the sharpened condition 3)′ from Definition 1.1. holds. Then it follows: For every number $0 < \varepsilon < 1$ and every closed subspace $\widetilde{\mathrm{Y}} \subseteq \mathrm{Y}$ with $\mathrm{Im}\, G_x(x^*, u^*) + \widetilde{\mathrm{Y}} = \mathrm{Y}$, the triple $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ forms a global minimizer of $(\mathrm{P})_\varepsilon$,* *the minimal value of* $(\mathrm{P})_\varepsilon$ *is zero, the problem* $(\mathrm{P})_\varepsilon$ *is relaxable at* $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ *with respect to* $\widetilde{\mathrm{U}}$*, and* $(\mathrm{P})_\varepsilon$ *satisfies together with* $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ *all assumptions 1) – 4) of Theorem 1.2.*

**Proof.** From Theorem 2.1. we know that $(\mathrm{P})_\varepsilon$ and $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ satisfy the assumptions 1), 3) and 4) of Theorem 1.2. It is only the relaxability property which remains to prove. In order to do this, we fix a subset $\{ u_1, \ldots, u_r \} \subset \mathrm{U}$ and a number $0 < \delta < 1$. We keep then the number $\eta > 0$ from the relaxability conditions for $(\mathrm{P})_0$ and define the mapping $\widetilde{v}(x, z, \alpha) \colon \mathrm{X} \times \widetilde{\mathrm{Y}} \times \mathbb{R}^m_+ \to \widetilde{\mathrm{U}}$ by $\widetilde{v}(x, z, \alpha) = v(x, \alpha)$ already given from $(\mathrm{P})_0$. Then for all $x', x'' \in \mathrm{B}(x^*, \eta) \subset \mathrm{X}$; $z', z'' \in \mathrm{B}(\mathfrak{o}_{\mathrm{Y}}, \eta) \cap \widetilde{\mathrm{Y}} \subset \mathrm{Y}$ and $\alpha', \alpha'' \in \mathrm{B}(\mathfrak{o}_r, \eta) \cap \mathbb{R}^r_+$ it holds:

1) $$\widetilde{v}(x', z', \mathfrak{o}_r) = v(x', \mathfrak{o}_r) = u^* \,; \tag{8}$$

2)
$$\begin{aligned}
&\Big\| \widetilde{G}(x', z', \widetilde{v}(x', z', \alpha')) - \widetilde{G}(x'', z'', \widetilde{v}(x'', z'', \alpha'')) - G_{(x,z)}(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)^{\mathrm{T}} \begin{pmatrix} x' - x'' \\ z' - z'' \end{pmatrix} \\
&\qquad - \sum_{s=1}^{r} (\alpha'_s - \alpha''_s) \big( \widetilde{G}(x^*, \mathfrak{o}_{\mathrm{Y}}, u_s) - \widetilde{G}(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*) \big) \Big\|_{\mathrm{Y}} \\
&= \Big\| G(x', v(x', \alpha')) + \varepsilon\, z' - G(x'', v(x'', \alpha'')) - \varepsilon\, z'' - G_x(x^*, u^*)\, (x' - x'') - \varepsilon\, (z' - z'') \\
&\qquad - \sum_{s=1}^{r} (\alpha'_s - \alpha''_s) \big( G(x^*, u_s) - G(x^*, u^*) \big) \Big\|_{\mathrm{Y}} \\
&\leqslant \delta \Big( \| x' - x'' \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s - \alpha''_s | \Big) \leqslant \delta \Big( \| x' - x'' \|_{\mathrm{X}} + \| z' - z'' \|_{\mathrm{Y}} + \sum_{s=1}^{r} | \alpha'_s - \alpha''_s | \Big) \,;
\end{aligned} \tag{9}$$

3)
$$\begin{aligned}
&\widetilde{F}(x', z', \widetilde{v}(x', z', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, \widetilde{F}(x', z', u_s) + \big( 1 - \sum_{s=1}^{m} a'_s \big)\, \widetilde{F}(x', z', u^*) \Big) \\
&= F(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, F(x', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, F(x', u^*) \Big) \\
&\leqslant \Big| F(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, F(x', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, F(x', u^*) \Big) \Big| \\
&\leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s | \Big) \leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \| z' \|_{\mathrm{Y}} + \sum_{s=1}^{r} | \alpha'_s | \Big) \,;
\end{aligned} \tag{10}$$

4)
$$\begin{aligned}
&\widetilde{H}_l(x', z', \widetilde{v}(x', z', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, \widetilde{H}_l(x', z', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, \widetilde{H}_l(x', z', u^*) \Big) \\
&= H_l(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, H_l(x', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, H_l(x', u^*) \Big) \\
&\leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s | \Big) \leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \| z' \|_{\mathrm{Y}} + \sum_{s=1}^{r} | \alpha'_s | \Big), \quad 1 \leqslant l \leqslant w \,;
\end{aligned} \tag{11}$$

Finally, from the sharpened condition 3)′ we may conclude that

$$\begin{aligned}
&\widetilde{H}_{w+1}(x', z', \widetilde{v}(x', z', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, \widetilde{H}_{w+1}(x', z', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, \widetilde{H}_{w+1}(x', z', u^*) \Big) \\
&= -\varepsilon \Big( F(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, F(x', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, F(x', u^*) \Big) \Big) \\
&\leqslant \varepsilon \Big| F(x', v(x', \alpha')) - \Big( \sum_{s=1}^{r} \alpha'_s\, F(x', u_s) + \big( 1 - \sum_{s=1}^{r} \alpha'_s \big)\, F(x', u^*) \Big) \Big| \\
&\leqslant \varepsilon\, \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \sum_{s=1}^{r} | \alpha'_s | \Big) \leqslant \delta \Big( \| x' - x^* \|_{\mathrm{X}} + \| z' \|_{\mathrm{Y}} + \sum_{s=1}^{r} | \alpha'_s | \Big) .
\end{aligned} \tag{12}$$

Summing up, we proved that the problem $(\mathrm{P})_\varepsilon$ is relaxable at $(x^*, \mathfrak{o}_{\mathrm{Y}}, u^*)$ with respect to $\widetilde{\mathrm{U}}$. □

**Remarks.** 1) From the proof it can be seen that, under assumptions of Theorem 2.2., the sharpened form of the relaxability condition 3)′ holds for $(\mathrm{P})_\varepsilon$ too.

2) If in Theorems 2.1. resp. 2.2. the closed subspace $\widetilde{Y} \subset Y$ is chosen in such a way that $\operatorname{Im} G_x(x^*, u^*) + \widetilde{Y} \neq Y$ has only finite codimension $> 0$ in Y then fulfillment of assumption 4) of Theorem 1.2. for $(P)_\varepsilon$ must be tested directly.

**b) The $\varepsilon$-extremal principle for** $(P)_0$**.** Now, let us consider a problem $(P)_0$ which satisfies the assumptions of Theorem 1.2. only in part and then apply IOFFE/TICHOMIROV's extremal principle to the related problem $(P)_\varepsilon$.

**Theorem 2.3.** *Let a problem* $(P)_0$ *together with its global minimizer* $(x^*, u^*)$ *satisfy only the assumption 1) of Theorem 1.2. Assume further that precisely the first* $k \leqslant w$ *of the mixed restrictions are active in* $(x^*, u^*)$*, i. e.* $H_l(x^*, u^*) = 0\,,\ 1 \leqslant l \leqslant k$*, and* $H_l(x^*, u^*) < 0\,,\ (k+1) \leqslant l \leqslant w$*. Let be given a number* $0 < \varepsilon < 1$ *and a closed subspace* $\widetilde{Y} \subseteq Y$ *with* $\operatorname{Im} G_x(x^*, u^*) + \widetilde{Y} = Y$*. If* $(P)_\varepsilon$ *is relaxable at its feasible solution* $(x^*, \mathfrak{o}_Y, u^*)$ *with respect to* $\widetilde{U}$ *then problem* $(P)_\varepsilon$ *satisfies together with* $(x^*, \mathfrak{o}_Y, u^*)$ *all assumptions 1) – 4) of Theorem 1.2. Then the set* $C_\varepsilon \subset \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times Y$ *defined by* $C_\varepsilon =$ (13)

$$\left\{ \begin{pmatrix} \tau_0 + \langle\, F_x(x^*, u^*)\,,\ \ x \,\rangle \\ \tau_l + \langle\, (H_l)_x(x^*, u^*)\,,\, x \,\rangle \\ \tau_{w+1} - \varepsilon \,\langle\, F_x(x^*, u^*)\,,\, x \,\rangle \\ \varepsilon\, z + G_x(x^*, u^*)\, x \end{pmatrix} \,\middle|\, \begin{array}{l} \tau_0 \geqslant 0 \\ \tau_l \geqslant 0\,,\ 1 \leqslant l \leqslant k \\ \tau_{w+1} \geqslant 0 \\ (x, z) \in X \times \widetilde{Y} \end{array} \right\} + \operatorname{co} \left\{ \begin{pmatrix} F(x^*, u) - F(x^*, u^*) \\ H_l(x^*, u) - H_l(x^*, u^*) \\ -\varepsilon \left( F(x^*, u) - F(x^*, u^*) \right) \\ G(x^*, u) - G(x^*, u^*) \end{pmatrix} \,\middle|\, u \in U \right\}$$

*is convex with nonempty interior. Further, the equality* $\inf (\widetilde{P})_\varepsilon = \inf (P)_\varepsilon$ *holds such that the intersection* $C_\varepsilon \cap R$ *with the following cone* R *is empty:*

$$R = \left\{ \begin{pmatrix} -\varrho_0 \\ -\varrho_l \\ -\varrho_{w+1} \\ \mathfrak{o}_Y \end{pmatrix} \,\middle|\, \begin{array}{l} \varrho_0 > 0 \\ \varrho_l > 0 \\ \varrho_{w+1} > 0 \end{array} \right\} \subset \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times Y. \tag{14}$$

**Proof.** By Theorem 2.1. we are allowed to apply Theorem 1.2. to problem $(P)_\varepsilon$. Since in $(P)_0$ precisely the first $k \leqslant w$ of the mixed restrictions are active in $(x^*, u^*)$, in $(P)_\varepsilon$ precisely the first $k \leqslant w$ and the last of the mixed restrictions are active in $(x^*, \mathfrak{o}_Y, u^*)$: $\widetilde{H}_l(x^*, \mathfrak{o}_Y, u^*) = 0\,,\ 1 \leqslant l \leqslant k\,,\ \widetilde{H}_l(x^*, \mathfrak{o}_Y, u^*) < 0\,,$ $(k+1) \leqslant l \leqslant w$, and $\widetilde{H}_{w+1}(x^*, \mathfrak{o}_Y, u^*) = 0$. The set $C_\varepsilon$ and the cone R from Theorem 1.2. take on the described shape for $(P)_\varepsilon$. Since the case $\inf (\widetilde{P})_\varepsilon < \inf (P)_\varepsilon$ cannot occur (see the remark after definition of $(P)_\varepsilon$ ), Theorem 1.2. tells out that the intersection $C_\varepsilon \cap R$ must be empty. □

**Theorem 2.4.** *Let a problem* $(P)_0$ *together with its global minimizer* $(x^*, u^*)$ *satisfy only assumptions 1) and 2) of Theorem 1.2.:* $(P)_0$ *is relaxable at* $(x^*, u^*)$ *with respect to* $\widetilde{U}$*. Assume further that the sharpened condition 3)′ from Definition 1.1. holds and that precisely the first* $k \leqslant w$ *of the mixed restrictions are active in* $(x^*, u^*)$*, i. e.* $H_l(x^*, u^*) = 0\,,\ 1 \leqslant l \leqslant k$*, and* $H_l(x^*, u^*) < 0\,,\ (k+1) \leqslant l \leqslant w$*. Now, let be given a number* $0 < \varepsilon < 1$ *and a closed subspace* $\widetilde{Y} \subseteq Y$ *with* $\operatorname{Im} G_x(x^*, u^*) + \widetilde{Y} = Y$*. Then problem* $(P)_\varepsilon$ *satisfies together with* $(x^*, \mathfrak{o}_Y, u^*)$ *all assumptions 1) – 4) of Theorem 1.2., and for the sets* $C_\varepsilon$ *and* R *defined above in Theorem 2.3. it holds:* $C_\varepsilon$ *is convex with nonempty interior, and the intersection* $C_\varepsilon \cap R$ *is empty since* $\inf (\widetilde{P})_\varepsilon = \inf (P)_\varepsilon$ .

**Proof.** This can be proven in complete analogy to Theorem 2.3. while the applicability of Theorem 1.2. to $(P)_\varepsilon$ is now justified by Theorem 2.2. □

Under the assumptions of Theorem 2.3. resp. Theorem 2.4., the separation theorem 1.3. can be applied to $(P)_\varepsilon$. In result of this, we receive the following set of necessary optimality conditions for $(P)_0$ and $(x^*, u^*)$ depending now on the choice of $0 < \varepsilon < 1$ and $\widetilde{Y} \subseteq Y$:

**Theorem 2.5.** (**$\varepsilon$-extremal principle for** $(\mathrm{P})_0$) *Let the assumptions of Theorem 2.3. or Theorem 2.4. be fulfilled for a problem* $(\mathrm{P})_0$ *and its global minimizer* $(x^*, u^*)$*. Assume further that* $\inf_{u \in \mathrm{U}} F(x^*, u)$ *is finite. Then there exists a nontrivial collection of multipliers* $\lambda_0(\varepsilon, \widetilde{\mathrm{Y}}) > 0,\ \lambda_1(\varepsilon, \widetilde{\mathrm{Y}}) \geqslant 0,\ \dots\ ,\ \lambda_w(\varepsilon, \widetilde{\mathrm{Y}}) \geqslant 0$ *and* $y^*(\varepsilon, \widetilde{\mathrm{Y}}) \in \mathrm{Y}^*$*, satisfying together with* $(x^*, u^*)$ *the* $\varepsilon$*-maximum condition* $(\mathcal{M})_\varepsilon$*, the canonical inequality* $(\mathcal{K})_\varepsilon$ *as well as the complementarity conditions* $(\mathcal{C})_\varepsilon$*:*

$$(\mathcal{M})_\varepsilon: \quad \varepsilon + \lambda_0(\varepsilon, \widetilde{\mathrm{Y}}) \big( F(x^*, u) - F(x^*, u^*) \big) + \sum_{l=1}^{w} \lambda_l(\varepsilon, \widetilde{\mathrm{Y}}) \big( H_l(x^*, u) - H_l(x^*, u^*) \big) + \langle y^*(\varepsilon, \widetilde{\mathrm{Y}}) , G(x^*, u) - G(x^*, u^*) \rangle \geqslant 0 \quad \forall u \in \mathrm{U};$$

$$(\mathcal{K})_\varepsilon: \quad \Big| \lambda_0(\varepsilon, \widetilde{\mathrm{Y}}) \langle F_x(x^*, u^*) , x \rangle + \sum_{l=1}^{w} \lambda_l(\varepsilon, \widetilde{\mathrm{Y}}) \langle (H_l)_x(x^*, u^*) , x \rangle + \langle y^*(\varepsilon, \widetilde{\mathrm{Y}}) , G_x(x^*, u^*)\, x \rangle \Big| \leqslant \varepsilon \, \| x \|_{\mathrm{X}} \quad \forall x \in \mathrm{X};$$

$$(\mathcal{C})_\varepsilon: \quad \langle y^*(\varepsilon, \widetilde{\mathrm{Y}}) , z \rangle = 0 \quad \forall z \in \widetilde{\mathrm{Y}}; \quad \lambda_l(\varepsilon, \widetilde{\mathrm{Y}})\, H_l(x^*, u^*) = 0 , \ 1 \leqslant l \leqslant w .$$

**Proof.** From Theorem 1.3. we deduce the existence of multipliers $\lambda_0 > 0,\ \lambda_1 \geqslant 0,\ \dots\ ,\ \lambda_w \geqslant 0$, $\lambda_{w+1} \geqslant 0$ and $y^* \in \mathrm{Y}^*$ which satisfy together with $(x^*, \mathfrak{o}_\mathrm{Y}, u^*)$ the maximum condition, canonical equation and complementarity conditions in relation to the problem $(\mathrm{P})_\varepsilon$. (These multipliers will already depend on $\varepsilon$ and $\widetilde{\mathrm{Y}}$ what is not reflected in the notation.) We define:

$$\lambda_0(\varepsilon, \widetilde{\mathrm{Y}}) = \lambda_0 \Big/ \Big( \lambda_0 + \sum_{l=1}^{w} \lambda_l + \lambda_{w+1} + \| y^* \|_{\mathrm{Y}^*} + \lambda_{w+1} \cdot \big| \inf_{u \in \mathrm{U}} F(x^*, u) \big| + \lambda_{w+1} \cdot \| F_x(x^*, u^*) \|_{\mathrm{X}} \Big);$$

$$\lambda_l(\varepsilon, \widetilde{\mathrm{Y}}) = \lambda_l \Big/ \Big( \lambda_0 + \sum_{l=1}^{w} \lambda_l + \lambda_{w+1} + \| y^* \|_{\mathrm{Y}^*} + \lambda_{w+1} \cdot \big| \inf_{u \in \mathrm{U}} F(x^*, u) \big| + \lambda_{w+1} \cdot \| F_x(x^*, u^*) \|_{\mathrm{X}} \Big), \quad 1 \leqslant l \leqslant w;$$

$$\lambda_{w+1}(\varepsilon, \widetilde{\mathrm{Y}}) = \lambda_{w+1} \Big/ \Big( \lambda_0 + \sum_{l=1}^{w} \lambda_l + \lambda_{w+1} + \| y^* \|_{\mathrm{Y}^*} + \lambda_{w+1} \cdot \big| \inf_{u \in \mathrm{U}} F(x^*, u) \big| + \lambda_{w+1} \cdot \| F_x(x^*, u^*) \|_{\mathrm{X}} \Big);$$

$$y^*(\varepsilon, \widetilde{\mathrm{Y}}) = y^* \Big/ \Big( \lambda_0 + \sum_{l=1}^{w} \lambda_l + \lambda_{w+1} + \| y^* \|_{\mathrm{Y}^*} + \lambda_{w+1} \cdot \big| \inf_{u \in \mathrm{U}} F(x^*, u) \big| + \lambda_{w+1} \cdot \| F_x(x^*, u^*) \|_{\mathrm{X}} \Big). \qquad (15)$$

Then from the maximum condition from Theorem 1.3. it follows:

$$\big( \lambda_0 - \varepsilon \, \lambda_{w+1} \big) \big( F(x^*, u) - F(x^*, u^*) \big) + \sum_{l=1}^{w} \lambda_l \big( H_l(x^*, u) - H_l(x^*, u^*) \big) + \langle y^* , G(x^*, u) - G(x^*, u^*) \rangle \geqslant 0 \quad \forall u \in \mathrm{U} \implies$$

$$\lambda_0 \big( F(x^*, u) - F(x^*, u^*) \big) + \sum_{l=1}^{w} \lambda_l \big( H_l(x^*, u) - H_l(x^*, u^*) \big) + \langle y^* , G(x^*, u) - G(x^*, u^*) \rangle \geqslant \varepsilon \cdot \lambda_{w+1} \cdot \inf_{u \in \mathrm{U}} F(x^*, u) \geqslant -\varepsilon \cdot \lambda_{w+1} \cdot \big| \inf_{u \in \mathrm{U}} F(x^*, u) \big| \quad \forall u \in \mathrm{U} \implies$$

$$\lambda_0(\varepsilon, \widetilde{\mathrm{Y}}) \big( F(x^*, u) - F(x^*, u^*) \big) + \sum_{l=1}^{w} \lambda_l(\varepsilon, \widetilde{\mathrm{Y}}) \big( H_l(x^*, u) - H_l(x^*, u^*) \big) + \langle y^*(\varepsilon, \widetilde{\mathrm{Y}}) , G(x^*, u) - G(x^*, u^*) \rangle \geqslant -\varepsilon \quad \forall u \in \mathrm{U}. \qquad (16)$$

This proves condition $(\mathcal{M})_\varepsilon$. The canonical equation from Theorem 1.3. reads as

$$\big( \lambda_0 - \varepsilon \, \lambda_{w+1} \big) \langle F_x(x^*, u^*) , x \rangle + \sum_{l=1}^{w} \lambda_l \langle (H_l)_x(x^*, u^*) , x \rangle + \langle y^* , G_x(x^*, u^*)\, x \rangle = 0 \quad \forall (x, z) \in \mathrm{X} \times \widetilde{\mathrm{Y}}. \qquad (17)$$

Inserting $z = \mathfrak{o}_{\mathrm{Y}}$ into (17), we arrive at the separate condition

$$\big(\lambda_0 - \varepsilon\,\lambda_{w+1}\big)\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,+\, \sum_{l=1}^{w} \lambda_l\,\langle\,(H_l)_x(x^*,u^*)\,,\,x\,\rangle \,+\, \langle\, y^*\,,\,G_x(x^*,u^*)\,x\,\rangle \,=\, 0 \quad \forall\, x \in \mathrm{X} \quad \Longrightarrow$$

$$\lambda_0\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,+\, \sum_{l=1}^{w} \lambda_l\,\langle\,(H_l)_x(x^*,u^*)\,,\,x\,\rangle \,+\, \langle\, y^*\,,\,G_x(x^*,u^*)\,x\,\rangle$$
$$=\, \varepsilon\,\lambda_{w+1}\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,\leqslant\, \varepsilon\cdot\lambda_{w+1}\cdot\big\|\,F_x(x^*,u^*)\,\big\|_{\mathrm{X}}\cdot\big\|\,x\,\big\|_{\mathrm{X}} \quad \forall\, x \in \mathrm{X} \qquad (18.1)$$

and, after replacing $x$ by $(-x)$

$$-\,\lambda_0\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,-\, \sum_{l=1}^{w} \lambda_l\,\langle\,(H_l)_x(x^*,u^*)\,,\,x\,\rangle \,-\, \langle\, y^*\,,\,G_x(x^*,u^*)\,x\,\rangle$$
$$\geqslant\, -\varepsilon\cdot\lambda_{w+1}\cdot\big\|\,F_x(x^*,u^*)\,\big\|_{\mathrm{X}}\cdot\big\|\,x\,\big\|_{\mathrm{X}} \quad \forall\, x \in \mathrm{X}\,. \qquad (18.2)$$

From both together, the following inequalities result:

$$\Big|\,\lambda_0\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,+\, \sum_{l=1}^{w} \lambda_l\,\langle\,(H_l)_x(x^*,u^*)\,,\,x\,\rangle \,+\, \langle\, y^*\,,\,G_x(x^*,u^*)\,x\,\rangle\,\Big|$$
$$\leqslant\, \varepsilon\cdot\lambda_{w+1}\cdot\big\|\,F_x(x^*,u^*)\,\big\|_{\mathrm{X}}\cdot\big\|\,x\,\big\|_{\mathrm{X}} \quad \forall\, x \in \mathrm{X} \quad \Longrightarrow$$

$$\Big|\,\lambda_0(\varepsilon,\widetilde{\mathrm{Y}})\,\langle\, F_x(x^*,u^*)\,,\,x\,\rangle \,+\, \sum_{l=1}^{w} \lambda_l(\varepsilon,\widetilde{\mathrm{Y}})\,\langle\,(H_l)_x(x^*,u^*)\,,\,x\,\rangle$$
$$+\,\langle\, y^*(\varepsilon,\widetilde{\mathrm{Y}})\,,\,G_x(x^*,u^*)\,x\,\rangle\,\Big| \,\leqslant\, \varepsilon\,\big\|\,x\,\big\|_{\mathrm{X}} \quad \forall\, x \in \mathrm{X}\,. \qquad (19)$$

This is the "canonical inequality" $(\mathcal{K})_\varepsilon$. The first condition in $(\mathcal{C})_\varepsilon$ comes again from (17) by inserting $x = \mathfrak{o}_{\mathrm{X}}$ while the other conditions can be derived from the complementarity conditions $(\mathcal{C})$ for the multipliers $\lambda_l$, $1 \leqslant l \leqslant w$, from Theorem 1.3. The proof is complete. □

**Remark.** From condition $(\mathcal{C})_\varepsilon$ it turns out that in order to find a multiplier $y^*(\varepsilon,\widetilde{\mathrm{Y}}) \neq \mathfrak{o}$, one has to satisfy the assumption $\mathrm{Im}\,G_x(x^*,u^*) + \widetilde{\mathrm{Y}} = \mathrm{Y}$ in Theorems 2.3. resp. 2.4. with a proper subspace $\widetilde{\mathrm{Y}} \neq \mathrm{Y}$ .

## 3. $\varepsilon$-extremal principles for control problems of Dieudonné-Rashevsky type.

**a) Statement of the problems.** In this Section, we investigate multidimensional control problems $(\mathrm{S})_0$ of following type:

$(\mathrm{S})_0$ $\quad F(x,u) = \displaystyle\int_\Omega f_0(t,x(t),u(t))\,dt + \int_\Omega f_1(t,x(t))\,d\gamma(t) \longrightarrow \mathrm{Min}\,!$ (20.1)

subject to $(x,u) \in \big(C^{0,n}(\Omega) \cap W_p^{1,n}(\Omega)\big) \times L_p^{nm}(\Omega)$ with (20.2)

$G(x,u) = \big(x_{i;\,t_j} - u_{ij}\big)_{ij} = \mathfrak{o}_{L_p^{nm}(\Omega)} \iff x_{i;\,t_j}(t) = u_{ij}(t) \quad (\forall)\,t \in \Omega \;\; \forall\, i,\,j\,;$ (20.3)

$u \in \mathrm{U} = \{\, u \in L_p^{nm}(\Omega) \;\big|\; u(t) \in \mathrm{K} \;\; (\forall)\,t \in \Omega\,\}\,;$ (20.4)

$x(t_0) = \mathfrak{o}_n$ for fixed $t_0 \in \partial\Omega\,.$ (20.5)

The corresponding Young measure relaxed problem is

$(\widetilde{\mathrm{S}})_0$ $\quad \overline{F}(x,\mu) = \displaystyle\int_\Omega \int_{\mathrm{K}} f_0(t,x(t),v)\,d\mu_t(v)\,dt + \int_\Omega f_1(t,x(t))\,d\gamma(t) \longrightarrow \mathrm{Min}\,!$ (21.1)

subject to $(x,\mu) \in \big(C^{0,n}(\Omega) \cap W_p^{1,n}(\Omega)\big) \times \mathfrak{Y}_{\mathrm{K}}$ with (21.2)

$\overline{G}(x,\mu) = \big(x_{i;\,t_j} - \displaystyle\int_{\mathrm{K}} v_{ij}\,d\mu_t(v)\big)_{ij} = \mathfrak{o}_{L_p^{nm}(\Omega)} \iff x_{i;\,t_j}(t) = \int_{\mathrm{K}} v_{ij}\,d\mu_t(v) \;\; (\forall)\,t \in \Omega \;\; \forall\, i,\,j\,;$ (21.3)

$\mu \in \widetilde{\mathrm{U}} = \mathfrak{Y}_{\mathrm{K}}\,;$ (21.4)

$x(t_0) = \mathfrak{o}_n$ for fixed $t_0 \in \partial\Omega\,.$ (21.5)

Problems of type $(\mathrm{S})_0$ resp. $(\widetilde{\mathrm{S}})_0$ were studied e. g. in [5] and [10] for the case $\gamma = \mathfrak{o}$ ("deposit problems") as well as in [8], [9] and [11] for the case $\gamma \neq \mathfrak{o}$ ("extended deposit problems"). We state the

**Basic assumptions about the data of** $(\mathrm{S})_0$ **resp.** $(\widetilde{\mathrm{S}})_0$**:**

(V1) Let $n \geqslant 1$, $m \geqslant 2$ and $m < p < \infty$. $\Omega \subset \mathbb{R}^m$ is a compact closure of a domain with $C^2$-boundary.

(V2) The functions $f_0(t, \xi, v)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ and $f_1(t, \xi)\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ are continuous with respect to all its arguments and continuously differentiable in $\xi$. $\gamma \in \mathit{rca}\,(\Omega\,, \mathfrak{B}\,)$ is a signed regular measure acting on the $\sigma$-algebra $\mathfrak{B}_\Omega$ of the Borel sets of $\Omega$.

(V3) K forms a convex, compact subset of $\mathbb{R}^{nm}$.

(V4) The problem $(\mathrm{S})_0$ admits a feasible solution.

**Remarks.** 1) The state variables $x$ of $(\mathrm{S})_0$ will be treated is individual functions, namely, by (V1), we can identify them with the uniquely determined continuous representatives of their $W^{1,n}_p(\Omega)$-equivalence classes.

2) We assumed higher smoothness of $\partial\Omega$ in order to get a proper subspace $\widetilde{\mathrm{Y}} \subset L^{nm}_p(\Omega)$ satisfying the condition from Theorems 2.1. resp. 2.2. via Helmholtz-Weyl decomposition of the space $L^{nm}_p(\Omega)$.

3) The problems $(\mathrm{S})_0$ and $(\widetilde{\mathrm{S}})_0$ belong to the class of abstract problems of type $(\mathrm{P})_0$ resp. $(\widetilde{\mathrm{P}})_0$. To see this, let be $\mathrm{X} = \{\, x \in C^{0,n}(\Omega) \cap W^{1,n}_p(\Omega) \mid x(t_0) = \mathfrak{o}_n \,\}$, $\mathrm{U} \subset L^{nm}_p(\Omega)$, $\widetilde{\mathrm{U}} = \mathfrak{Y}_{\mathrm{K}}$ and $\mathrm{Y} = L^{nm}_p(\Omega)$. Then the inclusion $\mathrm{U} \subseteq \widetilde{\mathrm{U}}$ holds in the sense of the embedding which associates with a function $u(t)\colon \Omega \to \mathbb{R}^{nm}$ the Young measure $\{\, \delta_{u(t)} \,\}$. The cost functionals $F(x, u)$ and $\overline{F}(x, \mu)$ as well as the mappings $G(x, u)$ and $\overline{G}(x, \mu)$ are in analogous correspondence.

Let us state now the enlarged problems $(\mathrm{S})_\varepsilon$ and $(\widetilde{\mathrm{S}})_\varepsilon$:

$$(\mathrm{S})_\varepsilon \quad \widetilde{F}(x, z, u) = \int_\Omega f_0(t, x(t), u(t))\, dt + \int_\Omega f_1(t, x(t))\, d\gamma(t) - \inf\,(\mathrm{S})_0 \longrightarrow \text{Min}\,! \tag{22.1}$$

$$\text{subject to } (x, z, u) \in \big( C^{0,n}(\Omega) \cap W^{1,n}_p(\Omega) \big) \times \widetilde{\mathrm{Y}} \times L^{nm}_p(\Omega) \text{ with} \tag{22.2}$$

$$\widetilde{G}(x, z, u) = \big( x_{i;\,t_j} + \varepsilon\, z_{ij} - u_{ij} \big)_{ij} = \mathfrak{o}_{L^{nm}_p(\Omega)} \iff x_{i;\,t_j}(t) + \varepsilon\, z_{ij}(t) = u_{ij}(t) \ \ (\forall)\, t \in \Omega \ \ \forall\, i,\, j\,; \tag{22.3}$$

$$\widetilde{H}_1(x, z, u) = -\varepsilon \Big( \int_\Omega f_0(t, x(t), u(t))\, dt + \int_\Omega f_1(t, x(t))\, d\gamma(t) - \inf\,(\mathrm{S})_0 \Big) \leqslant 0\,; \tag{22.4}$$

$$u \in \mathrm{U} = \{\, u \in L^{nm}_p(\Omega) \mid u(t) \in \mathrm{K} \ \ (\forall)\, t \in \Omega \,\}\,; \tag{22.5}$$

$$x(t_0) = \mathfrak{o}_n \text{ for fixed } t_0 \in \partial\Omega\,. \tag{22.6}$$

$$(\widetilde{\mathrm{S}})_\varepsilon \quad \overline{F}(x, z, \mu) = \int_\Omega \int_{\mathrm{K}} f_0(t, x(t), v)\, d\mu_t(v)\, dt + \int_\Omega f_1(t, x(t))\, d\gamma(t) - \inf\,(\mathrm{S})_0 \longrightarrow \text{Min}\,! \tag{23.1}$$

$$\text{subject to } (x, z, \mu) \in \big( C^{0,n}(\Omega) \cap W^{1,n}_p(\Omega) \big) \times \widetilde{\mathrm{Y}} \times \mathfrak{Y}_{\mathrm{K}} \text{ with} \tag{23.2}$$

$$\overline{G}(x, z, \mu) = \Big( x_{i;\,t_j} + \varepsilon\, z_{ij} - \int_{\mathrm{K}} v_{ij}\, d\mu_t(v) \Big)_{ij} = \mathfrak{o}_{L^{nm}_p(\Omega)} \iff x_{i;\,t_j}(t) + \varepsilon\, z_{ij}(t) = \int_{\mathrm{K}} v_{ij}\, d\mu_t(v) \quad (\forall)\, t \in \Omega \ \ \forall\, i,\, j\,; \tag{23.3}$$

$$\overline{H}_1(x, z, \mu) = -\varepsilon \Big( \int_\Omega \int_{\mathrm{K}} f_0(t, x(t), v)\, d\mu_t(v)\, dt + \int_\Omega f_1(t, x(t))\, d\gamma(t) - \inf\,(\mathrm{S})_0 \Big) \leqslant 0\,; \tag{23.4}$$

$$\mu \in \widetilde{\mathrm{U}} = \mathfrak{Y}_{\mathrm{K}}\,; \tag{23.5}$$

$$x(t_0) = \mathfrak{o}_n\,. \tag{23.6}$$

Here $\widetilde{\mathrm{Y}}$ ($\neq \{\, \mathfrak{o} \,\}$) is some closed subspace of $L^{nm}_p(\Omega)$.

**Proposition 3.1. (Coincidence of minimal values)** *Consider the problems* $(\mathrm{S})_0$, $(\widetilde{\mathrm{S}})_0$, $(\mathrm{S})_\varepsilon$ *and* $(\widetilde{\mathrm{S}})_\varepsilon$ *under assumptions* (V1) – (V4).

*1) For every number* $0 < \varepsilon < 1$, *for every closed subspace* $\widetilde{\mathrm{Y}} \subseteq L_p^{nm}(\Omega)$ *and for any* $n \geqslant 1$ *it holds:* $\inf\,(\mathrm{S})_0 = \inf\,(\mathrm{S})_\varepsilon = \inf\,(\widetilde{\mathrm{S}})_\varepsilon$.

*2) If the function* $f_0(t, \xi, v)$ *is convex in* $v$ *for all fixed* $(t, \xi) \in \Omega \times \mathbb{R}^n$ *then it holds for any* $n \geqslant 1$*:* $\inf\,(\mathrm{S})_0 = \inf\,(\widetilde{\mathrm{S}})_0$. *If convexity of* $f_0(t, \xi, v)$ *in* $v$ *fails then the equation* $\inf\,(\mathrm{S})_0 = \inf\,(\widetilde{\mathrm{S}})_0$ *still holds for* $n = 1$.

**Proof.** 1) is an immediate consequence of the definition of $(\mathrm{S})_\varepsilon$ resp. $(\widetilde{\mathrm{S}})_\varepsilon$. 2) is a well-known fact (cf. [ 7 : S. 127 ] ). □

**Proposition 3.2. (Relaxability)** *Under assumptions* (V1) – (V4), *both problems* $(\widetilde{\mathrm{S}})_0$ *and* $(\widetilde{\mathrm{S}})_\varepsilon$ *are relaxable at every feasible solution* $(x^*, \mu^*)$ *resp.* $(x^*, z^*, \mu^*)$ *with respect to their own control set* $\widetilde{\mathrm{U}} = \mathfrak{Y}_{\mathrm{K}}$. *In both cases, the sharpened condition 3)′ from Definition 1.1. holds.*

**Proof.** Consider at first a feasible solution $(x^*, \mu^*)$ of $(\widetilde{\mathrm{S}})_0$. Given a number $0 < \delta < 1$, a collection of generalized controls $\mu_1, \ldots, \mu_r \in \mathfrak{Y}_{\mathrm{K}}$ and numbers $0 \leqslant \alpha_s \leqslant \eta = 1/r$, $1 \leqslant s \leqslant r$, we define

$$v(x, \alpha) = \sum_{s=1}^{r} \alpha_s\, \mu_s + \Big(1 - \sum_{s=1}^{r} \alpha_s\Big)\, \mu^* \;\in\; \mathfrak{Y}_{\mathrm{K}}\,. \tag{24}$$

Then obviously the relaxibility condition 1) holds. Conditions 2) and 3)′ are satisfied since the left-hand sides of the inequalities vanish because of linearity of $\overline{F}(x, \mu)$ and $\overline{G}(x, \mu)$ in the generalized control variables.

In the same way, given some feasible solution $(x^*, z^*, \mu)$ of $(\widetilde{\mathrm{S}})_\varepsilon$, a number $0 < \delta < 1$, generalized controls $\mu_1, \ldots, \mu_r \in \mathfrak{Y}_{\mathrm{K}}$ and numbers $0 \leqslant \alpha_s \leqslant \eta = 1/r$, $1 \leqslant s \leqslant r$, we define

$$v(x, z, \alpha) = \sum_{s=1}^{r} \alpha_s\, \mu_s + \Big(1 - \sum_{s=1}^{r} \alpha_s\Big)\, \mu^* \;\in\; \mathfrak{Y}_{\mathrm{K}}\,. \tag{25}$$

Noting that $\overline{G}_{(x,z)}(x^*, z^*, \mu^*)(x', z') = \big(x'_{i;\,t_j} + \varepsilon\, z'_{ij}\big)_{ij}$, the relaxability conditions 1), 2) and 3)′ will follow as before. 4) can be derived from 3)′ since

$$\begin{aligned} &\Big|\, \overline{H}_1(x', z', v(x', \alpha')) \,-\, \Big( \sum_{s=1}^{r} \alpha'_s\, \overline{H}_1(x', z', \mu_s) + \Big(1 - \sum_{s=1}^{r} \alpha'_s\Big)\, \overline{H}_1(x', z', \mu^*) \Big) \Big| \\ &\quad = |\,{-\varepsilon}\,| \cdot \Big|\, \overline{F}(x', z', v(x', \alpha')) \,-\, \Big( \sum_{s=1}^{r} \alpha'_s\, \overline{F}(x', z', \mu_s) + \Big(1 - \sum_{s=1}^{r} \alpha'_s\Big)\, \overline{F}(x', z', \mu^*) \Big) \Big|\,. \quad \square \end{aligned} \tag{26}$$

**Remark.** In both problems, condition 3)′ still holds whether a constant is subtracted in the cost functionals.

**b) A complement for the subspaces** $\mathrm{Im}\, G_x(x^*, u^*)$ **resp.** $\mathrm{Im}\, \overline{G}_x(x^*, \mu^*)$**.** For any feasible solution $(x^*, u^*)$ of $(\mathrm{S})_0$ resp. $(x^*, \mu^*)$ of $(\widetilde{\mathrm{S}})_0$ it holds:

$$\mathrm{Im}\, G_x(x^*, u^*) = \mathrm{Im}\, \overline{G}_x(x^*, \mu^*) = \mathrm{cl}\, \Big( \{\, z \in L_p^{nm}(\Omega) \;\big|\; \exists\, x \in W_p^{1,n}(\Omega):\ x(t_0) = \mathfrak{o}_n\,,\ z_{ij} = x_{i;\,t_j}\ \ \forall\, i,\, j \,\} \Big)\,. \tag{27}$$

Then the space $L_p^{nm}(\Omega)$ admits the following decomposition:

**Proposition 3.3. (Helmholtz-Weyl decomposition of** $L_p^{nm}(\Omega)$ **)** *Assuming* (V1), *the space* $L_p^{nm}(\Omega)$ *can be decomposed into the direct sum* $\mathrm{Im}\, G_x(x^*, u^*) \,+\, \widetilde{\mathrm{Y}}$ *where*

$$\widetilde{\mathrm{Y}} = \Big( \mathrm{cl}_{\,L_p^m(\Omega)} \big( \{\, z \in C_\circ^{\infty, m}(\Omega) \;\big|\; \mathrm{div}\ z(t) = 0\ \ (\forall)\, t \in \Omega \,\} \big) \Big)^n\,. \tag{28}$$

[ 1 : p. 114, Theorem 1.2. ]  □

**c) The $\varepsilon$-extremal principles for $(\mathrm{S})_0$ and $(\widetilde{\mathrm{S}})_0$.**

**Theorem 3.4. ($\varepsilon$-maximum principle for $(\mathrm{S})_0$)** *Consider the problem $(\mathrm{S})_0$ together with its global minimizer $(x^*, u^*)$ under assumptions (V1) – (V4). After replacing the cost functional by $F(x,u)-F(x^*,u^*)$, the $\varepsilon$-extremal principle (Theorem 2.5.) can be applied: For every number $0<\varepsilon<1$ and for every closed subspace $\widetilde{\mathrm{Y}} \subset L_p^{nm}(\Omega)$ with $\operatorname{Im} G_x(x^*,u^*) + \widetilde{\mathrm{Y}} = L_p^{nm}(\Omega)$ there exist multipliers $\lambda_0(\varepsilon,\widetilde{\mathrm{Y}})>0$ and $y^*(\varepsilon,\widetilde{\mathrm{Y}}) \in L_q^{nm}(\Omega)$, $p^{-1}+q^{-1}=1$, satisfying*

$$(\mathcal{M})_\varepsilon: \quad \varepsilon + \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega \big( f_0(t,x^*(t),u(t)) - f_0(t,x^*(t),u^*(t)) \big)\, dt + \sum_{i,j} \int_\Omega \big( u_{ij}(t) - u_{ij}^*(t) \big)\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) \geqslant 0 \quad \forall\, u \in \mathrm{U};$$

$$(\mathcal{K})_\varepsilon: \quad \Big| \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega (f_0)_x(t,x^*(t),u^*(t))^{\mathrm{T}} x(t)\, dt + \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega (f_1)_x(t,x^*(t))^{\mathrm{T}} x(t)\, d\gamma(t) + \sum_{i,j} \int_\Omega x_{i;\,t_j}(t)\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) \Big| \leqslant \varepsilon \cdot \| x \|_{W_p^{1,n}(\Omega)} \quad \forall\, x \in W_p^{1,n}(\Omega):\ x(t_0) = \mathfrak{o}_n;$$

$$(\mathcal{C})_\varepsilon: \quad \sum_{i,j} \int_\Omega z_{ij}(t)\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) = 0 \quad \forall\, z \in \widetilde{\mathrm{Y}}.$$

**Proof.** After subtraction of the minimal value in the cost functional, $(\mathrm{S})_0$ satisfies condition 1) of Theorem 1.2. From Proposition 3.2. we know that $(\mathrm{S})_\varepsilon$ is relaxable at $(x^*,u^*)$ with respect to $\widetilde{\mathrm{U}}$. Therefore, Theorems 2.3. and 2.5. can be applied (the latter due to compactness of K and continuity of $f_0$ in $v$). Thus the above described set of necessary optimality conditions will hold. □

**Theorem 3.5. ($\varepsilon$-maximum principle for $(\widetilde{\mathrm{S}})_0$)** *Consider the problem $(\widetilde{\mathrm{S}})_0$ together with its global minimizer $(x^*,\mu^*)$ under assumptions (V1) – (V4). After replacing the cost functional by $\overline{F}(x,\mu)-\overline{F}(x^*,\mu^*)$, the $\varepsilon$-extremal principle (Theorem 2.5.) can be applied: For every number $0<\varepsilon<1$ and every closed subspace $\widetilde{\mathrm{Y}} \subset L_p^{nm}(\Omega)$ with $\operatorname{Im} G_x(x^*,\mu^*) + \widetilde{\mathrm{Y}} = L_p^{nm}(\Omega)$ there exist multipliers $\lambda_0(\varepsilon,\widetilde{\mathrm{Y}})>0$ and $y^*(\varepsilon,\widetilde{\mathrm{Y}}) \in L_q^{nm}(\Omega)$, $p^{-1}+q^{-1}=1$, satisfying*

$$(\mathcal{M})_\varepsilon: \quad \varepsilon + \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega \int_{\mathrm{K}} f_0(t,x^*(t),v)\, \big[ d\mu_t(v) - d\mu_t^*(v) \big]\, dt + \sum_{i,j} \int_\Omega \int_{\mathrm{K}} v_{ij}\, \big[ d\mu_t(v) - d\mu_t^*(v) \big]\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) \geqslant 0 \qquad \forall\, \mu \in \mathfrak{Y}_{\mathrm{K}};$$

$$(\mathcal{K})_\varepsilon: \quad \Big| \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega \int_{\mathrm{K}} (f_0)_x(t,x^*(t),v)\, d\mu_t^*(v)^{\mathrm{T}} x(t)\, dt + \lambda_0(\varepsilon,\widetilde{\mathrm{Y}}) \int_\Omega (f_1)_x(t,x^*(t))^{\mathrm{T}} x(t)\, d\gamma(t) + \sum_{i,j} \int_\Omega x_{i;\,t_j}(t)\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) \Big| \leqslant \varepsilon \cdot \| x \|_{W_p^{1,n}(\Omega)} \qquad \forall\, x \in W_p^{1,n}(\Omega):\ x(t_0) = \mathfrak{o}_n;$$

$$(\mathcal{C})_\varepsilon: \quad \sum_{i,j} \int_\Omega z_{ij}(t)\, dy_{ij}^*(\varepsilon,\widetilde{\mathrm{Y}})(t) = 0 \quad \forall\, z \in \widetilde{\mathrm{Y}}.$$

**Proof.** After changing the cost functional, $(\widetilde{\mathrm{S}})_0$ satisfies condition 1) of Theorem 1.2. By Proposition 3.2., condition 2) of Theorem 1.2. holds too, and the relaxability condition for the cost functional holds even in sharpened form. In order to enlarge the problem $(\widetilde{\mathrm{S}})_0$ one has to take from the beginning $\mathrm{U} = \widetilde{\mathrm{U}} = \mathfrak{Y}_{\mathrm{K}}$ and $\inf\, (\widetilde{\mathrm{S}})_0 = \overline{F}(x^*,\mu^*)$ (thus, with respect to $(\widetilde{\mathrm{S}})_0$, the problems $(\mathrm{S})_\varepsilon$ and $(\widetilde{\mathrm{S}})_\varepsilon$ become identical). Now, all assumptions of Theorem 2.4. and, for the same reasons as before, also of Theorem 2.5. are satisfied. □

**Remarks.** 1) Under assumptions of Theorem 3.1., 2), the existence of a global minimizer of $(\mathrm{S})_0$ is guaranteed.

2) Theorems 3.4. and 3.5. generalize the $\varepsilon$-maximum principles known before, e. g. [ 8 : p. 225, Theorem 3.1. ] , [ 9 : p. 313, Theorem 3.1. ] and [ 10 : p. 171, Theorem 2.3. ] since they contain the additional condition $(\mathcal{C})_\varepsilon$ for the multiplier $y^*$ associated with the equality restriction.

**Corollary 3.6.** *Choosing in Theorems 3.4. and 3.5. the subspace* $\widetilde{\mathrm{Y}} = \big( \mathrm{cl}_{\,L_p^m(\Omega)} \big( \{ z \in C_\circ^{\infty,m}(\Omega) \mid \mathrm{div}\; z(t) = 0 \;\; (\forall)\, t \in \Omega \} \big) \big)^n$*, the condition* $(\mathcal{C})_\varepsilon$ *reads as follows:*

$$y^*(\varepsilon, \widetilde{\mathrm{Y}}) \in \mathrm{cl}\, \big( \{ z \in L_q^{nm}(\Omega) \mid \exists\, x \in W_q^{1,n}(\Omega) :\; x(t_0) = \mathfrak{o}_n \,,\; z_{ij} = x_{i;\,t_j} \;\; \forall\, i,\, j \} \big) \text{ with } p^{-1} + q^{-1} = 1 \,. \tag{29}$$

**Proof.** By Proposition 3.3., the above described subspace $\widetilde{\mathrm{Y}}$ satisfies the assumption in Theorems 3.4. resp. 3.5. Property (29) is an immediate consequence of [ 1 : p. 116, Lemma 2.1. ] . □

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**Last modification:** 27. 05. 2002